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Some fundamental properties of complex geometry

A. Lesfari

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Abstract This paper gives sufficient conditions, which guarantee that a complex n-dimensional manifold is analytically isomorphic to a n-dimensional complex torus and a Kähler manifold. We discuss the relation with Hodge theory and an immediate consequence is that a complex manifold will complete to abelian variety by adjoining some divisors. Several examples are given.

Keywords Complex manifolds · Abelian varieties · Kähler manifolds

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1 Introduction

Let $\mathcal{D} \subset M$ be a divisor on a compact complex manifold M. In other words, an element of the form

$$\mathcal{D} = \sum n_i \mathcal{D}_i, \quad n_i \in \mathbb{Z},$$

where $\mathcal{D}_i$ are irreducible subvarieties of M. In particular a divisor on a curve is a finite formal sum $\sum n_i p_i$ where p_i are points of the curve and n_i integers. For example, one can associate a divisor to a meromorphic function f by taking p_i zeros and poles of f and n_i the order of p_i with a negative sign for the poles. We denote this divisor (f) and we have

A. Lesfari (✉)
Department of Mathematics, Faculty of Sciences,
University of Chouaïb Doukkali, B.P. 20, El Jadida, Morocco
e-mail: lesfariahmed@yahoo.fr

$$(f) = (\text{divisor of the zeros of } f) - (\text{divisor of poles of } f).$$

We say that a divisor $\mathcal{D}$ is positive and we write $\mathcal{D} \geq 0$, if the integers n_i involved in the sum are positive. Define

$$\mathcal{L}(\mathcal{D}) = \{f \text{ meromorphic on } M : (f) + \mathcal{D} \geq 0\},$$

i.e., a function $f \in \mathcal{L}(\mathcal{D})$ has at worst a n_i-fold pole along $\mathcal{D}_i$. For example, if the divisor $\mathcal{D}$ is positive, then $\mathcal{L}(\mathcal{D})$ is the set of holomorphic functions outside of $\mathcal{D}$ and having at most poles along $\mathcal{D}$.

Consider a basis $(1, f_1, \ldots, f_N)$ of the space $\mathcal{L}(\mathcal{D})$ and the map

$$F : M \longrightarrow \mathbb{P}^N(\mathbb{C}),\ p \longmapsto [1 : f_1(p) : \cdots : f_N(p)],$$

considered projectively. If F defines a smooth embedding of M into $\mathbb{P}^N(\mathbb{C})$, then by Chow's theorem (Griffiths and Harris 1978) (which states that any analytic submanifold of a projective space is algebraic), it is equivalent to say that the variety M is algebraic, i.e.,

$$M = \bigcap_i \left\{z \in \mathbb{P}^N(\mathbb{C}) : P_i(z) = 0\right\},$$

where $P_i(z)$ are homogeneous polynomials. Moreover, a theorem of Kodaira (Griffiths and Harris 1978) states that if $\mathcal{D} \subset M$ is a positive divisor, then for $k \in \mathbb{N}$, the mapping F defined by the functions of the space $\mathcal{L}(k\mathcal{D})$ embeds M into $\mathbb{P}^N(\mathbb{C})$ where

$$N = \dim \mathcal{L}(k\mathcal{D}) - 1.$$

Moreover, there exists a positive divisor if and only if M has a closed positive (1, 1)-form such that the cohomology class $[\omega] \in H^2(M, \mathbb{Z})$.

Now consider a n-dimensional complex torus

$$T^n = \mathbb{C}^n / L_\Omega, \quad L_\Omega \simeq H_1\left(T^n, \mathbb{Z}\right),$$

is the lattice generated by the $2n$ columns $\lambda_1, \ldots, \lambda_{2n}$ of the $n \times 2n$ period matrix $\Omega = (\lambda_1, \ldots, \lambda_{2n})$. The torus T^n is a smooth compact complex manifold of dimension n. A question arises: when a complex torus T^n can be embedded into a projective space and thus regarded as projective variety? The torus T^n will be embedded into projective space $\mathbb{P}^N(\mathbb{C})$, if there exists on $\mathbb{P}^N(\mathbb{C})$ a closed positive (1, 1)-form with integer cohomology class. This condition amounts to the Riemann conditions: there is an entire matrix Q (intersection matrix) of order $2n$ antisymmetric such that

$$\Omega Q \Omega^{\mathsf{T}} = 0, \quad i\Omega Q \overline{\Omega}^{\mathsf{T}} > 0.$$

Under these conditions, one can choose a new basis for L_Ω on $\mathbb{Z}$ of $2n$ column vectors $\lambda_1, \ldots, \lambda_{2n}$ such that:

$$Q = \begin{pmatrix} 0 & \Delta_\delta \\ -\Delta_\delta & 0 \end{pmatrix}, \quad \Omega = (\Delta_\delta, Z),$$

where

$$\Delta_\delta = \begin{pmatrix} \delta_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \delta_n \end{pmatrix},$$

and $\delta_1, \ldots, \delta_n \in \mathbb{N}^*$, $\delta_j \mid \delta_{j+1}$, $1 \leq j \leq n-1$, are elementary divisors and Z is a matrix satisfying $Z^{\mathsf{T}} = Z$, $\mathrm{Im} Z > 0$. The $(1, 1)$-form ω can then be expressed as

$$\omega = \sum_{j=1}^{n} \delta_j dx_j \wedge dx_{n+j},$$

where $x_1, \ldots, x_{2n}$ are coordinates on the base $(\lambda_1, \ldots, \lambda_{2n})$ such that:

$$\int_{\lambda_j} dx_k = \delta_{jk}.$$

There is then a positive line bundle $\mathcal{L}$ such that its Chern class $c_1(\mathcal{L}) = [\omega]$; corresponding to the line bundle $\mathcal{L}$ there is a linear system of equivalent divisors $\mathcal{D}$ all having $c_1(\mathcal{D}) = [\omega]$.

The divisor $\mathcal{D}$ is called ample when a basis $(f_0, \ldots, f_N)$ of $\mathcal{L}(k\mathcal{D})$ embeds M smoothly into $\mathbb{P}^N(\mathbb{C})$ for some k, via the map F, then $k\mathcal{D}$ is called very ample. A complex algebraic torus T^n is called an abelian variety. It is known that every positive divisor $\mathcal{D}$ on an irreducible abelian variety is ample and thus some multiple of $\mathcal{D}$ embeds M into $\mathbb{P}^N(\mathbb{C})$. By a theorem of Lefschetz, any $k \geq 3$ will work. The integers δ_j which provide the so-called polarization of the abelian variety M are then related to the divisor as follows: $\dim \mathcal{L}(\mathcal{D}) = \delta_1 \ldots \delta_n$.

Recall that a Kähler metric (Kähler form) is a hermitian metric (i.e., a 2-form of type (1,1)) whose imaginary part is closed. A Kähler manifold is a complex manifold equipped with a Kähler metric. Compact Kähler manifolds form a remarkable class of complex analytic manifolds. We will consider the class of Kähler manifolds, focusing on projective varieties. One reason is that they contain a lot of complex submanifolds while Kähler manifolds do not have them in general. We can find non-Kähler compact complex manifolds (for example Hopf's manifolds and Calabi–Eckmann's manifolds) but it is very difficult to build or to decide whether or not complex manifold is Kähler. The complex analytic projective varieties are particular examples of compact Kähler manifolds. Kodaira's theorem can still be stated as follows: a compact complex manifold admits a smooth embedding in $\mathbb{P}^N(\mathbb{C})$ if and only if it admits a Kähler metric whose Kähler form is of integral class. Another interesting result for Kähler varieties was obtained by Moishezon (1967) and Hartshorne (1977): a compact Kähler manifold of dimension n is projective if and only if it admits n algebraically independent meromorphic functions.

The purpose of this work is the study of some fundamental properties of complex geometry. The paper gives sufficient conditions, which guarantee that a complex n-dimensional manifold is analytically isomorphic to a n-dimensional complex torus and a Kähler manifold. Also, we discuss the relation with Hodge theory and an immediate consequence is that a complex manifold will complete to abelian variety by adjoining some divisors. Several important examples are given.

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2 Some properties of complex varieties

The following proposition, which will be used later, is a consequence of the following purely differential geometric fact: a compact and connected n-dimensional variety on which there exist n vector fields which commute and are independent at every point is diffeomorphic to an n-dimensional real torus.

Proposition 1 *A compact (connected) complex n-dimensional variety M on which there exist n holomorphic commuting vector fields $X_1, \ldots, X_n$ which are independent at every point is diffeomorphic to a complex torus $\mathbb{C}^n/L$ where L is a lattice in $\mathbb{C}^n$.*

Proof With every vector field $X_1, \ldots, X_n$, we associate a flow or one-parameter group of diffeomorphisms

$$g^{t_1}, \ldots, g^{t_n} : M \longrightarrow M, \quad (t_1, \ldots, t_n) \in \mathbb{C}^n.$$

The latter commute i.e.,

$$g^{t_1} \circ \cdots \circ g^{t_n}(p) = g^{t_n} \circ \cdots \circ g^{t_1}(p), \quad p \in M,$$

since by hypothesis $X_1, \ldots, X_n$ commute. It is therefore natural to consider the application $g^t : M \longrightarrow M$,

$$g^t = g^{t_1} \circ \cdots \circ g^{t_n}, \quad t = (t_1, \ldots, t_n) \in \mathbb{C}^n.$$

Obviously

$$g^{t+s} = g^t \circ g^s, \quad \forall t, s \in \mathbb{C}^n.$$

By the same argument as in the Arnold–Liouville theorem (Arnold 1978), one defines a holomorphic local diffeomorphism for a fixed origin $p \in M$:

$$G : \mathbb{C}^n \longrightarrow M, \quad t \longmapsto G(t) = g^t p.$$

To be precise, the point p moves along the trajectory of the first flow for time t_1, along the second flow for time t_2, etc. Let U be a sufficiently small neighborhood of the point $0 \in \mathbb{C}^n$ and let V be a neighborhood of the point $p \in M$. The composition of two holomorphic maps being holomorphic, we deduce that the resriction of G to U:

$$U \longrightarrow V, \quad (t_1, \dots, t_n) \longmapsto g^{t_1} \circ \cdots \circ g^{t_n}(p),$$

is holomorphic. Moreover, as $X_1, \dots, X_n$ are independent at each point of M, then the matrix

$$\begin{pmatrix} \frac{\partial}{\partial t_1} g^{t_1} \circ \cdots \circ g^{t_n}(p) \\ \vdots \\ \frac{\partial}{\partial t_n} g^{t_1} \circ \cdots \circ g^{t_n}(p) \end{pmatrix},$$

is invertible and by the local inversion theorem the mapping G is a local diffeomorphism. Note that G is surjective, i.e., for $q \in M$, there are $t \in \mathbb{C}^n$ such that $G(t) = g^t p = q$ where $p \in M$. Indeed, it suffices to connect a point $q \in M$ with p by a curve, cover the curve by a finite number of the neighborhoods V and define t as the sum of shifts t_i corresponding to peices of the curve. Therefore, the mapping G is surjective. On the other hand G is not injective because otherwise we would have a bijection between M a compact and a non-compact $\mathbb{C}^n$, which is absurd. To remedy this problem, we will examine the set of pre-images of $p \in M$. The stationary group of the point p is the set

$$L = \{t \in \mathbb{C}^n : G(t) = g^t p = p\},$$

of points $t \in \mathbb{C}^n$ for which $G(t) = p$. It is nonempty, closed under addition, the inverse of t is $-t$ and thus a subgroup of $\mathbb{C}^n$. It does not depend on p and its points lie in $\mathbb{C}^n$ discretely. Indeed, if $G(s) = p$ and $G(t) = p$, then $G(s+t) = g^s g^t p = g^s p = p$ and $g^{-1} p = g^{-t} g^t p = p$. Therefore, L is a subgroup of C^n. If $q = g^r p$ and $t \in L$, then $g^t q = g^{t+r} p = g^r g^t p = g^r p = q$. Therefore, L is a lattice of $\mathbb{C}^n$ (i.e., a discrete subgroup of $\mathbb{C}^n$ which spans the real vector space $\mathbb{R}^{2n}$). By taking the quotient of $\mathbb{C}^n$ by L, we obtain an injective mapping

$$\mathbb{C}^n / L \longrightarrow M, \quad [t] \longmapsto g^t p,$$

and hence a diffeomorphism. Therefore, M is conformal to a complex torus $\mathbb{C}^n/L$ as claimed. Note finally that the lattice L can be written as $L = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_k$, $1 \leq k \leq n$, where $e_1, \dots, e_n$ are linearly independent vectors. The proof of the proposition is thus complete. $\square$

Recall that in dimension one, any complex torus is an abelian variety. In this case the embedding is realized in a projective space of dimension two and we obtain models $\mathbb{C}/L$ as a projective plane curves. It is easier in this case to work with Weierstrass elliptic functions $\wp$ and $\wp'$.

In what follows, we will focus on the case where the dimension of the variety is greater than 1. Note that to show that M is to be the affine part of an abelian variety (for example), a naive guess would be to take the natural compactification $\overline{M}$ of M by projectiving the equations. Indeed, this can never work for a general reason: an abelian variety $\widetilde{M}$ of dimension bigger or equal than two is never a complete intersection, that is it can never be described in some projective space $\mathbb{P}^m(\mathbb{C})$ by m-dim $\widetilde{M}$ global polynomial homogeneous equations. In other words, if M is to be the affine part of an abelian variety, $\overline{M}$ must have a singularity somewhere along the locus at infinity $I = \overline{M} \cap \{Z_0 = 0\}$. When extended to $\mathbb{P}^m(\mathbb{C})$, affine varieties must be singular at infinity, because abelian varieties are not simply-connected and therefore cannot be projective complete intersections.

So from this result, if M is to be the affine part of an abelian variety, $\overline{M}$ must have a singularity somewhere along the locus at infinity. The theory of resolution of singularities of Hironaka (1964a, b) through the delicate procedure "blow-up, blow-down" allows at least theoretically resolve these singularities. The following result gives sufficient conditions for a complex manifold to be compact, connected, has an embedding in a projective space and diffeomorphic to a complex torus. In particular, we show that this is a Kähler manifold. We will show in the following some results on varieties of Hodge (these are compact Kähler varieties whose cohomology class of the Kähler form is a real multiple of a whole class) and that of abelian varieties whose applications are immense and important (Adler and van Moerbeke 1989; Adler et al. 2004; Lesfari 1988, 2007, 2008, 2009). In practice and in higher dimensions these problems are compounded considerably.

The idea of the proof we shall give here is closely related to the geometric spirit of the (real) Arnold–Liouville theorem (Adler et al. 2004; Arnold 1978).

Theorem 2 *Let* $Z = (Z_0, Z_1, \ldots, Z_n) \in \mathbb{P}^n(\mathbb{C})$ *and declare* $Z_0 \neq 0$ *to be affine part. Let*

$$M = \overline{M} \cap \{Z_0 \neq 0\},$$

be a smooth and irreducible variety and $\overline{M}$ *its closure in* $\mathbb{P}^n(\mathbb{C})$ *defined by*

$$\overline{M} = \bigcap_i \{Z \in \mathbb{P}^n(\mathbb{C}) : P_i(Z) = 0\},$$

involving a large number of homogeneous polynomials P_i. *Put* $\overline{M} \equiv M \cup \mathcal{D}$, *i.e.,* $\mathcal{D} = \overline{M} \cap \{Z_0 = 0\}$ *and consider the map*

$$f : \overline{M} \longrightarrow \mathbb{P}^N(\mathbb{C}), \quad Z \longmapsto f(Z).$$

Let

$$\mathcal{D} = \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_r,$$

where $\mathcal{D}_i$ are some codimension-one subvarieties and

$$\mathcal{S} \equiv f(\mathcal{D}) = f(\mathcal{D}_1) \cup \cdots \cup f(\mathcal{D}_r) \equiv \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_r.$$

Assume that:

(i) *f maps M smoothly and 1-1 onto $f(M)$.*
(ii) *There exist n holomorphic vector fields $X_1, \ldots, X_n$ on M which commute and are independent at every point. One vector field, say X_k, $1 \leq k \leq n$, extends holomorphically to a neighborhood of $\mathcal{S}_k$ in $\mathbb{P}^N(\mathbb{C})$.*
(iii) *For all $p \in \mathcal{S}_k$, the integral curve $f(t) \in \mathbb{P}^N(\mathbb{C})$ of the vector field X_k through $f(0) = p \in \mathcal{S}_k$ has the property that*

$$\{f(t) : 0 <| t |< \varepsilon, t \in \mathbb{C}\} \subset f(M).$$

Then the variety $\widetilde{M} = f(\overline{M}) = \overline{f(M)}$ is compact, connected and embeds smoothly into $\mathbb{P}^N(\mathbb{C})$ via f.

Proof Condition (iii) means that the orbits of X_k through $\mathcal{S}_k$ go immediately into the affine part and in particular, the vector field X_k does not vanish on any point of $\mathcal{S}_k$. A crucial step is to show that the orbits running through $\mathcal{S}_k$ form a smooth variety Σ_p, $p \in \mathcal{S}_k$ such that

$$\Sigma_p \backslash \mathcal{S}_k \subseteq M.$$

Let $p \in \mathcal{S}_k$, $\varepsilon > 0$ small enough, $g^t_{X_k}$ the flow generated by X_k on M and

$$\{g^t_{X_k} : t \in \mathbb{C}, 0 <| t |< \varepsilon\},$$

the orbit going through the point p. The vector field X_k is holomorphic in the neighborhood of any point $p \in \mathcal{S}_k$ and non-vanishing, by (ii) and (iii). Then the flow $g^t_{X_k}$ can be straightened out after a holomorphic change of coordinates. Let $\mathcal{H} \subset \mathbb{P}^N(\mathbb{C})$ be a hyperplane transversal to the direction of the flow at p and let Σ_p be the surface element formed by the divisor $\mathcal{S}_k$ and the orbits going through p. Consider the segment of $\mathcal{S}' \equiv \mathcal{H} \cap \Sigma_p$ and so locally, we have $\Sigma_p = \mathcal{S}' \times \mathbb{C}$. We shall show that Σ_p is smooth. Note that $\mathcal{S}'$ is smooth. Indeed, suppose that $\mathcal{S}'$ is singular at 0, then Σ_p would be singular along the trajectory (t-axis) which goes immediately into the affine $f(M)$, by condition (iii). Hence, the affine part would be singular which is impossible by condition (i). So, S' is smooth and by the implicit function theorem, Σ_p is smooth too. Consider now the map

$$\overline{M} \subset \mathbb{P}^n(\mathbb{C}) \longrightarrow \mathbb{P}^N(\mathbb{C}), \quad Z \longmapsto f(Z),$$

where $Z = (Z_0, Z_1, \ldots, Z_n) \in \mathbb{P}^n(\mathbb{C})$ and

$$\widetilde{M} = f(\overline{M}) = \overline{f(M)}.$$

Recall that the flow exists in a full neighborhood of p in $\mathbb{P}^N(\mathbb{C})$ and it has been straightened out. Therefore, near $p \in \mathcal{S}_k$, we have $\Sigma_p = \widetilde{M}$ and $\Sigma_p \backslash \mathcal{S}_k \subseteq M$. Otherwise, there would exist an element $\Sigma'_p \subset \widetilde{M}$ such that

$$\{g^t_{X_k} : t \in \mathbb{C}, 0 <\mid t \mid< \varepsilon\} = (\Sigma_p \cap \Sigma'_p)\backslash p \subset M,$$

by condition (iii). In other words, $\Sigma_p \cap \Sigma'_p$=t-axis and hence M would be singular along the t-axis which is impossible. Since the variety M is irreducible and since the generic hyperplane section $\mathcal{H}_{gen.}$ of $\widetilde{M}$ is also irreducible, all hyperplane sections are connected and hence $\mathcal{D}$ is also connected. Now consider the graph $G_f \subset \mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^N(\mathbb{C})$ of the map f, which is irreducible together with $\widetilde{M}$. It follows from the irreducibility of G_f that a generic hyperplane section $G_f \cap (\mathcal{H}_{gen.} \times \mathbb{P}^N(\mathbb{C}))$ is irreducible, hence the special hyperplane section $G_f \cap (\{Z_0 = 0\} \times \mathbb{P}^N(\mathbb{C}))$ is connected and therefore the projection map

$$Proj_{\mathbb{P}^N(\mathbb{C})}[G_f \cap (\{Z_0 = 0\} \times \mathbb{P}^N(\mathbb{C}))] = f(\mathcal{D}) \equiv \mathcal{S},$$

is connected. Hence, the variety

$$\widetilde{M} = M \cup \bigcup_{p \in \mathcal{S}_k} \Sigma_p = M \cup \mathcal{S}_k \subseteq \mathbb{P}^N(\mathbb{C}),$$

is compact, connected and admits an embedding into $\mathbb{P}^N(\mathbb{C})$. □

Corollary 3 *Under the same assumptions as the previous theorem, $\widetilde{M}$ is diffeomorphic to a n-dimensional complex torus. The vector fields $X_1, \ldots, X_n$ extend holomorphically and remain independent on $\widetilde{M}$.*

Proof Let g^{t_i} be the flow generated by X_i on M and let $p_1 \in \widetilde{M}\backslash M$. For small $\varepsilon > 0$ and for all $t_1 \in \mathbb{C}$ such that $0 < |t_1| < \varepsilon$, note that $q \equiv g^{t_1}(p_1)$ is well defined and $g^{t_1}(p_1) \in f(M)$, using condition (iii) (Theorem 2). Let $U(q) \subseteq M$ be a neighborhood of q and let

$$g^{t_2}(p_2) = g^{-t_1} \circ g^{t_2} \circ g^{t_1}(p_2), \quad \forall p_2 \in U(p_1) \equiv g^{-t_1}\left(U(q)\right),$$

which is well defined since by commutativity one can see that the right hand side is independent of t_1:

$$\begin{aligned} g^{-(t_1+\varepsilon)} \circ g^{t_2} \circ g^{t_1+\varepsilon}(p_2) &= g^{-(t_1+\varepsilon)} \circ g^{t_2} \circ g^{t_1} \circ g^{\varepsilon}(p_2), \\ &= g^{-(t_1+\varepsilon)} \circ g^{\varepsilon} \circ g^{t_2} \circ g^{t_1}(p_2), \\ &= g^{-t_1} \circ g^{t_2} \circ g^{t_1}(p_2). \end{aligned}$$

Note that $g^{t_2}(p_2)$ is a holomorphic function of p_2 and t_2, because in $U(p_1)$ the function g^{t_1} is holomorphic and its image is away from $\mathcal{S}$, i.e., in the affine, g^{t_2} is holomorphic. The same argument applies to $g^{t_3}(p_3), \ldots, g^{t_n}(p_n)$ where

$$g^{t_n}(p_n) = g^{-t_{n-1}} \circ g^{t_n} \circ g^{t_{n-1}}(p_n), \quad \forall p_n \in U(p_{n-1}) \equiv g^{-t_{n-1}}(U(q)).$$

Thus $X_1, \ldots, X_n$ have been holomorphically extended, remain independent and commuting on $\widetilde{M}$. Therefore, we can show along the same lines as in Proposition 1, that $\widetilde{M}$ is a complex torus $\mathbb{C}^n/lattice$. And that will be done, by considering the local diffeomorphism

$$\mathbb{C}^n \longrightarrow \widetilde{M}, \quad t = (t_1, \ldots, t_n) \longmapsto g^t p = g^{t_1} \circ \cdots \circ g^{t_n}(p),$$

for a fixed origin $p \in f(M)$. The additive subgroup

$$L = \{t \in \mathbb{C}^n : g^t p = p\},$$

is a lattice of $\mathbb{C}^n$ (spanned by $2n$ vectors in $\mathbb{C}^n$, independent over $\mathbb{R}$), hence $\mathbb{C}^n/L \longrightarrow \widetilde{M}$ is a biholomorphic diffeomorphism. □

Corollary 4 *Under the same assumptions as the previous theorem, $\widetilde{M}$ is a Kähler variety.*

Proof Let

$$ds^2 = \sum_{k=1}^{n} dt_k \otimes d\bar{t}_k,$$

be a hermitian metric on the complex variety $\widetilde{M}$ and let ω its fundamental $(1, 1)$-form. We have

$$\omega = -\frac{1}{2} \operatorname{Im} ds^2 = \frac{\sqrt{-1}}{2} \sum_{k=1}^{n} dt_k \wedge d\bar{t}_k.$$

So we see that ω is closed and the metric ds^2 is Kähler and consequently $\widetilde{M}$ is a Kähler variety. □

Corollary 5 *Under the same assumptions as the previous theorem, $\widetilde{M}$ is a Hodge variety. In particular, M is the affine part of an abelian variety $\widetilde{M}$.*

Proof On the Kähler variety $\widetilde{M}$ are defined periods of ω. If these periods are integers (possibly after multiplication by a number), we obtain a variety of Hodge. More specifically, integrals $\int_{\gamma_k} \omega$ of the form ω (where γ_k are cycles in $H_2(\widetilde{M}, \mathbb{Z})$) determine the periods ω. As they are integers, then $\widetilde{M}$ is a Hodge variety. The variety $\widetilde{M}$ is equipped with n holomorphic vector fields, independent and commuting. From Theorem 2 and Corollary 3, the variety $\widetilde{M}$ is both a projective variety and a complex torus and hence an abelian variety as a consequence of Chow theorem (Griffiths and Harris 1978). Another proof is to use the result that we just show since every Hodge torus is abelian, the converse is also true. Note also that by Moishezon's theorem (Moishezon 1967; Hartshorne 1977), a compact complex Kähler variety having as many independent meromorphic functions as its dimension is an abelian variety. □

A complex torus being a Kähler manifold, we deduce from Moishezon's theorem (Moishezon 1967; Hartshorne 1977) the following result:

Corollary 6 *A complex torus of dimension n is an abelian variety if and only if it admits n independent meromorphic functions.*

3 Examples

Example 7 The three quartic,

$$F_1 = \frac{1}{2}z_5 - z_1 z_2^2 + \frac{1}{2}z_3^2 - \frac{1}{4}z_1^2 - 2z_2^4,$$
$$F_2 = z_5^2 - z_1^2 z_5 + 4z_1 z_2 z_3 z_4 - z_1^2 z_3^2 + \frac{1}{4}z_1^4 - 4z_2^2 z_4^2,$$
$$F_3 = z_1 z_5 + z_1^2 z_2^2 - z_4^2,$$

are invariants of the following system of five differential equations in the unknowns $z_1, \dots, z_5 \in \mathbb{C}^5$,

$$\dot{z}_1 = 2z_4,$$
$$\dot{z}_2 = z_3,$$
$$\dot{z}_3 = z_2(3z_1 + 8z_2^2),$$
$$\dot{z}_4 = z_1^2 + 4z_1 z_2^2 + z_5,$$
$$\dot{z}_5 = 2z_1 z_4 + 4z_2^2 z_4 - 2z_1 z_2 z_3.$$

Let M be the complex affine variety defined by

$$M = \bigcap_{k=1}^{3} \{z = (z_1, \dots, z_5) \in \mathbb{C}^5 : F_k(z) = c_k\},$$

where $c_1, c_2, c_3 \in \mathbb{C}$. The main problem will be to complete M into a non singular compact complex algebraic variety $\widetilde{M} = M \cup \mathcal{D}$ in such a way that the vector fields generated respectively by F_1 and F_2, extend holomorphically along a divisor $\mathcal{D}$ and remain independent there. This is possible (for details see Lesfari 2007), $\widetilde{M}$ is an algebraic complex torus (an abelian variety). More precisely, the variety M generically is the affine part of an abelian surface $\widetilde{M}$. The reduced divisor at infinity $\widetilde{M} \backslash M = \mathcal{C}_1 + \mathcal{C}_{-1}$, consists of two copies $\mathcal{C}_1$ and $\mathcal{C}_{-1}$ of the same genus 7 Riemann surface.

Example 8 Let B be the affine variety defined by

$$B = \bigcap_{k=1}^{2} \{z = (q_1, q_2, p_1, p_2) \in \mathbb{C}^4 : H_k(z) = c_k\},$$

where $c_1, c_2 \in \mathbb{C}^2$ and

$$H_1 = \frac{1}{2}p_1^2 - \frac{3}{2}q_1^2 q_2^2 + \frac{1}{2}p_2^2 - \frac{1}{4}q_1^4 - 2q_2^4,$$
$$H_2 = p_1^4 - 6q_1^2 q_2^2 p_1^2 + q_1^4 q_2^4 - q_1^4 p_1^2 + q_1^6 q_2^2 + 4q_1^3 q_2 p_1 p_2 - q_1^4 p_2^2 + \frac{1}{4}q_1^8,$$

are invariants of the following system

$$\ddot{q}_1 = q_1\left(q_1^2 + 3q_2^2\right),$$
$$\ddot{q}_2 = q_2\left(3q_1^2 + 8q_2^2\right).$$

We show that the invariant surface B can be completed as a cyclic double cover $\overline{B}$ of the abelian surface $\widetilde{M}$ (Example 1), ramified along the divisor $\mathcal{C}_1 + \mathcal{C}_{-1}$. Moreover, $\overline{B}$ is smooth except at the point lying over the singularity (of type A_3) of $\mathcal{C}_1 + \mathcal{C}_{-1}$ and the resolution $\widetilde{B}$ of $\overline{B}$ is a surface of general type with Euler characteristic $\mathcal{X}(\widetilde{B}) = 1$ and geometric genus $p_g(\widetilde{B}) = 2$ (for details see Lesfari 2007).

Example 9 Another system similar to that of Example 1 is defined by

$$\begin{aligned}
F_1 &= \frac{1}{2}z_5 + 2z_1z_2^2 + \frac{1}{2}z_3^2 + \frac{1}{2}az_1 + 2az_2^2 + \frac{1}{4}z_1^2 + 4z_2^4,\\
F_2 &= az_1z_2 + z_1^2z_2 + 4z_1z_2^3 - z_2z_5 + z_3z_4,\\
F_3 &= z_1z_5 - 2z_1^2z_2^2 - z_4^2.
\end{aligned}$$

These three quartic are invariants of the following system of differential equations in the unknowns $z_1, \ldots, z_5 \in \mathbb{C}^5$,

$$\begin{aligned}
\dot{z}_1 &= 2z_4,\\
\dot{z}_2 &= z_3,\\
\dot{z}_3 &= -4az_2 - 6z_1z_2 - 16z_2^3,\\
\dot{z}_4 &= -az_1 - z_1^2 - 8z_1z_2^2 + z_5,\\
\dot{z}_5 &= -8z_2^2z_4 - 2az_4 - 2z_1z_4 + 4z_1z_2z_3,
\end{aligned}$$

where a is a constant. Let M be the complex affine variety defined by

$$M = \bigcap_{k=1}^{3}\{z = (z_1, \ldots, z_5) \in \mathbb{C}^5 : F_k(z) = c_k\},$$

where $c_1, c_2, c_3 \in \mathbb{C}$. This complex affine variety M defined by putting these invariants equal to generic constants, is a double cover of a Kummer surface defined by

$$p\,(z_1, z_2)\,z_5^2 + q\,(z_1, z_2)\,z_5 + r\,(z_1, z_2) = 0,$$

where

$$\begin{aligned}
p\,(z_1, z_2) &= z_2^2 + z_1,\\
q\,(z_1, z_2) &= \frac{1}{2}z_1^3 + 2az_1z_2^2 + az_1^2 - 2c_1z_1 + 2c_2z_2 - c_3,
\end{aligned}$$

$$r\left(z_1,z_2\right)=-8c_3z_2^4+\left(a^2+4c_1\right)z_1^2z_2^2-8c_2z_1z_2^3-2c_2z_1^2z_2-4c_3z_1z_2^2$$
$$-\frac{1}{2}c_3z_1^2-4ac_3z_2^2-2ac_2z_1z_2-ac_3z_1+c_2^2+2c_1c_3.$$

The variety M generically is the affine part of an abelian surface $\widetilde{M}$, more precisely the jacobian of a genus 2 curve. The reduced divisor at infinity

$$\widetilde{M}\backslash M=\mathcal{H}_1+\mathcal{H}_{-1},$$

consists of two smooth isomorphic genus 2 curves $\mathcal{H}_{\pm 1}$ (for details see Lesfari 2008).

Example 10 Let M be the variety defined by

$$M=\bigcap_{k=1}^{2}\left\{z=(q_1,q_2,p_1,p_2)\in\mathbb{C}^4,\ H_i(z)=c_i\right\},$$

where

$$H_1=\frac{1}{2}\left(p_1^2+p_2^2+Aq_1^2+Bq_2^2\right)+q_1^2q_2+6q_2^3,$$
$$H_2=q_1^4+4q_1^2q_2^2-4p_1\left(p_1q_2-p_2q_1\right)+4Aq_1^2q_2+(4A-B)\left(p_1^2+Aq_1^2\right),$$

are invaraints of the Hénon–Heiles system

$$\dot{q}_1=p_1,$$
$$\dot{q}_2=p_2,$$
$$\dot{p}_1=-Aq_1-2q_1q_2,$$
$$\dot{p}_2=-Bq_2-q_1^2-6q_2^2,$$

A and B, are constant parameters. The affine surface M completes into an abelian surface $\widetilde{M}$, by adjoining a curve $\mathcal{D}$. The latter determined by an eight-order equation is smooth, hyperelliptic and its genus is 3. More precisely, $\widetilde{M}=\mathbb{C}^2/Lattice\subseteq\mathbb{P}^7(\mathbb{C})$, where the lattice is generated by the period matrix $\begin{pmatrix}2&0&a&c\\0&4&c&b\end{pmatrix}$, $\operatorname{Im}\begin{pmatrix}a&c\\c&b\end{pmatrix}>0$, $(a,b,c\in\mathbb{C})$ (for details see Lesfari 2009).

Example 11 In $\mathbb{C}^6$, let M be the affine variety defined by

$$M=\bigcap_{k=1}^{4}\left\{z=(m_1,m_2,m_3,\gamma_1,\gamma_2,\gamma_3)\in\mathbb{C}^6: H_k(z)=c_k\right\},$$

where

$$
\begin{aligned}
H_1 &= \frac{1}{2}\left(m_1^2 + m_2^2\right) + m_3^2 + 2\gamma_1 = c_1,\\
H_2 &= m_1\gamma_1 + m_2\gamma_2 + m_3\gamma_3 = c_2,\\
H_3 &= \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = c_3 = 1,\\
H_4 &= \frac{1}{16}\left(m_2^2 + m_1^2\right)^2 - \frac{1}{2}\left(m_1^2 - m_2^2\right)\gamma_1 + \gamma_1^2 + \gamma_2^2 - m_1 m_2 \gamma_2 = c_4.
\end{aligned}
$$

are invariants for the Kowalewski's top and $c_k \in \mathbb{C}$, $1 \le k \le 4$. The invariant variety M can be completed via the flow into complex algebraic tori $\mathbb{C}^2/Lattice$ were the lattice is spanned by the columns of the period matrix $\begin{pmatrix} 1 & 0 & a & c \\ 0 & 2 & c & b \end{pmatrix}$, $\operatorname{Im}\begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0$. Here, the divisor $\mathcal{D}$ is a set of two isomorphic curves of genus 3, $\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_{-1}$. Each of the curve $\mathcal{D}_{\pm 1}$ is a 2-1 ramified cover of elliptic curves $\mathcal{D}^0_{\pm 1}$, ramified at four points. Each divisor $\mathcal{D}_{\pm 1}$ is ample and defines a polarization $(1, 2)$, whereas the divisor $\mathcal{D}$ of geometric genus 9 is very ample and defines a polarization $(2, 4)$. More precisely, the affine surface M defined by putting the four invariants of the Kowalewski flow equal to generic constants, is the affine part of an abelian surface $\widetilde{M}$ with

$\widetilde{M}\backslash M = \mathcal{D} =$ one genus 9 curve consisting of two genus 3 curves $\mathcal{D}_{\pm 1}$ intersecting in four points. Each $\mathcal{D}_{\pm 1}$ is a double cover of an elliptic curve $\mathcal{D}^0_{\pm 1}$ ramified at four points.

Moreover, $\widetilde{M} \simeq \mathbb{C}^2/Lattice$ admits an embedding in $\mathbb{P}^7(\mathbb{C})$ [for details see Lesfari (1988)].

Example 12 Let $\alpha_k, \beta_k, \gamma_k \in \mathbb{C}$, $1 \le k \le 3$, be given such that the α_k are distinct, non-zero and

$$
\det\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix} \neq 0.
$$

Let

$$
\begin{aligned}
\lambda_1 &= \frac{\beta_2 - \beta_3}{\alpha_2 - \alpha_3}, \lambda_2 = \frac{\beta_1 - \beta_3}{\alpha_1 - \alpha_3}, \lambda_3 = \frac{\beta_1 - \beta_2}{\alpha_1 - \alpha_2}, \lambda_4 = \frac{\beta_1}{\alpha_1}, \lambda_5 = \frac{\beta_2}{\alpha_2}, \lambda_6 = \frac{\beta_3}{\alpha_3},\\
\mu_1 &= \frac{\gamma_2 - \gamma_3}{\alpha_2 - \alpha_3}, \mu_2 = \frac{\gamma_1 - \gamma_3}{\alpha_1 - \alpha_3}, \mu_3 = \frac{\gamma_1 - \gamma_2}{\alpha_1 - \alpha_2}, \mu_4 = \frac{\gamma_1}{\alpha_1}, \mu_5 = \frac{\gamma_2}{\alpha_2}, \mu_6 = \frac{\gamma_3}{\alpha_3}.
\end{aligned}
$$

In $\mathbb{C}^6$, let M be the affine variety defined by

$$
M = \bigcap_{k=1}^{4} \left\{ z = (x_1, \ldots, x_6) \in \mathbb{C}^6 : Q_k(x) = c_k \right\},
$$

where

$$
\begin{aligned}
Q_1 &= z_1^2 + z_2^2 + \cdots + z_6^2,\\
Q_2 &= \lambda_1 z_1^2 + \lambda_2 z_2^2 + \cdots + \lambda_6 z_6^2,\\
Q_3 &= \mu_1 z_1^2 + \mu_2 z_2^2 + \cdots + \mu_6 z_6^2,\\
Q_4 &= z_1 z_4 + z_2 z_5 + z_3 z_6,
\end{aligned}
$$

are invariants of the geodesic flow on $SO(4)$ for a left invariant metric and $c_k \in \mathbb{C}$, $1 \le k \le 4$. Then for c_k's in a Zariski-open subset of $\mathbb{C}^4$, M is an affine open piece of an abelian surface $\widetilde{M}$. More precisely, $M = \widetilde{M} \backslash \mathcal{D}$, where $\mathcal{D}$ is a curve of genus 9, or $\widetilde{M} = \mathbb{C}^2/lattice \subseteq \mathbb{P}^7(\mathbb{C})$, having period matrix $\begin{pmatrix} 2 & 0 & a & c \\ 0 & 4 & c & b \end{pmatrix}$, $\operatorname{Im}\begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0$, $(a, b, c \in \mathbb{C})$ (for details see Adler et al. 2004).

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