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ORIGINAL PAPER

Characterizations of balls by sections and caps

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Abstract Among others, we prove that if a convex body $\mathcal{K}$ and a ball $\mathcal{B}$ have equal constant volumes of caps and equal constant areas of sections with respect to the supporting planes of a sphere, then $\mathcal{K} \equiv \mathcal{B}$.

Keywords Sections · Caps · Ball · Sphere · Characterization · Isoperimetric inequality · Floating body

Mathematics Subject Classification (2012) 52A40

1 Introduction

If the convex body $\mathcal{M}$, the *kernel*, contains the origin O, let $\hbar_{\mathcal{M}}(\boldsymbol{u})$ denote the supporting hyperplane of $\mathcal{M}$ that is perpendicular to the unit vector $\boldsymbol{u} \in \mathbb{S}^{n-1}$ and contains in its same half space $\hbar^{-}_{\mathcal{M}}(\boldsymbol{u})$ the origin O and the kernel $\mathcal{M}$. Its other half space is denoted by $\hbar^{+}_{\mathcal{M}}(\boldsymbol{u})$.

If the convex body $\mathcal{K}$ contains the kernel $\mathcal{M}$ in its interior, we define the functions

$$\mathrm{S}_{\mathcal{M};\mathcal{K}}(\boldsymbol{u}) = |\mathcal{K} \cap \hbar_{\mathcal{M}}(\boldsymbol{u})|, \quad (section\ function) \tag{1.1}$$

$$\mathrm{C}_{\mathcal{M};\mathcal{K}}(\boldsymbol{u}) = |\mathcal{K} \cap \hbar^{+}_{\mathcal{M}}(\boldsymbol{u})|, \quad (cap\ function) \tag{1.2}$$

where $|\cdot|$ is the appropriate Lebesgue measure.

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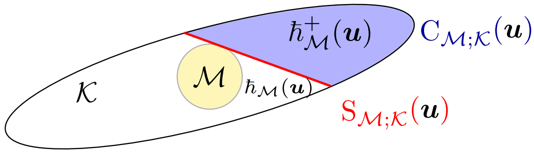

The goal of this article is to investigate the problem of determining $\mathcal{K}$ if some functions of the form (1.1) and (1.2) are given for a kernel $\mathcal{M}$.

Two convex bodies $\mathcal{K}$ and $\mathcal{K}'$ are called $\mathcal{M}$-*equicapped* if $\mathrm{C}_{\mathcal{M};\mathcal{K}} \equiv \mathrm{C}_{\mathcal{M};\mathcal{K}'}$, and they are $\mathcal{M}$-*equisectioned* if $\mathrm{S}_{\mathcal{M};\mathcal{K}} \equiv \mathrm{S}_{\mathcal{M};\mathcal{K}'}$. A convex body $\mathcal{K}$ is called $\mathcal{M}$-*isocapped* if $\mathrm{C}_{\mathcal{M};\mathcal{K}}$ is constant. It is said to be $\mathcal{M}$-*isosectioned* if $\mathrm{S}_{\mathcal{M};\mathcal{K}}$ is constant.

First we prove in the plane that

(a) two convex bodies coincide if they are $\mathcal{M}$-equicapped *and* $\mathcal{M}$-equisectioned, no matter what $\mathcal{M}$ is (Theorem 3.1), and
(b) any disc-isocapped convex body is a disc concentric to the kernel (Theorem 3.2).[1]

Then, in higher dimensions we consider only such convex bodies that are sphere-equisectioned and sphere-equicapped with a ball, and prove that

(1) a convex body that is sphere-equicapped and sphere-equisectioned with a ball, is itself a ball (Theorem 5.3);
(2) a convex body that is twice sphere-equicapped (for two different concentric spheres) with a ball is itself a ball (Theorem 5.1);
(3) a convex body that is twice sphere-equisectioned (for two different concentric spheres) with a ball is itself a ball (Theorem 5.2, but dimension $n = 3$ excluded).

For more information about the subject we refer the reader to [1,3] etc.

2 Preliminaries

We work with the n-dimensional real space $\mathbb{R}^n$, its unit ball is $\mathcal{B} = \mathcal{B}^n$ (in the plane the unit disc is $\mathcal{D}$), its unit sphere is $\mathbb{S}^{n-1}$ and the set of its hyperplanes is $\mathbb{H}$. The ball (resp. disc) of radius $\varrho > 0$ centred to the origin is denoted by $\varrho\mathcal{B} = \varrho\mathcal{B}^n$ (resp. $\varrho\mathcal{D}$).

Using the spherical coordinates $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_{n-1})$ every unit vector can be written in the form $\boldsymbol{u}_{\boldsymbol{\xi}} = (\cos\xi_1, \sin\xi_1\cos\xi_2, \sin\xi_1\sin\xi_2\cos\xi_3, \ldots)$, the ith coordinate of which is $u^i_{\boldsymbol{\xi}} = (\prod_{j=1}^{i-1}\sin\xi_j)\cos\xi_i$ ($\xi_n := 0$). In the plane we even use the $\boldsymbol{u}_\xi = (\cos\xi, \sin\xi)$ and $\boldsymbol{u}^\perp_\xi = \boldsymbol{u}_{\xi+\pi/2} = (-\sin\xi, \cos\xi)$ notations and in analogy to this latter one, we introduce the notation $\boldsymbol{\xi}^\perp = (\xi_1, \ldots, \xi_{n-2}, \xi_{n-1} + \pi/2)$ for higher dimensions.

A hyperplane $\hbar \in \mathbb{H}$ is parametrized so that $\hbar(\boldsymbol{u}_{\boldsymbol{\xi}}, r)$ means the one that is orthogonal to the unit vector $\boldsymbol{u}_{\boldsymbol{\xi}} \in \mathbb{S}^{n-1}$ and contains the point $r\boldsymbol{u}_{\boldsymbol{\xi}}$, where $r \in \mathbb{R}$.[2] For convenience we also frequently use $\hbar(P, \boldsymbol{u}_{\boldsymbol{\xi}})$ to denote the hyperplane through the point $P \in \mathbb{R}^n$ with normal vector $\boldsymbol{u}_{\boldsymbol{\xi}} \in \mathbb{S}^{n-1}$. For instance, $\hbar(P, \boldsymbol{u}_{\boldsymbol{\xi}}) = \hbar(\boldsymbol{u}_{\boldsymbol{\xi}}, \langle\overrightarrow{OP}, \boldsymbol{u}_{\boldsymbol{\xi}}\rangle)$, where $O = \mathbf{0}$ is the origin and $\langle ., . \rangle$ is the usual inner product.

[1] [1, Theorem 1] gives the same conclusion in the plane for disc-isosectioned convex bodies

[2] Although $\hbar(\boldsymbol{u}_{\boldsymbol{\xi}}, r) = \hbar(-\boldsymbol{u}_{\boldsymbol{\xi}}, -r)$ this parametrization is locally bijective.

On a convex body we mean a convex compact set $\mathcal{K} \subseteq \mathbb{R}^n$ with non-empty interior $\mathcal{K}^\circ$ and with piecewise C^1 boundary $\partial\mathcal{K}$. For a convex body $\mathcal{K}$ we let $p_{\mathcal{K}} \colon \mathbb{S}^{n-1} \to \mathbb{R}$ denote support function of $\mathcal{K}$, which is defined by $p_{\mathcal{K}}(\boldsymbol{u}_\xi) = \sup_{\boldsymbol{x}\in\mathcal{K}}\langle \boldsymbol{u}_\xi, x\rangle$. We also use the notation $\hbar_{\mathcal{K}}(\boldsymbol{u}) = \hbar(\boldsymbol{u}, p_{\mathcal{K}}(\boldsymbol{u}))$. If the origin is in $\mathcal{K}^\circ$, another useful function of a convex body $\mathcal{K}$ is its *radial function* $\varrho_{\mathcal{K}} \colon \mathbb{S}^{n-1} \to \mathbb{R}_+$ which is defined by $\varrho_{\mathcal{K}}(\boldsymbol{u}) = |\{r\boldsymbol{u} : r > 0\} \cap \partial K|$.

We need the special functions $I_x(a, b)$, the regularized incomplete beta function, $B(x; a, b)$, the incomplete beta function, $B(a, b)$, the beta function, and $\Gamma(y)$, Euler's Gamma function, where $0 < a, b \in \mathbb{R}$, $x \in [0, 1]$ and $y \in \mathbb{R}$. We introduce finally the notation $|\mathbb{S}^k| := 2\pi^{k/2}/\Gamma(k/2)$ as the standard surface measure of the k-dimensional sphere. For the special functions we refer the reader to [11,12].

We shall frequently use the utility function χ that takes relations as argument and gives 1 if its argument fulfilled. For example $\chi(1 > 0) = 1$, but $\chi(1 \le 0) = 0$ and $\chi(x > y)$ is 1 if $x > y$ and it is zero if $x \le y$. Nevertheless we still use χ also as the indicator function of the set given in its subscript.

A strictly positive integrable function $\omega \colon \mathbb{R}^n \setminus \mathcal{B} \to \mathbb{R}_+$ is called *weight* and the integral

$$V_\omega(f) := \int\limits_{\mathbb{R}^n\setminus\mathcal{B}} f(x)\omega(x)dx$$

of an integrable function $f \colon \mathbb{R}^n \to \mathbb{R}$ is called the *volume of f with respect to the weight ω* or simply the *ω-volume of f*. For the volume of the indicator function $\chi_{\mathcal{S}}$ of a set $\mathcal{S} \subseteq \mathbb{R}^n$ we use the notation $V_\omega(\mathcal{S}) := V_\omega(\chi_{\mathcal{S}})$ as a shorthand. If more weights are indexed by $i \in \mathbb{N}$, then we use the even shorter notation $V_i(\mathcal{S}) := V_{\omega_i}(\mathcal{S}) = V_i(\chi_{\mathcal{S}}) := V_{\omega_i}(\chi_{\mathcal{S}})$.

3 In the plane

We heard the following easy result from Kincses [5].

Theorem 3.1 *Assume that the border of the strictly convex plane bodies $\mathcal{M}$ and $\mathcal{K}$ are differentiable of class C^1 and we are given $\mathcal{M}$ and the functions $\mathrm{S}_{\mathcal{M};\mathcal{K}}$ and $\mathrm{C}_{\mathcal{M};\mathcal{K}}$. Then $\mathcal{K}$ can be uniquely determined.*

Proof Fix the origin $\mathbf{0}$ in $\mathcal{M}^\circ$. In the plane $\boldsymbol{u}_\xi = (\cos\xi, \sin\xi)$, therefore we consider the functions

$$f(\xi) := \mathrm{S}_{\mathcal{M};\mathcal{K}}(\boldsymbol{u}_\xi) = |\hbar(p_{\mathcal{M}}(\boldsymbol{u}_\xi), \boldsymbol{u}_\xi) \cap \mathcal{K}|$$
$$g(\xi) := \mathrm{C}_{\mathcal{M};\mathcal{K}}(\boldsymbol{u}_\xi) = |\hbar^+(p_{\mathcal{M}}(\boldsymbol{u}_\xi), \boldsymbol{u}_\xi) \cap \mathcal{K}|$$

where $\hbar^+$ is the appropriate half space bordered by $\hbar$.

Let $\boldsymbol{h}(\xi)$ be the point, where $\hbar(p_{\mathcal{M}}(\xi), \boldsymbol{u}_\xi)$ touches $\mathcal{M}$. Then, as it is well known, $\boldsymbol{h}(\xi) - p_{\mathcal{M}}(\xi)\boldsymbol{u}_\xi = p'_{\mathcal{M}}(\xi)\boldsymbol{u}_\xi^\perp$. Let $\boldsymbol{a}(\xi)$ and $\boldsymbol{b}(\xi)$ be the two intersections of

$\hbar(p_{\mathcal{M}}(\xi), \boldsymbol{u}_\xi)$ and $\partial\mathcal{K}$ taken so that $\boldsymbol{a}(\xi) = \boldsymbol{h}(\xi) + a(\xi)\boldsymbol{u}_\xi^\perp$ and $\boldsymbol{b}(\xi) = \boldsymbol{h}(\xi) - b(\xi)\boldsymbol{u}_\xi^\perp$, where $a(\xi)$ and $b(\xi)$ are positive functions.

Then $f(\xi) = a(\xi) + b(\xi)$.

In the other hand, we have

$$g(\xi) = \int\limits_{\mathcal{K}\setminus\mathcal{M}} \chi(\langle \boldsymbol{x}, \boldsymbol{u}_\xi\rangle \geq p_{\mathcal{M}}(\xi))\, d\boldsymbol{x} = \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int\limits_{0}^{\varrho_\xi(\zeta)} r\, dr\, d\zeta,$$

where $\boldsymbol{h}(\xi) + \varrho_\xi(\zeta)\boldsymbol{u}_\zeta \in \partial\mathcal{K}$. Since $\frac{d\varrho_\xi(\zeta)}{d\xi} = \frac{d\varrho_\xi(\zeta)}{d\zeta}$, this leads to

$$2g'(\xi) = \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d}{d\xi}\left(\int\limits_{0}^{\varrho_\xi(\zeta)} 2r\, dr\right) d\zeta = \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\varrho_\xi(\zeta)\varrho_\xi'(\zeta)\, d\zeta = a^2(\xi) - b^2(\xi)$$

that implies

$$a(\xi) = \frac{\frac{2g'(\xi)}{f(\xi)} + f(\xi)}{2} = \frac{2g'(\xi) + f^2(\xi)}{2f(\xi)}.$$

This clearly determines $\mathcal{K}$. □

If the kernel $\mathcal{M}$ is known to be a disc $\varrho\mathcal{D}$, then any one of the functions $\mathrm{S}_{\varrho\mathcal{D};\mathcal{K}}$ and $\mathrm{C}_{\varrho\mathcal{D};\mathcal{K}}$ can determine concentric discs by its constant value.

Theorem 3.2 *Assume that one of the functions* $\mathrm{S}_{\varrho\mathcal{D};\mathcal{K}}$ *and* $\mathrm{C}_{\varrho\mathcal{D};\mathcal{K}}$ *is constant, where* $\mathcal{D}$ *is the unit disc. Then* $\mathcal{K}$ *is a disc centred to the origin.*

Proof If $\mathrm{S}_{\varrho\mathcal{D};\mathcal{K}}$ is constant, then this theorem is [1, Theorem 1].

If $\mathrm{C}_{\varrho\mathcal{D};\mathcal{K}}$ is constant, the derivative of $\mathrm{C}_{\varrho\mathcal{D};\mathcal{K}}$ is zero, hence—using the notations of the previous proof—$a(\xi) = b(\xi)$ for every $\xi \in [0, 2\pi)$, that is, the point $\boldsymbol{h}(\xi)$ is the midpoint of the segment $\overline{\boldsymbol{a}(\xi)\boldsymbol{b}(\xi)}$ on $\hbar(\varrho, \boldsymbol{u}_\xi)$.

Let us consider the chord-map $C\colon \partial\mathcal{K} \to \partial\mathcal{K}$, that is defined by $C(\boldsymbol{b}(\xi)) = \boldsymbol{a}(\xi)$ for every $\xi \in [0, 2\pi)$. This is clearly a bijective map. If $\boldsymbol{\ell}_0 \in \partial\mathcal{K}$, then by $a(\xi) = b(\xi)$ the whole sequence $\boldsymbol{\ell}_i = C^i(\boldsymbol{\ell})$, where C^i means the i consecutive usage of C, are on a concentric circle of radius $|\boldsymbol{\ell}_0|$. Moreover, every point $\boldsymbol{\ell}_i$ $(i > 0)$ is the concentric rotation of $\boldsymbol{\ell}_{i-1}$ with angle $\lambda = 2\arccos(\frac{\varrho}{|\boldsymbol{\ell}_0|})$. It is well known [4, Proposition 1.3.3] that such a sequence is dense in $\partial\mathcal{K}$ if $\frac{\lambda}{\pi}$ is irrational, or it is finitely periodic in $\partial\mathcal{K}$ if $\frac{\lambda}{\pi}$ is rational. However, if $\mathcal{K}$ is not a disc, then there is surely a point $\boldsymbol{\ell} \in \partial\mathcal{K}$ for which $\frac{2\arccos(\frac{\varrho}{|\boldsymbol{\ell}_0|})}{\pi}$ is irrational, hence $\mathcal{K}$ must be a concentric disc. □

4 Measures of convex bodies

In this section the dimension of the space is $n = 2, 3, \ldots$. As a shorthand we introduce the notations

$$\mathrm{S}_{\varrho;\mathcal{K}}(\boldsymbol{u}) := \mathrm{S}_{\varrho\mathcal{B};\mathcal{K}}(\hbar(\varrho, \boldsymbol{u})) = |\mathcal{K} \cap \hbar(\varrho, \boldsymbol{u})|, \tag{4.1}$$

$$\mathrm{C}_{\varrho;\mathcal{K}}(\boldsymbol{u}) := \mathrm{C}_{\varrho\mathcal{B};\mathcal{K}}(\hbar(\varrho, \boldsymbol{u})) = |\mathcal{K} \cap \hbar^+(\varrho, \boldsymbol{u})|, \tag{4.2}$$

where $\varrho\mathcal{B}^n$ is the ball of radius $\varrho > 0$ centred to the origin and $\hbar^+$ is the appropriate half space bordered by $\hbar$.

Lemma 4.1 *If the convex body $\mathcal{K}$ in $\mathbb{R}^n$ contains in its interior the ball $\varrho\mathcal{B}^n$, then*

$$\int\limits_{\mathbb{S}^{n-1}} \mathrm{C}_{\varrho;\mathcal{K}}(\boldsymbol{u}_{\boldsymbol{\xi}}) d\boldsymbol{\xi} = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2})} \int\limits_{\mathcal{K}\setminus\varrho\mathcal{B}} I_{1-\frac{\varrho^2}{|\boldsymbol{x}|^2}} \left(\frac{n-1}{2}, \frac{1}{2}\right) d\boldsymbol{x}. \tag{4.3}$$

Proof We have

$$\begin{aligned}\int\limits_{\mathbb{S}^{n-1}} \mathrm{C}_{\varrho;\mathcal{K}}(\boldsymbol{u}_{\boldsymbol{\xi}}) d\boldsymbol{\xi} &= \int\limits_{\mathbb{S}^{n-1}} \int\limits_{\mathbb{R}^n} \chi_{\mathcal{K}}(\boldsymbol{x}) \chi\left(\langle \boldsymbol{x}, \boldsymbol{u}_{\boldsymbol{\xi}} \rangle \geq \varrho\right) d\boldsymbol{x} d\boldsymbol{\xi} \\ &= \int\limits_{\mathcal{K}\setminus\varrho\mathcal{B}} \int\limits_{\mathbb{S}^{n-1}} \chi\left(\left\langle \frac{\boldsymbol{x}}{|\boldsymbol{x}|}, \boldsymbol{u}_{\boldsymbol{\xi}} \right\rangle \geq \frac{\varrho}{|\boldsymbol{x}|}\right) d\boldsymbol{\xi} d\boldsymbol{x}\end{aligned}$$

The inner integral is the surface of the hyperspherical cap. The height of this hyperspherical cap is $h = 1 - \frac{\varrho}{|\boldsymbol{x}|}$, hence by the well-known formula [13] we obtain

$$\int\limits_{\mathbb{S}^{n-1}} \chi\left(\left\langle \frac{\boldsymbol{x}}{|\boldsymbol{x}|}, \boldsymbol{u}_{\boldsymbol{\xi}} \right\rangle \geq \frac{\varrho}{|\boldsymbol{x}|}\right) d\boldsymbol{\xi} = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2})} I_{\frac{|\boldsymbol{x}|^2-\varrho^2}{|\boldsymbol{x}|^2}} \left(\frac{n-1}{2}, \frac{1}{2}\right).$$

This proves the lemma. □

Note that the weight in (4.3) is $\frac{\pi}{\Gamma(1)} I_{1-\frac{\varrho^2}{|\boldsymbol{x}|^2}}(\frac{1}{2}, \frac{1}{2}) = 2\arccos(\frac{\varrho}{|\boldsymbol{x}|})$ for dimension $n = 2$, and it is $\frac{\pi^{3/2}}{\Gamma(\frac{3}{2})} I_{1-\frac{\varrho^2}{|\boldsymbol{x}|^2}}(1, \frac{1}{2}) = 2\pi(1 - \frac{\varrho}{|\boldsymbol{x}|})$ for dimension $n = 3$.

Lemma 4.2 *Let the convex body $\mathcal{K}$ contain in its interior the ball $\varrho\mathcal{B}^n$. Then the integral of the section function is*

$$\int\limits_{\mathbb{S}^{n-1}} \mathrm{S}_{\varrho;\mathcal{K}}(\boldsymbol{u}_{\boldsymbol{\xi}}) d\boldsymbol{\xi} = |\mathbb{S}^{n-2}| \int\limits_{\mathcal{K}\setminus\varrho\mathcal{B}^n} \frac{(\boldsymbol{x}^2 - \varrho^2)^{\frac{n-3}{2}}}{|\boldsymbol{x}|^{n-2}} d\boldsymbol{x}. \tag{4.4}$$

Proof Observe, that using (4.3) we have for any $\varepsilon > 0$ that

$$
\begin{aligned}
&\frac{\Gamma(\frac{n}{2})}{\pi^{n/2}} \int_0^{\varepsilon} \int_{\mathbb{S}^{n-1}} \mathrm{S}_{\varrho+\delta;\mathcal{K}}(\boldsymbol{u}_{\boldsymbol{\xi}}) d\boldsymbol{\xi} d\delta \\
&= \frac{\Gamma(\frac{n}{2})}{\pi^{n/2}} \int_{\mathbb{S}^{n-1}} \int_0^{\varepsilon} \mathrm{S}_{\varrho+\delta;\mathcal{K}}(\boldsymbol{u}_{\boldsymbol{\xi}}) d\delta d\boldsymbol{\xi} \\
&= \frac{\Gamma(\frac{n}{2})}{\pi^{n/2}} \int_{\mathbb{S}^{n-1}} \mathrm{C}_{\varrho;\mathcal{K}}(\boldsymbol{u}_{\boldsymbol{\xi}}) - \mathrm{C}_{\varrho+\varepsilon;\mathcal{K}}(\boldsymbol{u}_{\boldsymbol{\xi}}) d\boldsymbol{\xi} \\
&= \int_{\mathcal{K}\setminus\varrho\mathcal{B}} I_{\frac{|\boldsymbol{x}|^2-\varrho^2}{|\boldsymbol{x}|^2}}\left(\frac{n-1}{2}, \frac{1}{2}\right) d\boldsymbol{x} - \int_{\mathcal{K}\setminus(\varrho+\varepsilon)\mathcal{B}} I_{\frac{|\boldsymbol{x}|^2-(\varrho+\varepsilon)^2}{|\boldsymbol{x}|^2}}\left(\frac{n-1}{2}, \frac{1}{2}\right) d\boldsymbol{x} \\
&= \int_{(\varrho+\varepsilon)\mathcal{B}\setminus\varrho\mathcal{B}} I_{\frac{|\boldsymbol{x}|^2-\varrho^2}{|\boldsymbol{x}|^2}}\left(\frac{n-1}{2}, \frac{1}{2}\right) d\boldsymbol{x} \\
&\quad - \int_{\mathcal{K}\setminus(\varrho+\varepsilon)\mathcal{B}} I_{\frac{|\boldsymbol{x}|^2-(\varrho+\varepsilon)^2}{|\boldsymbol{x}|^2}}\left(\frac{n-1}{2}, \frac{1}{2}\right) - I_{\frac{|\boldsymbol{x}|^2-\varrho^2}{|\boldsymbol{x}|^2}}\left(\frac{n-1}{2}, \frac{1}{2}\right) d\boldsymbol{x},
\end{aligned}
$$

hence

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \frac{\Gamma(\frac{n}{2})}{\pi^{n/2}} \int_0^{\varepsilon} \int_{\mathbb{S}^{n-1}} \mathrm{S}_{\varrho+\delta;\mathcal{K}}(\boldsymbol{u}_{\boldsymbol{\xi}}) d\boldsymbol{\xi} d\delta \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{(\varrho+\varepsilon)\mathcal{B}\setminus\varrho\mathcal{B}} I_{\frac{|\boldsymbol{x}|^2-\varrho^2}{|\boldsymbol{x}|^2}}\left(\frac{n-1}{2}, \frac{1}{2}\right) d\boldsymbol{x} \\
&\quad - \int_{\mathcal{K}\setminus\varrho\mathcal{B}} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(I_{\frac{|\boldsymbol{x}|^2-(\varrho+\varepsilon)^2}{|\boldsymbol{x}|^2}}\left(\frac{n-1}{2}, \frac{1}{2}\right) - I_{\frac{|\boldsymbol{x}|^2-\varrho^2}{|\boldsymbol{x}|^2}}\left(\frac{n-1}{2}, \frac{1}{2}\right)\right) d\boldsymbol{x} \\
&= \lim_{\varepsilon\to 0} \frac{|\mathbb{S}^{n-1}|}{\varepsilon} \int_{\varrho}^{\varrho+\varepsilon} r^{n-1} I_{\frac{r^2-\varrho^2}{r^2}}\left(\frac{n-1}{2}, \frac{1}{2}\right) dr \\
&\quad - \int_{\mathcal{K}\setminus\varrho\mathcal{B}} \frac{d}{d\varrho}\left(I_{\frac{|\boldsymbol{x}|^2-\varrho^2}{|\boldsymbol{x}|^2}}\left(\frac{n-1}{2}, \frac{1}{2}\right)\right) d\boldsymbol{x}
\end{aligned}
$$

$$= |\mathbb{S}^{n-1}|\varrho^{n-1} I_{\frac{\varrho^2-\varrho^2}{\varrho^2}}\left(\frac{n-1}{2}, \frac{1}{2}\right)$$

$$- \frac{1}{B(\frac{n-1}{2}, \frac{1}{2})} \int\limits_{\mathcal{K}\setminus\varrho\mathcal{B}} \left(1 - \frac{\varrho^2}{|\boldsymbol{x}|^2}\right)^{\frac{n-3}{2}} \left(\frac{\varrho^2}{|\boldsymbol{x}|^2}\right)^{\frac{-1}{2}} \frac{-2\varrho}{|\boldsymbol{x}|^2}\, d\boldsymbol{x}$$

$$= \frac{2}{B(\frac{n-1}{2}, \frac{1}{2})} \int\limits_{\mathcal{K}\setminus\varrho\mathcal{B}} \left(1 - \frac{\varrho^2}{|\boldsymbol{x}|^2}\right)^{\frac{n-3}{2}} \frac{1}{|\boldsymbol{x}|}\, d\boldsymbol{x}.$$

As

$$\frac{\pi^{n/2}}{\Gamma(\frac{n}{2})} \frac{2}{B(\frac{n-1}{2}, \frac{1}{2})} = \frac{2\pi^{n/2}}{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})} = \frac{\frac{n-1}{2}}{\frac{n-1}{2}} \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} = \frac{(n-1)\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2}+1)} = |\mathbb{S}^{n-2}|,$$

the statement is proved. □

Note that the weight in (4.4) is $\frac{2}{\sqrt{x^2-\varrho^2}}$ in the plane, and $2\pi/|\boldsymbol{x}|$ in dimension $n = 3$, which is independent from ϱ!

A version of the following lemma first appeared in [9].

Lemma 4.3 *Let ω_i $(i = 1, 2)$ be weights and let $\mathcal{K}$ and $\mathcal{L}$ be convex bodies containing the unit ball $\mathcal{B}$. If $V_1(\mathcal{K}) \le V_1(\mathcal{L})$ and*

(1) *Either $\frac{\omega_2}{\omega_1}$ is a constant $c_{\mathcal{K}}$ on $\partial\mathcal{K}$ and $\frac{\omega_2}{\omega_1}(X)\begin{cases}\ge c_{\mathcal{K}}, \text{ if } X \notin \mathcal{K},\\ \le c_{\mathcal{K}}, \text{ if } X \in \mathcal{K},\end{cases}$ where equality may occur in a set of measure zero at most,*

(2) *or $\frac{\omega_2}{\omega_1}$ is a constant $c_{\mathcal{L}}$ on $\partial\mathcal{L}$ and $\frac{\omega_2}{\omega_1}(X)\begin{cases}\le c_{\mathcal{L}}, \text{ if } X \notin \mathcal{L},\\ \ge c_{\mathcal{L}}, \text{ if } X \in \mathcal{L},\end{cases}$ where equality may occur in a set of measure zero at most,*

then $V_2(\mathcal{K}) \le V_2(\mathcal{L})$, where equality is if and only if $\mathcal{K} = \mathcal{L}$.

Proof We have

$$V_2(\mathcal{L}) - V_2(\mathcal{K}) = V_2(\mathcal{L}\setminus\mathcal{K}) - V_2(\mathcal{K}\setminus\mathcal{L}) = \int\limits_{\mathcal{L}\setminus\mathcal{K}} \frac{\omega_2(x)}{\omega_1(x)}\omega_1(x)dx - \int\limits_{\mathcal{K}\setminus\mathcal{L}} \frac{\omega_2(x)}{\omega_1(x)}\omega_1(x)dx$$

$$\begin{cases} = 0, & \text{if } \mathcal{K}\triangle\mathcal{L} = \emptyset,\\ > c_{\mathcal{K}}(V_1(\mathcal{L}\setminus\mathcal{K}) - V_1(\mathcal{K}\setminus\mathcal{L})) = c_{\mathcal{K}}(V_1(\mathcal{L}) - V_1(\mathcal{K})), & \text{if } \mathcal{K}\triangle\mathcal{L} \ne \emptyset \text{ and } (1),\\ > c_{\mathcal{L}}(V_1(\mathcal{L}\setminus\mathcal{K}) - V_1(\mathcal{K}\setminus\mathcal{L})) = c_{\mathcal{L}}(V_1(\mathcal{L}) - V_1(\mathcal{K})), & \text{if } \mathcal{K}\triangle\mathcal{L} \ne \emptyset \text{ and } (2),\end{cases}$$

that proves the theorem. □

5 Ball characterizations

Although the following results are valid also in the plane, their points are for higher dimensions.

Theorem 5.1 *Let* $0 < \varrho_1 < \varrho_2 < \bar{r}$ *and let* $\mathcal{K}$ *be a convex body having* $\varrho_2\mathcal{B}$ *in its interior. If* $\mathrm{C}_{\varrho_1;\mathcal{K}} = \mathrm{C}_{\varrho_1;\bar{r}\mathcal{B}}$ *and* $\mathrm{C}_{\varrho_2;\mathcal{K}} = \mathrm{C}_{\varrho_2;\bar{r}\mathcal{B}}$*, then* $\mathcal{K} \equiv \bar{r}\mathcal{B}$*, where* $\mathcal{B}$ *is the unit ball.*

Proof Let $\bar{\omega}_1(r) = I_{\frac{r^2-\varrho_1^2}{r^2}}(\frac{n-1}{2}, \frac{1}{2})$ and $\bar{\omega}_2(r) = I_{\frac{r^2-\varrho_2^2}{r^2}}(\frac{n-1}{2}, \frac{1}{2})$ for every non-vanishing $r \in \mathbb{R}$, where I is the regularized incomplete beta function, and define $\omega_1(\boldsymbol{x}) := \bar{\omega}_1(|\boldsymbol{x}|)$ and $\omega_2(\boldsymbol{x}) := \bar{\omega}_2(|\boldsymbol{x}|)$.

By formula (4.3) in Lemma 4.1 we have

$$\int\limits_{\bar{r}\mathcal{B}\setminus\varrho_1\mathcal{B}^n} \omega_1(\boldsymbol{x})\,d\boldsymbol{x} = \frac{\Gamma(\frac{n}{2})}{\pi^{n/2}} \int\limits_{\mathbb{S}^{n-1}} \mathrm{C}_{\varrho_1;\mathcal{K}}(\boldsymbol{u}_{\boldsymbol{\xi}})d\boldsymbol{\xi} = \int\limits_{\mathcal{K}\setminus\varrho_1\mathcal{B}^n} \omega_1(\boldsymbol{x})\,d\boldsymbol{x},$$

and similarly

$$\int\limits_{\bar{r}\mathcal{B}\setminus\varrho_2\mathcal{B}^n} \omega_2(\boldsymbol{x})\,d\boldsymbol{x} = \frac{\Gamma(\frac{n}{2})}{\pi^{n/2}} \int\limits_{\mathbb{S}^{n-1}} \mathrm{C}_{\varrho_2;\mathcal{K}}(\boldsymbol{u}_{\boldsymbol{\xi}})d\boldsymbol{\xi} = \int\limits_{\mathcal{K}\setminus\varrho_2\mathcal{B}^n} \omega_2(\boldsymbol{x})\,d\boldsymbol{x}.$$

With the notations in Lemma 4.3, these mean $V_1(\mathcal{K}) = V_1(\bar{r}\mathcal{B})$ and $V_2(\mathcal{K}) = V_2(\bar{r}\mathcal{B})$.

Further, one can easily see that

$$1 < \frac{\omega_1(\boldsymbol{x})}{\omega_2(\boldsymbol{x})} = \frac{\bar{\omega}_1(|\boldsymbol{x}|)}{\bar{\omega}_2(|\boldsymbol{x}|)} =: q_n(|\boldsymbol{x}|), \qquad (n \text{ is the dimension})$$

is constant on every sphere, especially on $\bar{r}\mathbb{S}^{n-1}$.

As $\bar{\omega}_1$ and $\bar{\omega}_2$ are both strictly increasing, q_n is strictly decreasing if and only if

$$\frac{\bar{\omega}_1'(r)}{\bar{\omega}_2'(r)} < \frac{\bar{\omega}_1(r)}{\bar{\omega}_2(r)}. \tag{5.1}$$

First calculate for any $n \in \mathbb{N}$ that

$$\frac{\bar{\omega}_1'(r)}{\bar{\omega}_2'(r)} = \frac{\left(1 - \frac{\varrho_1^2}{r^2}\right)^{\frac{n-3}{2}} \left(\frac{\varrho_1^2}{r^2}\right)^{\frac{-1}{2}} \frac{2\varrho_1^2}{r^3}}{\left(1 - \frac{\varrho_2^2}{r^2}\right)^{\frac{n-3}{2}} \left(\frac{\varrho_2^2}{r^2}\right)^{\frac{-1}{2}} \frac{2\varrho_2^2}{r^3}} = \frac{(r^2 - \varrho_1^2)^{\frac{n-3}{2}}\varrho_1}{(r^2 - \varrho_2^2)^{\frac{n-3}{2}}\varrho_2},$$

then consider for $n \geq 4$ that

$$\frac{\bar{\omega}_1(r)B\left(\frac{n-1}{2}, \frac{1}{2}\right)}{\left(1 - \frac{\varrho_1^2}{r^2}\right)^{\frac{n-3}{2}}} = \left(1 - \frac{\varrho_1^2}{r^2}\right)^{\frac{3-n}{2}} \int\limits_0^{1-\frac{\varrho_1^2}{r^2}} t^{\frac{n-3}{2}}(1-t)^{\frac{-1}{2}}\,dt$$

$$= \int_0^1 s^{\frac{n-3}{2}} \left(1 - s\left(1 - \frac{\varrho_1^2}{r^2}\right)\right)^{\frac{-1}{2}} \left(1 - \frac{\varrho_1^2}{r^2}\right) ds$$

$$= -2 \int_0^1 s^{\frac{n-3}{2}} \frac{d}{ds} \left(\left(1 - s\left(1 - \frac{\varrho_1^2}{r^2}\right)\right)^{\frac{1}{2}}\right) ds$$

$$= -2 \left(\frac{\varrho_1}{r} - \frac{n-3}{2} \int_0^1 s^{\frac{n-5}{2}} \left(1 - s\left(1 - \frac{\varrho_1^2}{r^2}\right)\right)^{\frac{1}{2}} ds\right)$$

$$= \frac{2\varrho_1}{r} \left(\frac{n-3}{2} \int_0^1 s^{\frac{n-5}{2}} \left(\frac{r^2}{\varrho_1^2}(1-s) + s\right)^{\frac{1}{2}} ds - 1\right). \quad (5.2)$$

From the two equations above we deduce

$$\frac{\bar{\omega}_1(r)}{\bar{\omega}_2(r)} \frac{\bar{\omega}_2'(r)}{\bar{\omega}_1'(r)} = \frac{\frac{2\varrho_1}{r}\left(1 - \frac{\varrho_1^2}{r^2}\right)^{\frac{n-3}{2}} \left(\frac{n-3}{2} \int_0^1 s^{\frac{n-5}{2}} (\frac{r^2}{\varrho_1^2}(1-s)+s)^{\frac{1}{2}} ds - 1\right)}{\frac{2\varrho_2}{r}\left(1 - \frac{\varrho_2^2}{r^2}\right)^{\frac{n-3}{2}} \left(\frac{n-3}{2} \int_0^1 s^{\frac{n-5}{2}} (\frac{r^2}{\varrho_2^2}(1-s)+s)^{\frac{1}{2}} ds - 1\right)} \frac{(r^2-\varrho_2^2)^{\frac{n-3}{2}}\varrho_2}{(r^2-\varrho_1^2)^{\frac{n-3}{2}}\varrho_1}$$

$$= \frac{\frac{n-3}{2} \int_0^1 s^{\frac{n-5}{2}} \left(\frac{r^2}{\varrho_1^2}(1-s)+s\right)^{\frac{1}{2}} ds - 1}{\frac{n-3}{2} \int_0^1 s^{\frac{n-5}{2}} \left(\frac{r^2}{\varrho_2^2}(1-s)+s\right)^{\frac{1}{2}} ds - 1} \geq 1,$$

where in the last inequality we used $\varrho_1 < \varrho_2$. Thus, for $n \geq 4$ we have proved (5.1).

Assume now, that $n < 4$. It is easy to see that

$$\bar{\omega}_1(r) - \bar{\omega}_2(r) = \frac{1}{B\left(\frac{n-1}{2}, \frac{1}{2}\right)} \int_{1-\frac{\varrho_2^2}{r^2}}^{1-\frac{\varrho_1^2}{r^2}} t^{\frac{n-3}{2}} (1-t)^{\frac{-1}{2}} dt,$$

hence differentiation leads to

$$(\bar{\omega}_1'(r) - \bar{\omega}_2'(r)) B\left(\frac{n-1}{2}, \frac{1}{2}\right)$$

$$= \left(1 - \frac{\varrho_1^2}{r^2}\right)^{\frac{n-3}{2}} \left(\frac{\varrho_1^2}{r^2}\right)^{\frac{-1}{2}} \frac{2\varrho_1^2}{r^3} - \left(1 - \frac{\varrho_2^2}{r^2}\right)^{\frac{n-3}{2}} \left(\frac{\varrho_2^2}{r^2}\right)^{\frac{-1}{2}} \frac{2\varrho_2^2}{r^3}$$

$$= \frac{2}{r^{n-1}} \left((r^2 - \varrho_1^2)^{\frac{n-3}{2}} \varrho_1 - (r^2 - \varrho_2^2)^{\frac{n-3}{2}} \varrho_2\right).$$

This is clearly negative for all r if $n = 2$ and $n = 3$, hence

$$\frac{\bar{\omega}_1(r)}{\bar{\omega}_2(r)}\frac{\bar{\omega}_2'(r)}{\bar{\omega}_1'(r)} = \frac{\bar{\omega}_1(r)}{\bar{\omega}_2(r)}\left(\frac{\bar{\omega}_2'(r) - \bar{\omega}_1'(r)}{\bar{\omega}_1'(r)} + 1\right) \geq \frac{\bar{\omega}_1(r)}{\bar{\omega}_2(r)} \geq 1$$

proving (5.1) for $n \leq 3$.

Thus, $\frac{\bar{\omega}_1(r)}{\bar{\omega}_2(r)}$ is strictly monotone decreasing in any dimension, hence $\mathcal{K} \equiv \bar{r}\mathcal{B}$ follows from Lemma 4.3. □

Theorem 5.2 *Let* $0 < \varrho_1 < \varrho_2 < \bar{r}$ *and the dimension be* $n \neq 3$. *If* $\mathcal{K}$ *is a convex body having* $\varrho_2\mathcal{B}$ *in its interior, and* $\mathrm{S}_{\varrho_1;\mathcal{K}} \equiv \mathrm{S}_{\varrho_1;\bar{r}\mathcal{B}}$, $\mathrm{S}_{\varrho_2;\mathcal{K}} \equiv \mathrm{S}_{\varrho_2;\bar{r}\mathcal{B}}$, *then* $\mathcal{K} \equiv \bar{r}\mathcal{B}$.

Proof Let $\bar{\omega}_1(r) = (r^2 - \varrho_1^2)^{\frac{n-3}{2}} r^{2-n}$ and $\bar{\omega}_2(r) = (r^2 - \varrho_2^2)^{\frac{n-3}{2}} r^{2-n}$ for every non-vanishing $r \in \mathbb{R}$, and define $\omega_1(\boldsymbol{x}) := \bar{\omega}_1(|\boldsymbol{x}|)$ and $\omega_2(\boldsymbol{x}) := \bar{\omega}_2(|\boldsymbol{x}|)$.

By formula (4.4) in Lemma 4.2 we have

$$\int_{\bar{r}\mathcal{B}\setminus\varrho_1\mathcal{B}^n} \omega_1(\boldsymbol{x})\, d\boldsymbol{x} = \frac{1}{|\mathbb{S}^{n-2}|}\int_{\mathbb{S}^{n-1}} \mathrm{S}_{\varrho_1;\mathcal{K}}(\boldsymbol{u}_{\boldsymbol{\xi}})d\boldsymbol{\xi} = \int_{\mathcal{K}\setminus\varrho_1\mathcal{B}^n} \omega_1(\boldsymbol{x})\, d\boldsymbol{x},$$

and similarly

$$\int_{\bar{r}\mathcal{B}\setminus\varrho_2\mathcal{B}^n} \omega_2(\boldsymbol{x})\, d\boldsymbol{x} = \frac{1}{|\mathbb{S}^{n-2}|}\int_{\mathbb{S}^{n-1}} \mathrm{S}_{\varrho_2;\mathcal{K}}(\boldsymbol{u}_{\boldsymbol{\xi}})d\boldsymbol{\xi} = \int_{\mathcal{K}\setminus\varrho_2\mathcal{B}^n} \omega_2(\boldsymbol{x})\, d\boldsymbol{x}.$$

With the notations in Lemma 4.3, these mean $V_1(\mathcal{K}) = V_1(\bar{r}\mathcal{B})$ and $V_2(\mathcal{K}) = V_2(\bar{r}\mathcal{B})$.

The ratio $\frac{\omega_1(\boldsymbol{x})}{\omega_2(\boldsymbol{x})} = \frac{\bar{\omega}_1(|\boldsymbol{x}|)}{\bar{\omega}_2(|\boldsymbol{x}|)}$ is obviously constant on every sphere, especially on $\bar{r}\mathbb{S}^{n-1}$, and it is

$$\frac{\bar{\omega}_1(r)}{\bar{\omega}_2(r)} = \begin{cases} \frac{\sqrt{r^2-\varrho_2^2}}{\sqrt{r^2-\varrho_1^2}} = \sqrt{1 - \frac{\varrho_1^2-\varrho_2^2}{r^2-\varrho_1^2}}, & \text{if } n = 2,\\ 1, & \text{if } n = 3,\\ \left(1 + \frac{\varrho_2^2-\varrho_1^2}{r^2-\varrho_2^2}\right)^{\frac{n-3}{2}}, & \text{if } n > 3. \end{cases}$$

Thus, $\frac{\bar{\omega}_1(r)}{\bar{\omega}_2(r)}$ is strictly monotone if the dimension $n \neq 3$, hence $\mathcal{K} \equiv \bar{r}\mathcal{B}$ follows from Lemma 4.3 for dimensions other than 3. □

This theorem leaves the question open in dimension 3 if $\mathrm{S}_{\varrho_1;\mathcal{K}} \equiv \mathrm{S}_{\varrho_1;\bar{r}\mathcal{B}}$ and $\mathrm{S}_{\varrho_2;\mathcal{K}} \equiv \mathrm{S}_{\varrho_2;\bar{r}\mathcal{B}}$ imply $\mathcal{K} \equiv \bar{r}\mathcal{B}$. We have not yet tried to find an answer.

The following generalizes Theorem 3.1 for most dimensions, but only for spheres.

Theorem 5.3 *Let* $\varrho_1, \varrho_2 \in (0, \bar{r})$ *and let* $\mathcal{K}$ *be a convex body in* $\mathbb{R}^n$ *having* $\max(\varrho_1, \varrho_2)\mathcal{B}$ *in its interior. If* $\mathrm{S}_{\varrho_1;\mathcal{K}} \equiv \mathrm{S}_{\varrho_1;\bar{r}\mathcal{B}}$ *and* $\mathrm{C}_{\varrho_2;\mathcal{K}} \equiv \mathrm{C}_{\varrho_2;\bar{r}\mathcal{B}}$, *and*

(1) $n = 2$ *or* $n = 3$, *or*

(2) $n \geq 4$ *and* $\varrho_1 \leq \varrho_2$,
then $\mathcal{K} \equiv \bar{r}\mathcal{B}$.

Proof Let $\bar{\omega}_1(r) = (r^2 - \varrho_1^2)^{\frac{n-3}{2}} r^{2-n}$ and and $\bar{\omega}_2(r) = I_{\frac{r^2-\varrho_2^2}{r^2}}(\frac{n-1}{2}, \frac{1}{2})$ for every non-vanishing $r \in \mathbb{R}$, and define $\omega_1(\boldsymbol{x}) := \bar{\omega}_1(|\boldsymbol{x}|)$ and $\omega_2(\boldsymbol{x}) := \bar{\omega}_2(|\boldsymbol{x}|)$.

By formula (4.4) in Lemma 4.2 we have

$$\int\limits_{\bar{r}\mathcal{B}\setminus\varrho_1\mathcal{B}^n} \omega_1(\boldsymbol{x})\, d\boldsymbol{x} = \frac{1}{|\mathbb{S}^{n-2}|} \int\limits_{\mathbb{S}^{n-1}} \mathrm{S}_{\varrho_1;\mathcal{K}}(\boldsymbol{u}_{\boldsymbol{\xi}}) d\boldsymbol{\xi} = \int\limits_{\mathcal{K}\setminus\varrho_1\mathcal{B}^n} \omega_1(\boldsymbol{x})\, d\boldsymbol{x},$$

and by formula (4.3) in Lemma 4.1 we have

$$\int\limits_{\bar{r}\mathcal{B}\setminus\varrho_2\mathcal{B}^n} \omega_2(\boldsymbol{x})\, d\boldsymbol{x} = \frac{\Gamma(\frac{n}{2})}{\pi^{n/2}} \int\limits_{\mathbb{S}^{n-1}} \mathrm{C}_{\varrho_2;\mathcal{K}}(\boldsymbol{u}_{\boldsymbol{\xi}}) d\boldsymbol{\xi} = \int\limits_{\mathcal{K}\setminus\varrho_2\mathcal{B}^n} \omega_2(\boldsymbol{x})\, d\boldsymbol{x}.$$

With the notations in Lemma 4.3, these mean $V_1(\mathcal{K}) = V_1(\bar{r}\mathcal{B})$ and $V_2(\mathcal{K}) = V_2(\bar{r}\mathcal{B})$.

The ratio $\frac{\omega_2(\boldsymbol{x})}{\omega_1(\boldsymbol{x})} = \frac{\bar{\omega}_2(|\boldsymbol{x}|)}{\bar{\omega}_1(|\boldsymbol{x}|)}$ is obviously constant on every sphere, especially on $\bar{r}\mathbb{S}^{n-1}$, and it is

$$\begin{aligned}
\frac{\bar{\omega}_2(r)}{\bar{\omega}_1(r)} &= \frac{\int_0^{1-\frac{\varrho_2^2}{r^2}} t^{\frac{n-3}{2}} (1-t)^{\frac{-1}{2}}\, dt}{(r^2 - \varrho_1^2)^{\frac{n-3}{2}} r^{2-n}} \\
&= \frac{\frac{2\varrho_2}{r}\left(1 - \frac{\varrho_2^2}{r^2}\right)^{\frac{n-3}{2}} \left(\frac{n-3}{2} \int_0^1 s^{\frac{n-5}{2}} \left(\frac{r^2}{\varrho_2^2}(1-s) + s\right)^{\frac{1}{2}} ds - 1\right)}{\frac{1}{r}\left(1 - \frac{\varrho_1^2}{r^2}\right)^{\frac{n-3}{2}}} \qquad \text{by} \quad (5.2) \\
&= 2\varrho_1 \left(\frac{r^2 - \varrho_2^2}{r^2 - \varrho_1^2}\right)^{\frac{n-3}{2}} \left(\frac{n-3}{2} \int\limits_0^1 s^{\frac{n-5}{2}} \left(\frac{r^2}{\varrho_2^2}(1-s) + s\right)^{\frac{1}{2}} ds - 1\right) \\
&= 2\varrho_1 \left(1 + \frac{\varrho_1^2 - \varrho_2^2}{r^2 - \varrho_1^2}\right)^{\frac{n-3}{2}} \left(\frac{n-3}{2} \int\limits_0^1 s^{\frac{n-5}{2}} \left(\frac{r^2}{\varrho_2^2}(1-s) + s\right)^{\frac{1}{2}} ds - 1\right)
\end{aligned}$$

if $n > 3$. For other values of n we have

$$\begin{aligned}
\frac{\bar{\omega}_2(r)}{\bar{\omega}_1(r)} &= \frac{\int_0^{1-\frac{\varrho_2^2}{r^2}} t^{\frac{n-3}{2}} (1-t)^{\frac{-1}{2}}\, dt}{(r^2 - \varrho_1^2)^{\frac{n-3}{2}} r^{2-n}} \\
&= \begin{cases} (r^2 - \varrho_1^2)^{\frac{1}{2}} \int_0^{1-\frac{\varrho_2^2}{r^2}} t^{\frac{-1}{2}} (1-t)^{\frac{-1}{2}}\, dt, & \text{if } n = 2, \\ r \int_0^{1-\frac{\varrho_2^2}{r^2}} (1-t)^{\frac{-1}{2}}\, dt, & \text{if } n = 3. \end{cases}
\end{aligned}$$

Thus, $\frac{\bar{\omega}_2(r)}{\bar{\omega}_1(r)}$ is strictly monotone increasing if $n = 2, 3$ and it is also strictly monotone increasing if $n > 3$ and $\varrho_1 \leq \varrho_2$. In these cases Lemma 4.3 implies $\mathcal{K} \equiv \bar{r}\mathcal{B}$. □

This theorem leaves open the case when $\varrho_1 > \varrho_2$ in dimensions $n > 3$. We have not yet tried to complete our theorem.

6 Discussion

Barker and Larman conjectured in [1, Conjecture 2] that in the plane $\mathcal{M}$-equisectioned convex bodies coincide, but they were unable to justify this in full.[3] Nevertheless they proved, among others, that a $\mathcal{D}$-isosectioned convex body $\mathcal{K}$ in the plane is a disc concentric to the disc $\mathcal{D}$.

Having a convex body $\mathcal{K}$ that is sphere-isocapped with respect to two concentric spheres raises the problem if there is a concentric ball $\bar{r}\mathcal{B}$—obviously sphere-isocapped with respect to that two concentric spheres—that is sphere-equicapped to $\mathcal{K}$ with respect to that two concentric spheres. The very same problem exists also for bodies that are sphere-isosectioned with respect to two concentric spheres. So we have the following *range characterization* problems: Let $0 < \varrho_1 < \varrho_2$ and let $c_1 > c_2 > 0$ be positive constants. Is there a convex body $\mathcal{K}$ containing the ball $\varrho_2\mathcal{B}$ in its interior and satisfying

(i) $c_1 \equiv \mathrm{C}_{\varrho_1;\mathcal{K}}$ and $c_2 \equiv \mathrm{C}_{\varrho_2;\mathcal{K}}$ (raised by Theorem 5.1)?
(ii) $c_1 \equiv \mathrm{S}_{\varrho_1;\mathcal{K}}$ and $c_2 \equiv \mathrm{S}_{\varrho_2;\mathcal{K}}$ (raised by Theorem 5.2)?
(iii) $c_1 \equiv \mathrm{S}_{\varrho_1;\mathcal{K}}$ and $c_1 \equiv \mathrm{C}_{\varrho_1;\mathcal{K}}$ (raised by Theorem 5.3)?

In the plane if $\mathcal{M}$ is allowed to shrink to a point (empty interior), then $\mathrm{S}_{\mathcal{M};\mathcal{K}}$ is the X-ray picture at a point source [3] investigated by Falconer in [2]. The method used in Falconer's article made Barker and Larman mention in [1] that in dimension 2 the convex body $\mathcal{K}$ can be determined from $\mathrm{S}_{\mathcal{M};\mathcal{K}}$ and $\mathrm{S}_{\mathcal{M}';\mathcal{K}}$ if $\partial\mathcal{M}$ and $\partial\mathcal{M}'$ are intersecting each other in a suitable manner. The method in the anticipated proof presented in [1] decisively depends on the condition of proper intersection.

Finally we note that determining a convex body by its constant width and constant brightness [8] sounds very similar a problem as the ones investigated in this paper. Moreover also the result is analogous to Theorem 5.3.

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