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Circle packings as differentiable manifolds

David Bauer · Kenneth Stephenson · Elias Wegert

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Abstract Circle packings are configurations of circles satisfying specified patterns of tangency and have emerged as the foundation for a fairly comprehensive theory of discrete analytic functions. Though many classical results found their counterpart in circle packing, other concepts have not yet been transferred, particularly those which require a linear structure. This paper puts circle packings in a framework of smooth manifolds, providing access to linear structures in their tangent spaces. Since we are especially interested in applications to boundary value problems (of Beurling and Riemann–Hilbert type), we do not only investigate the manifolds of circle packings and packing labels, but also manifolds formed by the centers of the boundary circles. The approach is elementary and rests on a detailed analysis of the contact equations which govern the tangency relation between neighboring circles.

Keywords Circle packing · Discrete analytic functions · Conformal geometry

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D. Bauer
Max Planck Institute MIS Leipzig, Inselstr. 22, 04103 Leipzig, Germany
e-mail: david.bauer@mis.mpg.de

K. Stephenson
Department of Mathematics, University of Tennessee at Knoxville,
Knoxville, TN 37996, USA
e-mail: kens@math.utk.edu

E. Wegert (✉)
Institute of Applied Analysis, TU Bergakademie Freiberg, Akademiestr. 6, 09596 Freiberg, Germany
e-mail: wegert@math.tu-freiberg.de

1 Introduction

During the last decades the theory of analytic functions found its discrete counterpart in circle packing. These configurations of circles form a quantum complex analysis—they mimic and approximate the behavior of classical analytic functions. The whole topic was launched with a talk of William Thurston in 1985 on the occasion of the proof of the Bieberbach conjecture. The book of Stephenson (2005) gives an overview of the present theory and lists a wide range of interrelations between circle packing and complex analysis. There are many open problems and some questions have not yet even been posed, particularly those which are related to the linear structure of analytic functions.

The aim of this paper is to describe circle packings in the framework of differentiable manifolds. We expect that this approach will provide the basis for a deeper understanding of circle packings and bring forth new aspects of the theory. In particular it will be inevitable for subsequent investigations, among which we mention univalence criteria, packings of specified regions, and discrete boundary value problems of Beurling and Riemann–Hilbert type. More generally, manifolds provide computational mechanisms for the type of open-ended experiments that have characterized the discrete theory to date.

We point out that the foundation of numerical methods is also one reason why we favored an approach based on the contact equations against more abstract topological concepts: As a useful side product we obtain the invertibility of certain matrices involved in the numerical algorithms for solving discrete boundary value problems [see Wegert and Bauer (2009); Wegert et al. (2011)].

The plan of the paper is as follows. In Sect. 2 we fix notations and summarize some facts from circle packing; an illustration may aid the reader unfamiliar with the topic. In Sect. 3 we introduce the manifolds $\mathcal{D}$ of circle packings and $\mathcal{D}^*$ of packing labels with fixed combinatorics, which are the fundamental objects of this paper. In Sects. 4 and 5 we prove that $\mathcal{D}$ and $\mathcal{D}^*$ are indeed differentiable submanifolds of appropriately chosen ambient spaces, and we parametrize their components by global charts. Since we are especially interested in describing circle packings by their boundary elements, we study the corresponding manifolds $\mathcal{C}$ and $\mathcal{C}^*$ of boundary centers in Section 6. Although attention is restricted in this paper to Euclidean geometry, the results generalize in straightforward fashion to hyperbolic and spherical geometry.

2 Definitions and notations

The study of circle packings in the context of conformal mapping was initiated by Koebe (1936), but quickly dropped out of sight. The success of the topic started with William Thurston's conjecture in 1985 of a discrete version of the famous Riemann Mapping Theorem via circle packing, which yields the continuous setting in the limit. This is now known as the Rodin–Sullivan Theorem and was firstly proven in Rodin and Sullivan (1987) two years after Thurston's talk. Apparently, we had to wait for the coming of the computer age to appreciate the beauty and mathematical richness inherent in circle packing.

The story starts with the observation that an analytic function maps infinitesimal circles to infinitesimal circles. If a function F is analytic in a domain, then it maps any circle c of small radius r centered at z to a curve $F(c)$ which approximates the circle of radius $|F'(z)|\,r$ centered at $F(z)$. Essentially, in circle packing these infinitesimal circles are replaced by real ones.

Each circle packing has a prescribed pattern of tangencies encoded in the underlying combinatorics which can be described in terms of simplicial complexes. The underlying complex K of a circle packing is an abstract simplicial 2-complex which is a triangulation of an oriented topological surface.

In the following we will assume that K is simply connected. If K is additionally finite and with nonempty boundary, it is called a *combinatorial closed disc*. Other possibilities include *combinatorial spheres*, *combinatorial open discs*, when K is simply connected, infinite, and without boundary, and mixed cases when K is simply connected, infinite, with non-empty boundary. However, in this paper we shall exclusively study circle packings for combinatorial closed discs. We denote the sets of vertices, edges and faces of K by V, E, F, respectively, and assume that

$$V = \{v_1, \dots, v_n\}, \quad E = \{e_1, \dots, e_p\}, \quad F = \{f_1, \dots, f_q\}.$$

The edge adjacent to v_i and v_j is denoted by $\langle v_i, v_j \rangle$, and faces are written as $\langle v_i, v_j, v_k \rangle$, where the vertices v_i, v_j, v_k are ordered according to their positive orientation. We further suppose that K has m boundary vertices and assume that these are $v_1, \dots, v_m$.

Since K is a combinatorial closed disc we have $n + q = p + 1$ by Euler's Theorem. By counting the edges of K in two different ways we obtain that $3q = 2p - m$, and elimination of q leads to the fundamental relation

$$p = 3n - m - 3. \tag{1}$$

Each vertex v_i has an associated *combinatorial flower* $\langle v_i; v'_1, v'_2, \dots, v'_k \rangle$, which consists of v_i and all its neighbors. Usually we assume that $v'_1, \dots, v'_k$ are arranged in positive order, so that $\langle v_i, v'_j, v'_{j+1} \rangle \in F$ for $j = 1, \dots, k-1$. If also $\langle v_i, v'_k, v'_1 \rangle \in F$ the flower is said to be *closed*, otherwise it is called *open*.

The standard geometries in which circle packings are studied are the complex plane, $\mathbb{C}$, with the Euclidean metric, the hyperbolic plane, $\mathbb{D}$, with the Poincaré metric, and the Riemann Sphere, $\mathbb{P}$, with the spherical metric.

Definition 2.1 (*Circle packing*). Let $\mathcal{G}$ be one of the standard geometries, $\mathbb{C}$, $\mathbb{D}$, or $\mathbb{P}$. A collection $P = \{c_v\}$ of circles in $\mathcal{G}$ is a circle packing for a complex K, if it satisfies the following:

(i) P has a circle c_v associated with each vertex v of K.
(ii) Two circles c_u, c_v are externally tangent whenever $\langle u, v \rangle$ is an edge of K.
(iii) If $\langle u, v, w \rangle$ is a (positively oriented) face of K, then the centers of the circles c_u, c_v, c_w form a positively oriented triangle.

Circles corresponding to the boundary or interior of K are termed *boundary circles* and *interior circles*, respectively. A circle packing consisting of a circle with all boundary circles tangent to it is called a *flower*.

A circle packing in which the circles have mutually disjoint interiors is said to be *univalent*. It is called *locally univalent* if the neighbors of each interior circle wrap once around it; circle packings which are not locally univalent contain one or more *branch points*, i.e. interior circles whose chain of neighboring circles wraps more than once around it. A precise definition is deferred to Sect. 3.

Maximal packings play a special role. For combinatorial closed discs the term maximal means that the packing is univalent and each boundary circle is internally tangent to $\mathbb{T} = \partial\mathbb{D}$.

The maximal packings for a fixed complex K are *essentially unique*, that is unique up to conformal automorphisms of the underlying space. The next theorem is the Discrete Uniformization Theorem for combinatorial closed discs. For a proof we refer to Chapters 6 and 8 of Stephenson (2005).

Theorem 2.2 (Maximal packings for combinatorial closed discs). *For every combinatorial closed disc K there exists an essentially unique univalent circle packing P_K in the unit disc $\mathbb{D}$ such that every boundary circle is internally tangent to the unit circle $\mathbb{T}$.*

If P_K denotes a maximal packing for a complex K, then functions of the form $f : P_K \to P$, can be considered as discrete counterparts of classical analytic functions on the unit disc $\mathbb{D}$. Figure 1 illustrates the maximal packing serving as domain on the left. Two Euclidean packings on the right share the same combinatorics. The first is univalent, and the discrete analytic function $f : P_K \to P_1$ is a discrete analogue of $z \mapsto (z + 1.1)^2$. The second has one branch point and the discrete analytic function $g : P_K \to P_2$ is a discrete analogue of $z \mapsto (z + 0.2)^2$. The boundary circles are shown to help visualize the mapping behaviors.

While the holomorphic functions in $\mathbb{D}$ form a linear space $\mathcal{O}(\mathbb{D})$, their discrete counterparts (circle packings) lack a linear structure. As a substitute we shall prove that all circle packings for a fixed finite complex K form a differentiable manifold and that every component of this manifold can be parameterized by a global chart. It is the tangent bundle of this manifold which provides circle packings with a *locally* linear structure useful both in the theory and the computation of circle packings.

3 Circle packings and packing labels

Throughout the paper K will always refer to a fixed combinatorial closed disc with n vertices, p edges and q faces. With a circle packing P for K we associate the radii $r = (r_1, \dots, r_n)$ and the centers $z = (z_1, \dots, z_n)$ of its circles. Thus the packing can be considered as an element of the ambient space $\mathbb{R}^n_+ \times \mathbb{C}^n$. The subset of circle packings (for the fixed combinatorial disc K) in $\mathbb{R}^n_+ \times \mathbb{C}^n$ is denoted by $\mathcal{D}$.

A convenient description of $\mathcal{D}$ in $\mathbb{R}^n_+ \times \mathbb{C}^n$ is based on the contact function, which checks if the two circles adjacent to every edge of K are mutually tangent.

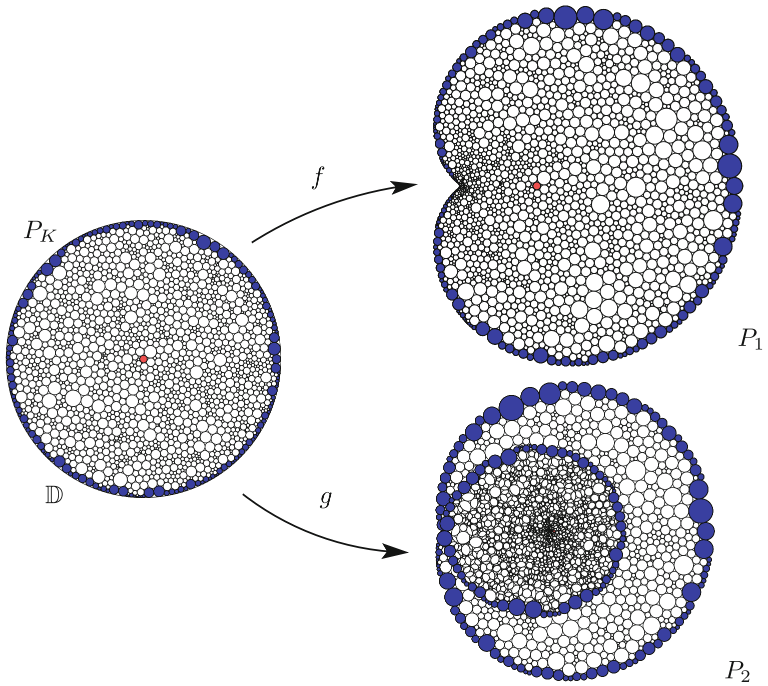

Fig. 1 Discrete analytic functions $f : z \mapsto (z + 1.1)^2$ and $g : z \mapsto (z + 0.2)^2$

Definition 3.1 (*Contact function*). The contact function $\omega : \mathbb{R}^n_+ \times \mathbb{C}^n \to \mathbb{R}^p$ is defined by $\omega = (\omega_1, \dots, \omega_p)$, with components ω_i given by

$$\omega_i(r, z) := (x_j - x_k)^2 + (y_j - y_k)^2 - (r_j + r_k)^2 \quad \text{for } e_i = \langle v_j, v_k \rangle \in E. \tag{2}$$

Here and in the following $x_j := \operatorname{Re} z_j$ and $y_j := \operatorname{Im} z_j$ denote the real and imaginary part of z_j, respectively.

Clearly, each circle packing (r, z) is a zero of ω. Conversely, each zero of ω corresponds to a configuration of circles in $\mathbb{C}$ satisfying (i) and (ii) in Definition 2.1, but ω does not feel orientations, and hence condition (iii) may be violated. In order to guarantee the correct orientation of the triples we consider the set

$$\mathcal{U} := \left\{ (r, z) \in \mathbb{R}^n_+ \times \mathbb{C}^n : 0 < \sphericalangle(z_i; z_j, z_k) < \pi \quad \text{for all } \langle v_i, v_j, v_k \rangle \in F \right\}, \tag{3}$$

where $\sphericalangle(z_i; z_j, z_k)$ denotes the oriented angle of the triple z_i, z_j, z_k at z_i,

$$\sphericalangle(z_i; z_j, z_k) := \arg \frac{z_k - z_i}{z_j - z_i} \quad \text{for } z_i \neq z_j.$$

The location of the centers $z_1, \dots, z_n$ of a circle packing in the plane implies $\mathcal{D} \subset \mathcal{U}$. Moreover, letting

$$\mathcal{Z} := \left\{(r, z) \in \mathbb{R}_+^n \times \mathbb{C}^n : \omega(r, z) = 0\right\} \tag{4}$$

we obtain that $\mathcal{D} = \mathcal{U} \cap \mathcal{Z}$.

Lemma 3.2 (Local characterization of $\mathcal{D}$). *There exists a neighborhood $\mathcal{U}$ of $\mathcal{D}$ in $\mathbb{R}_+^n \times \mathbb{C}^n$ such that for $(r, z) \in \mathcal{U}$ the conditions $(r, z) \in \mathcal{D}$ and $\omega(r, z) = 0$ are equivalent.*

It is important that there is another characterization of circle packings which mainly works with the radii. To describe this approach we start with the projection

$$\upsilon : \mathbb{R}_+^n \times \mathbb{C}^n \to \mathbb{R}_+^n, \ (r, z) \mapsto r$$

which forgets about the centers. In this context we refer to the elements of $\mathbb{R}_+^n$ as *labels* and the elements of the image set $\mathcal{D}^* := \upsilon(\mathcal{D})$ of circle packings $\mathcal{D}$ are called *packing labels*.

The reconstruction of a circle packing (r, z) from its packing label (radii) r according to the combinatorics of K goes with the name *layout*. In particular, one starts by placing one circle and an immediate (tangent) neighbor, a process with three degrees of freedom. One then places the remaining circles in turn, with each placement requiring that two contiguous neighbors already be in place. The following result ensures that this process does not run into a dead end.

Theorem 3.3 (Monodromy theorem, Stephenson (2005), Theorem 5.4). *For any packing label $r \in \mathcal{D}^*$ there exists a vector $z \in \mathbb{C}^n$ such that $(r, z) \in \mathcal{D}$. The vector z is unique up to a plane rigid motion of $\mathbb{C}^n$.*

By a plane rigid motion of $z \in \mathbb{C}^n$ we understand a rigid motion of $\mathbb{C}$ which acts on all components of z,

$$g_{\xi,\eta,\rho} : z \mapsto \mathrm{e}^{\mathrm{i}\rho} z + (\xi + \mathrm{i}\eta) \cdot \mathbb{1}, \quad \mathbb{1} := (1, \ldots, 1). \tag{5}$$

In order to distinguish a special representative of all possible layouts we designate an α-vertex v_α of K and one of its neighbors v_β, and place the centers z_α and z_β of the corresponding circles at the origin and on the positive real line, respectively.

We refer to the result of this procedure as the *standard layout*, $z = \gamma_0(r)$. An arbitrary layout $\gamma_{\xi,\eta,\rho}(r)$ of r is then obtained by post-composing the standard layout with a plane rigid motion $g_{\xi,\eta,\rho}$ of $\mathbb{C}^n$. The *layout parameters* ξ, η and ρ are related to the centers of the α and β circles of the packing by

$$\xi + \mathrm{i}\eta := z_\alpha, \quad \mathrm{e}^{\mathrm{i}\rho} := \frac{z_\beta - z_\alpha}{|z_\beta - z_\alpha|}.$$

In order to find an intrinsic description of packing labels we consider the angle sums of the triangles (faces) which meet at one vertex. For circle packings $P = (r, z) \in \mathcal{D}$ we have two options to compute angle sums. The standard definition involves only the labels (radii) r_i, while an alternative makes use of the centers z_i. Both definitions coincide on $\mathcal{D}$ but their extensions to the ambient space $\mathbb{R}^n_+ \times \mathbb{C}^n$ fall apart. For the second definition we further have to exclude the *exceptional set*

$$\mathcal{E} := \{(z_1, \ldots, z_n) \in \mathbb{C}^n : z_j = z_k \quad \text{for some } \langle v_j, v_k \rangle \in E\}. \tag{6}$$

Definition 3.4 (*Angle sum maps*). Let $(r, z) \in \mathbb{R}^n_+ \times \mathbb{C}^n$. Then for vertices $v_i, v_j, v_k \in V$ the *radial angle* at v_i from v_j to v_k is defined as

$$\varphi(v_i; v_j, v_k) := \arccos\left(\frac{(r_i + r_j)^2 + (r_i + r_k)^2 - (r_j + r_k)^2}{2(r_i + r_j)(r_i + r_k)}\right).$$

For pairwise distinct points $z_i, z_j, z_k \in \mathbb{C}^n$ the *vertex angle* at v_i is given by

$$\psi(v_i; v_j, v_k) := \arccos\left(\frac{|z_i - z_j|^2 + |z_i - z_k|^2 - |z_j - z_k|^2}{2|z_i - z_j||z_i - z_k|}\right).$$

Let $\langle v_i; v'_1, v'_2, \ldots, v'_k\rangle$ denote the combinatorial flower at $v_i \in V$ and set

$$\Phi_i(r, z) := \sum_{j=1}^{l} \varphi(v_i; v'_j, v'_{j+1}), \qquad \Psi_i(r, z) := \sum_{j=1}^{l} \psi(v_i; v'_j, v'_{j+1}), \tag{7}$$

where $l = k$ and $r_{k+1} = r_1$ if the flower is closed, and $l = k - 1$ otherwise. Then the *radial angle sum map* and the *vertex angle sum map* are

$$\Phi := (\Phi_1, \ldots, \Phi_n) : \mathbb{R}^n_+ \times \mathbb{C}^n \mapsto \mathbb{R}^n_+, \;\; \Psi := (\Psi_1, \ldots, \Psi_n) : \mathbb{R}^n_+ \times \left(\mathbb{C}^n \backslash \mathcal{E}\right) \mapsto \mathbb{R}^n.$$

In the following we shall frequently omit the irrelevant variables and simply write $\Phi(r)$ and $\Psi(z)$. The angle sums at the interior vertices are of special importance and we define the (radial and central) *interior angle sum maps* $\widetilde{\Phi} : \mathbb{R}^n_+ \to \mathbb{R}^{n-m}_+$ and $\widetilde{\Psi} : \mathbb{C}^n \backslash \mathcal{E} \to \mathbb{R}^{n-m}$ by

$$\widetilde{\Phi}_i(r) := \Phi_{m+i}(r), \quad \widetilde{\Psi}_i(z) := \Psi_{m+i}(z), \qquad i = 1, \ldots, n - m, \tag{8}$$

respectively. The function $\widetilde{\Phi}$ gives the desired description of packing labels.

Lemma 3.5 (Packing labels and branching). *A vector $r \in \mathbb{R}^n_+$ is a packing label if and only if*

$$\widetilde{\Phi}(r) = 2\pi\,(b + 1),$$

with a vector $b \in \mathbb{Z}^{n-m}_+$ of nonnegative integers and $1 := (1, \ldots, 1) \in \mathbb{Z}^{n-m}_+$.

The vector $b = (b_1, \ldots, b_{n-m})$ is called the *branching* of the packing label r and its components b_j are referred to as the *branch orders* at the corresponding inner vertices. If all entries of b are zero, then the packing label is *unbranched* or *locally univalent*.

The set of all packing labels with branching b is denoted by $\mathcal{D}_b^*$, and $\mathcal{D}_b := \upsilon^{-1}(\mathcal{D}_b^*)$ denotes the related set of circle packings.

It is not difficult to see that the number of non–empty sets $\mathcal{D}_b^*$ is finite. Somewhat surprising, there is even a necessary and sufficient condition for $\mathcal{D}_b^* \neq \emptyset$. To formulate it we need the following definition.

Definition 3.6 (*Branch structure*). A vector $b \in \mathbb{Z}_+^{n-m}$ is said to be a *branch structure* for K, if it satisfies the following condition:
If $\gamma = \{e_1, e_2, \ldots, e_k\}$ is any simple closed edge path in K, and N denotes the sum of the branch orders b_i of all vertices in K which are interior to γ, then $k > 2N + 2$.

The next result is a combination of Theorem 11.5 and Theorem 11.6 in Stephenson (2005) and belongs to the miracles of circle packing. It does not only answer the question about the existence of packing labels with prescribed branching, but parametrizes the sets $\mathcal{D}_b^*$ by their boundary labels.

Theorem 3.7 (Packing labels with specified branching). *The set $\mathcal{D}_b^*$ of packing labels with branching b is non-void if and only if b is a branch structure for K. In this case each vector $(r_1, \ldots, r_m) \in \mathbb{R}_+^m$ of boundary labels for K admits a unique extension to a packing label $r = (r_1, \ldots, r_n) \in \mathbb{R}_+^n$ in $\mathcal{D}_b^*$.*

It is useful to rephrase the result. For a given branch structure b the theorem shows that there is a unique function

$$\varrho_b^* : \mathbb{R}_+^n \to \mathcal{D}_b^*, \ (r_1, \ldots, r_m) \mapsto (r_1, \ldots, r_n)$$

which extends an arbitrary boundary label $(r_1, \ldots, r_m)$ to a packing label with branching b. Denoting by $\widetilde{\varrho}_b^*$ the projection of ϱ_b^* onto its last $n - m$ components, we obtain that $\mathcal{D}_b^*$ is the graph of $\widetilde{\varrho}_b^*$. In the next section we shall prove that the functions ϱ_b^* are smooth.

4 The manifold of packing labels

Let us briefly fix some notations. If $f : M \to N$ is a differentiable map between smooth manifolds the differential of f is denoted by $\mathrm{d}f : TM \to TN$ and the differential of f at $x \in M$ is $\mathrm{d}_x f : T_x M \to T_{f(x)} N$. We also use the notation $\mathrm{d}f(x) := \mathrm{d}_x f$ and identify $\mathrm{d}f(x)$ with the Jacobian $\mathrm{D}f(x)$ if this is appropriate.

A parameter line on M is a mapping of an interval into M. The tangent vectors along a parameter line ξ are denoted by $\partial\xi$.

Usually M and N are submanifolds of appropriate ambient Euclidean spaces which are composed from several distinct subspaces. In this context integer subscripts indicate components or coordinates of an object. If, for example the ambient space is $\mathbb{R}^n \times \mathbb{R}^n$, its elements are written as (x, y) with $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$.

In order to distinguish between vectors in the base space and tangent vectors we write the latter as (dx, dy) with $dx = (dx_1, \ldots, dx_n)$, $dy = (dy_1, \ldots, dy_n)$. These notations are convenient since they allow to keep track of the geometric meaning of the variables. They are further motivated by the fact that the coordinates ξ_j of a tangent vector $\partial\xi$ in the standard basis x_i are the values of the coordinate differential $\mathrm{d}x_j$ on $\partial\xi$,

$$\partial\xi \simeq (\xi_1, \ldots, \xi_n) := \big(\mathrm{d}x_1(\partial\xi), \ldots, \mathrm{d}x_n(\partial\xi)\big) \equiv (dx_1, \ldots, dx_n).$$

The differential of a mapping $f : x \mapsto y$ is then written as

$$\mathrm{d}_x f : (dx_1, \ldots, dx_n) \mapsto \mathrm{D}f \cdot (dx_1, \ldots, dx_n).$$

The following lemma is the key to the proof that $\mathcal{D}^*$ is a smooth embedded m-dimensional submanifold of $\mathbb{R}^n_+$.

Lemma 4.1 (Jacobian of the radial angle sum map). *For any label $r \in \mathbb{R}^n_+$ the kernel of the Jacobian $\mathrm{D}\Phi(r)$ of the radial angle sum map Φ at r is one dimensional and consists of the scalar multiples of r.*

Proof Suppose the assertion was not true, i.e., $\ker \mathrm{D}\Phi(r)$ contains a vector dr which is not a scalar multiple of r. Then we can find a vertex v_J, such that

$$\frac{dr_J}{r_J} = \max\left\{\frac{dr_i}{r_i} : i = 1, \ldots, n\right\} \text{ and } \frac{dr_J}{r_J} > \frac{dr_j}{r_j} \tag{9}$$

for some neighbor v_j of v_J in K. Let $\langle v_J; v_1, v_2, \ldots, v_k\rangle$ denote the combinatorial flower at v_J. Differentiating (7) shows that

$$\begin{aligned}\mathrm{D}\Phi_J(r)\, dr &= \sum_{i=1}^{l} \mathrm{d}\varphi(r_J; r_i, r_{i+1})(r)\, dr \\ &= \sum_{i=1}^{l} \sqrt{\frac{r_J r_i r_{i+1}}{r_J + r_i + r_{i+1}}} \left(\frac{1}{r_J + r_i}\left(\frac{dr_i}{r_i} - \frac{dr_J}{r_J}\right) + \frac{1}{r_J + r_{i+1}}\left(\frac{dr_{i+1}}{r_{i+1}} - \frac{dr_J}{r_J}\right)\right),\end{aligned} \tag{10}$$

where $l = k$ and $r_{k+1} = r_1$ if the flower is closed, and $l = k - 1$ otherwise. Since all radii are strictly positive, we must have $\mathrm{D}\Phi_J(r)\, dr < 0$ by (9), contradicting the assumption that $\mathrm{D}\Phi(r)\, dr = 0$. □

Lemma 4.2 (Linearly independent rows of $\mathrm{D}\Phi$). *For every $r \in \mathbb{R}^n_+$ any $n - 1$ rows of the Jacobian $\mathrm{D}\Phi(r)$ are linearly independent.*

Proof Pick any vertex v_j. To simplify notations we renumber the vertices such that its combinatorial flower is given by $\langle v_j; v_1, v_2, \ldots, v_k\rangle$. A straightforward computation using (10) yields that the derivatives of Φ_i satisfy

$$-\frac{\partial\Phi_j}{\partial r_j}(r) = \sum_{i=1}^{l} \sqrt{\frac{r_j r_i r_{i+1}}{r_j + r_i + r_{i+1}}} \left(\frac{1}{(r_j + r_i)r_j} + \frac{1}{(r_j + r_{i+1})r_j}\right) = \sum_{i=1}^{k} \frac{\partial\Phi_i}{\partial r_j}(r),$$

where $l = k$ and $r_{k+1} = r_1$ if the flower is closed, and $l = k - 1$ otherwise. Moreover, we have $(\partial \Phi_i / \partial r_j)(r) = 0$ if $i \neq j$ and $i > k$. Thus the sum of the elements in each column of $\mathrm{D}\Phi(r)$ is zero, so that the vector $(1, \ldots, 1)$ belongs to the kernel of $(\mathrm{D}\Phi)^\top(r)$. By Lemma 4.1 we have

$$\dim \ker (\mathrm{D}\Phi)^\top(r) = \dim \ker \mathrm{D}\Phi(r) = 1,$$

which then implies that the kernel of $(\mathrm{D}\Phi)^\top(r)$ is spanned by $(1, \ldots, 1)$. If $n - 1$ rows of $\mathrm{D}\Phi(r)$ were linearly dependent, then the kernel of $(\mathrm{D}\Phi)^\top(r)$ would contain a nonzero vector $(\lambda_1, \ldots, \lambda_n)$ with at least one entry λ_i equal zero, which is a contradiction. □

Recall that the interior radial angle sum map $\widetilde{\Phi}$ is the projection of Φ onto its last $n - m$ components. By Lemma 4.2 the Jacobian $\mathrm{D}\widetilde{\Phi}$, which consists of the last $n - m$ rows of $\mathrm{D}\Phi$, has maximal rank.

Corollary 4.3 *The map* $\widetilde{\Phi} : \mathbb{R}_+^n \to \mathbb{R}_+^{n-m}$ *is a submersion.*

Lemma 4.4 (Linearly independent columns of $\mathrm{D}\widetilde{\Phi}$). *For every* $r \in \mathbb{R}_+^n$ *the last* $n - m$ *columns of the Jacobian* $\mathrm{D}\widetilde{\Phi}(r)$ *are linearly independent.*

Proof We use a similar approach as in Lemma 4.1. Suppose $\ker \mathrm{D}\widetilde{\Phi}(r)$ contains a nonzero vector dr satisfying $dr_1 = \cdots = dr_m = 0$. Without loss of generality we may assume that at least one entry of dr is positive, otherwise consider $-dr$. Then we can choose an interior vertex v_J with $J > m$ such that (9) holds with some neighbor v_j of v_J in K. Using (10) leads to the contradiction $\mathrm{D}\widetilde{\Phi}_{J-m}(r)\, dr = \mathrm{D}\Phi_J(r)\, dr < 0$. Hence $dr = 0$ follows. □

After these preparations the first main result follows immediately. Recall that $\mathcal{D}_b^*$ denotes the set of all packing labels with prescribed branching b.

Theorem 4.5 (The manifold of packing labels). *The set* $\mathcal{D}^*$ *of packing labels is a* m*-dimensional submanifold of* $\mathbb{R}_+^n$. *For every branch structure* b *the projection*

$$\pi_b^* : \mathcal{D}_b^* \to \mathbb{R}_+^m, \; (r_1, \ldots, r_n) \mapsto (r_1, \ldots, r_m),$$

which assigns to $P^* \equiv (r_1, \ldots, r_n) \in \mathcal{D}^*$ *its boundary labels* $r_1, \ldots, r_m$ *is a diffeomorphism.*

Proof The set $\mathcal{D}^*$ is a finite union of $\mathcal{D}_b^*$ over all branch structures b for the complex K. Since

$$\mathcal{D}_b^* = \widetilde{\Phi}^{-1}\left(2\pi(b + 1)\right), \quad \text{with } 1 := (1, \ldots, 1)$$

is the preimage of a point in $\mathbb{R}_+^{n-m}$ with respect to the submersion $\widetilde{\Phi}$, it is a smooth closed submanifold of $\mathbb{R}_+^n$ and

$$T_r \mathcal{D}_b^* = \ker \mathrm{D}\widetilde{\Phi}(r) \tag{11}$$

(see Diedonné (1972), Theorem 16.8.8). Moreover, we already know that the inverse of π_b^*,

$$\varrho_b^* : \mathbb{R}_+^m \to \mathcal{D}_b^*, \ (r_1, \ldots, r_m) \mapsto (r_1, \ldots, r_n),$$

which extends a boundary label $(r_1, \ldots, r_m) \in \mathbb{R}^+$ to a packing label in $\mathcal{D}_b^*$, is bijective, and Lemma 4.4 allows an application of the Implicit Function Theorem showing its differentiability. □

Corollary 4.6 (Parametrization of packing labels). *The mapping π_b^* is a global chart on $\mathcal{D}_b^*$. Its inverse ϱ_b^* is a regular parametrization of $\mathcal{D}_b^*$ and for every $r \in \mathcal{D}$ the tangent vectors $\partial r_1, \ldots, \partial r_m$ to the parameter lines of $r_1, \ldots, r_m$ span the tangent space $T_r\mathcal{D}^*$ at every point $r \in \mathcal{D}^*$.*

5 The manifold of circle packings

In this section we show that the circle packings $\mathcal{D}$ form a smooth $m + 3$–dimensional submanifold of $\mathbb{R}_+^n \times \mathbb{C}^n$. Every component $\mathcal{D}_b$ can be parametrized by the boundary radii $r_1, \ldots, r_m$ and three additional real parameters ξ, η, ρ associated with plane rigid motions.

In fact we shall give two (almost) independent proofs based on Theorem 4.5. The first (short) one uses the smoothness of the layout. The second proof gives the extra information that the contact function ω has maximal rank, which is also of practical importance in computations with circle packings.

Lemma 5.1 (Smoothness of layout). *The standard layout $\gamma_0 : \mathcal{D}^* \to \mathbb{C}^n$ is a smooth map.*

Proof Obviously the centers of the α-circle and the β-circle depend smoothly on r. Proceeding by induction we assume that two centers z_i, z_j depend smoothly on r and show that the same holds for z_k if the corresponding vertices v_i, v_j, v_k form a face of K. Differentiating the contact equations

$$\omega_i(x_k, y_k) := (x_i - x_k)^2 + (y_i - y_k)^2 - (r_i + r_k)^2 = 0,$$
$$\omega_j(x_k, y_k) := (x_j - x_k)^2 + (y_j - y_k)^2 - (r_j + r_k)^2 = 0$$

for the edges $v_i v_k$ and $v_j v_k$ with respect to x_k and y_k yields the Jacobian

$$J = \begin{bmatrix} 2\,(x_k - x_i), & 2\,(y_k - y_i) \\ 2\,(x_k - x_j), & 2\,(y_k - y_j) \end{bmatrix}.$$

This matrix is regular since the centers z_i, z_j, z_k of a face are not collinear, and the Implicit Function Theorem gives the desired result. □

Recall that the general layout of a packing label $P^* \equiv r$ is the composition of the standard layout with the plane rigid motion $g_{\xi,\eta,\rho}$ introduced in (5). More precisely we define

$$\gamma : \mathcal{D}^* \times \mathbb{C} \times \mathbb{T} \to \mathbb{R}^n_+ \times \mathbb{C}^n_+,\ (r, \xi + \mathrm{i}\eta, \mathrm{e}^{\mathrm{i}\rho}) \mapsto \big(r, (g_{\xi,\eta,\rho} \circ \gamma_0)(r)\big).$$

It is clear that $g_{\xi,\eta,\rho}$ depends smoothly on the layout parameters ξ, η, ρ. The (tangent) vectors $\partial\xi$, $\partial\eta$, $\partial\rho$ to the parameter lines ξ, η and ρ of γ at r are

$$\partial\xi = (0, 1), \quad \partial\eta = (0, \mathrm{i}\,1), \quad \partial\rho = (0, \mathrm{i}\, e^{\mathrm{i}\rho}\, \gamma_0(r))$$

with $1 := (1, \ldots, 1)$. They are linearly independent (as elements of $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \equiv \mathbb{R}^n \times \mathbb{C}^n$) and belong to the subspace $\{0\} \times \mathbb{C}^n$, and hence we can lift the parametrization ϱ^*_b of $\mathcal{D}^*_b$ to a regular parametrization ϱ_b of $\mathcal{D}_b$:

$$\varrho_b : \mathbb{R}^m \times \mathbb{C} \times \mathbb{T} \to \mathcal{D}_b,\ (r_1, \ldots, r_m, \xi + \mathrm{i}\eta, \mathrm{e}^{\mathrm{i}\rho}) \mapsto \gamma\big(\varrho^*_b(r_1, \ldots, r_m), \xi + \mathrm{i}\eta, \mathrm{e}^{\mathrm{i}\rho}\big).$$

By virtue of Theorem 4.5 the mapping ϱ_b is a diffeomorphism. Denoting by π_b its inverse we summarize the results in the following theorem.

Theorem 5.2 (The manifold of circle packings). *The set $\mathcal{D}$ of circle packings for a combinatorial closed disc K is a smooth submanifold of $\mathbb{R}^n_+ \times \mathbb{C}^n$ of dimension $m+3$. For every branch structure b for K the mapping*

$$\pi_b : \mathcal{D}_b \to \mathbb{R}^m_+ \times \mathbb{C} \times \mathbb{T},\ (r_1, \ldots, r_n, z_1, \ldots, z_n) \mapsto (r_1, \ldots, r_m, \xi + \mathrm{i}\eta, \mathrm{e}^{\mathrm{i}\rho})$$

with $\xi + \mathrm{i}\eta := z_\alpha$, $\mathrm{e}^{\mathrm{i}\rho} := (z_\beta - z_\alpha)/(|z_\beta - z_\alpha|)$ is a diffeomorphism.

Corollary 5.3 (Parametrization of circle packings). *For every branch structure b the mapping $\pi_b : \mathcal{D}_b \to \mathbb{R}^m_+ \times \mathbb{C} \times \mathbb{T}$ is a global chart on $\mathcal{D}_b$. Its inverse ϱ_b is a regular parametrization of $\mathcal{D}_b$ and the tangent vectors $\partial r_1, \ldots, \partial r_m, \partial\rho, \partial\xi, \partial\eta$ to the parameter lines of ϱ_b span the tangent space $T_{(r,z)}\mathcal{D}$ at every point (r, z) of $\mathcal{D}$.*

For an alternative approach to Theorem 5.2 we show that the contact function ω has maximal rank. The components of ω are given by (2) and differentiation leads to

$$\mathrm{d}\omega_i = 2(x_j - x_k)(\mathrm{d}x_j - \mathrm{d}x_k) + 2(y_j - y_k)(\mathrm{d}y_j - \mathrm{d}y_k) - 2(r_j + r_k)(\mathrm{d}r_j + \mathrm{d}r_k)$$

with $(v_j, v_k) = e_i \in E$. Then the Jacobian $\mathrm{D}\omega(r, z)$ of ω at (r, z) is a real matrix of dimension $p \times 3n$ with the entries $2\omega_{ij}$, where

$$\omega_{ij} := \begin{cases} -(r_j + r_k) & \text{for } e_i = v_j v_k \quad \text{and } 1 \le j \le n, \\ x_j - x_k & \text{for } e_i = v_{j-n} v_k \quad \text{and } n+1 \le j \le 2n, \\ y_j - y_k & \text{for } e_i = v_{j-2n} v_k \quad \text{and } 2n+1 \le j \le 3n \end{cases} \tag{12}$$

and $\omega_{ij} := 0$ otherwise. The submatrices $\mathrm{D}_r\omega$, $\mathrm{D}_z\omega$, and $\mathrm{D}_e\omega$ composed from the first n columns, the last $2n$ columns and the last $3n - m$ columns of $\mathrm{D}\omega$ are referred to as the *radial part*, the *central part*, and the *essential part* of $\mathrm{D}\omega$, respectively.

Lemma 5.4 (Rank and kernel of $\mathrm{D}_z\omega$). *Let* $(r, z) \in \mathcal{D}$ *with* $z = x + \mathrm{i}y$. *Then the rank of the central part* $\mathrm{D}_z\omega(r, z)$ *is* $2n - 3$ *and*

$$\ker \mathrm{D}_z\omega(r, z) = \operatorname{span}\{(1, 0), (0, 1), (-y, x)\} \tag{13}$$

Proof If (dx, dy) lies in the kernel of $\mathrm{D}_z\omega$ we have

$$(x_j - x_k)(dx_j - dx_k) + (y_j - y_k)(dy_j - dy_k) = 0$$

for all $(j, k) \in E$. Geometrically this means that for every edge $v_j v_k \in E$ the lines through $\mathrm{i}z_j, \mathrm{i}z_k$ and dz_j, dz_k are parallel. For any face $\langle v_i, v_j, v_k\rangle \in F$ the triangles formed by $\mathrm{i}z_i, \mathrm{i}z_j, \mathrm{i}z_k$ and dz_i, dz_j, dz_k must be similar with ratio

$$r(\langle v_i, v_j, v_k\rangle) := \frac{|z_i - z_j|}{|dz_i - dz_j|}.$$

Since the ratios corresponding to neighboring faces must be the same, and K is connected, all ratios coincide. Hence the tuple $dz := dx + \mathrm{i}\,dy \in \mathbb{C}^n$ is obtained from $\mathrm{i}z \in \mathbb{C}^n$ by dilation and translation, $dz = \mathrm{i}z\,d\rho + (d\xi + \mathrm{i}\,d\eta)\cdot 1$ with $d\rho,\ d\xi,\ d\eta \in \mathbb{R}$. This yields (13), and $\dim\ker \mathrm{D}_z\omega(r, z) = 3$ implies that $\operatorname{rank} \mathrm{D}_z\omega(r, z) = 2n - 3$. □

The image of $\mathrm{D}_z\omega(r, z)$ equals the orthogonal complement of the kernel of $(\mathrm{D}_z\omega(r, z))^\top$ and hence

$$\dim\left(\ker\left(\mathrm{D}_z\omega(r, z)^\top\right)\right) = p - \operatorname{rank}\ (\mathrm{D}_z\omega(r, z))) = p - (2n - 3) = n - m.$$

We point out that there is a basis of the kernel of $(\mathrm{D}_z\omega(r, z))^\top$, where every basis vector corresponds to an associated interior vertex and all entries are expressed explicitly in geometric terms of the flower of that vertex. For details we refer to Chapter 4.4 of Bauer (2009).

The next lemma is an auxiliary result about the central angle sum and will be needed in the proof of Lemma 5.6.

Lemma 5.5 *If* $(r, z) \in \mathcal{D}$, *the function* $\widetilde{\Psi}$ *is locally constant at* z.

Proof If $(r, z) \in \mathcal{D}$, then the value $\widetilde{\Psi}_i(z)$ is equal to $2\pi(b_i + 1)$, where b_i is the branching order of the packing label at the vertex v_{i+m}. Geometrically, $b_i + 1$ is the winding number of the polygonal line formed by the chain of neighboring centers of z_{i+m} about z_{i+m}. This line does not meet z_{i+m}, and hence its winding number is stable with respect to small perturbations of the centers z_j. □

Lemma 3.2 tells us that $\omega = 0$ defines $\mathcal{D}$ locally as a submanifold of $\mathbb{C}^n \times \mathbb{R}^n_+$. This alone does not guarantee that the kernel of $\mathrm{D}\omega$ is the tangent space of $\mathcal{D}$, but the following lemma is a first step to prove this fact.

Lemma 5.6 (Kernel of differential of contact function). *Let* $(r, z) \in \mathcal{D}$ *and* $(dr, dx, dy) \in \mathbb{R}^{3n}$. *Then* $(dr, dx, dy) \in \ker \mathrm{D}\omega(r, z)$ *implies that* $dr \in T_r\mathcal{D}^*$.

Proof Differentiating the function $f(s,t,u) := \arccos\left((s+t-u)/(2\sqrt{st})\right)$ yields

$$\begin{aligned} \mathrm{d}f &= \frac{t-u-s}{2s\sqrt{4su-(u+s-t)^2}}\,\mathrm{d}s + \frac{s-t-u}{2t\sqrt{4tu-(t+u-s)^2}}\,\mathrm{d}t \\ &\quad + \frac{1}{\sqrt{4st-(s+t-u)^2}}\,\mathrm{d}u. \end{aligned} \tag{14}$$

Suppose we have $s = a^2, t = b^2, u = c^2$, where a, b, c are the sides of a triangle. Then the law of cosines implies

$$4st - (s+t-u)^2 = 4a^2b^2 - \left(a^2+b^2-c^2\right)^2 = 4a^2b^2 - (2ab\cos\gamma)^2 > 0.$$

Similarly, $4su-(u+s-t)^2 > 0$ and $4tu-(t+u-s)^2 > 0$ follows.

To shorten notation let $h := (dr, dx, dy)$. From $\mathrm{D}\omega(r,z)\,h = 0$ we obtain for every $e_i = v_j v_k \in E$ that

$$\begin{aligned} &\mathrm{d}\left(|z_j - z_k|^2\right)(r,z)h = \mathrm{d}\left((x_j-x_k)^2 + (y_j-y_k)^2\right)(r,z)h \\ &\quad = 2(x_j-x_k)(\mathrm{d}x_j(r,z)h - \mathrm{d}x_k(r,z)h) + 2(y_j-y_k)(\mathrm{d}y_j(r,z)h - \mathrm{d}y_k(r,z)h) \\ &\quad = 2(r_j+r_k)(\mathrm{d}r_j(r,z)h + \mathrm{d}r_k(r,z)h) = \mathrm{d}\left((r_j+r_k)^2\right)(r,z)h. \end{aligned}$$

Let v_i, v_j, v_k be the vertices of a triangle in K. Plugging the last line into (14) gives

$$\begin{aligned} \mathrm{d}\varphi(v_i; v_j, v_k)(r,z)h &= \mathrm{d}f\left((r_i+r_j)^2, (r_i+r_k)^2, (r_j+r_k)^2\right)(r,z)h \\ &= \frac{\partial f}{\partial s}(r,z)\mathrm{d}\left((r_i+r_j)^2\right)(r,z)h + \frac{\partial f}{\partial t}(r,z)\mathrm{d}\left((r_i+r_k)^2\right)(r,z)h \\ &\quad + \frac{\partial f}{\partial u}(r,z)\mathrm{d}\left((r_j+r_k)^2\right)(r,z)h \\ &= \frac{\partial f}{\partial s}(r,z)\mathrm{d}\left(|z_i-z_j|^2\right)(r,z)h + \frac{\partial f}{\partial t}(r,z)\mathrm{d}\left(|z_i-z_k|^2\right)(r,z)h \\ &\quad + \frac{\partial f}{\partial u}(r,z)\mathrm{d}\left(|z_j-z_k|^2\right)(r,z)h \\ &= \mathrm{d}f\left(|z_i-z_j|^2, |z_i-z_k|^2, |z_j-z_k|^2\right)(r,z)h \\ &= \mathrm{d}\psi(v_i; v_j, v_k)(r,z)h. \end{aligned} \tag{15}$$

Note that the partial derivatives $\partial f/\partial s$, $\partial f/\partial t$ and $\partial f/\partial u$ are well-defined at (r,z), since $|z_i - z_j| = r_i + r_j$, $|z_i - z_k| = r_i + r_k$ and $|z_j - z_k| = r_j + r_k$ are the lengths of sides of a triangle.

Finally, we pick an $i \in \{1, 2, \ldots, n-m\}$ and consider the combinatorial flower $\langle v_{m+i}; v'_1, \ldots, v'_k\rangle$ at v_{m+i}. By virtue of (15) and Lemma 5.5 we obtain

$$\mathrm{D}\widetilde{\Phi}_i(r)dr = \mathrm{D}\Phi_{m+i}(r,z)h = \sum_{j=1}^{k} \mathrm{d}\varphi(v_{m+i}; v'_j, v'_{j+1})(r,z)h$$

$$= \sum_{j=1}^{k} \mathrm{d}\psi(v_{m+i}; v'_j, v'_{j+1})(r,z)h = \mathrm{D}\Psi_{m+i}(r,z)h = \mathrm{D}\widetilde{\Psi}_i(z)(dx,dy) = 0.$$

By (11) the kernel of $\mathrm{D}\widetilde{\Phi}(r)$ coincides with $T_r\mathcal{D}^*$, which completes the proof. □

After these preparations it becomes fairly easy to show that the rank of $\mathrm{D}\omega$ is maximal—it even suffices to consider its essential part.

Lemma 5.7 *For* $(r,z) \in \mathcal{D}$ *the essential part* $\mathrm{D}_e\omega(r,z)$ *of* $\mathrm{D}\omega(r,z)$ *has rank* $3n - m - 3$.

Proof The essential part $\mathrm{D}_e\omega(r,z)$ is a matrix of size $p \times 3n - m$. In order to verify that it has column rank $3n - m - 3$ we pick any vector $(dr, dx, dy) \in \ker \mathrm{D}\omega(r,z)$ with $dr_1 = \ldots = dr_m = 0$. Then we obtain from Lemma 5.6 that $dr \in T_r\mathcal{D}^*$. By Corollary 4.6 this tangent space is spanned by the tangent vectors $\partial r_1, \ldots, \partial r_m$ so that $dr_1 = \ldots = dr_m = 0$ implies $dr = 0$. It then follows that $(0, dx, dy) \in \ker \mathrm{D}\omega(r,z)$, i.e. (dx, dy) belongs to the kernel of $\mathrm{D}_z\omega(r,z)$, which has dimension 3 (see Lemma 5.4). Since this conclusion also works the other way around we obtain $\dim\ker \mathrm{D}_e\omega(r,z) = 3$, which proves the claim. □

The next result describes the manifold of circle packings locally as the zero set of the contact function ω.

Theorem 5.8 (Circle packings via contact function). *There exists a neighborhood* $\mathcal{U}$ *of* $\mathcal{D}$ *in* $\mathbb{R}^n \times \mathbb{C}^n$ *such that*

$$\mathcal{D} = \{(r,z) \in \mathcal{U} : \omega(r,z) = 0\}$$

and $\omega : \mathcal{U} \to \mathbb{R}^p$ *is a submersion, i.e.,* $\mathrm{d}\omega$ *has rank* p.

Proof The Jacobian of ω has p rows and $3n$ columns and by (1) we have $p = 3n - m - 3$. Thus Lemma 5.7 shows that the rank is maximal on $\mathcal{D}$. This remains valid in a neighborhood $\mathcal{U}$ of $\mathcal{D}$, which can be chosen so small that the assertion of Lemma 3.2 holds. □

Invoking a standard result we again obtain that $\mathcal{D} = \mathcal{U} \cap \omega^{-1}(0)$ is a smooth closed submanifold of $\mathbb{R}^n \times \mathbb{C}^n$ of dimension $3n - p$, which is equal to $m + 3$.

In order to explore the relation between the manifolds of circle packings and packing labels a little further, we consider the three-dimensional Lie group G of rigid motions of the plane $\mathbb{R}^2 \equiv \mathbb{C}$. The element g of G which acts as $g : z \mapsto \mathrm{e}^{\mathrm{i}\varphi}z + a$ is represented as $g = (\mathrm{e}^{\mathrm{i}\varphi}, a) \in \mathbb{T} \times \mathbb{C}$. In this representation the product of two elements $g = (\mathrm{e}^{\mathrm{i}\varphi}, a)$ and $h = (\mathrm{e}^{\mathrm{i}\psi}, b)$ is

$$g\,h = (\mathrm{e}^{\mathrm{i}(\varphi+\psi)}, \mathrm{e}^{\mathrm{i}\varphi}b + a).$$

For $g = (e^{i\varphi}, a) \in G$ the left translation of G on $\mathbb{C}^n$ is a plane rigid motion of $z \in \mathbb{C}^n$,

$$z \mapsto e^{i\varphi} z + a \cdot 1.$$

It is not difficult to see that this action of G on $\mathbb{C}^n$ is smooth and proper. If we remove the exceptional set $\mathcal{E}$ defined in (6), then $\mathbb{C}^n \backslash \mathcal{E}$ is G-stable and the action of G on $\mathbb{C}^n \backslash \mathcal{E}$ is free. The same holds for two actions α and β of G on $\mathbb{R}^n_+ \times (\mathbb{C}^n \backslash \mathcal{E})$ and on $\mathbb{R}^m_+ \times \mathbb{T} \times \mathbb{C}$, which are defined by

$$\begin{aligned} \alpha &: (g, r, z) \mapsto \big(r, e^{i\varphi} z + a \cdot 1\big), \\ \beta &: \big(g, r, e^{i\rho}, \xi + i\eta\big) \mapsto \big(r, e^{i(\rho+\varphi)}, e^{i\varphi}(\xi + i\eta) + a\big), \end{aligned}$$

respectively, where $g = \big(e^{i\varphi}, a\big)$. If $(r, z) \in \mathbb{R}^n \times (\mathbb{C}^n \backslash \mathcal{E})$ is a circle packing, then $\alpha(g, (r, z))$ describes the packing rotated by the action of g. Consequently all components $\mathcal{D}_b$ are G-stable. The action β has a similar interpretation for the vector $\big(r, e^{i\rho}, \xi + i\eta\big)$ which consists of the boundary labels and (potential) layout parameters. Then the diagram

$$\begin{array}{ccc} \mathcal{D}_b & \xrightarrow[\cong]{\pi_b} & \mathbb{R}^m_+ \times \mathbb{T} \times \mathbb{C} \\ {\scriptstyle \alpha_g}\downarrow & & \downarrow{\scriptstyle \beta_g} \\ \mathcal{D}_b & \xrightarrow[\cong]{\pi_b} & \mathbb{R}^m_+ \times \mathbb{T} \times \mathbb{C} \end{array}$$

commutes, so that the mapping π_b is a G-equivariant diffeomorphism between $\mathcal{D}_b$ and $\mathbb{R}^m_+ \times \mathbb{T} \times \mathbb{C}$. It induces a diffeomorphism π_b^* between the corresponding factor spaces $\mathcal{D}_b/G$ and $(\mathbb{R}^m_+ \times \mathbb{T} \times \mathbb{C})/G$ with inverse ϱ_b^*. The latter space is diffeomorphic to $\mathbb{R}^m_+$ and hence to $\mathcal{D}_b^*$. This confirms the intuitive understanding that the manifold of packing labels can be interpreted as the orbit space of the manifold of circle packings with respect to the group of plane rigid motions, $\mathcal{D}_b^* \cong \mathcal{D}_b/G$.

6 Boundary center manifolds

One purpose of this paper is to prepare the investigation of boundary value problems for circle packings, which are discrete counterparts of the classical (linear and nonlinear) Riemann–Hilbert problems [see Wegert and Bauer (2009)]. The last section is therefore devoted to manifolds formed by boundary values of circle packings and packing labels.

It should be noted that the concept of boundary values in circle packing is a little ambiguous. The simplest and most straightforward approach uses the (Euclidean) centers of the boundary circles, but it is not clear if this is the best approach. Since we lack anything better we use the following definition.

Definition 6.1 (*Boundary values for circle packings*). The *boundary values* of a circle packing $P = (r, z)$ are the Euclidean centers $z_1, \ldots, z_m$ of its boundary circles.

The function

$$\zeta : \mathcal{D} \to \mathbb{C}^m \quad \text{with} \quad \zeta(r, z) := (z_1, \dots, z_m).$$

is said to be the *boundary center map*. The subset of $\mathbb{C}^n$ which consists of all possible boundary values is denoted by $\mathcal{C}$, i.e. $\mathcal{C} := \zeta(\mathcal{D})$, and $\mathcal{C}_b := \zeta(\mathcal{D}_b)$ stands for the set of boundary values of circle packings with branching b.

We will show that $\mathcal{C}$ is a $m+3$-dimensional real submanifold of $\mathbb{C}^m$, which is done by proving that the projection ζ has maximal rank and is injective on every component $\mathcal{D}_b$.

If the number m of boundary circles is odd, the injectivity of ζ on $\mathcal{D}_b$ is easy to see. For given boundary centers, the equations $|z_i - z_{i+1}| = r_i + r_{i+1}$, with $1 \le i \le m$ and indices running modulo m, form a linear system for the boundary radii. If m is odd it has a unique solution and then Theorem 3.7 shows that there exists a unique packing label in $\mathcal{D}_b^*$. Thus two circle packings in $\mathcal{D}_b$ with the same boundary values can differ just by a rigid motion, and then they must coincide.

In order to prove that ζ is injective on $\mathcal{D}_b$ in general, we need a different argument which will make use of Lemma 11.11 in Stephenson (2005).

Lemma 6.2 (Maximum principle for circle packings). *Consider two packing labels $r, r' \in \mathcal{D}^*$ such that r is not a scalar multiple of r'. If $\widetilde{\Phi}(r) \le \widetilde{\Phi}(r')$ holds componentwise, and if there exists an $i \in \{1, \dots, n\}$ such that*

$$\frac{r'_i}{r_i} \ge \frac{r'_j}{r_j} \quad \text{for} \quad 1 \le j \le n,$$

then necessarily $i \in \{1, \dots, m\}$, i.e. r'_j/r_j cannot attain its maximum at an interior vertex.

The assumption $\widetilde{\Phi}(r) \le \widetilde{\Phi}(r')$ cannot be dropped, but if (r, z) is a univalent circle packing we have $\widetilde{\Phi}(r) = 0$ and then it is automatically satisfied. Note that the ratio r'/r plays the role of the modulus of the derivative in the discrete analytic function which maps the packing (r, z) to (r', z').

The following lemma is an elementary result about monotonicity of angles in an Euclidean triangle and can easily be verified.

Lemma 6.3 (Monotonicity of Euclidean angles). *For any $v_i, v_j, v_k \in V$ the angle $\varphi(v_i; v_j, v_k)$ is strictly monotone increasing in r_j and in r_k.*

The next result shows that circle packings with prescribed branching are uniquely determined by their boundary centers.

Lemma 6.4 (Injectivity of ζ on components of $\mathcal{D}$). *For every branch structure b the boundary center map ζ is injective on $\mathcal{D}_b$.*

Proof Suppose there are two circle packings (r, z) and (r', z') in $\mathcal{D}_b$ satisfying $\zeta(r, z) = \zeta(r', z')$. Then the corresponding packing labels r and r' in $\mathcal{D}_b^*$ must share the same angle sums at their boundary vertices. Since their angle sums at the interior vertices coincide by assumption, we obtain

$$\Phi_i(r) = \Phi_i(r') \quad \text{for } 1 \le i \le n. \tag{16}$$

Assume that r' is not a scalar multiple of r. Applying Lemma 6.2 yields that r'/r can attain its maximum only on the boundary. Consequently, we can choose a boundary vertex v_J such that

$$C := \frac{r'_J}{r_J} = \max\left\{\frac{r'_i}{r_i} : i = 1, \ldots, n\right\} \quad \text{and} \quad \frac{r'_J}{r_J} > \frac{r'_j}{r_j}$$

for some neighbor v_j of v_J in K. Then $r' \le Cr$ holds componentwise with equality at the J-th and strict inequality at the j-th component. Using the monotonicity result from Lemma 6.3 leads to

$$\Phi_J(r') < \Phi_J(Cr) = \Phi_J(r),$$

contradicting (16). Therefore r' must be a scalar multiple of r. Let $r' = \lambda r$ with $\lambda \in \mathbb{R}_+$. It follows that

$$|z_1 - z_2| = |z'_1 - z'_2| = r'_1 + r'_2 = \lambda(r_1 + r_2) = \lambda|z_1 - z_2|,$$

which implies $\lambda = 1$ and $r = r'$. Hence the circle packings (r, z) and (r', z') differ just by a rigid motion of the plane. Since their boundary values coincide, we must have $z = z'$. □

Note that the proof of Lemma 6.4 only requires the weaker assumption that $\widetilde{\Phi}(r) \le \widetilde{\Phi}(r')$. Thus ζ is actually injective on any set

$$\mathcal{D}_{b_1} \cup \mathcal{D}_{b_2} \cup \mathcal{D}_{b_3} \cup \cdots$$

where $b_1 \le b_2 \le b_3 \le \cdots$ is a componentwise monotone sequence. If ζ is indeed injective on the whole manifold $\mathcal{D}$ of circle packings remains a challenging open problem. The validation of the following conjecture is important for the existence of a discrete counterpart to the Cauchy integral formula in circle packing.

Conjecture 6.5 *The boundary center map $\zeta : \mathcal{D} \to \mathbb{C}^m$ is injective.*

For proving that the rank of $d\zeta$ is maximal we shall apply arguments similar to those used in Lemmas 5.5 and 5.6.

Lemma 6.6 (Local independence of boundary angle sum). *Fix $(r, z) \in \mathcal{D}$. Let v_i be a boundary vertex of K and denote its combinatorial (open) flower by $\langle v_i; v'_1, v'_2, \ldots, v'_k\rangle$. Then the angle sum Ψ_i at v_i is locally constant as a function of $z'_2, \ldots, z'_{k-1}$.*

Proof A little thought shows that the central angle sum Ψ_i at v_i depends locally only on the location of the boundary centers z_i, z'_1, z'_k. □

Lemma 6.7 (Kernel of differential of ω). *Let* $(r, z) \in \mathcal{D}$ *and consider* $(dr, dx, dy) \in \ker \mathrm{D}\omega(r, z)$. *Then* $dx_1 = \cdots = dx_m = dy_1 = \cdots = dy_m = 0$ *implies that* $dr = 0$.

Proof From Lemma 5.6 and (11) we conclude that $dr \in T_r(\mathcal{D}^*) = \ker \mathrm{D}\widetilde{\Phi}(r)$, which means that

$$\mathrm{D}\Phi_i(r)\,dr = 0, \qquad i = m+1, \ldots, n.$$

In order to prove that $\mathrm{D}\Phi_i(r)\,dr$ also vanishes for $i = 1, \ldots, m$ we denote by $\langle v_i; v'_1, v'_2, \ldots, v'_k \rangle$ the combinatorial open flower of the boundary vertex v_i. Since $(dr, dx, dy) \in \ker \mathrm{D}\omega(r, z)$ we can apply (15) and obtain

$$\begin{aligned}
\mathrm{D}\Phi_i(r)\,dr &= \mathrm{D}\Phi_i(r, z)(dr, dx, dy) = \sum_{j=1}^{k-1} \mathrm{d}\varphi(v_i; v'_j, v'_{j+1})(r, z)(dr, dx, dy) \\
&= \sum_{j=1}^{k-1} \mathrm{d}\psi(v_i; v'_j, v'_{j+1})(r, z)(dr, dx, dy) = \mathrm{D}\Psi_i(r, z)(dr, dx, dy) \\
&= \left(\frac{\partial \Psi_i}{\partial x_i}(z)\,dx_i + \frac{\partial \Psi_i}{\partial y_i}(z)\,dy_i \right) + \sum_{j=1}^{k} \left(\frac{\partial \Psi_i}{\partial x'_j}(z)\,dx'_j + \frac{\partial \Psi_i}{\partial y'_j}(z)\,dy'_j \right).
\end{aligned} \tag{17}$$

Since Ψ_i is locally independent of $z'_2, \ldots, z'_{k-1}$ by Lemma 6.6, we must have

$$\frac{\partial \Psi_i}{\partial x'_j}(z) = \frac{\partial \Psi_i}{\partial y'_j}(z) = 0$$

for $1 < j < k$. The (remaining) vertices v_i, v'_1, v'_k belong to the boundary of K and consequently $dx_i = dy_i = dx'_1 = dy'_1 = dx'_k = dy'_k = 0$ by assumption. Substituting this into (17) results in $\mathrm{D}\Phi_i(r)\,dr = 0$ for $1 \le i \le m$.

Once we have $dr \in \ker \mathrm{D}\Phi(r)$, Lemma 4.1 tells us that $dr = \lambda r$ with $\lambda \in \mathbb{R}$. Finally, let $e_i = \langle v_j v_k \rangle$ be a boundary edge of K. Then we get

$$\begin{aligned}
\lambda(r_j + r_k)^2 &= (r_j + r_k)(dr_j + dr_k) \\
&= (x_j - x_k)(dx_j - dx_k) + (y_j - y_k)(dy_j - dy_k).
\end{aligned}$$

and $dx_j = dx_k = dy_j = dy_k = 0$ leads to $\lambda(r_j + r_k)^2 = 0$. Since $r_j, r_k > 0$, we obtain $\lambda = 0$, which completes the proof. □

Theorem 6.8 (Properties of boundary center map). *The boundary center map* $\zeta : \mathcal{D} \to \mathbb{C}^m$ *is injective on each component* $\mathcal{D}_b$ *and* $\mathrm{d}\zeta$ *has maximal rank at every* $(r, z) \in \mathcal{D}$, *i.e.* $\operatorname{rank} \mathrm{D}\zeta(r, z) = m + 3$.

Proof It only remains to show that dζ has maximal rank. This is equivalent to the injectivity of the projection

$$T_{(r,z)}\mathcal{D} \to \mathbb{R}^{2m}, \ (dr, dx, dy) \mapsto (dx_1, \ldots, dx_m, dy_1, \ldots, dy_m)$$

at each $(r, z) \in \mathcal{D}$. So we take (dx, dy, dr) in $T_{(r,z)}\mathcal{D} \equiv \ker \mathrm{D}\omega(r, z)$ and assume that $dx_1 = \cdots = dx_m = dy_1 = \cdots = dy_m = 0$. Then Lemma 6.7 implies $dr = 0$ and thus (dx, dy) belongs to the kernel of the central part $\mathrm{D}_z\omega(dr, dz)$ of the Jacobian $\mathrm{D}\omega(r, z)$. From Lemma 5.4 we now infer that

$$dz := dx + \mathrm{i}\, dy = d\rho \cdot \mathrm{i}z + (d\xi + \mathrm{i}\, d\eta) \cdot 1$$

with $d\rho,\ d\xi,\ d\eta \in \mathbb{R}$. Since $dz_1 = \cdots = dz_m = 0$, it follows that $d\rho = d\xi = d\eta = 0$, and finally $dx = dy = dr = 0$. □

Corollary 6.9 (Parametrization of boundary centers). *For every branch structure b the set $\mathcal{C}_b := \zeta\left(\mathcal{D}_b\right)$ is a differentiable submanifold of $\mathbb{C}^m$ of dimension $m + 3$, the mapping*

$$\zeta \circ \varrho_b : \ \mathbb{R}^m_+ \times \mathbb{T} \times \mathbb{C} \to \mathcal{C}_b$$

is a regular parametrization, and the tangent vectors $\partial r_1, \ldots, \partial r_m, \partial\xi, \partial\eta, \partial\rho \in \mathbb{C}^m$ to the parameter lines of $\zeta \circ \varrho_b$ span the tangent space of $\mathcal{C}_b$.

A challenging problem is an intrinsic characterization of the tangent space $T_P\mathcal{C}$ at a packing P as subspace of the ambient space $\mathbb{C}^m$ from geometric properties of the packing P.

We finally consider boundary values of packing labels, which are equivalence classes $[z_1, \ldots, z_m]$ of points $(z_1, \ldots, z_m)$ in $\mathbb{C}^m$ with respect to the Lie group G of plane rigid motions.

Most of the results follow from the previous considerations for boundary values of circle packings. Similar to the definition of the exceptional set $\mathcal{E}$ we set

$$\widetilde{\mathcal{E}} := \{(z_1, \ldots, z_m) \in \mathbb{C}^m : z_j = z_k \quad \text{for some } \langle v_j, v_k\rangle \in E\}. \tag{18}$$

The action α induces an action $\widetilde{\alpha}$ of G on $\mathbb{C}^m \backslash \widetilde{\mathcal{E}}$ which is smooth, proper and free, and factorization of $\mathbb{C}^m \backslash \widetilde{\mathcal{E}}$ with respect to G yields the orbit manifold $\mathbb{C}^m_* := (\mathbb{C}^m \backslash \widetilde{\mathcal{E}})\ /\ G$ with a smooth projection

$$\pi_* : \ \mathbb{C}^m \backslash \widetilde{\mathcal{E}} \to \mathbb{C}^m_*.$$

With the standard layout mapping γ_0 we now introduce the *boundary center map for packing labels*,

$$\zeta^* : \ \mathcal{D}^* \to \mathbb{C}^m_*, \ r \mapsto (\pi_* \circ \zeta)(r, \circ\gamma_0(r)),$$

and define the boundary value(s) of a packing label $r \in \mathcal{D}^*$ as the equivalence class $[z_1, \dots, z_m] := \zeta^*(r)$. The set $\mathcal{C}_b^* := \zeta^*(\mathcal{D}_b^*)$ then consists of boundary values of packing labels with branching b. The diagram

$$\begin{array}{ccc} \mathbb{R}_+^n \times (\mathbb{C}^n \backslash \mathcal{E}) & \xrightarrow{\zeta} & \mathbb{C}^m \backslash \widetilde{\mathcal{E}} \\ \downarrow{\scriptstyle \alpha_g} & & \downarrow{\scriptstyle \widetilde{\alpha}_g} \\ \mathbb{R}_+^n \times (\mathbb{C}^n \backslash \mathcal{E}) & \xrightarrow{\zeta} & \mathbb{C}^m \backslash \widetilde{\mathcal{E}} \end{array}$$

commutes and $\mathcal{C}_b$ is G-stable. For $(r, z) \in \mathcal{D}_b$ we obtain

$$(\pi_* \circ \widetilde{\alpha})\big(G, \zeta(r, z)\big) = (\pi_* \circ \widetilde{\alpha})\big(G, \zeta(r, \gamma_0(r))\big) = \pi_*\big(\zeta(r, \gamma_0(r))\big) = \zeta^*(r).$$

Therefore the equivalence classes of G in $\mathcal{C}_b$ are the elements of $\mathcal{C}_b^*$ and we have the factorization $\mathcal{C}_b^* = \mathcal{C}_b / G$.

Theorem 6.10 (Boundary center map for packing labels). *The boundary center map $\zeta^* : \mathcal{D}^* \to \mathbb{C}_*^m$ is injective on each component $\mathcal{D}_b^*$ and has maximal rank at every $(r, z) \in \mathcal{D}^*$.*

Proof The actions $\widetilde{\alpha}$ and β of G on $\mathcal{C}_b$ and $\mathbb{R}_+^m \times \mathbb{T} \times \mathbb{C}$ are smooth, proper and free. Since the diagram

$$\begin{array}{ccc} \mathbb{R}_+^m \times \mathbb{T} \times \mathbb{C} & \xrightarrow[\cong]{\zeta \circ \varrho_b} & \mathcal{C}_b \\ \downarrow{\scriptstyle \beta_g} & & \downarrow{\scriptstyle \widetilde{\alpha}_g} \\ \mathbb{R}_+^m \times \mathbb{T} \times \mathbb{C} & \xrightarrow[\cong]{\zeta \circ \varrho_b} & \mathcal{C}_b \end{array}$$

commutes for each $g \in G$, the mapping $\zeta \circ \varrho_b$ is a G-equivariant diffeomorphism, and the induced map $(\zeta \circ \varrho_b)^* \equiv \zeta^* \circ \varrho_b^*$ is a diffeomorphism as well,

$$\mathbb{R}_+^m \cong \frac{\mathbb{R}_+^m \times \mathbb{T} \times \mathbb{C}}{G} \xrightarrow[\cong]{\zeta^* \circ \varrho_b^*} \frac{\mathcal{C}_b}{G} = \mathcal{C}_b^* .$$

Thus ζ^* must be injective on $\mathcal{D}_b^*$ with maximal rank. □

Theorem 6.11 (Boundary centers of packing labels). *For each branch structure b the set $\mathcal{C}_b^*$ is a smooth real submanifold of $\mathbb{C}_*^m$ with dimension m and the mapping*

$$\varrho_b^* : \mathbb{R}_+^m \to \mathcal{C}_b^*, \; r \mapsto [z_1, \dots, z_m] := (\pi_* \circ \zeta)\big(\varrho_b(r, 1, 0)\big)$$

is a diffeomorphism.

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