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ORIGINAL PAPER

Inductive schemes for the complete classification of affine hypersurfaces with parallel second fundamental form

Salvador Gigena

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Abstract We present in this article inductive schemes that allow to classify all those affine hypersurfaces with affine normal parallel second fundamental (cubic) form, for every possible dimensional case greater or equal than two. The solution proposed is a follow up to previous works on the same topic by this author and it uses, firstly, the reduction of the problem, eminently geometric, to the classification of a certain class of solutions to an equation of Monge-Ampère type. Then it is applied the so-called "method of algorithmic sequence of coordinate changes", in order to achieve the latter.

Keywords Affine normal · Parallel second fundamental (cubic) form · Monge-Ampère equations

Mathematics Subject Classification (2000) Primary 53A15; Secondary 35J60

1 Introduction

The problem of classifying all those hypersurfaces with affine normal parallel second fundamental (cubic) form, which are not hyperquadrics, was first considered, and solved for dimension $n = 2$, by Nomizu and Pinkall (1989). See also the book by

S. Gigena (✉)
Departamento de Matemática, Facultad de Ciencias Exactas, Ingeniería y Agrimensura,
Universidad Nacional de Rosario, Avenida Pellegrini 250, 2000 Rosario, Argentina
e-mail: sgigena@fceia.unr.edu.ar

S. Gigena
Departamento de Matemática, Facultad de Ciencias Exactas, Físicas y Naturales,
Universidad Nacional de Córdoba, Avenida Velez Sarsfield 1600, 5000 Córdoba, Argentina
e-mail: sgigena@efn.uncor.edu

Nomizu and Sasaki (1994), where a different method of proof was presented. Vrancken (1988) treated the case of dimension $n = 3$.

In our first article dedicated to the problem (Gigena 2002) we introduced a new method of approaching the solution, different to the ones previously used by the other mentioned authors. We called it *algorithmic sequence of coordinate changes* and consists, basically, in referring the hypersurface to a suitable linear coordinate system of the ambient space, and then making algorithmic adjustments into the Hessian matrix so that this can be integrated, fairly easily, to obtain the graph function representing the hypersurface. With this approach the classification depends strongly on two integer constants, that we labeled k and r, with $1 \leq k \leq n/2$, $1 \leq r \leq n-1$, where $n =$ dimension of the immersed manifold (≥ 2). Moreover, in that article we presented new proofs of the previously stated results and, then, extended the classification to dimension $n = 4$. In that work we also started to rename the *cubic form* as *second fundamental form*, on the basis of our earlier articles dedicated to the field.

More recently (Gigena 2003) we extended the classification to the case of dimension $n = 5$.

In both of our articles there appear values of k and r for which there are no solutions. Thus, it is the purpose of this article to present the necessary tools which will allow to achieve the complete classification of the class of hypersurfaces under consideration. Two questions are treated and developed in the present work:

(1) Verification, by means of an inductive process, of the

Conjecture There exist no solutions to the problem for those classificatory values where $r > 2k$.

(2) Implementation of an inductive scheme on the classificatory values n , k and r which permits, from the classification process developed so far, to obtain all remaining possible cases of solution.

While most of the background material has been presented in our previous articles, some complementary results are needed in order to reach our goal. Thus, in the next section we present those. The main results of the article are then presented in Sects. 3 and 4.

2 Summary of complementary properties

By keeping notation and terminology as in our previous articles Gigena (2002, 2003), we present in this section two more auxiliary results that shall be useful in the development of the present work. These allow to construct new solutions to the problem from an existing one. In the first place we extend solutions to higher dimensions by proving

Lemma 2.1 *Let $X : M^n \to E^{n+1}$ be a nondegenerate hypersurface with parallel cubic form, $\nabla (II_{ua}) = \nabla C = 0$, which is not a hyperquadric, i.e., $II_{ua} = C$ not vanishing identically, with corresponding graph function f, and $n \times n$ Hessian matrix*

$H(f)$ having characterizing values k and r, according to Lemma 2.1 in Gigena (2002). Then, while keeping fixed the value of the maximal rank r, we can construct new solutions where the number of blocks of the form $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and also the number of elements equal to 1, both have been increased and are in diagonal position, the rest of entries being equal to 0 within the new, higher dimensional Hessian matrix.

Proof We write $f\left(t^1, t^2, \ldots, t^n\right)$ for the original graph function and define $F : U \to R$, where U is a suitable open subset of R^{n+2l+m}, by

$$F\left(t^1, \ldots, t^{n+2l+m}\right) = f\left(t^1, t^2, \ldots, t^n\right) + \sum_{k=n+1}^{n+2l-1} t^k t^{k+1} + \frac{1}{2} \sum_{i=1}^{m} \left(t^{n+2l+i}\right)^2 .$$

It is obvious that this function satisfies all of the requirements stated in the lemma, its graph providing a nondegenerate hypersurface with parallel cubic form which is not a hyperquadric, dimension equal to $n + 2l + m$, $k + l$ blocks of the form $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and maximal rank of the complementary matrix still equal to r. □

Moreover, it is not difficult to see that a kind of converse to Lemma 2.1 above is also valid.

Lemma 2.2 *Suppose that a solution to the problem is known to exist in dimension $n + 2l + m$ and that the function $F\left(t^1, \ldots, t^n, t^{n+1}, \ldots, t^{n+2l+m}\right)$ representing the graph has been written in the form described by the equation in the proof of Lemma 2.1 above. Then the function $f\left(t^1, t^2, \ldots, t^n\right)$ is also a solution to the problem in dimension n.*

Proof Obvious. □

3 Nonexistence scheme

In this section we expose the arguments that allow to prove the conjecture, stated in the introduction, asserting that for those classificatory values where $r > 2k$ these exist no solution to the problem, i.e., there are no affine hypersurfaces satisfying the required geometrical properties and having those values for their Hessian matrices. In fact, the procedural result stated as Lemma 2.1 of Gigena (2002) established the dependence of the classificatory problem on three integer numbers: the dimension n of the hypersurface, in the first place; the number k of blocks of the form $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, once the Hessian matrix had been reduced to the form $H(f) = (f_{ij}) = J_k + (x_{ij})$, and the maximal rank r of the so-called complementary matrix (x_{ij}). There was no apparent relation between the ranges of variation for the classificatory values k and r, as far as the existence of solutions is concerned. However, as we shall see, there is a relation and this is made explicit in the next result.

Theorem 3.1 *Let $X : M^n \to E^{n+1}$ be a nondegenerate hypersurface with parallel second fundamental (cubic) form, $\nabla(II_{ua}) = \nabla C = 0$, which is not a hyperquadric, i.e., $II_{ua} = C$ does not vanish identically, and suppose that it has already been expressed in the form indicated in Lemma 2.1 of Gigena (2002), i.e., by means of the graph function f with corresponding $n \times n$ Hessian matrix $H(f)$, and values k for the number of blocks of the form $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, where $1 \le k \le \frac{n}{2}$, and r for the maximal rank of the complementary matrix $\left(x_{ij}\right)$, with $1 \le r \le n-1$. Then, the classificatory values k and r must satisfy, besides, the condition that r be less or equal to $2k$, i.e., there exist no solution to the classificatory problem in those cases where $r > 2k$.*

Proof We proceed by induction on the dimension n, from the knowledge that the result is true for $n = 4, 5$ (see (Gigena 2002, 2003)).

Thus, we assume presently that the result stated in the theorem is true for all dimensional values up to $n-1$, and suppose that there exists a solution to the problem for some k and r such that $2k < r < n$.

The procedure that follows is general, covers all possibilities of remaining cases and shall also be used in the course of the proof of the existence theorem, presented in next Sect. 4. In the present, nonexistence context we shall see at the end of the proof that the assumption $2k < r < n$ leads to a contradiction.

First, we express the Hessian matrix of the supposedly existing solution in the form indicated in Lemma 2.1 of Gigena (2002), define the vectors $X_i := (x_{i1}, x_{i2}, \ldots, x_{ir}, \ldots, x_{in})$, $i = 1, \ldots, n$, by using the entries of the complementary matrix, and assume to have that the ones labeled $X_{j_1}, \ldots, X_{j_r}$, are linearly independent on an open, dense subset of the domain, while the rest are expressible as linear combination of these, i.e., $X_i = \sum_{h=1}^{r} a_{ij_h} X_{j_h}$, for $i \neq j_1, \ldots, j_r$. Then, the elementary operations $R_i - \sum_{h=1}^{r} a_{ij_h} R_{j_h}$, $C_i - \sum_{h=1}^{r} a_{ij_h} C_{j_h}$, with $i \neq j_1, \ldots, j_r$, followed by suitable row (and column) interchanges allow to represent the Hessian matrix by

$$\begin{bmatrix} b_{11}+x_{11} & \cdots & b_{1r}+x_{1r} & a'_{1r+1} & \cdots & a'_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b_{1r}+x_{1r} & \cdots & b_{rr}+x_{rr} & a'_{rr+1} & \cdots & a'_{rn} \\ a'_{1r+1} & \cdots & a'_{rr+1} & b_{r+1r+1} & \cdots & b_{r+1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a'_{1n} & \cdots & a'_{rn} & b_{r+1n} & \cdots & b_{nn} \end{bmatrix}, \tag{3.1}$$

where the left superior square $r \times r$ submatrix

$$\begin{bmatrix} b_{11}+x_{11} & \cdots & b_{1r}+x_{1r} \\ \vdots & \ddots & \vdots \\ b_{1r}+x_{1r} & \cdots & b_{rr}+x_{rr} \end{bmatrix}$$

is formed with those entries which remained unchanged in the process, so that all scalars in it are zeros or ones, $\det[x_{ij}]$ being, by hypothesis, a polynomial function in the ambient space variables of degree exactly equal to r, which is non vanishing on an open dense subset of the domain. On the other hand, the right inferior submatrix

$$B := \left[b_{ij}\right] = \begin{bmatrix} b_{r+1r+1} & \cdots & b_{r+1n} \\ \vdots & \ddots & \vdots \\ b_{r+1n} & \cdots & b_{nn} \end{bmatrix}$$

is formed with scalars which depend on the previously introduced a_{ij_h} , $h = 1, \ldots r$, $i \neq j_1, \ldots, j_r$, and we must have $\det[b_{ij}] = 0$, since we arrive at a contradiction otherwise. Finally, the right superior $r \times (n - r)$ submatrix

$$\begin{bmatrix} a'_{1r+1} & \cdots & a'_{1n} \\ \vdots & \ddots & \vdots \\ a'_{rr+1} & \cdots & a'_{rn} \end{bmatrix}$$

is formed with scalars which are equal to (minus) some of the scalars a_{ij_h}, $h = 1, \ldots r$, $i \neq j_1, \ldots, j_r$.

Now, since $\det[b_{ij}] = 0$ we may perform further elementary operations in the last $n - r$ rows (and columns) of the Hessian matrix so that the submatrix $[b_{ij}]$ is diagonalized into the form

$$\begin{bmatrix} b_{r+1r+1} & \cdots & b_{r+1n} \\ \vdots & \ddots & \vdots \\ b_{r+1n} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & \cdots & & & & & & & 0 \\ \vdots & \ddots & & & & & & & & & \\ 0 & & 0 & & & & & & & & \\ \vdots & & & 0 & 1 & & & & & & \\ & & & 1 & 0 & & & & & & \vdots \\ & & & & & \ddots & & & & & \\ & & & & & & 0 & 1 & & & \\ & & & & & & 1 & 0 & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & 0 \\ 0 & & & & \cdots & & & & & 0 & 1 \end{bmatrix},$$

i.e., with l ($\geqslant 0$) blocks of the form $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and with m ($\geqslant 0$) entries equal to 1 in diagonal position, the rest of entries being equal to zero. Next, if there actually exist some of those blocks and/or entries, we can annihilate the elements of the form a'_{ij} belonging to the same rows (and columns), in the Hessian, as those with blocks $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $1's$ in diagonal position in the latter submatrix. Thus, by integrating partially the Hessian we would obtain a solution with the corresponding part expressed as $\ldots t^{p+1}t^{p+2} + \cdots + t^{p+2l-1}t^{p+2l} + \frac{1}{2}\sum_{i=1}^{m}\left(t^{p+2l+i}\right)^2$, $p > r$.

It follows, by using Lemma 2.2 above, that the rest of this solution is also, independently, a solution to the problem in a lesser dimension, so that the result of the theorem follows by induction.

Next, we consider the case where the last square submatrix is the null one, i.e.,

$$\begin{bmatrix} b_{r+1r+1} & \cdots & b_{r+1n} \\ \vdots & \ddots & \vdots \\ b_{r+1n} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}.$$

Then, since the r-vector $\begin{pmatrix} a'_{1n} & \cdots & a'_{rn} \end{pmatrix}$ must be different from zero we can perform further elementary operations in the first r rows (and columns) and transform the Hessian matrix into the form

$$\begin{bmatrix} b_{11}+x_{11} & \cdots & b_{1r}+x_{1r} & a'_{1r+1} & \cdots & a'_{1n-1} & 1 \\ \vdots & \ddots & \vdots & \vdots & & \vdots & 0 \\ b_{1r}+x_{1r} & & b_{rr}+x_{rr} & a'_{rr+1} & \cdots & a'_{rn-1} & \vdots \\ a'_{1r+1} & \cdots & a'_{rr+1} & 0 & \cdots & & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & & \vdots \\ a'_{1n-1} & \cdots & a'_{rn-1} & & & & \\ 1 & 0 & \cdots & 0 & \cdots & & 0 \end{bmatrix}. \tag{3.2}$$

Now the entries of the form $b_{ij} + x_{ij}$ and those of the form a'_{ij} may have changed in the process, but we keep the notation for reasons of simplicity. Moreover, further elementary operations using the entries equal to 1 at the corners allow to annihilate all of the rest of scalar values in the first row (and column), i.e., $b_{11} = \cdots = b_{1r} = a'_{1r+1} = \cdots = a'_{1n-1} = 0$.

By same token, since the $(r-1)$-vector $\begin{pmatrix} a'_{2n-1} & \cdots & a'_{rn-1} \end{pmatrix}$ must be different from zero we may proceed in a similar fashion as with the previous case. This procedure may be repeated a finite number of times until the Hessian is transformed into the form

$$\begin{bmatrix} x_{11} & \cdots & x_{1n-r} & \cdots & & x_{1r} & 0 & \cdots & 1 \\ \vdots & \ddots & & & & & & \diagup & 0 \\ x_{1n-r} & & x_{n-rn-r} & \cdots & & x_{n-rr} & 1 & & 0 \\ \vdots & & \vdots & b_{ss}+x_{ss} & \cdots & b_{sr}+x_{sr} & 0 & & \vdots \\ & & & \vdots & \ddots & \vdots & \vdots & & \\ x_{1r} & & x_{n-rr} & b_{sr}+x_{sr} & \cdots & b_{rr}+x_{rr} & & & \\ 0 & & 1 & 0 & \cdots & & 0 & \cdots & 0 \\ \vdots & \diagup & & & & & \vdots & \ddots & \\ 1 & & 0 & & & & 0 & & 0 \end{bmatrix}, \tag{3.3}$$

with $s := n - r + 1$.

In writing the latter expression for the Hessian matrix we have assumed that $n-r \leq k$. In fact, this is precisely the case because if it were, firstly, $n-r > r$ the procedure indicated would lead us to write the Hessian into the form

$$\begin{bmatrix} x_{11} & \cdots & x_{1r} & 0 & \cdots & 0 & 0 & \cdots & 1 \\ \vdots & \ddots & & & & & & \diagup & 0 \\ x_{1n-r} & & x_{rr} & 0 & & & 1 & & 0 \\ 0 & & 0 & 0 & \cdots & & 0 & & \vdots \\ & & & \vdots & \ddots & & \vdots & & \\ 0 & & 1 & 0 & \cdots & & 0 & \cdots & 0 \\ \vdots & \diagup & & & & & \vdots & \ddots & \\ 1 & & 0 & & & & 0 & & 0 \end{bmatrix}, \tag{3.4}$$

i.e., it would have at least one row (and column) equal to zero, which is impossible. Second, if it were $n-r=r$ the Hessian would look like

$$\begin{bmatrix} x_{11} & \cdots & x_{1r} & 0 & \cdots & 1 \\ \vdots & \ddots & \vdots & & \diagup & 0 \\ x_{13} & \cdots & x_{rr} & 1 & & \\ 0 & & 1 & 0 & & 0 \\ \vdots & \diagup & & & & \\ 1 & & & 0 & & 0 \end{bmatrix}. \tag{3.5}$$

But then, by making further elementary operations, this would have $n-r=r$ blocks of the form $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which contradicts the fact that we are assuming that it has k blocks and $r > 2k > k$.

It is also clear now, from the above, that indeed it must hold that $n-r \leq k$ in the expression (3.3) for the Hessian matrix. Let us observe that, for $k=1$, we have $n-r=1$, since $n-r \geq 1$, too. Hence it holds $n=r+1>3$. Recall that the exact value $n=r+1=4$, was proven as a part of Theorem 3.1 of Gigena (2002), initiating the inductive process. The next value $n=r+1=5$, was also obtained previously as part of the Main Theorem in Gigena (2003).

It is also obvious from the same expression, or by making further interchanges of rows (and columns) that, since the determinant of the Hessian is equal $(-1)^k$, we must have that the determinant of the submatrix $\left(b_{ij}+x_{ij}\right)$ equals $(-1)^{k-(n-r)}$, i.e.,

$$\det \begin{bmatrix} b_{ss}+x_{ss} & \cdots & b_{sr}+x_{sr} \\ \vdots & \ddots & \vdots \\ b_{sr}+x_{sr} & \cdots & b_{rr}+x_{rr} \end{bmatrix} = (-1)^{k-(n-r)}. \tag{3.6}$$

Therefore such a submatrix, with $n-2(n-r)$ rows (and columns) transforms, by means of elementary operations practiced to the Hessian matrix, involving only rows (and columns) from $n-r+1=s$ to r, into a matrix having $k-(n-r)$ blocks of the form $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. By the inductive hypothesis its maximal rank r' must satisfy $r' \le 2(k-(n-r))$. (It is clear too, that this even holds in the possible limiting case where $n-r=k$. In this instance we would obviously have $r'=0$, meaning that the corresponding complementary matrix is the null one). But then the main $r \times r$ submatrix of the complementary matrix of the Hessian may be written

$$\begin{bmatrix}
x_{11} & \cdots & x_{1n-r} & x_{1s} & \cdots & x_{1s'} & x_{1s'+1} & \cdots & x_{1r} \\
\vdots & \ddots & & & & \vdots & & & \vdots \\
x_{1n-r} & & x_{n-rn-r} & \cdots & & x_{n-rs'} & x_{n-rs'+1} & \cdots & x_{n-rr} \\
x_{1s} & & \vdots & x_{ss} & \cdots & x_{ss'} & 0 & & 0 \\
\vdots & & & \vdots & \ddots & \vdots & \vdots & & \\
x_{1s'} & & x_{n-rs'} & x_{ss'} & \cdots & x_{s's'} & 0 & \cdots & 0 \\
x_{1s'+1} & & x_{n-rs'+1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & & & \vdots & \vdots & \ddots & \vdots \\
x_{1r} & & x_{n-rr} & 0 & & 0 & 0 & \cdots & 0
\end{bmatrix}, \tag{3.7}$$

with $s := n-r+1$ and $s' := n-r+r'$.

Let us recall that we are assuming that the latter is non-singular. However, if we compute its determinant by using the Laplace development for the last $r-(n-r)=2r-n$ columns, i.e., computing the determinants of the corresponding square $(2r-n)\times(2r-n)$ matrices, we find that all of these determinants vanish since the maximum number of rows different from zero in every one of them is $n-r+r' \le n-r+2(k-(n-r)) = r+2k-n < r$. Hence the determinant of the matrix represented in (3.7) vanishes everywhere and this is a contradiction to our original hypothesis, in which we assumed that the maximal rank was exactly equal to r. □

4 Inductive existence process

In this section we expose the inductive scheme which allows to find all possible cases of solutions to the problem considered, from the knowledge of the previously obtained results, by proving the following theorem.

Theorem 4.1 *Consider the class of nondegenerate affine hypersurfaces,* $\{X : M^n \to E^{n+1}, n \ge 2\}$, *and the problem of finding and classifying those objects in the class with parallel second fundamental (cubic) form,* $\nabla(II_{ua}) = \nabla C = 0$, *which are not hyperquadrics, i.e., with* $II_{ua} = C$ *not vanishing identically. Then, there always exists a solution to the problem for every one of those classificatory values expressed by using Lemma* 2.1 *of* Gigena (2002)*, with the further restriction imposed by Theorem* 3.1 *above, i.e., for* $1 \le k \le \frac{n}{2}$ *and* $1 \le r \le \min\{n-1, 2k\}$. *Moreover, all* *of those solutions can be obtained and classified, directly and/or inductively, from the knowledge of the previously classified ones, exposed in Theorems* 2.2, 2.3, 3.1 *of* Gigena (2002) *and the Main Theorem of* Gigena (2003).

Proof We proceed by induction on the dimension n, the number of blocks k, the value r for the maximal rank of the complementary matrix $\left(x_{ij}\right)$, and from the knowledge that the result is true for $n = 2, 3, 4, 5$ (see (Gigena 2002, 2003).

Thus, we assume firstly that the result stated in the theorem is true for every case of those values, up to the ones determined by the dimensional value equal to $n - 1$ and retake, for reasons of simplicity and economy, the same notation and terminology as in the proof of Theorem 3.1, only that we assume here and now that $1 \leq r \leq \min\{n-1, 2k\}$.

Besides, in the present context of existence, we shall get all possible solutions if the choice of the linearly independent vectors, described as $\left\{X_{j_1}, \ldots, X_{j_r}\right\}$, run over all non equivalent subsets of the whole set of vectors participating in the complementary matrix $\{X_1, \ldots, X_n\}$.

Thus, for every one of those possible choices, we have the same expression for the right inferior submatrix

$$B = \begin{bmatrix} b_{r+1r+1} & \cdots & b_{r+1n} \\ \vdots & \ddots & \vdots \\ b_{r+1n} & \cdots & b_{nn} \end{bmatrix}$$

which, for the same reasons as before can be diagonalized to have a certain number of zeros, l ($\geqslant 0$) blocks of the form $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and m ($\geqslant 0$) entries equal to 1 in diagonal position, the rest of entries being equal to zero. Again, if there actually exists some of these non-zero blocks, we can annihilate the elements of the form a'_{ij} belonging to the same rows (and columns), in the Hessian, as those with blocks $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $1's$ in diagonal position in the latter submatrix.

Let us consider first the case where there are no blocks different from zero, i.e., the submatrix is the null one. In that instance we arrive again at Eq. (3.3), with $n - r \leq k$ because, for the same reasons as before, the case $n - r > r$ leads to a contradiction. On the other hand the case $n - r = r$ reproduces Eq. (3.5) and, in the present situation, it is not difficult to see that the simple process of interchanges of rows and columns, starting for example with R_{2n}, C_{2n}, allow to reduce the matrix to one having $n - r = r = k$ blocks of the form $\begin{bmatrix} x_{ii} & 1 \\ 1 & 0 \end{bmatrix}$ in diagonal position. This kind of setting allows to carry out integration of the Hessian matrix in a direct and straightforward way.

We consider next the case $n - r < r$. Here we observe that Eq. (3.6) is again valid, $\det\left[b_{ij} + x_{ij}\right] = (-1)^{k-(n-r)}$, number of rows and columns in the submatrix $\left[b_{ij} + x_{ij}\right] = r - 2(n-r) = 2r - n$, with $0 < 2r - n < r \leq 2k$. Now, further elementary operations practiced in the rows and columns of the Hessian matrix, from

$s := n - r + 1$ to r, allow to reduce the submatrix to a regular, integrable form. In particular, with $k - (n - r)$ blocks of the form $\begin{bmatrix} x_{ii} & 1 \\ 1 & 0 \end{bmatrix}$ in diagonal position. Finally, further operations of interchange of row and columns, practiced on the rest of entries of the Hessian matrix, allow to reduce the latter to a form which is manageable for direct integration, in a similar fashion to the preceding case.

In what follows let us consider the case where the submatrix B is not the null one. Hence, if we assume that there is at least one block 1×1 of the form $[1]$ we can, after making operations to annihilate the rest of elements in the last row and column of the Hessian, integrate precisely that part concerning the last row and column and, by also using Lemma 2.2, we get the problem reduced to dimension $n - 1$. By inductive hypothesis, the latter has solution for those values of k and r up to $\frac{n-1}{2}$, $n - 1$, if n is odd; or $\frac{n-2}{2}$, $n - 2$, if n is even.

If we assume next that there is no block with a single element, i.e., one block 1×1 of the form $[1]$, but there is at least one of the form $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we have again, by the same token, that the problem has solution for the those values of k and r up to $\frac{n-3}{2}$, $n - 3$, if n is odd; or $\frac{n-2}{2}$, $n - 2$, if n is even. Thus, in all of these cases there is always a solution for $r \leq n - 3$.

As a consequence, we can assert to have solved the problem, at least for all possible cases in which the latter inequality holds, and it remains only to analyze it, in the sense of providing full details, for the cases where $n - 2 \leq r \leq 2k \leq n$. This kind of final procedure is needed in order to ensure that the full classification is obtained, step by step, via the increasing values of the dimension, from the already known results exposed in our previous articles (Gigena 2002, 2003), where the full classifications for dimensions $n = 4$ and $n = 5$ were achieved respectively.

In that context, we are led to consider the next two situations: (I) n is even (II) n is odd.

- (I) If n is even, say $n = 2q$, the classification is known up to dimension $n' := n - 1$, for the number of blocks up to $k' = q - 1$, and for the maximal rank of the complementary matrix up to $r' = n' - 1 = 2k' = n - 2$. By Lemma 2.1 above every one of those solutions provides a solution for the present dimensional case. Thus, it remains to analyze the cases where:
- (I_1) dimension equal to n, number of blocks equal to $k = q$, maximal rank equal to $r = n - 1$; and
- (I_2) dimension equal to n, number of blocks equal to $k = q$, maximal rank equal to $r = n - 2$.
- (II) If n is odd, $n = 2q + 1$, let $n' := n - 1 = 2q$. Here, for the dimensional value n', the classification is supposed to be known up to the number of blocks $k' = q$, and up to the maximal rank $r' = n' - 1 = 2k' - 1 = n - 2$. Again, the use of Lemma 2.1 above allows to conclude that every one of those solutions provides a solution for the present, dimensional case n, remaining to analyze the cases where:
- (II_1) dimension equal to n, number of blocks equal to $k = q$, maximal rank equal to $r = n - 1$; and

(II_2) dimension equal to n, number of blocks equal to $k = q$, maximal rank equal to $r = n - 2$.

We proceed to analyze next the four cases listed above:

(I_1) Here we have that $n = 2k = r + 1$ and can assume, by Lemma 2.1 of Gigena (2002), that the Hessian matrix has already been reduced to the form

$$\begin{bmatrix} x_{11} & 1+x_{12} & x_{13} & \cdots & x_{1r} & x_{1n} \\ 1+x_{12} & x_{22} & x_{23} & \cdots & x_{2r} & x_{2n} \\ x_{13} & x_{23} & x_{33} & \cdots & \cdots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ x_{1r} & x_{2r} & \vdots & & x_{2k-12k-1} & 1+x_{2k-1n} \\ x_{1n} & x_{2n} & x_{3n} & \cdots & 1+x_{2k-1n} & x_{nn} \end{bmatrix}. \tag{4.1}$$

If we define the vectors $X_i := (x_{i1}, x_{i2}, \ldots, x_{ir}, x_{in})$ we can further assume that the vectors $X_1, X_2, \ldots, X_r$ are linearly independent on an open, dense subset of the domain while $X_n = \sum_{i=1}^{r} a_{ni} X_i$. Then, it is easy to see, by means of suitable elementary operations, that the Hessian matrix can be transformed into

$$\begin{bmatrix} x_{11} & 1+x_{12} & x_{13} & \cdots & x_{1r} & -a_{n2} \\ 1+x_{12} & x_{22} & x_{23} & \cdots & x_{2r} & -a_{n1} \\ x_{13} & x_{23} & x_{33} & \cdots & \cdots & -a_{n4} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ x_{1r} & x_{2r} & \vdots & & x_{rr} & 1 \\ -a_{n2} & -a_{n1} & -a_{n4} & \cdots & 1 & b \end{bmatrix}, \tag{4.2}$$

where $b := 2\,(a_{n1}a_{n2} + a_{n3}a_{n4} + \cdots + a_{nr-1}a_{nr-2} - a_{nr}) = 0$.

Next, by means of further elementary operations, we can reduce it to

$$\begin{bmatrix} x_{11} & 1+x_{12} & x_{13} & \cdots & x_{1r} & 0 \\ 1+x_{12} & x_{22} & x_{23} & \cdots & x_{2r} & 0 \\ x_{13} & x_{23} & x_{33} & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & & 0 \\ x_{1r} & x_{2r} & \vdots & & x_{rr} & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}. \tag{4.3}$$

Now, we analyze the situation of the left superior square submatrix of order $n-2$. It is obvious that the determinant of this must equal $(-1)^{k-1}$ and, besides, $n-2 = 2(k-1)$. Moreover, if the maximal rank of the latter, say r', were strictly less than $r-2$, i.e. $r' < r-2$, then we could reduce it, by means of further elementary operations performed on the first $n-2$ rows (and columns), to have two rows (and columns) equal to zero. But then, the maximal rank of the original Hessian matrix would be less than r, which is a contradiction. Then, the maximal rank of that matrix must equal $r' = r-2$.

Hence we can write, besides, $n-2=2(k-1)=r'+1$ and we can proceed, by induction, to represent the original Hessian matrix by

$$\begin{bmatrix} x_{11} & 1 & x_{13} & 0 & \cdots & x_{1r-2} & 0 & x_{1r} & 0 \\ 1 & 0 & x_{23} & 0 & \cdots & x_{2r-2} & 0 & x_{2r} & 0 \\ x_{13} & x_{23} & x_{33} & 1 & \cdots & \vdots & 0 & \vdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & & 0 & & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & & 0 & & 0 \\ x_{1r-2} & x_{2r-2} & \cdots & & & x_{r-2r-2} & 1 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & x_{r-1r} & 0 \\ x_{1r} & x_{2r} & \cdots & & & \cdots & x_{r-1r} & x_{rr} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \tag{4.4}$$

with the condition that the determinant of the complementary matrix, which equals $x_{11}x_{23}^2x_{45}^2\cdots x_{r-1r}^2$, be different from zero. In each particular, dimensional case integration of the above Hessian matrix is straightforward and can be executed in a way totally similar to the procedure shown in our previous articles.

(II_1) The next case we analyze is $n=2k+1=r+1$, where we have two possible subcases to be labeled as (a) and (b):

(a) We assume first, again by Lemma 2.1 of Gigena (2002), that the Hessian matrix has already been reduced to the form

$$\begin{bmatrix} x_{11} & 1+x_{12} & x_{13} & \cdots & x_{12k} & x_{1n} \\ 1+x_{12} & x_{22} & x_{23} & \cdots & x_{22k} & x_{2n} \\ x_{13} & x_{23} & \ddots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & x_{2k-12k-1} & 1+x_{2k-12k} & x_{2k-1n} \\ x_{12k} & x_{22k} & \cdots & 1+x_{2k-12k} & x_{2k2k} & x_{2kn} \\ x_{1n} & x_{2n} & \cdots & x_{2k-1n} & x_{2kn} & 1+x_{nn} \end{bmatrix}, \tag{4.5}$$

that the vectors $X_1, X_2, \ldots, X_r$ are linearly independent on an open, dense subset of the domain, and that $X_n=\sum_{i=1}^{r} a_{ni}X_i$. Hence, the Hessian matrix can be transformed into

$$\begin{bmatrix} x_{11} & 1+x_{12} & x_{13} & \cdots & x_{1r} & -a_{n2} \\ 1+x_{12} & x_{22} & x_{23} & \cdots & x_{2r} & -a_{n1} \\ x_{13} & x_{23} & \ddots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & x_{r-1r-1} & 1+x_{r-1r} & -a_{nr} \\ x_{1r} & x_{2r} & \cdots & 1+x_{r-1r} & x_{rr} & -a_{nr-1} \\ -a_{n1} & -a_{n1} & \cdots & -a_{nr} & -a_{nr-1} & b \end{bmatrix}, \tag{4.6}$$

where $b:=1+2\left(a_{n1}a_{n2}+a_{n3}a_{n4}+\cdots+a_{nr-1}a_{nr}\right)=0$. Then, some of the summands in the last expression must be non-vanishing and we may assume, for example, that $a_{n1}a_{n2} \neq 0$. Further elementary operations allow to reduce the Hessian matrix to

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1r} & -a_{n2} \\ x_{12} & b_{22}+x_{22} & b_{23}+x_{23} & \cdots & b_{2r}+x_{2r} & 0 \\ x_{13} & b_{23}+x_{23} & \ddots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & x_{r-1r-1} & 1+x_{r-1r} & 0 \\ x_{1r} & b_{2r}+x_{2r} & \cdots & 1+x_{r-1r} & x_{rr} & 0 \\ -a_{n2} & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}. \tag{4.7}$$

Next, by evaluating the determinant of the latter we conclude that we must have $b_{22} \neq 0$, and we may assume that $b_{22} = 1$, $-a_{n2} = 1$. Further operations allow to make $b_{2i} = 0$, for every $i = 3, \ldots, r$ and transform the Hessian matrix into

$$\begin{bmatrix} x_{11} & 1 & x_{13} & \cdots & x_{1r} & x_{1n} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ x_{13} & 0 & b_{33}+x_{33} & \cdots & b_{3r}+x_{3r} & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{1r} & 0 & b_{3r}+x_{3r} & \cdots & b_{rr}+x_{rr} & x_{rn} \\ x_{1n} & 0 & x_{3n} & \cdots & x_{rn} & 1+x_{nn} \end{bmatrix}. \tag{4.8}$$

For the square submatrix of order $n-2$ formed by the last rows and columns we obviously have the conditions that its determinant equals $(-1)^{k-1}$, and that the inductive hypothesis described by equalities $n-2 = 2(k-2)+1 = (r-2)+1$ hold, where the fact that the complementary matrix must have maximal rank exactly equal to $r-2$ is proven by the same token as in the previous case.

Therefore, we conclude that the original Hessian matrix can be transformed finally into the form

$$\begin{bmatrix} x_{11} & 1 & x_{13} & x_{14} & \cdots & x_{1n} \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ x_{13} & 0 & x_{33} & 1 & \cdots & x_{3n} \\ x_{14} & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ x_{1n} & 0 & x_{3n} & 0 & & B_{33} \end{bmatrix}, \tag{4.9}$$

with the 3×3 submatrix B_{33} equal to $\begin{bmatrix} x_{r-1r-1} & 1 & x_{r-1n} \\ 1 & 0 & 0 \\ x_{r-1n} & 0 & 1 \end{bmatrix}$. Integration of the latter expression for the Hessian is straightforward.

(b) We assume next that, in the expression (4.5) for the Hessian matrix, we have that the vectors $X_2, X_3, \ldots, X_n$ are linearly independent on an open, dense subset of the domain, while $X_1 = \sum_{i=2}^{n} a_{1i} X_i$. Then, the Hessian matrix can be transformed into

$$\begin{bmatrix} b_{11} & 1 & -a_{14} & \cdots & -a_{1n-2} & -a_{1n} \\ 1 & x_{22} & x_{23} & \cdots & x_{2r} & x_{2n} \\ -a_{14} & x_{23} & \ddots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & x_{r-1r-1} & 1 + x_{r-1r} & x_{r-1n} \\ -a_{1n-2} & x_{2r} & \cdots & 1 + x_{r-1r} & x_{rr} & x_{rn} \\ -a_{1n} & x_{2n} & \cdots & x_{r-1n} & x_{rn} & 1 + x_{nn} \end{bmatrix}, \qquad (4.10)$$

where we must have that $b_{11} = 0$ and can execute further elementary operations to make $a_{14} = a_{13} = \cdots = a_{1n} = 0$ and obtain the expression

$$\begin{bmatrix} x_{11} & 1 & x_{13} & \cdots & x_{1r} & x_{1n} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ x_{13} & 0 & x_{33} & \cdots & x_{3r} & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{1r} & 0 & x_{3r} & \cdots & x_{rr} & x_{rn} \\ x_{1n} & 0 & x_{3n} & x_{r-1n} & x_{rn} & 1 + x_{nn} \end{bmatrix}, \qquad (4.11)$$

so that this case is the same as the previous one.

(I_2) We analyze now the case where $n = 2k = r + 2$, and also have here two possible subcases, that we label as (a) and (b):

(a) In the expression (4.1) for the Hessian matrix we suppose that the vectors $X_1, X_2, \ldots, X_{n-3}, X_{n-1}$ are linearly independent on an open, dense subset of the domain, while $X_i = \sum_{j \neq n-2,n} a_{ij} X_j$ for $i = n-2, n$. Then, by means of the obviously suggested elementary operations, we transform the Hessian matrix into

$$\begin{bmatrix} B_{n-4n-4} & & x_{1n-3} & 0 & x_{1n-1} & 0 \\ & & \vdots & \vdots & \vdots & \vdots \\ x_{1n-3} & \cdots & x_{n-3n-3} & 1 & x_{n-1n-3} & 0 \\ 0 & \cdots & 1 & b_{n-2n-2} & 0 & b_{n-2n} \\ x_{1n-1} & \cdots & x_{n-1n-3} & 0 & x_{n-1n-1} & 1 \\ 0 & \cdots & 0 & b_{n-2n} & 1 & b_{nn} \end{bmatrix}. \qquad (4.12)$$

In the latter expression the square submatrix B_{n-4n-4}, of order $n-4$, remains unchanged while for the square submatrix of order 2 indicated as (b_{ij}) we must have det $(b_{ij}) = 0$. Then, we have two subcases, labeled as (a_1) and (a_2), that we analyze next:

(a$_1$) We assume first that $(b_{ij}) = 0$. Then, the right inferior submatrix B_{44} is given by

$$B_{44} := \begin{bmatrix} x_{n-3n-3} & 1 & x_{n-1n-3} & 0 \\ 1 & 0 & 0 & 0 \\ x_{n-1n-3} & 0 & x_{n-1n-1} & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Hence, $\det B_{n-4n-4} = (-1)^{k-2}$ and the maximal rank of the complementary matrix of B_{n-4n-4} must be strictly less than $n-4$. This gives rise to the inductive process of reduction, integrating first the submatrix B_{44}, followed by reduction and integration of B_{n-4n-4} and the rest of blocks.

(a$_2$) We assume next that $(b_{ij}) \neq 0$, and we may further assume that $b_{n-2n-2} \neq 0$. Then, there exists $\alpha \in R$ such that $(b_{n-2n}, b_{nn}) = \alpha(b_{n-2n-2}, b_{n-2n})$, we can reduce B_{44} to

$$\begin{bmatrix} x_{n-3n-3} & 1 & x_{n-1n-3} & 0 \\ 1 & 0 & \alpha x_{n-1n-3} & 0 \\ x_{n-1n-3} & \alpha x_{n-1n-3} & x_{n-1n-1} & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and again apply the reduction and integration process as before.

(b) We suppose now that the vectors $X_1, X_2, \ldots, X_{n-3}, X_{n-2}$ are linearly independent on an open, dense subset of the domain, while $X_i = \sum_{j \neq n-1,n} a_{ij} X_j$ for $i = n-1, n$. Then, the expression (4.1) for the Hessian matrix may be reduced, first to

$$\begin{bmatrix} B_{mm} & & x_{1n-3} & x_{1n-2} & -a_{n-12} & -a_{n2} \\ & & x_{2n-3} & x_{2n-2} & -a_{n-11} & -a_{n1} \\ & & \vdots & \vdots & \vdots & \vdots \\ x_{1n-3} & \cdots & x_{n-3n-3} & 1 + x_{n-3n-2} & -a_{n-1n-2} & -a_{nn-2} \\ x_{1n-2} & \cdots & 1 + x_{n-3n-2} & x_{n-2n-2} & -a_{n-1n-3} & -a_{nn-3} \\ -a_{n-12} & \cdots & -a_{n-1n-2} & -a_{n-1n-3} & b_{n-1n-1} & b_{n-1n} \\ -a_{n2} & \cdots & -a_{nn-2} & -a_{nn-3} & b_{n-1n} & b_{nn} \end{bmatrix}, \quad (4.13)$$

where the left superior square submatrix B_{mm}, of order $m := n-4$, has remained unchanged and for the 2×2 submatrix (b_{ij}) we must have $\det (b_{ij}) = 0$. Hence we have again two subcases, labeled as (b$_1$) and (b$_2$):

(b$_1$) We assume first that $(b_{ij}) = 0$. Then, some of the 2×2 submatrices of the form $\begin{pmatrix} a_{n-1i} & a_{ni} \\ a_{n-1j} & a_{nj} \end{pmatrix}$ is nonsingular, we may even assume this is the last one and, by further elementary operations, reduce the right inferior square submatrix of order 4 to the form

$$\begin{bmatrix} x_{n-3n-3} & x_{n-3n-2} & 1 & 0 \\ x_{n-3n-2} & x_{n-2n-2} & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Therefore, by means of further elementary operations, we can reduce the Hessian matrix to the same expression as in the previous subcase (a_1) and repeat the procedure.

(b_2) We consider next the case where $(b_{ij}) \neq 0$. Then, we may assume that $b_{nn} \neq 0$ and perform further elementary operations so that the expression of the Hessian matrix in Eq. (4.13) is transformed, first into

$$\begin{bmatrix} B_{mm} & & x_{1n-3} & x_{1n-2} & a_{1n-1} & a_{1n} \\ & & x_{2n-3} & x_{2n-2} & a_{2n-1} & a_{2n} \\ & & \vdots & \vdots & \vdots & \vdots \\ x_{1n-3} & \cdots & x_{n-3n-3} & 1 + x_{n-3n-2} & a_{n-3n-1} & a_{n-3n} \\ x_{1n-2} & \cdots & 1 + x_{n-3n-2} & x_{n-2n-2} & a_{n-2n-1} & a_{n-2n} \\ a_{1n-1} & \cdots & a_{n-3n-1} & a_{n-2n-1} & 0 & 0 \\ a_{1n} & \cdots & a_{n-3n} & a_{n-2n} & 0 & 1 \end{bmatrix}, \quad (4.14)$$

then into

$$\begin{bmatrix} b_{11} + x_{11} & \cdots & b_{1n-2} + x_{1n-2} & a_{1n-1} & 0 \\ & & & a_{2n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ b_{1n-2} + x_{1n-2} & \cdots & b_{n-2n-2} + x_{n-2n-2} & a_{n-2n-1} & 0 \\ a_{1n-1} & \cdots & a_{n-2n-1} & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix}, \quad (4.15)$$

and since we may assume that $a_{n-3n-1} \neq 0$, perform further elementary operations transforming the latter into

$$\begin{bmatrix} B_{mm} & & & x_{1n-3} & x_{1n-2} & 0 & 0 \\ & & & x_{2n-3} & x_{2n-2} & \vdots & 0 \\ & & & \vdots & \vdots & 0 & \vdots \\ x_{1n-3} & & \cdots & x_{n-3n-3} & x_{n-3n-2} & 1 & 0 \\ x_{1n-2} & & \cdots & x_{n-3n-2} & b_{n-2n-2} + x_{n-2n-2} & 0 & a_{n-2n} \\ 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & a_{n-2n} & 0 & 1 \end{bmatrix}. \quad (4.16)$$

Next, by evaluating the determinant we conclude that we must have $x_{n-2n-2} = 0$ and $b_{n-2n-2} = -1$, so that we may transform the above, first into the form

$$\begin{bmatrix} B_{mm} & & & x_{1n-3} & 0 & x_{1n-1} & 0 \\ & & & \vdots & \vdots & \vdots & 0 \\ & & & x_{n-4n-3} & 0 & x_{n-4n-1} & \vdots \\ x_{1n-3} & \cdots & x_{n-4n-3} & x_{n-3n-3} & 1 & x_{n-3n-1} & 0 \\ 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\ x_{1n-1} & \cdots & x_{n-4n-1} & x_{n-3n-1} & 0 & -1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.17}$$

and then, by means of further elementary operations, so that the right inferior 4×4 submatrix is taken to the form

$$\begin{bmatrix} x_{n-3n-3} & 1 & x_{n-1n-3} & 0 \\ 1 & 0 & -\frac{1}{2}x_{n-1n-3} & 0 \\ x_{n-1n-3} & -\frac{1}{2}x_{n-1n-3} & x_{n-1n-1} & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

which is particular subcase of the one labeled as (a_2). The inductive process works again.

(II_2) We consider next the case where $n = 2k + 1 = r + 2$, which can be analyzed by means of three possible subcases labeled as (a), (b) and (c):

(a) In the expression (4.5) for the Hessian matrix we assume first that the vectors $X_1, X_2, \ldots, X_r$, are linearly independent on an open, dense subset of the domain, while $X_i = \sum_{j=1}^{r} a_{ij} X_j$ for $i = n-1, n$. Then, by means of elementary operations, we transform the Hessian matrix into

$$\begin{bmatrix} B_{n-3n-3} & \cdots & & x_{1n-2} & -a_{n-12} & -a_{n2} \\ \vdots & \ddots & & \vdots & \vdots & \vdots \\ & & & x_{n-3n-2} & -a_{n-1n-4} & -a_{nn-4} \\ x_{1n-2} & \cdots & x_{n-3n-2} & x_{n-2n-2} & 1 & 0 \\ -a_{n-12} & \cdots & -a_{n-1n-4} & 1 & b_{n-1n-1} & b_{n-1n} \\ -a_{n2} & \cdots & -a_{nn-4} & 0 & b_{n-1n} & b_{nn} \end{bmatrix}, \tag{4.18}$$

where the square submatrix B_{n-3n-3}, of order $n-3$, has remained unchanged and where for the 2×2 submatrix (b_{ij}) we must necessarily have $\det(b_{ij}) = 0$, $(b_{ij}) \neq 0$.

Then, by performing further elementary operations we may transform the latter into

$$\begin{bmatrix} a_{11}+x_{11} & \cdots & & a_{1n-3}+x_{1n-3} & x_{1n-2} & 0 & 0 \\ & & & \vdots & \vdots & \vdots & \vdots \\ \vdots & & \ddots & & \vdots & 0 & 0 \\ a_{1n-3}+x_{1n-3} & \cdots & \cdots & a_{n-3n-3}+x_{n-3n-3} & x_{n-3n-2} & 0 & 0 \\ x_{1n-2} & \cdots & \cdots & x_{n-3n-2} & x_{n-2n-2} & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.19}$$

This expression gives place to the inductive process since we may integrate, first, the right inferior 3×3 submatrix $\begin{bmatrix} x_{n-2n-2} & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and, then, since we must have for the left superior square matrix of order $n-3$ the two conditions $\det(a_{ij}+x_{ij}) = (-1)^{k-1}$ and maximal rank of the complementary matrix less than $n-3$, i.e., maximal $\operatorname{rank}(x_{ij}) < n-3$, we may proceed to reduce and integrate the rest of the Hessian matrix.

(b) In the representation (4.5) of the Hessian we assume now that the vectors $X_1, X_2, \ldots, X_{r-2}, X_r, X_n$, are linearly independent on an open, dense subset of the domain, while $X_i = \sum_{j \neq r-1, r+1} a_{ij} X_j$ for $i = r-1, r+1$. Then, we can transform the Hessian matrix into

$$\begin{bmatrix} B_{n-5n-5} & & x_{1n-4} & -a_{r-12} & x_{1r} & -a_{r+12} & x_{1n} \\ & & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{1n-4} & \cdots & x_{n-4n-4} & 1 & x_{r-2r} & 0 & x_{r-2n} \\ -a_{r-12} & \cdots & 1 & b_{r-1r-1} & 0 & b_{r-1r+1} & -a_{r-1n} \\ x_{1r} & \cdots & x_{r-2r} & 0 & x_{rr} & 1 & x_{rn} \\ -a_{r+12} & \cdots & 0 & b_{r-1r+1} & 1 & b_{r+1r+1} & -a_{r+1n} \\ x_{1n} & \cdots & x_{r-2n} & -a_{r-1n} & x_{rn} & -a_{r+1n} & 1+x_{nn} \end{bmatrix}. \tag{4.20}$$

It is obvious that in the above expression we can annihilate all elements of the form a_{ij} by means of suitable elementary operations. Besides, we must necessarily have $\det(b_{ij}) = 0$ and this gives place to two subcases, labeled as (b_1) and (b_2), to be considered next:

(b_1) Let us assume first that $(b_{ij}) = 0$. Then, the above Hessian matrix is equal to

$$\begin{bmatrix} B_{n-5n-5} & & x_{1n-4} & 0 & x_{1r} & 0 & x_{1n} \\ & & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{1n-4} & \cdots & x_{n-4n-4} & 1 & x_{r-2r} & 0 & x_{r-2n} \\ 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\ x_{1r} & \cdots & x_{r-2r} & 0 & x_{rr} & 1 & x_{rn} \\ 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\ x_{1n} & \cdots & x_{r-2n} & 0 & x_{rn} & 0 & 1+x_{nn} \end{bmatrix}. \tag{4.21}$$

Thus, by evaluating the determinant of the latter we obtain the condition that $x_{nn} = 0$ and, hence, implement again the inductive process by integrating, first, the right inferior 5×5 submatrix and observing that the left superior square superior submatrix B_{n-5n-5} must have determinant equal to $(-1)^{k-2} = (-1)^k$.

(b$_2$) If $\det(b_{ij}) = 0$, $(b_{ij}) \neq 0$, we can assume that $(b_{r-1r-1}, b_{r-1r+1}) = c\,(b_{r-1r+1}, b_{r+1r+1})$, for some $c \in R$, and perform elementary operations to express the Hessian matrix as

$$\begin{bmatrix} B_{n-5n-5} & & x_{1n-4} & 0 & x_{1r} & 0 & x_{1n} \\ & & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{1n-4} & \cdots & x_{n-4n-4} & 1 & x_{r-2r} & 0 & x_{r-2n} \\ 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\ x_{1r} & \cdots & x_{r-2r} & 0 & x_{rr} & 1 & x_{rn} \\ 0 & \cdots & 0 & 0 & 1 & b_{r+1r+1} & 0 \\ x_{1n} & \cdots & x_{r-2n} & 0 & x_{rn} & 0 & 1+x_{nn} \end{bmatrix}. \tag{4.22}$$

If we calculate the determinant, by rows $r-2$, $r-1$, we obtain

$$\det H(f) = -\det B_{n-5n-5} \cdot \det \begin{bmatrix} x_{rr} & 1 & x_{rn} \\ 1 & b_{r+1r+1} & 0 \\ x_{rn} & 0 & 1+x_{nn} \end{bmatrix} = (-1)^k .$$

Hence, we must have

$$\det \begin{bmatrix} x_{rr} & 1 & x_{rn} \\ 1 & b_{r+1r+1} & 0 \\ x_{rn} & 0 & 1+x_{nn} \end{bmatrix} = -1 \text{ and } \det B_{n-5n-5} = (-1)^{k-2} = (-1)^k .$$

It follows that we have to consider two subcases:

(b$_{21}$) If $b_{r+1r+1} := b > 0$, we can perform further elementary operations on the last 5 rows (and columns), in a similar fashion to the case g$_{22}$), in Gigena (2003), so that the Hessian matrix is reduced to

$$\begin{bmatrix} B_{n-5n-5} & & x_{1n-4} & 0 & x_{1r} & x_{1n-1} & 0 \\ & & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{1n-4} & \cdots & x_{n-4n-4} & 1 & x_{r-2r} & x_{r-2n-1} & 0 \\ 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\ x_{1r} & \cdots & x_{r-2r} & 0 & x_{rr} & 1 & 0 \\ x_{1n-1} & \cdots & x_{r-2n-1} & 0 & 1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.23}$$

This gives place to the inductive process by integrating, first, the 5×5 right inferior submatrix and proceeding, afterwards, to integrate the rest of the Hessian matrix.

(b$_{22}$) In case $b_{r+1r+1} := b < 0$, we must necessarily have that $x_{rr} = x_{rn} = x_{nn} = 0$ and the Hessian matrix can be reduced to

$$\begin{bmatrix} B_{n-5n-5} & & x_{1n-4} & 0 & x_{1r} & -\frac{b}{2}x_{1r} & x_{1n} \\ & & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{1n-4} & \cdots & x_{n-4n-4} & 1 & x_{r-2r} & -\frac{b}{2}x_{r-2r} & x_{r-2n} \\ 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\ x_{1r} & \cdots & x_{r-2r} & 0 & 0 & 1 & 0 \\ -\frac{b}{2}x_{1r} & \cdots & -\frac{b}{2}x_{r-2r} & 0 & 1 & 0 & 0 \\ x_{1n} & \cdots & x_{r-2n} & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.24}$$

Thus, the inductive process works again.

(c) We assume, finally, that in the representation (4.5) of the Hessian the vectors $X_3, X_4, \ldots, X_r, X_{r+1}, X_n$, are linearly independent on an open, dense subset of the domain, while $X_i = \sum_{j=3}^{n} a_{ij} X_j$ for $i = 1, 2$. Then, we can transform the Hessian matrix into

$$\begin{bmatrix} b_{11} & b_{12} & -a_{14} & -a_{13} & \cdots & -a_{1r} & -a_{1n} \\ b_{12} & b_{22} & -a_{24} & -a_{23} & \cdots & -a_{2r} & -a_{2n} \\ -a_{14} & -a_{24} & x_{33} & & & & x_{3n} \\ -a_{13} & -a_{23} & & \ddots & & & \\ \vdots & \vdots & & & x_{rr} & 1+x_{rn-1} & x_{rn} \\ -a_{1r} & -a_{2r} & & & 1+x_{rn-1} & x_{n-1n-1} & x_{n-1n} \\ -a_{1n} & -a_{2n} & x_{3n} & & x_{rn} & x_{n-1n} & 1+x_{nn} \end{bmatrix}. \tag{4.25}$$

Since we must have $\det \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} = 0$ we may diagonalize the submatrix (b_{ij}), by means of suitable elementary operations on the above Hessian, and have two possible subcases:

(c$_1$) Suppose, first, that $(b_{ij}) = 0$. Then, it turns out that some of the submatrices $\begin{pmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{pmatrix}$ must be nonsingular and we can assume that $\begin{pmatrix} a_{14} & a_{13} \\ a_{24} & a_{23} \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence, by means of further elementary operations the Hessian matrix may be transformed into

$$\begin{bmatrix} x_{11} & 1 & x_{13} & 0 & x_{15} & \cdots & & & x_{1n} \\ 1 & 0 & 0 & 0 & 0 & \cdots & & & 0 \\ x_{13} & 0 & x_{33} & 1 & x_{35} & \cdots & & & x_{3n} \\ 0 & 0 & 1 & 0 & 0 & \cdots & & & 0 \\ x_{15} & 0 & x_{35} & 0 & x_{55} & \cdots & & & x_{5n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & \vdots \\ & & & & & & x_{rr} & 1+x_{rn-1} & x_{rn} \\ & & & & & & 1+x_{rn-1} & x_{n-1n-1} & x_{n-1n} \\ x_{1n} & 0 & x_{3n} & 0 & x_{5n} & & x_{rn} & x_{n-1n} & 1+x_{nn} \end{bmatrix}. \quad (4.26)$$

We obviously have, for the submatrix

$$B_{n-4n-4} := \begin{bmatrix} x_{55} & \cdots & & & x_{5n} \\ \vdots & \ddots & & & \vdots \\ & & x_{rr} & 1+x_{rn-1} & x_{rn} \\ & & 1+x_{rn-1} & x_{n-1n-1} & x_{n-1n} \\ x_{5n} & & x_{rn} & x_{n-1n} & 1+x_{nn} \end{bmatrix},$$

$\det B_{n-4n-4} = (-1)^{k-2}$. Thus, we may implement again the inductive process by integrating, first, the 4×4 left superior submatrix, reducing then the right inferior submatrix B_{n-4n-4}, and integrating finally the rest of the Hessian matrix.

(c$_2$) We suppose, next, that $(b_{ij}) \neq 0$ and we may further assume that $b_{11} = 1$, $b_{12} = b_{22} = 0$. Then, the Hessian matrix may be taken to the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -a_{24} & -a_{23} & \cdots & -a_{2r} & -a_{2n} \\ 0 & -a_{24} & x_{33} & & & & x_{3n} \\ 0 & -a_{23} & & \ddots & & & \\ \vdots & \vdots & & & x_{rr} & 1+x_{rn-1} & x_{rn} \\ 0 & -a_{2r} & & & 1+x_{rn-1} & x_{n-1n-1} & x_{n-1n} \\ 0 & -a_{2n} & x_{3n} & & x_{rn} & x_{n-1n} & 1+x_{nn} \end{bmatrix}. \quad (4.27)$$

We may consider now for the latter expression two subcases:

(c_{21}) Let us assume, first, that $a_{24} \neq 0$. Then, by means of additional elementary operations, we can transform the Hessian into

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & x_{33} & & & & x_{3n} \\ 0 & 0 & & \ddots & & & \\ \vdots & \vdots & & & & & \\ 0 & 0 & & & & & \\ 0 & 0 & x_{3n} & & & & B_{n-3n-3} \end{bmatrix}, \tag{4.28}$$

where we obviously have, for the right inferior square submatrix

$$B_{n-3n-3} := \begin{bmatrix} x_{44} & & & x_{4n} \\ & \ddots & & \\ & & x_{rr} & 1 + x_{rn-1} \\ x_{4n} & & 1 + x_{rn-1} & x_{n-1n-1} \end{bmatrix},$$

the condition $\det B_{n-3n-3} = (-1)^{k-1}$. Thus, by means of further elementary operations on the last $n-3$ rows (and columns) of the Hessian matrix, we may inductively reduce the submatrix B_{n-3n-3}. Finally, additional elementary operations allow to represent the Hessian by the expression

$$\begin{bmatrix} x_{11} & 1 & x_{13} & & \cdots & x_{1n-1} & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ x_{13} & 0 & & & & & \\ & 0 & & \ddots & & & \\ \vdots & \vdots & & & B_{n-3n-3} & & \\ x_{1n-1} & 0 & & & & \ddots & \\ 0 & 0 & & & & & 1 \end{bmatrix}. \tag{4.29}$$

The inductive process works again.

(c_{22}) Let us assume, next, that $a_{2n} \neq 0$. Then, the Hessian may be reduce to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -a_{2n} \\ 0 & 0 & x_{33} & & & & x_{3n} \\ 0 & 0 & & \ddots & & & \\ \vdots & \vdots & & & x_{rr} & 1 + x_{rn-1} & x_{rn} \\ 0 & 0 & & & 1 + x_{rn-1} & x_{n-1n-1} & x_{n-1n} \\ 0 & -a_{2n} & x_{3n} & & x_{rn} & x_{n-1n} & x_{nn} \end{bmatrix}. \tag{4.30}$$

By evaluating the determinant we conclude that $-a_{2n} = 1$, so that by further elementary operations the Hessian may again be taken to the form (4.29), and to the inductive process as prescribed. □

Remarks (1) Similar comments and observations as those expressed in our previous articles Gigena (2002, 2003) may be made in the present context, where we have developed the present method allowing to obtain all solutions to the classificatory problem. Thus, and except for lower dimensions $n = 2, 3$ where one may claim to have unique solutions, in each of the general classes classified by n, k and r, different values of the parameters appearing as coefficients of the third degree terms, in a given solution, may give rise to different classes of equivalence under the action of the unimodular affine group $ASL\,(n+1, \mathbb{R})$.

(2) Moreover, if in a given solution one sets equal to zero some of those mentioned coefficients, but not all of them, the result is still a solution to the problem, perhaps with a lesser value for the maximal rank r. In practice, one may manipulate the third degree polynomial, solution to the problem, by adding some third degree terms and setting equal to zero others, so that solutions are obtained with maximal ranks $r-1, r-2, \ldots, 1$.

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