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ORIGINAL PAPER

Impact of prior perception on bridge health diagnosis

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Abstract We use Bayesian logic in reproducing how a rational agent, called Ernest in the paper, analyses monitoring data and infers structural condition. The case study is Adige Bridge, a 260 m-long statically indeterminate structure with a deck supported by 12 stay cables. Bridge structural redundancy, possible relaxation losses and an as-built condition differing from design suggest that long-term load redistribution between cables can be expected. Therefore, the bridge owner installed a monitoring system, including fiberoptic sensors that allow measurement of deformation with an accuracy of a few microstrains. After 1 year of system operation, which included maintenance of the interrogation unit, the data analysis showed an apparent contraction of the cable lengths. This result is in contrast with the expected behavior. We analyze how a rational agent analyzes the observed response, and, in particular, we discuss to what extent he is prone to accept the sensor response as a result of the real mechanical behavior of the bridge versus a mere malfunction of the interrogation unit. In this analysis, we consider four psychological profiles, which vary based on their personal trust in the reliability of the instrumentation and on their knowledge of the structural behavior of the bridge. Using Bayesian logic as a tool to combine prior belief with sensor data, we explore how the extent of prior knowledge can alter the final engineering perception of the current state of the bridge and we demonstrate how the engineer's posterior judgment is predictable with a mathematical model. Formal reproduction of the human decision-making process can have strong impact in the field of structural health monitoring, as it may enable: (1) quantification of probabilities that engineers attribute to various events based on their subjective experience (which is currently an important challenge); (2) better understanding and improvement of the decision-making process itself; (3) embedding of decision making into structural health-monitoring methods for the full benefit of the latter.

Keywords Bayesian analysis · Cable-stayed bridge · Data interpretation · Fiberoptic sensors · Heuristic reasoning · Structural health monitoring

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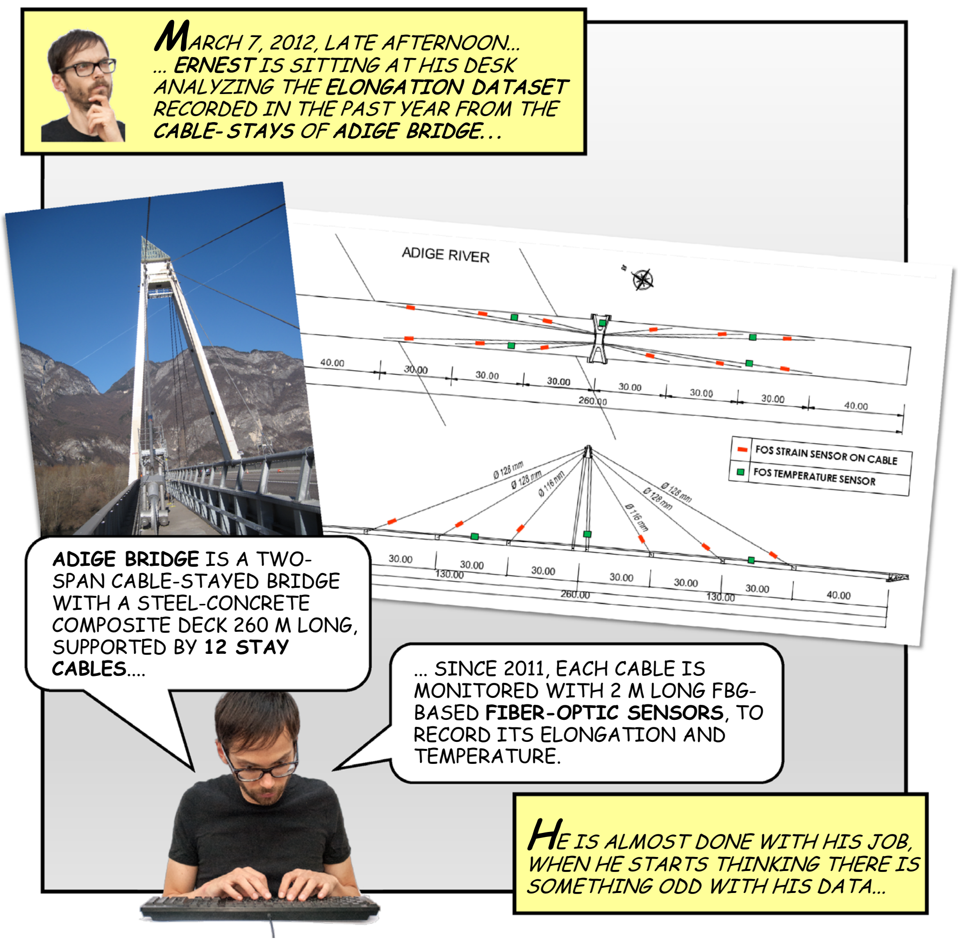
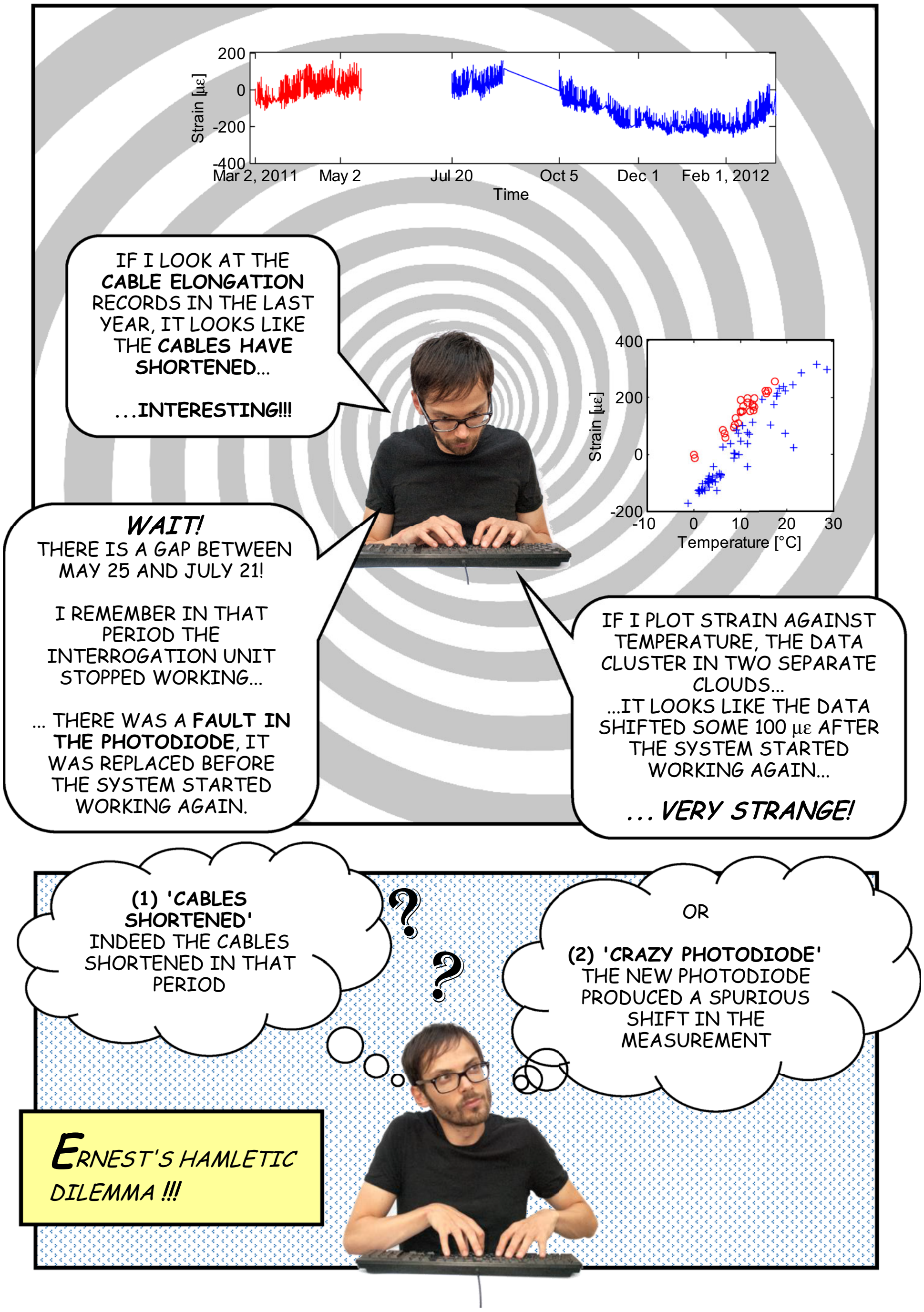
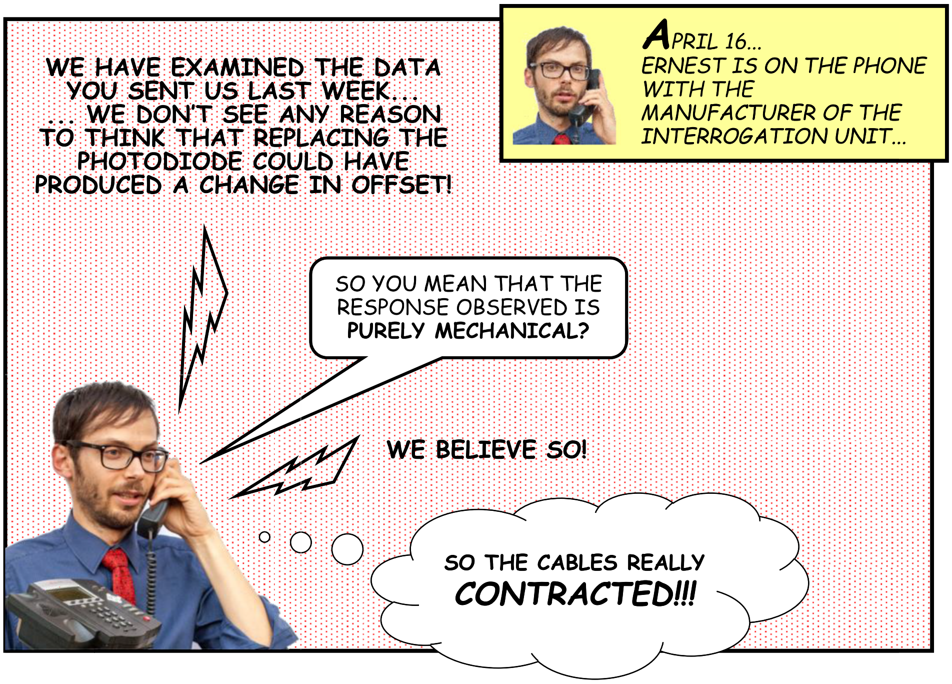
IMPACT of PRIOR PERCEPTION on BRIDGE HEALTH DIAGNOSIS

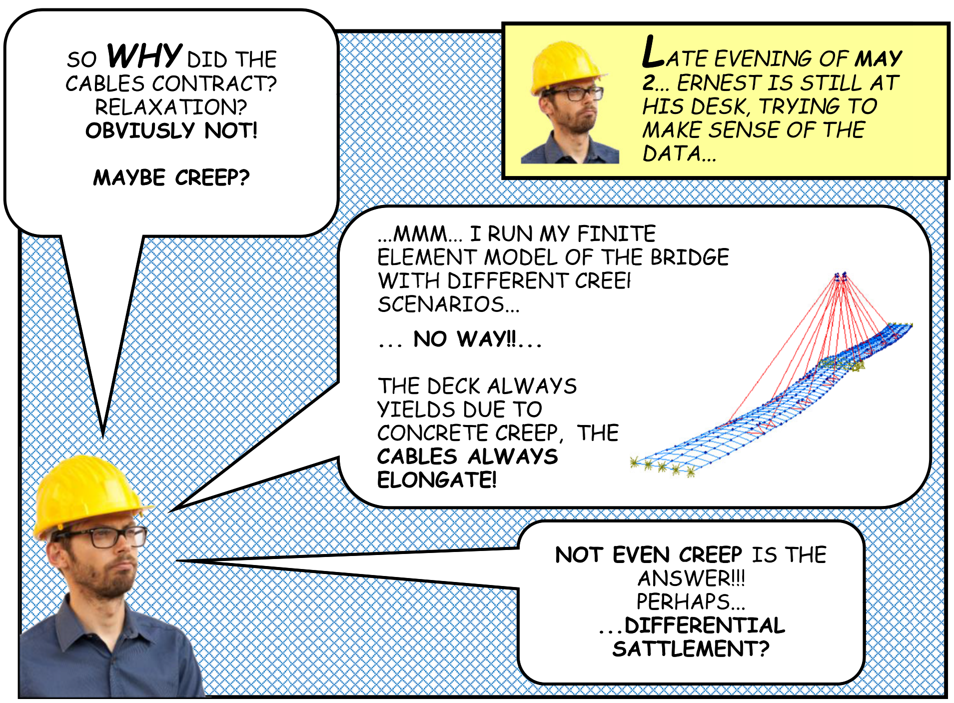
by

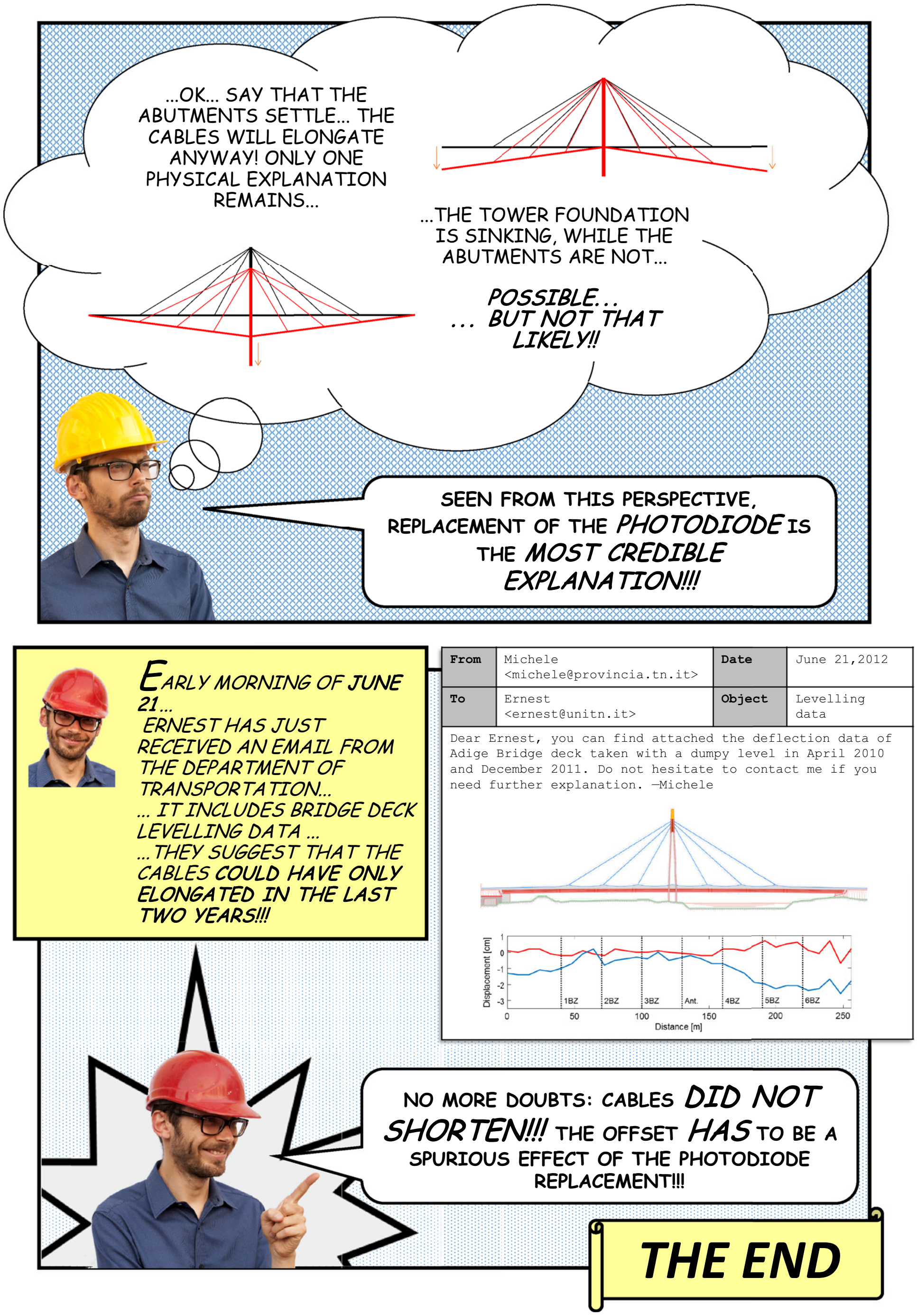
C. Cappello, D. Zonta, B. Glisic, M. Pozzi & R. Zandonini




200
0
-200
-400
Strain [με]
Mar 2, 2011
May 2
Jul 20
Oct 5
Dec 1
Feb 1, 2012
Time
IF I LOOK AT THE **CABLE ELONGATION** RECORDS IN THE LAST YEAR, IT LOOKS LIKE THE **CABLES HAVE SHORTENED**... ...**INTERESTING!!!**
400
200
0
-200
Strain [με]
-10
0
10
20
30
Temperature [°C]
WAIT! THERE IS A GAP BETWEEN MAY 25 AND JULY 21! I REMEMBER IN THAT PERIOD THE INTERROGATION UNIT STOPPED WORKING... ... THERE WAS A **FAULT IN THE PHOTODIODE**, IT WAS REPLACED BEFORE THE SYSTEM STARTED WORKING AGAIN.
IF I PLOT STRAIN AGAINST TEMPERATURE, THE DATA CLUSTER IN TWO SEPARATE CLOUDS... ...IT LOOKS LIKE THE DATA SHIFTED SOME 100 με AFTER THE SYSTEM STARTED WORKING AGAIN... ***... VERY STRANGE!***
(1) 'CABLES SHORTENED' INDEED THE CABLES SHORTENED IN THAT PERIOD
OR
(2) 'CRAZY PHOTODIODE' THE NEW PHOTODIODE PRODUCED A SPURIOUS SHIFT IN THE MEASUREMENT
ERNEST'S HAMLETIC DILEMMA !!!


APRIL 16... ERNEST IS ON THE PHONE WITH THE MANUFACTURER OF THE INTERROGATION UNIT...
WE HAVE EXAMINED THE DATA YOU SENT US LAST WEEK... ... WE DON'T SEE ANY REASON TO THINK THAT REPLACING THE PHOTODIODE COULD HAVE PRODUCED A CHANGE IN OFFSET!
SO YOU MEAN THAT THE RESPONSE OBSERVED IS PURELY MECHANICAL?
WE BELIEVE SO!
SO THE CABLES REALLY CONTRACTED!!!


LATE EVENING OF MAY 2... ERNEST IS STILL AT HIS DESK, TRYING TO MAKE SENSE OF THE DATA...
SO WHY DID THE CABLES CONTRACT? RELAXATION? OBVIUSLY NOT! MAYBE CREEP?
...MMM... I RUN MY FINITE ELEMENT MODEL OF THE BRIDGE WITH DIFFERENT CREEI SCENARIOS... ... NO WAY!!... THE DECK ALWAYS YIELDS DUE TO CONCRETE CREEP, THE CABLES ALWAYS ELONGATE!
NOT EVEN CREEP IS THE ANSWER!!! PERHAPS... ...DIFFERENTIAL SATTLEMENT?



1 Introduction

1.1 Episode 1: ‘The Geek’

In the late afternoon of March 7, 2012, Ernest is sitting at his desk analyzing the elongation dataset recorded in the past year from the stay cables of Adige Bridge. He is almost done with his job, when he starts thinking there may be something odd in his data.

Adige Bridge was built in 2008 10 km north of the city of Trento, Italy, and is owned by the transportation agency of the Autonomous Province of Trento [1–3]. Ernest has in front of him a scheme of the bridge, the same reported to the benefit of our Reader in Fig. 1. It is a two-span cable-stayed bridge with a steel–concrete composite deck 260 m long. The composite deck is made from four “I” section steel girders and a 25 cm cast-on-site concrete slab; the deck is supported by 12 stay cables, 6 per side, with diameters of 116 mm and 128 mm. Both ends of the deck

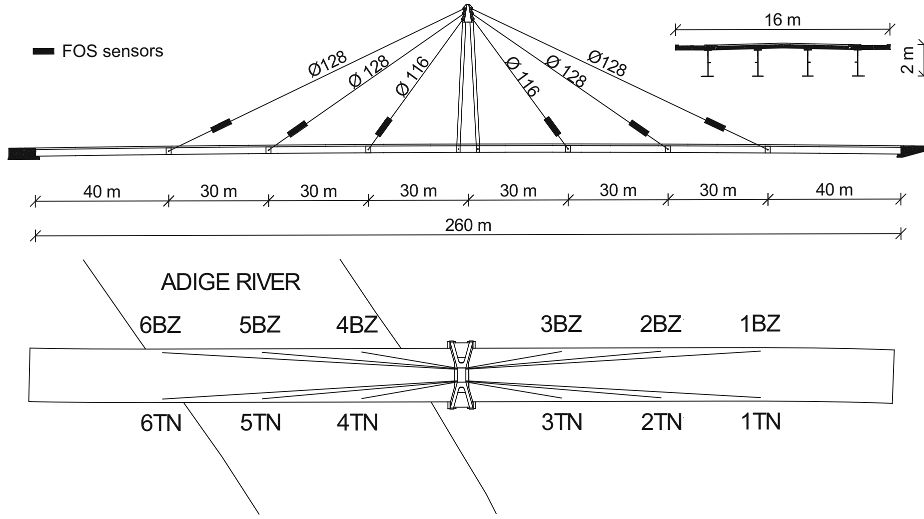


Fig. 1 Adige Bridge: longitudinal section, cross section and plan view with cable labels

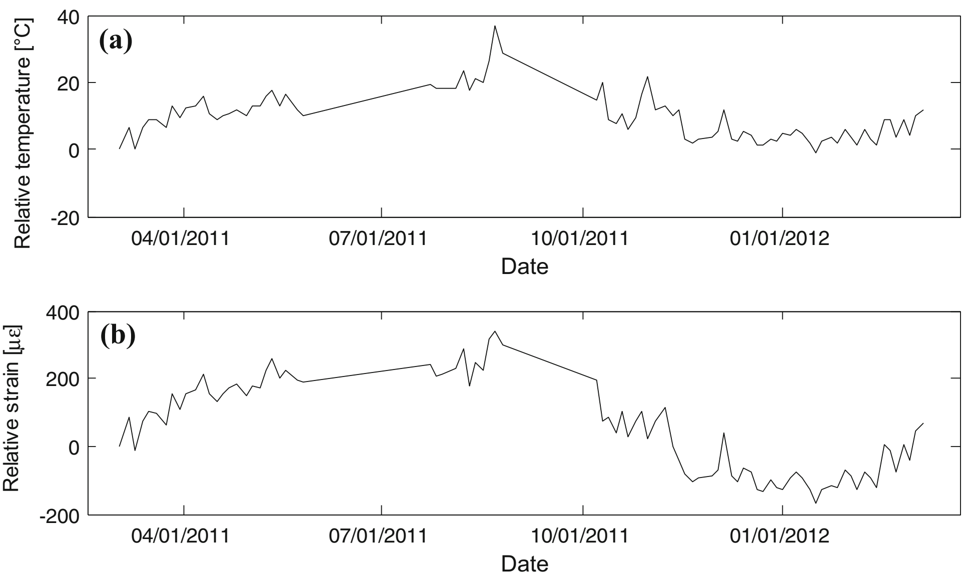


Fig. 2 Data recorded by the FOS monitoring system at cable 1TN, from March 2011 to March 2012

are fully restrained by abutments, which are supported by a micropile underpinning foundation system. The cables are strung to the central bridge tower, consisting of four steel pylons. The pylons are 45 m high and their foundation consists of six piles, which have a diameter of 150 cm and a length of 34 m.

In 2011, the owner instrumented Adige Bridge with a monitoring system to record possible changes in tensions and elongation of the 12 cables. The system includes 2 m-long FBG-based fiberoptic sensors (FOS) [4], located on each cable to record its elongation and temperature. The values are relative to the instant of setup. The whole FOS network is configured to record one set of strain values and one set of temperature values at the same instant, every 15 min.

Ernest observes the temperature and elongation records (Fig. 2), collected for cable 1TN from March 8, 2011, when acquisition started, to now, March 7, 2012. Something does not make sense. There is a gap between May 25 and July 21, but this is not the problem: Ernest knows that in late May the interrogation unit stopped working; he clearly remembers there was a fault in the photodiode (a component of the interrogation unit) that was eventually replaced with a new one before the system started working again, on July 21. Ernest plots strain against temperature and gets the graph of Fig. 3: the data are clustered in two separate clouds, one for samples acquired before the interruption 'o', another for those taken after '+'. It looks like the data shifted some 100 με after the system started working again. A similar change in the offset can be observed for the measurements collected from the sensors installed on the other cables. Something has happened during that time interval, and Ernest has two possible explanations: (1) indeed, the cables shortened in that period or (2) for reasons that he does not understand, the new photodiode produced a spurious shift in the measurements. What has really happened? Ernest is confused: he really does not know.

1.2 Episode 2: 'The Company Man'

"We have examined the data you sent us last week." It is the early afternoon of April 16, and Ernest is on the phone with an SHM system manufacturer, who keeps speaking. "We don't see any reason to think that replacing the photodiode could have produced a change in offset."

"So you mean that the response observed is purely mechanical?"

"Yeah, we believe so!"

Therefore, the cables, thinks Ernest, really contracted.

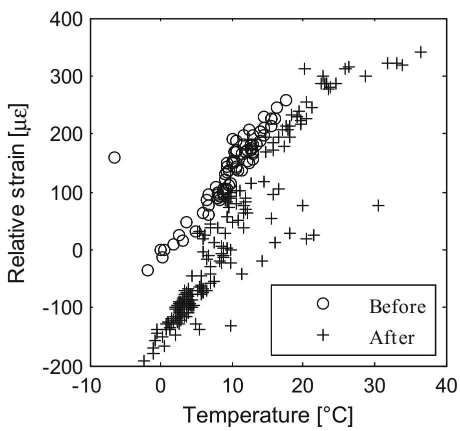


Fig. 3 Correlation between strain and temperature before and after the interruption for cable 1TN

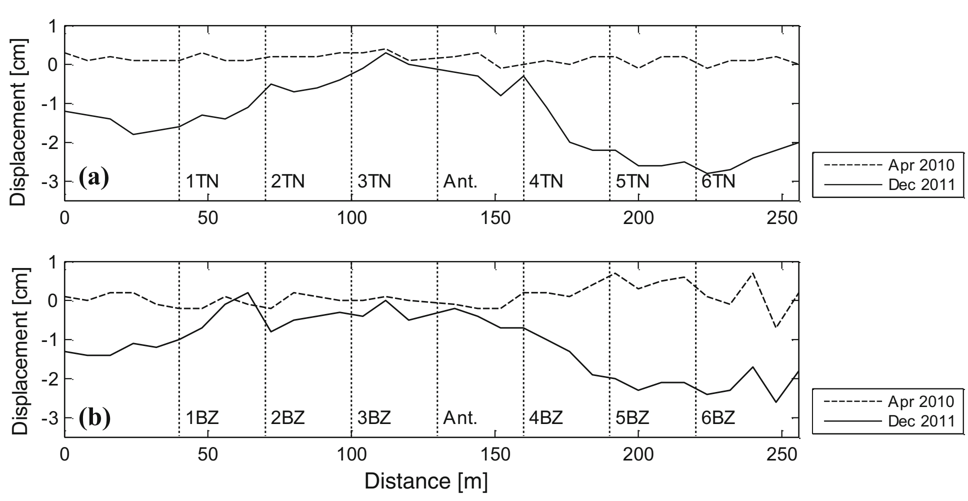
1.3 Episode 3: 'The Modeler'

In the late evening of May 2, Ernest is still working at his desk, trying to make sense of the data observed. A change in strain was expected by design, but this would have to be elongation, rather than contraction. So why did the cables contract? Steel relaxation? Obviously not; cable relaxation always results in elongation. Creep of concrete slab, perhaps? Ernest runs a finite element model (FEM) of the bridge with two different creep scenarios, before realizing that, regardless of the scenario, the deck always yields due to concrete creep and therefore the cables always elongate. Using nominal design values the elongation should be about 20 $\mu\varepsilon{\cdot}\text{year}^{-1}$. No way to explain cable contraction with creep. To Ernest, only one physical explanation remains: the tower foundation is sinking, while the abutments are not: possible, but not that likely. Seeing things from this perspective, Ernest starts thinking again that the replacement of the photodiode is the most credible explanation. However, he needs further evidence before drawing a conclusion.

1.4 Episode 4: 'The Engineer'

It is in the early morning of June 21 that this story comes to a turning point: Ernest has just received an email from the technical office of APT's Department of Transportation, showing the graph of Fig. 4. This graph plots the deflection of the bridge deck taken with a dumpy level at two different times, April 2010 and December 2011. Comparison of the plots shows no settlement of the tower foundation has occurred, while it looks like the abutments have sunk and the deck has deflected downward over time suggesting that the cables could have only elongated in the last 2 years. At this point, Ernest has no more doubts: indeed the cable did not shorten, and the offset observed in the

Fig. 4 Bridge deck deflection recorded by topographic surveys: southern (**a**) and northern (**b**) sides



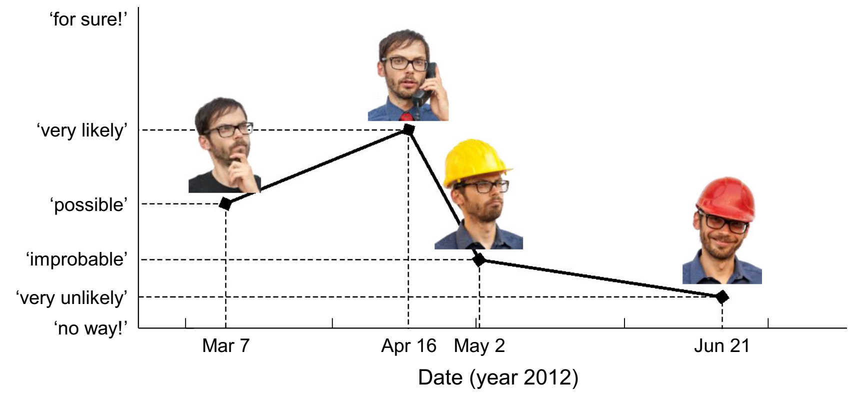


Fig. 5 Qualitative representation of Ernest's perception of cable contraction

data has to be a spurious effect of the photodiode replacement.

1.5 Epilogue

In short, this is the story of an engineer repeatedly changing his mind about the state of a structure: not an extraordinary story, yet true. To some extent, Ernest is the personification of a good engineer having trouble drawing inference from ambiguous information and using personal experience, third party expert judgment, and a lot of common sense to find a solution.

A question arising from this story is the following: is Ernest a rational individual?

At first sight, we could claim that he is not, because, faced with the problem of interpreting a set of objective data, he keeps changing his judgment. From March 7 to August 31, 2012, Ernest's perceived probability of having a cable contraction has swung from 'possible', to 'very likely', to 'improbable', to 'very unlikely'. We can even plot Ernest's personal perception, in rough qualitative terms, in a graph similar to that depicted by Fig. 5.

However, at any stage of the story, Ernest has always drawn a judgment consistent with the background knowledge available to him at that time. The different judgments he expresses simply reflect how his background information changes. Although Ernest is undoubtedly always the same individual, throughout the course of the story he wears different hats. First, he behaves as a solitary geek, who fiddles with data regardless of their physical meaning. Then, his thinking conforms to that of a company, which sees the problem through the looking glasses of its technical experience and expertise in fiberoptic devices. Next, he is a modeler who tries to make sense of the data through finite element analysis. Finally, he wears the hard hat of the structural engineer, who combines present and past information to summarize an answer. In summary, Ernest's personal profile changes as he acquires new information or new awareness about the problem, and his outcome changes along with this background. Since rationality is not about the trueness of the background, but about using logic in inference, we can say that Ernest's behavior is rational.

Based on a set of uncertain facts, we cannot establish with certainty the state of the bridge. Yet we can still use logic, and mathematics, to predict what the judgment of a rational engineer will be. The tool for making inference based on uncertain information is well established and is commonly referred to as Bayesian logic. The name goes back to Bayes' well-known essay [5]. Reference works on the subject are those by Jaynes [6] and Skilling [7], while many modern specialized textbooks provide the reader with a critical review and applications of this theory to data analysis (see for instance [8–10]). Principles of Bayesian trend fitting are reported by Bishop [11], while Bayesian model updating is presented by Mackay [12] and applied to structural identification by Beck and Yuen [13]. A general overview of application to dynamic models is given by Yuen and Kuok [14]. There are also many applications to structural health monitoring. Among the many, we wish to mention the seminal work by Beck's group [15–17], which defined a consistent framework for probabilistic data processing, but see also [18–22]. Finally, an example of integration of Bayesian updating in maintenance planning is provided by Memarzadeh et al. [22].

What we wish to show in this paper is that it is possible to predict an engineer's judgment with Bayesian inference. We will reproduce with a quantitative model how Ernest perceives, and believes plausible, the cable contraction. In doing this, we first state, in Sect. 2, the general tools of Bayesian inference, and we illustrate how this applies to Ernest's case. Section 3 explains how the engineer's background information can be translated into mathematical terms. Next, in Sect. 4, we simulate, through a Bayesian model, Ernest's perception of cable shortening, and we compare the results with his actual heuristic conclusions, as told in the story above. A comprehensive discussion and remarks are offered at the end of the paper. Formal reproduction of the human decision-making process can have strong impact in the field of structural health monitoring as it may enable: (1) quantification of probabilities that engineers attributes to various events based on their subjective experience (which is currently an important challenge); (2) better understanding and improvement of the decision-making process itself; (3) embedding of decision making into structural health-monitoring methods, for the full benefit of the latter.

The stay cable 1TN (Fig. 1b) is taken as a representative to show the numerical results throughout the manuscript, once verified that the analyses carried out for the other cables lead to similar outcomes and conclusions.

2 Problem formulation

Before flooding the reader with mathematics, let us recap Ernest's way of thinking. He has acquired strain and temperature records from the monitoring system of Adige Bridge (those shown in Fig. 2) and he wants to make sense of them. He has two ways to look at them.

Scenario S_1: the strain measurements reflect the actual mechanical elongation/contraction of the cable as an effect of temperature changes, settlement, steel relaxation, concrete shrinkage and creep.

Scenario S_2: the data reflect the mechanical behavior as above, but replacing the photodiode in the interrogation unit has produced a change in offset.

These two scenarios are mutually exclusive and exhaustive, meaning that, in the mind of Ernest, either the first scenario is true or the second is true. The simultaneous trueness of the two and further possibilities are excluded. In principle, either of the two could be the right answer and, based on the information available, there is objectively no overwhelming evidence supporting either one or the other. So Ernest's fundamental question is: which of the two is the most likely?

To answer, the time has arrived to introduce a little math: denote with $\boldsymbol{\varepsilon}$ the set of strain measurements and with $\mathbf{T}$ the set of temperature measurements. Ernest wishes to estimate the probability of a scenario, S_1 or S_2, *after* the monitoring observations have been acquired, in formulas: $P(S_i|\boldsymbol{\varepsilon},\mathbf{T})$, with i being either 1 or 2. We denote this quantity the posterior probability (posterior to the data acquisition). In contrast, for each scenario S_i, the prior probability $P(S_i)$ represents what Ernest believes before acquiring any data. We have stressed that Ernest is a rational individual, who judges based on logic. When dealing with uncertainties, the logical inference process is encoded in the well-known Bayes rule, which states that the posterior probability is related to the prior through

$$P(S_i|\boldsymbol{\varepsilon},\mathbf{T}) = \frac{p(\boldsymbol{\varepsilon},\mathbf{T}|S_i)\cdot P(S_i)}{p(\boldsymbol{\varepsilon},\mathbf{T})}, \tag{1}$$

where $p(\ldots)$ generally indicates a probability density function, $p(\boldsymbol{\varepsilon},\mathbf{T}|S_i)$ is a quantity known as *likelihood*, and $p(\boldsymbol{\varepsilon},\mathbf{T})$ is the evidence, a normalization term which makes sure that the sum of the two posterior probabilities is one:

$$p(\boldsymbol{\varepsilon},\mathbf{T}) = p(\boldsymbol{\varepsilon},\mathbf{T}|S_1)\cdot P(S_1) + p(\boldsymbol{\varepsilon},\mathbf{T}|S_2)\cdot P(S_2). \tag{2}$$

Useful readings about Bayesian logic are, for instance, Sivia's [9] and Bolstad's [23] textbooks. In rough terms, Bayes' theorem says that the posterior probability of a scenario (what Ernest believes after observing the data) is inferred by updating the prior probability (what Ernest believed before observing the data) through the *likelihood*, a quantity that expresses how well the data observed match Ernest's expectations, when he assumes that the scenario is true.

The prior probability is something depending on Ernest's subjective knowledge and changes depending on his individual's background and even his character or mood. In the case of Ernest, this changes with the hat he is wearing (the profile) and this will be discussed for the present case in Sect. 4. The *likelihood* depends, of course, on the data, but also on the interpretation model assumed by Ernest, and this is what we discuss now, in the next section.

3 Interpretation models

We need a model to make sense of the data observed for each cable, and this model changes with the scenario. Measurements at any time t_j are related to true values by including additive random noise:

$$\begin{aligned} \varepsilon_j &= \bar{\varepsilon}_i(t_j) + n_j(\sigma_\varepsilon), \\ T_j &= \bar{T}(t_j) + n_j(\sigma_T), \end{aligned} \tag{3}$$

where ε_j and T_j indicate the strain and temperature measurements, respectively, $\bar{T}$ and $\bar{\varepsilon}$ the corresponding true physical values, and $n_j(\sigma)$ the realization of an independent white noise process, i.e., a zero-mean normal random variable with standard deviation σ. So σ_ε and σ_T are the standard deviations for strain and temperature, respectively. The accuracy of fiberoptic sensors used in the Adige Bridge monitoring system is discussed in [1] and [2]; strain gauge standard uncertainty is estimated at $\sigma_\varepsilon = 5\ \mu\varepsilon$, while the uncertainty of temperature is $\sigma_T = 0.5\ ^\circ\mathrm{C}$. Additional information about physical principles of FBGs and their performance can be found in Glisic and Inaudi [4] and Measures [24]. In (3), we add the subscript j to n to stress that noise is different for each measurement (for each sensor and time). Furthermore, the subscript i indicates that assumptions on the true strain values depend on scenario i. No prior assumption is made on the true temperature process, which can be interpreted as a white noise with wide variance.

To model the time dependence of the true strain, let us start with scenario 1, 'no offset'. In this case, the strain observed is expected to change due to temperature and possibly due to settlement, steel relaxation, concrete shrinkage and creep. In formulas, the true values satisfy the following:

$$\bar{\varepsilon}_1(t_j) = \varepsilon_{0,1} + \alpha_1 \cdot \bar{T}(t_j) + m_1 \cdot t_j, \tag{4}$$

where $\varepsilon_{0,1}$ is an offset representing the ideal strain at time $t = 0$ and temperature $T = 0$; α_1 is the apparent thermal expansion coefficient; m_1 is the slope of a linear trend of elongation which takes into account all long-term effects (i.e., concrete creep and shrinkage, steel relaxation and possible settlement). Herein we assume that the long-term effects can be modeled with a linear trend, because the measurements that Ernest considers are acquired during a relatively short time interval. In practice, the mechanical model that connects temperatures and strains is controlled by three unknown parameters, which we can cluster and indicate with the vector

$$\boldsymbol{\theta}_1 = \{\varepsilon_{0,1}, \alpha_1, m_1\}. \tag{5}$$

The subscript 1, which accompanies these parameters, is to remember that they are specific to scenario 1 and are not necessarily the same as Ernest would estimate assuming scenario 2.

In (3) we have stated a relationship between the true physical values of strain and temperature, $\bar{\varepsilon}(t_j)$ and $\bar{T}(t_j)$. Equations (3) and (4) can be merged into a single equation:

$$\varepsilon_j = \varepsilon_{0,1} + \alpha_1 \cdot T_j + m_1 \cdot t_j + n_j(\sigma_1), \tag{6}$$

where $n_j(\sigma_1)$ indicates a zero-mean Gaussian error with standard deviation $\sigma_1 = 300\ \mu\varepsilon$, which includes the sensor noise σ_ε, σ_T, and the uncertainty of the model defined in (4). Let us label $z_{1,j}$ the residual between observation ε_j and the nominal value of the model:

$$z_{1,j} = \varepsilon_j - (\varepsilon_{0,1} + \alpha_1 \cdot T_j + m_1 \cdot t_j). \tag{7}$$

From (6), we immediately observe that the probability distribution of residual $z_{1,j}$ is exactly that of n_j, which is to say:

$$p(z_{1,j}) = \mathcal{N}(0, \sigma_1; z_{1,j}), \tag{8}$$

where, in general, the notation $\mathcal{N}(\mu, \sigma; x)$ indicates a normal distribution with mean μ and standard deviation σ, calculated in x.

The second way to interpret the recorded data is that of scenario 2. In scenario 2, before the photodiode replacement, which occurs at time t_s, the strain still depends on temperature and time as in scenario 1, but there is a change in the offset of $\Delta\varepsilon_{0,2}$ after the photodiode is replaced. Seen from this standpoint, strains $\bar{\varepsilon}_2$ and temperatures $\bar{T}$ satisfy the equation model.

$$\bar{\varepsilon}_2(t_j) = \begin{cases} \varepsilon_{0,2} + \alpha_2 \cdot \bar{T}(t_j) + m_2 \cdot t_j, & t_j < t_s, \\ (\varepsilon_{0,2} + \Delta\varepsilon_{0,2}) + \alpha_2 \cdot \bar{T}(t_j) + m_2 \cdot t_j, & t_j \geq t_s. \end{cases} \tag{9}$$

In this case, the unknown parameters $\boldsymbol{\theta}_2$ are those presented for scenario 1 plus the shift $\Delta\varepsilon_{0,2}$ introduced by the maintenance of the monitoring system:

$$\boldsymbol{\theta}_2 = \{\varepsilon_{0,2}, \alpha_2, m_2, \Delta\varepsilon_{0,2}\}. \tag{10}$$

Following the same approach already used for scenario 1, we obtain:

$$\varepsilon_j = \begin{cases} \varepsilon_{0,2} + \alpha_2 \cdot T_j + m_2 \cdot t_j + n_j(\sigma_2), & t_j < t_s, \\ (\varepsilon_{0,2} + \Delta\varepsilon_{0,2}) + \alpha_2 \cdot T_j + m_2 \cdot t_j + n_j(\sigma_2), & t_j \geq t_s, \end{cases} \tag{11}$$

where, similarly to $n_j(\sigma_1)$, $n_j(\sigma_2)$ indicates a zero-mean Gaussian error with standard deviation $\sigma_2 = 300\ \mu\varepsilon$, which includes the sensor noise σ_ε, σ_T, and the uncertainty of the model defined in (9). In this case, we define the residual $z_{2,j}$ of the model as:

$$z_{2,j} = \begin{cases} \varepsilon_j - \{\varepsilon_{0,2} + \alpha_2 \cdot T_j + m_2 \cdot t_j\} & t_j < t_s, \\ \varepsilon_j - \{(\varepsilon_{0,2} + \Delta\varepsilon_{0,2}) + \alpha_2 \cdot T_j + m_2 \cdot t_j\} & t_j \geq t_s. \end{cases} \tag{12}$$

and once again we recognize that

$$p(z_{2,j}) = \mathcal{N}(0, \sigma_2; z_{2,j}). \tag{13}$$

Having defined the interpretation models, we can calculate their corresponding *likelihood*. First, observe that when we choose a model, $i = 1$ or 2, the likelihood function is proportional, in the domain of the parameters, to the probability of the residuals:

$$p(\varepsilon_j, T_j | S_i, \theta_i) \propto p(z_{i,j}) = \mathcal{N}(0, \sigma_1; z_{1,j}). \tag{14}$$

Assuming uncorrelated noise, the probability $p(\boldsymbol{\varepsilon}, \mathbf{T} | S_i, \boldsymbol{\theta}_i)$ is simply given by

$$p(\varepsilon, T | S_i, \theta_i) \propto \prod_{j=1}^{N} \mathcal{N}(0, \sigma_2; z_{2,j}), \tag{15}$$

where $\mathcal{N}$ is the number of measurements recorded, and the constant of proportionality $p(\mathbf{T})$ is immaterial for the inference process. Now, the *likelihood* of scenario i, $p(\boldsymbol{\varepsilon}, \mathbf{T} | S_i)$, can be calculated by marginalization of parameters $\boldsymbol{\theta}_i$ using the prior distribution of the parameters $p(\boldsymbol{\theta}_i|S_i)$ as a weighting function:

$$p(\boldsymbol{\varepsilon}, \mathbf{T}|S_i) = \int\limits_{D\boldsymbol{\theta}_i} p(\boldsymbol{\varepsilon}, \mathbf{T}|S_i, \boldsymbol{\theta}_i) \cdot p(\boldsymbol{\theta}_i|S_i) \cdot \mathrm{d}\boldsymbol{\theta}_i, \quad (16)$$

where $D\boldsymbol{\theta}_i$ is the domain of the parameters for scenario i. Bayes' theorem is also used to calculate the posterior distribution of the parameters $\boldsymbol{\theta}_i$ as

$$p(\boldsymbol{\theta}_i|\boldsymbol{\varepsilon}, \mathbf{T}, S_i) = \frac{p(\boldsymbol{\varepsilon}, \mathbf{T}|\boldsymbol{\theta}_i, S_i) \cdot p(\boldsymbol{\theta}_i|S_i)}{p(\boldsymbol{\varepsilon}, \mathbf{T}|S_i)}. \quad (17)$$

Equations (3) to (17) enable us to calculate the posterior probability of each scenario, as well as the posterior distribution of the parameters, for each of Ernest's profiles. Before proceeding with the calculation, we need to set the priors, and this is the topic of the next section.

4 Prior knowledge and profiles

We saw in the introduction how Ernest changes his mind as to the possibility of cable contraction based on the information acquired. We can say that Ernest acts differently at the different stages of this story, characterized by the different profiles: 'Geek', 'Company Man', 'Modeler' and 'Engineer'. Each profile is in turn driven by different background knowledge. Here, we will use formal Bayesian logic to reproduce Ernest's perception. We will represent Ernest's background knowledge with prior information as explained in the following.

4.1 Profile A: the 'Geek'

At the beginning of this story, Ernest simply analyzes the data statistically (he just calculates correlation) without attempting to provide mechanical interpretation. Because for this profile, his judgment does not involve any mechanical knowledge, and he has no particular reason to believe or not believe that the cables have really shortened. This suggests that we should reproduce Ernest's knowledge by assigning for each cable equal prior credibility to each scenario, or $P(S_1) = P(S_2) = 0.50$. For the same reason, we can assign a distribution uniform in $\mathbb{R}$ to the parameters m, α and ε_0, as shown in Table 1. Conversely, for the shift $\Delta\varepsilon_{0,2}$ we choose a normal distribution with mean zero and standard deviation 1000 $\mu\varepsilon$.

4.2 Profile B: the 'Company Man'

Following the phone call with the SHM company, Ernest updates his judgment by acknowledging the sensor provider's belief. The SHM company's answer is justly based on their experience, for a change in offset due to photodiode replacement has never been reported, nor is there any physical explanation that could directly justify such a behavior. Since after the call Ernest considers actual shortening as the most likely explanation for the dataset, we can formalize this belief by assigning prior probabilities of, say, 90 % to scenario 1, and 10 % to scenario 2. It seems likewise reasonable that the discussion with the SHM company did not change Ernest's prior expectations about the parameters. Therefore, we will use for this profile the same prior distributions as for profile A.

Table 1 Prior information for the four Ernest's profiles

Profile	A Geek	B Company man	C Modeler	D Engineer
Prob (S_1)	0.50	0.90	0.90	0.90
Prob (S_2)	0.50	0.10	0.10	0.10
α_1, α_2				
Distribution function	Uniform	Uniform	Normal	Normal
Mean value ($\mu\varepsilon$ °C^{-1})	–	–	12.00	12.00
Standard deviation ($\mu\varepsilon$ °C^{-1})	–	–	3.00	3.00
m_1, m_2				
Distribution function	Uniform	Uniform	See 4.3	See 4.4
Mean value ($\mu\varepsilon$ year^{-1})	–	–	18.50	14.50
Standard deviation ($\mu\varepsilon$ year^{-1})	–	–	5.10	4.40
$\Delta\varepsilon_{0,2}$				
Distribution function	Normal	Normal	Normal	Normal
Mean value ($\mu\varepsilon$)	0.00	0.00	0.00	0.00
Standard deviation ($\mu\varepsilon$)	1000.00	1000.00	1000.00	1000.00

Table 2 Components of linear drift m predicted with FEM for cable 1TN and Ernest's profile C, 'Modeler'

	Creep m_ϕ	Relaxation m_r	Settlement m_s
Probability distribution function	Lognormal	Lognormal	Normal
Mean value ($\mu\varepsilon$ year^{-1})	12.00	6.50	0.00
Standard deviation ($\mu\varepsilon$ year^{-1})	4.80	5.00	4.00
Coefficient of variation	0.40	0.77	–

4.3 Profile C: the 'Modeler'

Ernest's knowledge is updated once more as he analyzes the structure using the FEM. Now, Ernest reconsiders SHM company's judgment based on the outcomes of the analysis, as a mechanical shortening of the cables is a very unlikely possibility. We can reproduce Ernest's reasoning as follows: Ernest still believes that the prior probability of a mechanical contraction is 90 %; however, his analysis suggests that parameter m cannot take just any value. The strain trend slope m is seen as the sum of three independent components: m_ϕ due to creep, m_r due to steel relaxation and m_s due to pile settlement. Ernest studies these components with the aid of the FEM. He sets material proprieties based on the design information, when available, or the literature. The concrete resistance, f_c, and elastic modulus, E_c, are those obtained from concrete sampling as reported in the design documentation; the creep coefficient, φ, is then calculated based on the Eurocode 2 creep model. Using these values in the FEM he estimates that for cable 1TN the elongation owing to creep is in the order of 12 $\mu\varepsilon$ year^{-1}. Ernest is aware that this number is very uncertain: using his own words, the correct value could well be, say, 7 or 17 $\mu\varepsilon$·year^{-1}, but in any case not lower than zero. More quantitatively, we can model his perception as a lognormal distribution with a mean value equal to the nominal value of 12 $\mu\varepsilon$·year^{-1} and coefficient of variation of 40 %. Similarly, he uses the Eurocode 2 model to estimate the steel relaxation coefficient, ρ, and then, with the initial cable load found in the design documentation, he calculates with the FEM an elongation trend of cable 1TN of about $m_r = 6.5$ $\mu\varepsilon$·year^{-1}. He again recognizes that this value is very uncertain, with an error of, say, 5 $\mu\varepsilon$·year^{-1}, but never less than zero. Similarly, we can model this expectation with a lognormal distribution with a mean value of 6.5 $\mu\varepsilon$·year^{-1} and standard deviation of 5 $\mu\varepsilon$·year^{-1}.

Then there is the settlement. In general, Ernest accepts there could be a differential settlement between abutments and the central pier, and based on his experience he also knows that this difference could be just a few millimeters, say up to 5 mm in the last year. However, he has no clue about the sign of the differential settlement, which is to say he has no preference between the scenario where the pier sinks with respect to the abutments and the opposite case. We can model this prior expectation of settlement as a normal distribution with zero mean and 5 mm standard deviation. Using the FEM, Ernest calculates that when the pier sinks 5 mm with respect to the abutments, cable 1TN contracts 4 mm, and similarly he estimates a 4 mm elongation in the opposite case. Thus, we can finally define the prior of the elongation due to settlement for cable 1TN as a normal with zero mean and 4 mm standard deviation (Table 2).

In conclusion, parameter m, the elongation trend slope, is seen as the sum of the three components above:

$$m = m_\phi + m_r + m_s. \tag{18}$$

The resulting prior distribution of m, $p_C(m)$, consistent with profile C, is not normal or lognormal, and has a mean value of +18.5 $\mu\varepsilon$·year^{-1} (elongation) and standard deviation of 5.1 $\mu\varepsilon$·year^{-1} (Fig. 6).

After Ernest analyzes the structure, he also expects a specific value of α. Therefore, we assign a normal distribution with mean 12 $\mu\varepsilon$· °C^{-1} and the standard deviation 3 $\mu\varepsilon$· °C^{-1} to α, where the value of 12 $\mu\varepsilon$· °C^{-1} is the average thermal expansion coefficient of steel.

4.4 Profile D: the 'Engineer'

The survey data obtained by Ernest on June 21 provides him with an independent way to estimate the cable extension trend m. In detail, if we consider an individual cable on April 8, 2010—the time of the first leveling—its length L can be estimated as

$$L = \sqrt{H^2 + B^2}, \tag{19}$$

where B is the distance between the pier and the cable bottom anchorage and H is the difference in level between the top and bottom anchorages, as illustrated in Fig. 7a. At the time of the last leveling, December 20, 2011, the bottom anchorage is found to be deflected by a quantity we denote δ, the cable is elongated by a quantity ΔL, and similarly the drop H changes by a quantity ΔH. The geometrical relationship between these new quantities is (Fig. 7b):

$$L + \Delta L = \sqrt{(H + \Delta H + \delta)^2 + B^2}. \tag{20}$$

Quantity ΔH depends on the thermal expansion of the tower and therefore:

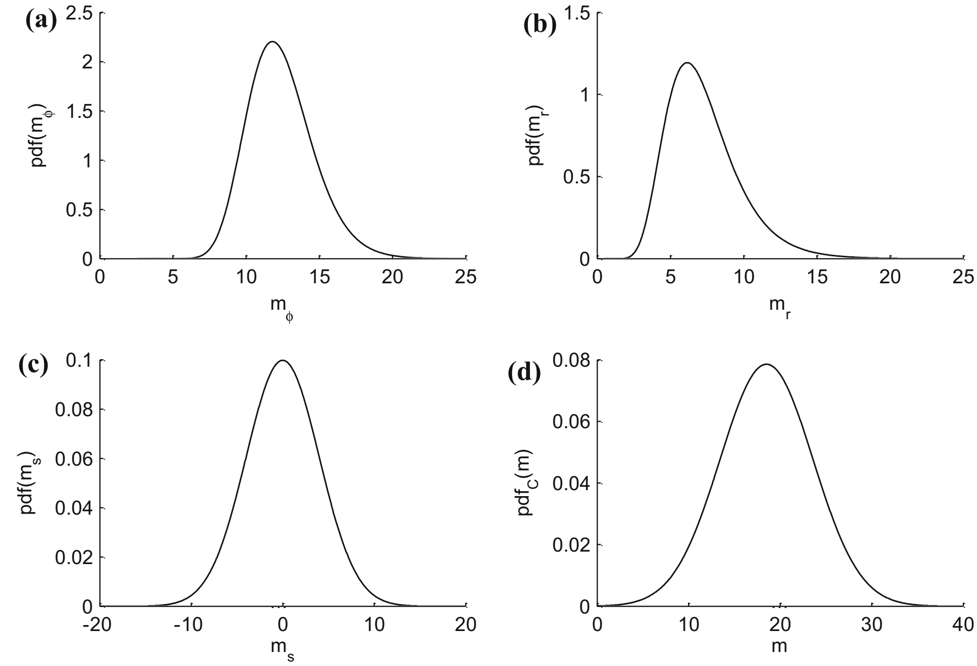


Fig. 6 Prior distributions of linear drift m for cable 1TN according to profile C, 'Modeler': creep component m_ϕ, relaxation component m_r, settlement component m_s and overall combination of the above m

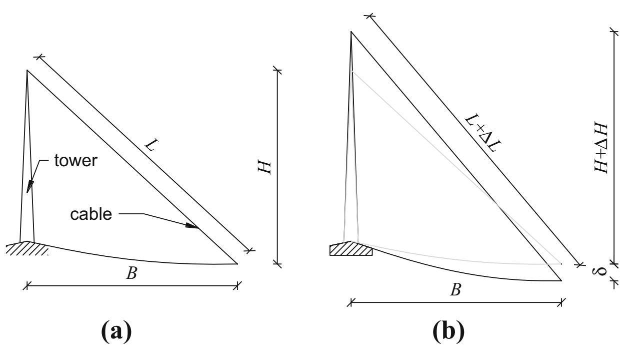


Fig. 7 Schematic geometry of cables and tower as interpreted by Ernest's profile D, 'Engineer'

$$\Delta H = H \cdot \alpha \cdot \Delta T. \tag{21}$$

Similarly, ΔL can be expressed by

$$\Delta L = L(m \cdot \Delta t + \alpha \cdot \Delta T), \tag{22}$$

where m is our unknown cable elongation trend and Δt is the time interval between the two surveys, i.e., 621 days. Combining equations from (19) to (22), the relationship between m and δ is finally expressed through:

$$m = \frac{1}{\Delta t}\left\{\sqrt{\frac{[H(1+\alpha\cdot\Delta T)+\delta]^2+B^2}{H^2+B^2}} - (1+\alpha\cdot\Delta T)\right\}. \tag{23}$$

Using this expression, knowing $\Delta T = 18\ °C$ and assuming $\alpha = 12\ \mu\varepsilon\ °C^{-1}$, Ernest estimates a trend of elongation of 1.8 $\mu\varepsilon$ year^{-1} for cable 1TN. Again, he recognizes that this estimate is uncertain. He noticed a scatter in δ of $\sigma_\delta = 5.3$ mm year^{-1}. We use this value and, following a standard linear error propagation analysis [25], we can estimate a standard deviation of m of 9 $\mu\varepsilon$ year^{-1}. Therefore, we can model the *likelihood* $p(\delta|m)$ of the trend slope m, based on the deflection observation δ, with a normal distribution with a mean value of 1.8 $\mu\varepsilon$ year^{-1} and standard deviation of 9 $\mu\varepsilon$ year^{-1} (Table 3 and Fig. 8).

As we said, the distribution $p(\delta|m)$ reflects in statistical terms the information on m obtained through an independent measurement method, unknown to profile C. The new prior distribution of m according to profile D, $p_D(m)$, can be formally calculated by updating the prior of m of profile C, $p_C(m)$, with this new information, $p(\delta|m)$, through Bayes' rule:

$$p_D(m) = \frac{p(\delta|m)\cdot p_C(m)}{\int\limits_{-\infty}^{+\infty} p(\delta|m)\cdot p_C(m)dm}. \tag{24}$$

The parameters of the prior distribution for cable 1TN corresponding to profile D are reported, along with the other profiles, in Table 1.

5 Results and discussion

Having defined the prior information, (3) to (17) allow us to calculate the posterior probability of each scenario, as well as the posterior distribution of the parameters, for each of Ernest's profiles. We applied the Bayesian approach,

Table 3 Components of linear drift m for cable 1TN and Ernest's profile D, 'Engineer'

	Creep m_{ϕ}	Relaxation m_r	Settlement m_s	Surveys m_{ts}
Probability distribution function	Lognormal	Lognormal	Normal	Normal
Mean value ($\mu\varepsilon$ year^{-1})	12.00	6.50	0.00	1.80
Standard deviation ($\mu\varepsilon$ year^{-1})	4.80	5.00	4.00	9.00
Coefficient of variation	0.40	0.77	–	5.00

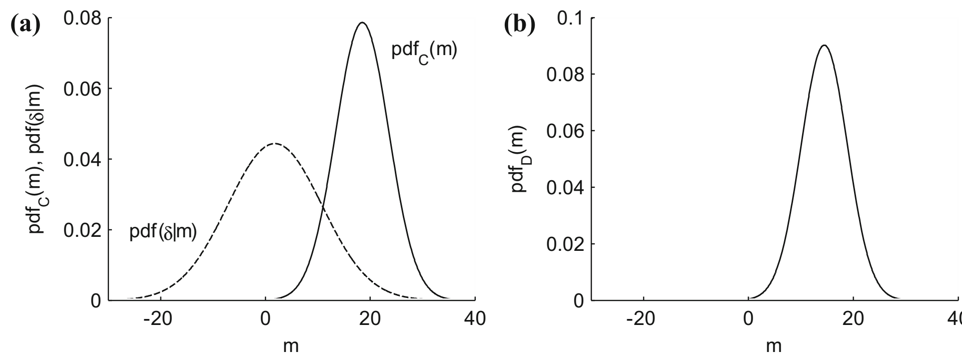


Fig. 8 Distributions of linear drift m for cable 1TN according to Ernest's profile C, 'Modeler', and profile D, 'Engineer'

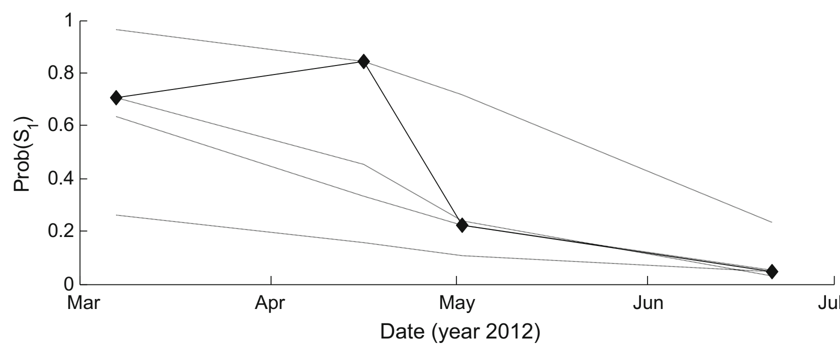


Fig. 9 Posterior probabilities of scenario 1

using all the measurements, recorded from March 2011 to the date by which Ernest is expected to give his judgment. This allows us to observe how Ernest's perception of the bridge state changes over time.

When, as in this case, the prior distributions of the parameters or the likelihood function are not Gaussian, the analytical solutions of (16) and (17) may not exist, and we may have to rely on numerical methods to carry out Bayesian inference. Herein, we used the Metropolis–Hastings algorithm [26, 27] to calculate the expected value and the uncertainty of the parameters a posteriori. The Metropolis–Hastings algorithm does not, in fact, provide an estimation of the posterior distribution, but a set of samples that can be considered random samples from the posterior. Then, we can make inference from those samples to obtain the mean and the variance of the parameters.

In addition, we used a Monte Carlo algorithm with importance sampling [28] to solve the integral of (16). To reduce the number of samples required to calculate the probability of each scenario, we defined the sampling distribution of each parameter as a Gaussian probability density function with mean and standard deviation equal to those of the corresponding posterior distribution, obtained using the Metropolis–Hasting algorithm.

Figure 9 shows, day by day, the result of the Bayesian updating for cable 1TN. The four dashed lines represent Ernest's perception under each of the four profiles, while the solid line reproduces the actual history of Ernest's perception. Ernest *changes his hat* three times during 2012, on April 16, May 2 and June 21, corresponding in the graph to jump from one profile line to another. The principal numerical results are also summarized in Table 4.

We observe that, for any of the four lines, the probability of scenario 1 always decreases with time, meaning that the fresh monitoring data arriving from the bridge tend to support the credibility of scenario 2. Therefore, in principle, if we provided enough monitoring data to Ernest,

Table 4 Bayesian inference results for Ernest's four profiles

Profile	A Geek	B Company man	C Modeler	D Engineer
Prob (S_1)	0.71	0.84	0.22	0.04
Prob (S_2)	0.29	0.16	0.78	0.96
α_1				
Mean value ($\mu\varepsilon$ °C^{-1})	13.11	15.12	14.86	15.48
Standard deviation ($\mu\varepsilon$ °C^{-1})	1.65	1.60	1.39	
m_1				
Mean value ($\mu\varepsilon$ year^{-1})	–175.58	–115.15	–89.80	–50.99
Standard deviation ($\mu\varepsilon$ year^{-1})	36.01	29.98	29.02	21.86
$\varepsilon_{0,1}$				
Mean value ($\mu\varepsilon$)	1,309,991.12	1,309,954.43	1,309,944.59	1,309,920.70
Standard deviation ($\mu\varepsilon$)	31.67	29.24	27.46	21.85
α_2				
Mean value ($\mu\varepsilon$ °C^{-1})	14.23	16.76	15.87	15.58
Standard deviation ($\mu\varepsilon$ °C^{-1})	1.93	1.69	1.42	1.14
m_2				
Mean value ($\mu\varepsilon$ year^{-1})	2.87	18.38	24.36	39.64
Standard deviation ($\mu\varepsilon$ year^{-1})	84.46	64.51	49.27	31.40
$\varepsilon_{0,2}$				
Mean value ($\mu\varepsilon$)	1,309,986.61	1,309,942.09	1,309,948.24	1,309,949.03
Standard deviation ($\mu\varepsilon$)	34.22	30.09	25.90	23.29
$\Delta\varepsilon_{0,2}$				
Mean value ($\mu\varepsilon$)	–40.26	–115.83	–114.74	–120.90
Standard deviation ($\mu\varepsilon$)	55.36	48.03	38.90	31.58

at some point he would always arrive at the conclusion that the bridge cables are not shortening.

The perceived probability of cable contraction (scenario 1) is about 71 % when Ernest acts as a 'Geek' (profile A). It grows to 84 % for the 'Company Man' (profile B) under the authoritarian influence of the company's opinion, but then falls again to 22 % after the 'Modeler' (profile C) carries out a structural analysis which demonstrates that the data are not compatible with cable shortening. The probability of scenario 1 finally reduces to only 4 % after Ernest 'The Engineer' (profile D) observes that old data from deck leveling are consistent with cable elongation. Overall, it is interesting to compare the numerical results of Fig. 9, calculated with a formal Bayesian approach, with Fig. 5, which illustrates in qualitative terms Ernest's perception as told in the introductory story: although the origin of the two is obviously different in nature, they overlap surprisingly well.

In addition, formal Bayesian inference allows calculation of the expected values of the parameters involved in the data interpretation. Evidently, they change with the profile and with the interpretation model accepted, S_1 or S_2. Take for example parameter m, the linear elongation drift. At the beginning of the story the 'Geek' does not have a clear idea on what the correct interpretation model could be (the probabilities are 71 and 29 %, respectively), so the best he can say is: I don't know whether the cable is really shortening or not, but, if it does, I estimate $E[m_1] = -175.58$ $\mu\varepsilon\cdot$year^{-1}. However, if the issue is in the photodiode, I guess its elongation trend $E[m_2]$ is only $+2.87$ $\mu\varepsilon\cdot$year^{-1}.

The 'Company man' is almost convinced of the mechanical explanation, and says it looks like the cable is really contracting, with $E[m_1] = -115.15$ $\mu\varepsilon\cdot$year^{-1}. Then the 'Modeler' estimates -89.80 and $+24.36$ $\mu\varepsilon\cdot$year^{-1}, for scenario 1 and 2, respectively. Eventually, 'The Engineer' estimates an elongation trend of $E[m_2] = +39.64$ $\mu\varepsilon\cdot$year^{-1}. Similarly, Ernest adapts his best estimate of the apparent measurement shift $\Delta\varepsilon_{0,2}$ with the role he plays. Thus, $E[\Delta\varepsilon_{0,2}]$ is -40.26 $\mu\varepsilon$ for the 'Geek', -115.83 for the 'Company Man', -114.74 $\mu\varepsilon$ for the 'Modeler' and finally -120.90 $\mu\varepsilon$ for the 'Engineer'.

The reader might have noted that, in this analysis, we gave up on attempting to model the physical state of the structure: we just modeled the engineer's perception of the state. This is simply because there is no other logical solution to the problem of describing the physical state of the bridge when the information is uncertain in nature. To better understand this concept, we have to focus on the nature of Bayes' theorem (1), the keystone of logical inference. Put simply, Bayes' theorem says that the posterior probability is a combination of the prior probability and the *likelihood*. The prior probability represents the background knowledge, something inherently subjective by definition. The *likelihood* is the way we make sense of the observations through the interpretation model: while observations are undoubtedly objective, the interpretation model, i.e., the way these observations are connected to the structural state, is again subjective to the individual who judges. In summary, when information on the state is offered by a set of uncertain clues, and physical state cannot be demonstrated by overwhelming evidence, the best we can do is to provide an estimation through an interpretation process. Any interpretation process, although perfectly rational, is epistemic in nature, and thus inherently subjective, so the objective description of this process is necessarily bonded to the individual who makes inference. The reader who wishes to look more in depth at the fundamentals of subjective probability can refer to [29] or [30].

6 Conclusions

Above, we used Bayesian logic to reproduce an engineer's interpretation of elongation data recorded by a monitoring system installed on the stay cables of a bridge. The sensors have recorded an apparent shortening of the cables, and the engineer's dilemma is whether this shortening is (1) purely physical behavior or (2) just a spurious effect of the interrogation unit. We showed that Ernest, our scrupulous engineer, tends to believe one or the other option based on information that he acquires over time. This information is of multiple natures: sensor data first of all, but also personal experience, background knowledge and even verbal suggestion from other experts. We showed how Bayesian logic is suitable for reproducing the engineer's perception: we modeled the information from the monitoring system using a likelihood function and the knowledge that characterizes each psychological profile using different priors. The application of Bayesian framework enabled us to integrate uncertain information of different kinds: uncertain performance of the sensing technology, technical judgment expressed by an SHM manufacturer, outcomes of structural analysis and survey data. Using formal Bayesian inference, we estimated that the final posterior perception is 4 % for option (1) and 96 % for option (2), fully consistent with the engineer's final conclusion. The integration of multiple sources of information is always done in practice, often heuristically and without quantitative analysis. Herein, we showed how Bayesian logic allows rigorous formalization of this process and possibly detection of judgment inconsistency.

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