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The adjoint of an operator on a Banach space

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Abstract
Self-adjoint operators in smooth Banach spaces have been already defined in recent works. Here, we extend the concept of adjoint of an operator to the scope of (non-necessarily Hilbert) Banach spaces, obtaining in particular the notion of self-adjoint operator in the non-smooth case. As a consequence, we define the probability density operator on Banach spaces and verify most of its well-known properties.

Keywords Smooth Banach space · Duality mapping · Self-adjoint operator · Probability density operator

Mathematics Subject Classification Primary 47A05 · Secondary 46B20

Introduction

The duality mapping of a normed space X is the set-valued map defined as

$$\begin{aligned} J : X &\to \mathcal{P}(X^*) \\ x &\mapsto J(x) := \{x^* \in X^* : \|x^*\| = \|x\| \text{ and } x^*(x) = \|x^*\|\|x\|\}. \end{aligned}$$

A selection (or selector) of the duality mapping is simply a map $J_0 : X \to X^*$ such that $J_0(x) \in J(x)$ for all $x \in X$. The Axiom of Choice, together with the Hahn-Banach Theorem, guarantees the existence of plenty of selections. A selection J_0 of the duality mapping is called a supporting map provided that it is positive-homogeneous, that is, $J_0(\lambda x) = \lambda J_0(x)$ for all $x \in X$ and all $\lambda \geq 0$, and J_0 is called a strong supporting map if it is conjugate-homogeneous, that is, $J_0(\lambda x) = \overline{\lambda} J_0(x)$ for all $x \in X$ and all $\lambda \in \mathbb{C}$. We refer the reader to [6, 7, 12, 18, 27] for a wider perspective on duality mappings and supporting maps.

A point x in the unit sphere S_X of X is said to be a smooth point of the unit ball B_X of X provided that $J(x)$ is a singleton. The subset of smooth points of B_X is typically denoted by

Dedicated to the beloved memory of Vladimir Gurariy

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$\mathrm{smo}(\mathsf{B}_X)$. Notice that if J_0, J_1 are selections of the duality mapping, then $J_0(x) = J_1(x)$ for all $x \in \mathrm{smo}(\mathsf{B}_X)$. See [11, 23] for a better explanation of smoothness and related geometrical notions.

A vector x of a normed space X is said to be a supporting vector of a continuous linear operator $T : X \to Y$, where Y is another normed space, if $\|T(x)\| = \|T\|\|x\|$. According to [10], the set of supporting vectors of T is denoted by $\mathrm{suppv}(T)$, that is,

$$\mathrm{suppv}(T) := \{x \in X : \|T(x)\| = \|T\|\|x\|\}.$$

By relying on the duality mapping, in [14] the notion of self-adjoint operator was transported to the scope of smooth Banach spaces. Here, we will consider general Banach spaces and we will define the adjoint of a general operator between Banach spaces, generalizing this way the results in [14].

The probability density operator is an important observable magnitude in a quantum mechanical system [19, Section 6]. Recall that a quantum mechanical system is represented by an infinite-dimensional separable complex Hilbert space [25]. The states of the system are the rays passing through a unit vector. The observable magnitudes are the bounded or unbounded self-adjoint operators. The unsharp observations are given by the bounded positive self-adjoint operators lying below the identity (this is the first example of effect algebra [13]). Here, we will reproduce this situation by modeling a quantum mechanical system with a general infinite-dimensional Banach space.

This manuscript is divided in three sections. The first section deals with selectors of the duality mapping and we will use them to characterize classical geometrical properties of Banach spaces. The second section is aimed at introducing the notion of adjoint of a general operator between Banach spaces. Finally, in the third section, we will deal with modeling quantum mechanical systems in general Banach spaces and show that the probability density operator satisfies similar properties as for classical systems on Hilbert spaces, improving this way [17, Subsection 6.1] and [15, Example 5 & Theorem 10].

1 Selectors of the duality mapping

This section is aimed at characterizing several geometrical properties of the unit ball of a normed space by means of the selectors in the duality mapping. However, before introducing the precise setting, it is useful to review several basic properties of the duality mapping.

Remark 1.1 Let X be a normed space. The following holds for every $x, y \in X$ and every $\lambda \in \mathbb{C}$:

(i) If $J(x) \cap J(y) \neq \varnothing$, then $\|x\| = \|y\|$.
(ii) $J(\lambda x) = \overline{\lambda} J(x)$.
(iii) If $x \in \mathrm{smo}(\mathsf{B}_X)$ and J_0 is any selection of J, then $J_0(\lambda x) = \overline{\lambda} J_0(x)$.
(iv) If $x \in \mathrm{smo}(\mathsf{B}_X)$ and J_0, J_1 are selections of the duality mapping, then $J_0(x) = J_1(x)$.

The first result of this section is a characterization of smoothness in terms of selectors of the duality mapping. This result improves [18, Proposition 2.3].

Theorem 1.2 *Let X be a normed space. The following conditions are equivalent:*

(i) *X is smooth.*
(ii) *Every selection of the duality mapping is a strong supporting map.*

(iii) *Every selection of the duality mapping is a supporting map.*

Proof Suppose first that X is smooth. Then $J(x)$ is a singleton for all $x \in X$, hence all selections coincide with the duality map. By Remark 1.1, the duality mapping is conjugate-homogeneous. Conversely, suppose that every selection of the duality mapping is a supporting map. Assume to contrary that X is not smooth. Let $x \in \mathsf{S}_X \setminus \mathrm{smo}(\mathsf{B}_X)$. Fix any arbitrary selection J_1 of the duality mapping. There exists $g \in J(x) \setminus \{J_1(x)\}$. The following is a well-defined selection of the duality mapping which is not positive-homogeneous:

$$\begin{aligned} J_0 : X &\to X^* \\ z &\mapsto J_0(z) := \begin{cases} J_1(z) & \text{if } z \neq 2x, \\ 2g & \text{if } z = 2x. \end{cases} \end{aligned} \tag{1.1}$$

□

Our next result is a characterization of strict convexity in terms of selections of the duality mapping. For this, a technical lemma is needed.

Lemma 1.3 *Let X be a complex normed space. Let $f \in X^*$. If there exists $x \in \mathsf{S}_X$ such that $|\Re f(x)| = \|f\|$, then $f(x) = \Re f(x)$.*

Proof Simply observe that

$$\|f\|^2 \geq |f(x)|^2 = |\Re f(x)|^2 + |\Im f(x)|^2 \geq \|f\|^2,$$

meaning that $\Im f(x) = 0$, hence $f(x) = \Re f(x)$. □

Theorem 1.4 *Let X be a normed space. The following conditions are equivalent:*

(i) *X is strictly convex.*
(ii) *Every selection of the duality mapping is one-to-one.*

Proof Suppose first that X is strictly convex. Fix any arbitrary selection J_0 of the duality mapping. Suppose that $x, y \in X$ are so that $J_0(x) = J_0(y)$. We will show that $x = y$. In the first place, note that $\|x\| = \|J_0(x)\| = \|J_0(y)\| = \|y\|$. Thus, if either $x = 0$ or $y = 0$, then both are also 0. So, we may assume that neither x nor y are 0. Denote $t := \|x\| = \|J_0(x)\| = \|J_0(y)\| = \|y\|$ and $f := \frac{J_0(x)}{t} = \frac{J_0(y)}{t}$. Then $\frac{x}{t}, \frac{y}{t} \in f^{-1}(\{1\}) \cap \mathsf{B}_X$. Since X is strictly convex, $f^{-1}(\{1\}) \cap \mathsf{B}_X$ is a singleton, meaning that $\frac{x}{t} = \frac{y}{t}$, hence $x = y$. As a consequence, J_0 is one-to-one. Conversely, suppose that every selection of the duality mapping is one-to-one. Assume to the contrary that X is not strictly convex. Then we can find $x \neq y$ both in S_X satisfying that $[x, y] \subseteq \mathsf{S}_X$. In view of the Hahn-Banach Separation Theorem, there exists $f \in \mathsf{S}_{X^*}$ such that $\Re f([x, y]) = \{1\}$. According to Lemma 1.3, $f([x, y]) = \{1\}$. Fix any arbitrary selection J_1 of the duality mapping. The following is a well-defined selection of the duality mapping which is not one-to-one:

$$\begin{aligned} J_0 : X &\to X^* \\ z &\mapsto J_0(z) := \begin{cases} J_1(z) & \text{if } z \notin [x, y], \\ f & \text{if } z \in [x, y]. \end{cases} \end{aligned} \tag{1.2}$$

□

Notice that, under the settings of Theorem 1.4, the selection J_0 constructed in (1.2) is not necessarily a (strong) supporting map. There is a way of redefining J_0 to become a (strong) supporting map provided that so is J_1. Indeed, if J_1 is a supporting map, then

$$\begin{aligned} J_0 : X &\to X^* \\ z &\mapsto J_0(z) := \begin{cases} J_1(z) & \text{if } z \notin \mathbb{R}^+[x, y], \\ \|z\| f & \text{if } z \in \mathbb{R}^+[x, y], \end{cases} \end{aligned} \tag{1.3}$$

is a supporting map, where $\mathbb{R}^+[x, y] := \{tu : t \geq 0, u \in [x, y]\}$. If J_1 is a strong supporting map, then

$$\begin{aligned} J_0 : X &\to X^* \\ z &\mapsto J_0(z) := \begin{cases} J_1(z) & \text{if } z \notin \mathbb{C}[x, y], \\ \overline{\lambda} f & \text{if } z = \lambda u \text{ for } \lambda \in \mathbb{C}, u \in [x, y], \end{cases} \end{aligned} \tag{1.4}$$

is a strong supporting map, where $\mathbb{C}[x, y] := \{\lambda u : \lambda \in \mathbb{C}, u \in [x, y]\}$.

Theorem 1.5 *Let X be a real normed space with* $\dim(X) > 1$. *If B_X has a proper face C with nonempty interior relative to S_X, then no selection of the duality mapping is one-to-one.*

Proof Let $J_0 : X \to X^*$ be a selection of the duality mapping. According to [9, Lemma 5(2)], there exists a unique $x^* \in \mathsf{S}_{X^*}$ such that $x^*(C) = \{1\}$. On the other hand, by bearing in mind [9, Lemma 5(4)], we have that $\operatorname{int}_{\mathsf{S}_X}(C) \subseteq \operatorname{smo}(\mathsf{B}_X)$. As a consequence, $J(c) = \{x^*\}$ for all $c \in \operatorname{int}_{\mathsf{S}_X}(C)$. Since $\dim(X) > 1$, we have that $\operatorname{int}_{\mathsf{S}_X}(C)$ is not a singleton. Finally, $J_0(c) = x^*$ for all $c \in \operatorname{int}_{\mathsf{S}_X}(C)$, meaning that J_0 is not injective. □

In virtue of [16, Theorem 2.1], no complex Banach space has a convex subset in its unit sphere with non-empty interior relative to the unit sphere. This is why Theorem 1.5 only works for real spaces. On the other hand, in [1, Theorem 3.12] it is shown that every real normed space can be equivalently renormed in such a way that its unit sphere contains a facet, that is, a face with non-empty interior relative to the unit sphere. As a direct consequence of this fact together with Theorem 1.5, we obtain the following corollary.

Corollary 1.6 *Let X be a real normed space with* $\dim(X) > 1$. *Then X can be equivalently renormed so that no selection of the duality mapping is one-to-one*

To finalize this section, we will discuss how much surjectivity of the selectors of the duality mapping affects the reflexivity condition of a normed space. In the first place, it is worth mentioning the existence of non-complete normed spaces on which every functional attains its norm [21]. In view of the famous James' Characterization of Reflexivity [20], it is easy to understand that a smooth Banach space is reflexive if and only if the duality mapping is surjective. We will show in the last theorem of this section that the previous equivalence fails if we drop the smoothness hypothesis. Nevertheless, we first need to recall the notion of rotund point [5]: a point $x \in \mathsf{S}_X$ in the unit sphere of a normed space X is said to be a rotund point of the unit ball of X if x is contained in no non-trivial segment of the unit sphere, in other words, $\{x\}$ is a maximal proper face of B_X. The set of rotund points of B_X is denoted by $\operatorname{rot}(\mathsf{B}_X)$.

Theorem 1.7 *Let X be a normed space. If there exists $x \in \operatorname{rot}(\mathsf{B}_X) \setminus \operatorname{smo}(\mathsf{B}_X)$, then no selector of the duality mapping is surjective.*

Proof Let $J_0 : X \to X^*$ be any selector of the duality mapping. Since x is not a smooth point of B_X, there exists $f \in J(x) \setminus \{J_0(x)\}$. We will show that $f \notin J_0(X)$. Indeed, suppose on the contrary that $f = J_0(x')$ for some $x' \in X$. Observe that $\|x'\| = \|J_0(x')\| = \|f\| = \|x\| = 1$. Then $f(x) = 1 = f(x')$, meaning that $[x, x'] \subseteq \mathsf{S}_X$. Since $x \in \operatorname{rot}(\mathsf{B}_X)$, we reach the contradiction that $x = x'$ because $f \neq J_0(x)$. □

2 The adjoint of an operator

A semi-scalar product on a complex vector space is a function $(\bullet|\bullet) : X \times X \to \mathbb{C}$ such that:

(i) $(\bullet|\bullet)$ is linear on the first component.
(ii) It is strictly positive, that is, $(x|x) > 0$ for all $x \in X \setminus \{0\}$.
(iii) It verifies the Cauchy–Swartz inequality: $|(x|y)| \leq \sqrt{(x|x)}\sqrt{(y|y)}$ for all $x, y \in X$.

$(\bullet|\bullet)$ is called positive-homogeneous (Hermitian) or conjugate-homogeneous if so it is in the second component, respectively. Every semi-scalar product defines a norm $\|x\| := \sqrt{(x|x)}$ [22, Theorem 2]. A semi-scalar product in a normed space is said to be (topologically) consistent with the norm if the norm induced by the semi-scalar product (is equivalent to) coincides with the original norm. The following theorem, whose proof is omitted, describes a correspondence between semi-scalar products and selectors of the duality mapping [8].

Theorem 2.1 *Let X be a normed space with duality mapping $J : X \to \mathcal{P}(X^*)$. Then:*

(i) *Every selection $J_0 : X \to X^*$ of the duality mapping defines a semi-scalar product on X consistent with the norm:*

$$\begin{aligned} (\bullet|\bullet) : X \times X &\to \mathbb{C} \\ (x, y) &\mapsto (x|y) := J_0(y)(x). \end{aligned} \tag{2.1}$$

(ii) *Every semi-scalar product $(\bullet|\bullet)$ consistent with the norm induces a selection of the duality mapping:*

$$\begin{aligned} J_0 : X &\to X^* \\ x &\mapsto \begin{aligned} J_0(x) : X &\to \mathbb{C} \\ y &\mapsto J_0(x)(y) := (y|x) \end{aligned} \end{aligned} \tag{2.2}$$

In this identification, the Hermitian semi-scalar products correspond with the supporting maps and the conjugate-homogeneous semi-scalar products correspond with the strong supporting maps.

We will be obviously interested in semi-scalar products consistent with the norm. However, next definition will be extensive to general semi-scalar products generalizing this way [14, Definition 4.1].

Definition 2.2 Let X be a normed space endowed with a semi-scalar product $(\bullet|\bullet)$. Let $T \in \mathcal{B}(X)$. Then T is said to be:

- Self-adjoint if $(T(x)|y) = (x|T(y))$ for all $x, y \in X$,
- Hermitian if $(T(x)|x) \in \mathbb{R}$ for all $x \in X$,
- positive if $(T(x)|x) \geq 0$ for all $x \in X$,
- strongly normal if $T = S^2$ for some self-adjoint $S \in \mathcal{B}(X)$,
- unitary if $(T(x)|T(y)) = (x|y)$ for all $x, y \in X$.

If $R \in \mathcal{B}(X)$, then we will say that $R \leq T$ provided that $T - R$ is positive. If $S := (e_k)_{k\in\mathbb{N}} \subseteq \mathsf{S}_X$ is a sequence of orthogonal vectors, then the S-trace of T is defined as $\operatorname{tr}_S(T) := \sum_{k=1}^{\infty}(T(e_k)|e_k)$ if this series converges.

The notion of adjoint operator in the context of non-Hilbert Banach spaces is, as far as we know, undeveloped yet. Here, we propose the following definition, which consistently generalizes [14, Definition 4.1].

Definition 2.3 (Adjoint) Let X, Y be normed spaces. Fix selectors $J_X : X \to X^*$ and $J_Y : Y \to Y^*$ of the duality mappings on X and Y, respectively. The set of adjoints of an operator $T \in \mathcal{B}(X, Y)$ is defined as the set

$$T' := \left\{S \in \mathcal{B}(Y, X) : J_X \circ S = T^* \circ J_Y.\right\}$$

Notice that, under the settings of the previous definition, an operator $S \in \mathcal{B}(Y, X)$ is an element of T' if and only if $J_Y(y)(T(x)) = J_X(S(y))(x)$ for every $x \in X$ and every $y \in Y$. By bearing in mind the previous definition, an operator T is self-adjoint in the sense of [14, Definition 4.1] if and only if $T \in T'$. Note that T' might be empty unless, for instance, X, Y are Hilbert spaces. According to [14, Theorems 2.3 and 4.3], there are examples of non-Hilbert Banach spaces admitting non-trivial operators for which the set of adjoints is not empty. The following results unveil the most characteristic properties satisfied by the adjoint a of an operator.

Proposition 2.4 *Let X, Y be normed spaces. Fix selectors $J_X : X \to X^*$ and $J_Y : Y \to Y^*$ of the duality mappings on X and Y, respectively. Let $T \in \mathcal{B}(X, Y)$. Then:*

(i) $\|S\| = \|T\|$ *for every* $S \in T'$.
(ii) *If J_X is a strong supporting map, then* $(\lambda T)' = \overline{\lambda} T'$ *for all* $\lambda \in \mathbb{C}$.
(iii) *If J_X is a supporting map, then* $(\lambda T)' = \lambda T'$ *for all* $\lambda \geq 0$.

Proof (i) Fix an arbitrary $S \in T'$. For every $y \in \mathsf{B}_Y$,

$$\|S(y)\| = \|J_X(S(y))\| = \|J_Y(y) \circ T\| \leq \|J_Y(y)\| \, \|T\| \leq \|T\|,$$

meaning that $\|S\| \leq \|T\|$. If $T = 0$, then we obtain that $S = 0$, so we may assume that $T \neq 0$. For every $x \in \mathsf{B}_X$,

$$\begin{aligned} \|T(x)\|^2 &= \|J_Y(T(x))(T(x))\| \\ &= \|J_X(S(T(x)))(x)\| \\ &\leq \|J_X(S(T(x)))\| \\ &= \|S(T(x))\| \\ &\leq \|S\|\|T\|, \end{aligned}$$

meaning that $\|T\|^2 \leq \|S\|\|T\|$, in other words, $\|T\| \leq \|S\|$.

(ii) If $\lambda = 0$, then $(0 \cdot T)' = 0' = \{0\} = 0 \cdot T'$, so let us assume that $\lambda \neq 0$. Take any $S \in (\lambda T)'$. We will show that $\frac{\lambda}{|\lambda|^2} S \in T'$. Indeed, for every $x \in X$ and every $y \in Y$, we have that

$$\begin{aligned} J_X\left(\frac{\lambda}{|\lambda|^2} S(y)\right)(x) &= \frac{\overline{\lambda}}{|\lambda|^2} J_X(S(y))(x) \\ &= \frac{\overline{\lambda}}{|\lambda|^2} J_Y(y)(\lambda T(x)) \\ &= \frac{\overline{\lambda}}{|\lambda|^2} \lambda J_Y(y)(T(x)) \\ &= J_Y(y)(T(x)). \end{aligned}$$

Therefore, $\frac{\lambda}{|\lambda|^2}S \in T'$, hence $S = \overline{\lambda}\frac{\lambda}{|\lambda|^2}S \in \overline{\lambda}T'$. As a consequence, $(\lambda T)' \subseteq \overline{\lambda}T'$. A similar argument shows the reverse inclusion.

(iii) Follows a similar proof as the item above.

□

The definition of unitary operators can also be extended to operators between different normed spaces.

Definition 2.5 (Unitary operator) Let X, Y be normed spaces. An operator $T \in \mathcal{B}(X, Y)$ is said to be unitary provided that $(T(x_1)|T(x_2))_Y = (x_1|x_2)_X$ for all $x_1, x_2 \in X$, where $(\bullet|\bullet)_X$ and $(\bullet|\bullet)_Y$ are fixed semi-scalar products on X and Y, respectively.

The following proposition is the last result of this section and serves to characterized unitary operators.

Proposition 2.6 *Let X, Y be normed spaces. Fix selectors $J_X : X \to X^*$ and $J_Y : Y \to Y^*$ of the duality mappings on X and Y, respectively, and consider the corresponding semi-scalar products $(\bullet|\bullet)_X$ and $(\bullet|\bullet)_Y$. Let $T \in \mathcal{B}(X, Y)$. Then:*

(i) *If T is unitary, then T is an isometry.*
(ii) *If T is unitary and surjective, then $T^{-1} \in T'$.*
(iii) *If T is unitary and surjective and J_X is one-to-one, then $T' = \{T^{-1}\}$.*
(iv) *If T is an isomorphism such that $T^{-1} \in T'$, then T is unitary.*

Proof (i) Simply observe that

$$\|T(x)\|^2 = J_Y(T(x))(T(x)) = (T(x)|T(x))_Y = (x|x)_X = J_X(x)(x) = \|x\|^2$$

for all $x \in X$.

(ii) Fix arbitrary elements $x \in X$ and $y \in Y$. Since T is surjective, there exists $x' \in X$ such that $T(x') = y$. Then

$$\begin{aligned} J_Y(y)(T(x)) &= J_Y\left(T(x')\right)(T(x)) = \left(T(x)|T(x')\right)_Y \\ &= \left(x|x'\right)_X = J_X(x')(x) = J_X\left(T^{-1}(y)\right)(x). \end{aligned}$$

As a consequence, $T^{-1} \in T'$.

(iii) By hypothesis, for every $x_1, x_2 \in X$,

$$J_Y(T(x_2))(T(x_1)) = (T(x_1)|T(x_2))_Y = (x_1|x_2)_X = J_X(x_2)(x_1).$$

Take any $S \in T'$. Note that

$$J_X(S(T(x_2)))(x_1) = J_Y(T(x_2))(T(x_1)) = J_X(x_2)(x_1)$$

for every $x_1, x_2 \in X$. In other words, $J_X(S(T(x))) = J_X(x)$ for all $x \in X$. By hypothesis, J_X is one-to-one, meaning that $S(T(x)) = x$ for all $x \in X$. As a consequence, S is a left-inverse for T. However, T is invertible because T is a surjective isometry, hence $S = T^{-1}$.

(iv) Fix arbitrary elements $x_1, x_2 \in X$. Then

$$\begin{aligned} (T(x_1)|T(x_2))_Y &= J_Y\left(T(x_2)\right)\left(T(x_1)\right) = \\ &= J_X\left(T^{-1}(T(x_2))\right)(x_1) = J_X(x_2)(x_1) = (x_1|x_2)_X\,, \end{aligned}$$

meaning that T is unitary.

□

3 Probability density operator

In quantum mechanics, the probability density operator of a quantum system in a mixed state represented by the infinite dimensional separable complex Hilbert space H is given by

$$\begin{aligned} D : H &\to H \\ x &\mapsto D(x) := \sum_{n=1}^{\infty} t_n (x|x_n) x_n, \end{aligned} \tag{3.1}$$

where $(x_n)_{n\in\mathbb{N}}$ is a sequence of states in H, that is, a sequence of unit vectors of H such that $\mathsf{S}_{\mathbb{C}} x_n \cap \mathsf{S}_{\mathbb{C}} x_m = \varnothing$ if $n \neq m$, and $\sum_{n=1}^{\infty} t_n$ is a convex series, that is, $t_n \geq 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} t_n = 1$. In quantum mechanics, if a quantum system is in a mixed state described by a density operator D, then t_n indicates the probability that the system is at the state x_n. This type of operators can also be defined in Banach spaces by means of either semiscalar products or selections of the duality mapping.

Definition 3.1 (Probability density operator) Let X be a smooth Banach space. Let $(\bullet|\bullet)$ be the Hermitian semiscalar product on X induced by the duality mapping like (2.1). Let $(x_n)_{n\in\mathbb{N}} \subseteq \mathsf{S}_X$ be a sequence. Let $(\rho_n)_{n\in\mathbb{N}} \in \ell_\infty(\mathbb{C})$. The probability density operator is defined as

$$\begin{aligned} D : Y &\to X \\ x &\mapsto D(x) := \sum_{n=1}^{\infty} \rho_n (x|x_n) x_n, \end{aligned} \tag{3.2}$$

where $Y := \left\{x \in X : \sum_{n=1}^{\infty} \rho_n (x|x_n) x_n \text{ is convergent}\right\}$ is the domain of convergence.

Under the settings of the previous definition, notice that Y is clearly a (non-necessarily closed) subspace of X. Also, note that if $(\rho_n)_{n\in\mathbb{N}} \in \ell_1(\mathbb{C})$, then $Y = X$. In fact, if $(\rho_n)_{n\in\mathbb{N}} \in \ell_1(\mathbb{C})$, then $D(x)$ is absolutely convergent for all $x \in X$.

Proposition 3.2 *Let X be a smooth Banach space. Let $(\bullet|\bullet)$ be the Hermitian semi-scalar product on X induced by the duality mapping like (2.1). Let $(x_n)_{n\in\mathbb{N}} \subseteq \mathsf{S}_X$ be a sequence. Let $(\rho_n)_{n\in\mathbb{N}} \in \ell_\infty(\mathbb{C})$. Consider the probability density operator D given in Equation (3.2). Then:*

- (i) *If $(x_n)_{n\in\mathbb{N}}$ is a binormalized unconditional Schauder basis for X, then the convergence domain Y of D in (3.2) is the whole of X.*
- (ii) *If $(\rho_n)_{n\in\mathbb{N}} \in \ell_1(\mathbb{C})$, then the domain of convergence of D is the whole of X, $\|D\| \leq \|(\rho_n)_{n\in\mathbb{N}}\|_1$, and D can be approximated by finite-rank operators in the norm topology of $\mathcal{B}(X)$.*

Proof Note that D can be rewritten as $D(x) = \sum_{n=1}^{\infty} \rho_n J(x_n)(x) x_n$ for all $x \in X$, where $J : X \to X^*$ is the duality mapping.

- (i) In the first place, observe that since $(x_n)_{n\in\mathbb{N}} \subseteq \mathsf{S}_X$ is a binormalized Schauder basis, the basic sequence of coordinate functionals is normalized. Since X is smooth, then this basic sequence of coordinate functionals is precisely the sequence $(J(x_n))_{n\in\mathbb{N}} \subseteq \mathsf{S}_{X^*}$. Therefore, for every $x \in X$, $\sum_{n=1}^{\infty} J(x_n)(x) x_n$ is an unconditional convergent series in X whose summation is precisely x, meaning that $\sum_{n=1}^{\infty} \rho_n J(x_n)(x) x_n$ is convergent because $(\rho_n)_{n\in\mathbb{N}} \in \ell_\infty(\mathbb{C})$ [3, 4, 24]. As a consequence, $D(x)$ exists in X for all $x \in X$.

(ii) For every $x \in X$, $D(x)$ is absolutely convergent and

$$\|D(x)\| \leq \sum_{n=1}^{\infty} |\rho_n| \|J(x_n)(x)x_n\| \leq \|x\| \sum_{n=1}^{\infty} |\rho_n| = \|x\| \, \|(\rho_n)_{n\in\mathbb{N}}\|_1 ,$$

meaning that $\|D\| \leq \|(\rho_n)_{n\in\mathbb{N}}\|_1$. Finally, D can be approximated in the operator norm of $\mathcal{B}(X)$ by the sequence $(D_k)_{k\in\mathbb{N}}$ of finite-rank operators, where for every $k \in \mathbb{N}$

$$\begin{aligned} D_k : X &\to \operatorname{span}\{x_1, \dots, x_k\} \\ x &\mapsto D(x) := \sum_{n=1}^{k} \rho_n J(x_n)(x)x_n. \end{aligned}$$

Indeed, for every $x \in \mathsf{B}_X$ we have that

$$\|D(x) - D_k(x)\| = \left\| \sum_{n=k+1}^{\infty} \rho_n J(x_n)(x)x_n \right\| \leq \sum_{n=k+1}^{\infty} |\rho_n| \to 0$$

as $k \to \infty$ because $\sum_{n=k+1}^{\infty} |\rho_n|$ is the rest of a convergent series.

□

In order to achieve more properties of the probability density operator, it is sufficient to rely on another geometrical notion: L^2-summand vectors [2].

Remark 3.3 (L^2-summand subspace) Let X be a Banach space. A closed subspace M of X is said be an L^2-summand subspace of X provided that there exists another closed subspace N of X satisfying that M and N are L^2-complemented in X, that is, $X = M \oplus_2 N$, or equivalently, $\|m + n\|^2 = \|m\|^2 + \|n\|^2$ for all $m \in M$ and all $n \in N$.

An L^2-summand vector of a Banach space is simply a vector whose linear span is an L^2-summand subspace.

Remark 3.4 Let X be a smooth Banach space. In accordance with [2, Theorem 3.2], if $x \in X$ is an L^2-summand vector of X, then $J(x + y) = J(x) + J(y)$ and $J(x)(y) = \overline{J(y)(x)}$ for all $y \in X$. As a consequence, if $x_1, \dots, x_k$ are L^2-summand vectors, then $J\left(\sum_{n=1}^{k} x_n\right) = \sum_{n=1}^{k} J(x_n)$.

Lemma 3.5 *Let X be a Banach space and $\sum_{n=1}^{\infty} x_n$ a convergent series in X of L^2-summand vectors. Then:*

(i) *If X is smooth, then $J\left(\sum_{n=1}^{\infty} x_n\right) = w^* \sum_{n=1}^{\infty} J(x_n)$.*
(ii) *If X is strongly smooth, then $J\left(\sum_{n=1}^{\infty} x_n\right) = \sum_{n=1}^{\infty} J(x_n)$.*
(iii) *If X is smooth and $\sum_{n=1}^{\infty} J(x_n)$ is convergent, then $J\left(\sum_{n=1}^{\infty} x_n\right) = \sum_{n=1}^{\infty} J(x_n)$.*
(iv) *If X is smooth and $\sum_{n=1}^{\infty} x_n$ is absolutely convergent, then $J\left(\sum_{n=1}^{\infty} x_n\right) = \sum_{n=1}^{\infty} J(x_n)$.*

Proof Observe that $\left(\sum_{n=1}^{k} x_n\right)_{k\in\mathbb{N}}$ converges to $\sum_{n=1}^{\infty} x_n$.

(i) The duality mapping is $J : X \to X^*$ norm-w^* continuous, therefore by Remark 3.4, $\left(\sum_{n=1}^{k} J(x_n)\right)_{n\in\mathbb{N}}$ is w^*-convergent to $J\left(\sum_{n=1}^{\infty} x_n\right)$, that is, $J\left(\sum_{n=1}^{\infty} x_n\right) = w^* \sum_{n=1}^{\infty} J(x_n)$.

(ii) The duality mapping $J : X \to X^*$ is norm-norm continuous, therefore $\left(J\left(\sum_{n=1}^{k} x_n\right)\right)_{k\in\mathbb{N}}$ converges to $J\left(\sum_{n=1}^{\infty} x_n\right)$. Finally, by applying Remark 3.4 we conclude that $J\left(\sum_{n=1}^{\infty} x_n\right) = \sum_{n=1}^{\infty} J(x_n)$.
(iii) We know that $J\left(\sum_{n=1}^{\infty} x_n\right) = w^* \sum_{n=1}^{\infty} J(x_n)$. Since $\sum_{n=1}^{\infty} J(x_n)$ is convergent, we conclude that $J\left(\sum_{n=1}^{\infty} x_n\right) = w^* \sum_{n=1}^{\infty} J(x_n) = \sum_{n=1}^{\infty} J(x_n)$.
(iv) Since $\|J(x_n)\| = \|x_n\|$ for every $n \in \mathbb{N}$, we have that $\sum_{n=1}^{\infty} J(x_n)$ is also absolutely convergent. Thus, we just need to call on the previous item.

□

Lemma 3.5 allows the following characterization of separable Hilbert spa-ces.

Lemma 3.6 *Let X be a separable Banach space. The following conditions are equivalent:*

(i) *X is strongly smooth and there exists a sequence of L^2-summand vectors which is a Scahuder basis for X.*
(ii) *X is a Hilbert space.*

Proof If X is a Hilbert space, then every closed subspace of X is L^2-comple-mented, hence every vector of X is an L^2-summand vector. Also, Hilbert spaces are uniformly smooth, hence strongly smooth. Since X is separable by assumption, any orthonormal basis is a Schauder basis. Conversely, assume that X is strongly smooth and there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of L^2-summand vectors which is a Scahuder basis for X. We will rely on [18, Theorem 2.8] and prove that the duality mapping is additive. Indeed, fix arbitrary elements $x, y \in X$. Since $(x_n)_{n\in\mathbb{N}}$ is a Scahuder basis for X, there are unique sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ in such a way that $x = \sum_{n=1}^{\infty} \alpha_n x_n$ and $y = \sum_{n=1}^{\infty} \beta_n x_n$. In view of Lemma 3.5, we have that

$$\begin{aligned} J(x+y) &= J\left(\sum_{n=1}^{\infty}(\alpha_n + \beta_n)x_n\right) = \sum_{n=1}^{\infty} J((\alpha_n + \beta_n)x_n) \\ &= \sum_{n=1}^{\infty}(\overline{\alpha_n} + \overline{\beta_n})J(x_n) = \sum_{n=1}^{\infty}\overline{\alpha_n}J(x_n) + \sum_{n=1}^{\infty}\overline{\beta_n}J(x_n) \\ &= \sum_{n=1}^{\infty} J(\alpha_n x_n) + \sum_{n=1}^{\infty} J(\beta_n x_n) = J(x) + J(y). \end{aligned}$$

□

Recall that in a smooth Banach space X, two elements $x, y \in X$ are said to be orthogonal provided that $J(x)(y) = J(y)(x) = 0$.

Lemma 3.7 *Let X be a smooth Banach space. Let $(e_k)_{k\in\mathbb{N}} \subseteq \mathsf{S}_X$ be a sequence of orthogonal L^2-summand vectors. Let $x \in X$. Then:*

(i) $\sum_{k=1}^{\infty} |J(e_k)(x)|^2 \leq \|x\|^2$.
(ii) *If $x \in \operatorname{span}\{e_1, \ldots, e_l\}$, then $\sum_{k=1}^{l} |J(e_k)(x)|^2 = \|x\|^2$.*
(iii) *If $\sum_{k=1}^{\infty} J(e_k)(x)e_k$ converges, then $\sum_{k=1}^{\infty} |J(e_k)(x)|^2 = \left\|\sum_{k=1}^{\infty} J(e_k)(x)e_k\right\|^2$.*

Proof We can write $x = J(e_1)(x)e_1 + m_1$ with $m_1 \in \ker(J(e_1))$. Then $m_1 = J(e_2)(m_1)e_2 + m_2$ for some $m_2 \in \ker(J(e_2))$. Next, $m_2 = J(e_3)(m_2)e_3 + m_3$ for some $m_3 \in \ker(J(e_3))$ and we keep going on. Notice that the orthogonality yields

$$x = J(e_1)(x)e_1 + m_1$$

$$
\begin{aligned}
&= J(e_1)(x)e_1 + J(e_2)(m_1)e_2 + m_2 \\
&= J(e_1)(x)e_1 + J(e_2)(x)e_2 + m_2 \\
&= J(e_1)(x)e_1 + J(e_2)(x)e_2 + J(e_3)(m_2)e_3 + m_3 \\
&= J(e_1)(x)e_1 + J(e_2)(x)e_2 + J(e_3)(x)e_3 + m_3 \\
&\vdots \\
&= J(e_1)(x)e_1 + J(e_2)(x)e_2 + J(e_3)(x)e_3 + \cdots + J(e_l)(m_{l-1})e_l + m_l \\
&= J(e_1)(x)e_1 + J(e_2)(x)e_2 + J(e_3)(x)e_3 + \cdots + J(e_l)(x)e_l + m_l \\
&\vdots
\end{aligned}
$$

Observe also that

$$
\begin{aligned}
\|x\|^2 &= |J(e_1)(x)|^2 + \|m_1\|^2 \\
&= |J(e_1)(x)|^2 + |J(e_2)(m_1)|^2 + \|m_2\|^2 \\
&= |J(e_1)(x)|^2 + |J(e_2)(x)|^2 + \|m_2\|^2 \\
&= |J(e_1)(x)|^2 + |J(e_2)(x)|^2 + |J(e_3)(m_2)|^2 + \|m_3\|^2 \\
&= |J(e_1)(x)|^2 + |J(e_2)(x)|^2 + |J(e_3)(x)|^2 + \|m_3\|^2 \\
&\vdots \\
&= |J(e_1)(x)|^2 + |J(e_2)(x)|^2 + |J(e_3)(x)|^2 + \cdots + |J(e_l)(m_{l-1})|^2 + \|m_l\|^2 \\
&= |J(e_1)(x)|^2 + |J(e_2)(x)|^2 + |J(e_3)(x)|^2 + \cdots + |J(e_l)(x)|^2 + \|m_l\|^2 \\
&\vdots
\end{aligned}
$$

As a consequence, $\sum_{k=1}^{l} |J(e_k)(x)|^2 \leq \|x\|^2$, meaning that $\sum_{k=1}^{\infty} |J(e_k)(x)|^2 \leq \|x\|^2$ by the arbitrariness of $l \in \mathbb{N}$. Next, assume that $x \in \operatorname{span}\{e_1, \ldots, e_l\}$. Then $m_l \in \operatorname{span}\{e_1, \ldots, e_l\}$. The equation

$$
x = J(e_1)(x)e_1 + J(e_2)(x)e_2 + J(e_3)(x)e_3 + \cdots + J(e_l)(x)e_l + m_l
$$

allows that $J(e_k)(x) = J(e_k)(x) + J(e_k)(m_l)$ for all $k \in \{1, \ldots, l\}$, meaning that $J(e_k)(m_l) = 0$ for all $k \in \{1, \ldots, l\}$, that is, $m_l = 0$. As a consequence,

$$
\|x\|^2 = |J(e_1)(x)|^2 + |J(e_2)(x)|^2 + |J(e_3)(x)|^2 + \cdots + |J(e_l)(x)|^2.
$$

Finally, if $\sum_{k=1}^{\infty} J(e_k)(x)e_k$ is a convergent series, then $\left(\sum_{k=1}^{l} J(e_k)(x)e_k\right)_{l\in\mathbb{N}}$ converges to $\sum_{k=1}^{\infty} J(e_k)(x)e_k$, hence

$$
\sum_{k=1}^{l} |J(e_k)(x)|^2 = \left\| \sum_{k=1}^{l} J(e_k)(x)e_k \right\|^2 \to \left\| \sum_{k=1}^{\infty} J(e_k)(x)e_k \right\|^2
$$

as $l \to \infty$, that is, $\sum_{k=1}^{\infty} |J(e_k)(x)|^2 = \left\| \sum_{k=1}^{\infty} J(e_k)(x)e_k \right\|^2$. □

Lemma 3.7 yields a new characterization of separable Hilbert spaces.

Corollary 3.8 *Let X be a smooth Banach space. If there exists a binormalized Schauder basis* $(e_n)_{n\in\mathbb{N}} \subseteq \mathsf{S}_X$ *of* L^2*-summand vectors, then X is a Hilbert space.*

Proof Since X is smooth and $(e_n)_{n\in\mathbb{N}} \subseteq \mathsf{S}_X$ is a binormalized Schauder basis, the basic sequence of coordinate functionals is precisely $(J(e_n))_{n\in\mathbb{N}} \subseteq \mathsf{S}_{X^*}$. Therefore, for every $x \in X$, $x = \sum_{n=1}^{\infty} J(e_n)(x)e_n$. Fix an arbitrary $x \in X$. For every $k \in \mathbb{N}$, let $x_k := \sum_{n=1}^{k} J(e_n)(x)e_n$. Note that, for every $n \in \{1, \ldots, k\}$, $J(e_n)(x_k) = J(e_n)(x)$, therefore $x_k = \sum_{n=1}^{k} J(e_n)(x_k)e_n$. According to Lemma 3.7,

$$\|x_k\|^2 = \sum_{n=1}^{k} |J(e_n)(x_k)|^2 = \sum_{n=1}^{k} |J(e_n)(x)|^2.$$

Since $(x_k)_{k\in\mathbb{N}}$ converges to x, we have that $\left(\|x_k\|^2\right)_{k\in\mathbb{N}}$ converges to $\|x\|^2$ and so $\|x\|^2 = \sum_{n=1}^{\infty} |J(e_n)(x)|^2$. This shows that the embedding

$$\begin{aligned} X &\to \ell_2 \\ x &\mapsto (J(e_n)(x))_{n\in\mathbb{N}} \end{aligned} \tag{3.3}$$

is a linear isometry. □

The following technical lemma serves to assure sufficient conditions for the summations of a double series of positive terms to be switched.

Lemma 3.9 *Let X be a Banach space. Let $(a_{nm})_{n,m\in\mathbb{N}} \subseteq X$ be a double sequence. Suppose that:*

- *For every $m \in \mathbb{N}$, $\sum_{n=1}^{\infty} a_{nm}$ is absolutely convergent.*
- *$\sum_{m=1}^{\infty} \beta_m$ is convergent, where $\beta_m := \sum_{n=1}^{\infty} \|a_{nm}\|$ for all $m \in \mathbb{N}$.*

Then:

(i) *For every $n \in \mathbb{N}$, $\sum_{m=1}^{\infty} a_{nm}$ is absolutely convergent.*
(ii) *$\sum_{n=1}^{\infty} \gamma_n$ is convergent, where $\gamma_n := \sum_{m=1}^{\infty} \|a_{nm}\|$ for all $n \in \mathbb{N}$.*
(iii) *$\sum_{m=1}^{\infty} \beta_m = \sum_{n=1}^{\infty} \gamma_n$.*

Proof (i) Fix an arbitrary $n \in \mathbb{N}$. For every $k \in \mathbb{N}$, $\sum_{m=1}^{k} \|a_{nm}\| \le \sum_{m=1}^{\infty} \beta_m \le \|(\beta_m)_{m\in\mathbb{N}}\|_1$. This shows that $\sum_{m=1}^{\infty} a_{nm}$ is absolutely convergent.
(ii) For every $k \in \mathbb{N}$, $\sum_{n=1}^{k} \gamma_n = \sum_{n=1}^{k} \sum_{m=1}^{\infty} \|a_{nm}\| = \sum_{m=1}^{\infty} \sum_{n=1}^{k} \|a_{nm}\| \le \sum_{m=1}^{\infty} \beta_m \le \|(\beta_m)_{m\in\mathbb{N}}\|_1$. This shows that $\sum_{n=1}^{\infty} \gamma_n$ is convergent and $\sum_{n=1}^{\infty} \gamma_n \le \sum_{m=1}^{\infty} \beta_m$.
(iii) By repeating the same argument as above starting from the assumption that $\sum_{n=1}^{\infty} \gamma_n$ is convergent, we conclude that $\sum_{m=1}^{\infty} \beta_m \le \sum_{n=1}^{\infty} \gamma_n$.

□

We are finally ready to state and prove the main theorem of this section, which unveils the main properties satisfied by the probability density operator and generalizes [17, Subsection 6.1] and [15, Example 5 & Theorem 10]. We will be assuming that the domain of convergence of the probability density operator is the whole space. Proposition 3.2 reveals two sufficient conditions for this.

Theorem 3.10 *Let X be a smooth Banach space. Let $(\bullet|\bullet)$ be the Hermitian semiscalar product on X induced by the duality mapping like (2.1). Let $(x_n)_{n\in\mathbb{N}} \subseteq \mathsf{S}_X$ be a sequence of L^2-summand vectors. Let $(\rho_n)_{n\in\mathbb{N}} \in \ell_\infty(\mathbb{C})$. Consider the probability density operator D given in Equation (3.2) and suppose that the domain of convergence of D is the whole of X. Then:*

(i) *If* $(x_n)_{n\in\mathbb{N}}$ *is orthogonal, then* $\|D(x)\|^2 = \sum_{n=1}^{\infty} |\rho_n|^2 |J(x_n)(x)|^2$ *for every* $x \in X$ *and* $\|D\| = \|(\rho_n)_{n\in\mathbb{N}}\|_\infty$.
(ii) *If* $\rho_n \in \mathbb{R}$ *for each* $n \in \mathbb{N}$, *then* D *is self-adjoint.*
(iii) *If* $\rho_n \geq 0$ *for each* $n \in \mathbb{N}$, *then* D *is positive.*
(iv) *If* $\rho_n \geq 0$ *for each* $n \in \mathbb{N}$ *and* $(x_n)_{n\in\mathbb{N}}$ *is a binormalized unconditional Schauder basis, then* D *is strongly normal.*
(v) *If* $\rho_n \leq 1$ *for each* $n \in \mathbb{N}$ *and* $(x_n)_{n\in\mathbb{N}}$ *is orthogonal, then* $D \leq I$.
(vi) *If* $\rho_n \geq 0$ *for each* $n \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty} \rho_n \leq 1$, *then* $0 \leq D^2 \leq D$.
(vii) *If* $\rho_n \geq 0$ *for each* $n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \rho_n < \infty$ *and* $S := (e_k)_{k\in\mathbb{N}} \subseteq \mathsf{S}_X$ *is a sequence of orthogonal* L^2*-summand vectors, then* $\operatorname{tr}_S(D) \leq \|(\rho_n)_{n\in\mathbb{N}}\|_1$.
(viii) *If* $\rho_n \geq 0$ *for each* $n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \rho_n < \infty$ *and* $S := (e_k)_{k\in\mathbb{N}} \subseteq \mathsf{S}_X$ *is a binormalized Schauder basis of* L^2*-summand vectors, then* $\operatorname{tr}_S(D) = \|(\rho_n)_{n\in\mathbb{N}}\|_1$.
(ix) *If* $(x_n)_{n\in\mathbb{N}}$ *is orthogonal, then* $\sigma_p(T) \supseteq \{\rho_n : n \in \mathbb{N}\}$ *and* $\sigma_p(T)\setminus\{0\} \subseteq \{\rho_n : n \in \mathbb{N}\}$. *If* $(x_n)_{n\in\mathbb{N}}$ *is a binormalized Schauder basis, then* $\sigma_p(T) = \{\rho_n : n \in \mathbb{N}\}$.
(x) *If* $(x_n)_{n\in\mathbb{N}}$ *is orthogonal, then we have* $\overline{\operatorname{span}}\left(\left\{x_m : |\rho_m| = \left\|(\rho_n)_{n\in\mathbb{N}}\right\|_\infty\right\}\right) \subseteq \operatorname{suppv}(D) \subseteq \left\{x \in X : \exists m \in \mathbb{N}\ J(x_m)(x) \neq 0 \Rightarrow |\rho_m| = \left\|(\rho_n)_{n\in\mathbb{N}}\right\|_\infty\right\}$. *If* $(x_n)_{n\in\mathbb{N}}$ *is a binormalized Schauder basis, then the previous inclusions are equalities.*

Proof Let us rewrite D as $D(x) = \sum_{n=1}^{\infty} \rho_n J(x_n)(x)x_n$ for all $x \in X$, where $J : X \to X^*$ is the duality mapping.

(i) Fix an arbitrary $x \in X$. For every $k \in \mathbb{N}$, let $x_k := \sum_{n=1}^{k} \rho_n J(x_n)(x)x_n$. Note that, for every $n \in \{1, \ldots, k\}$, $J(x_n)(x_k) = \rho_n J(x_n)(x)$, therefore $x_k = \sum_{n=1}^{k} J(x_n)(x_k)x_n$. According to Lemma 3.7,

$$\|x_k\|^2 = \sum_{n=1}^{k} |J(x_n)(x_k)|^2 = \sum_{n=1}^{k} |\rho_n|^2 |J(x_n)(x)|^2.$$

Since $(x_k)_{k\in\mathbb{N}}$ converges to $D(x)$, we have that $\left(\|x_k\|^2\right)_{k\in\mathbb{N}}$ converges to $\|D(x)\|^2$ and so $\|D(x)\|^2 = \sum_{n=1}^{\infty} |\rho_n|^2 |J(x_n)(x)|^2$. By using again Lemma 3.7, for every $x \in X$,

$$\|D(x)\|^2 = \sum_{n=1}^{\infty} |\rho_n|^2 |J(x_n)(x)|^2 \leq \|(\rho_n)_{n\in\mathbb{N}}\|_\infty^2 \|x\|^2,$$

meaning that $\|D\| \leq \|(\rho_n)_{n\in\mathbb{N}}\|_\infty$. Since $D(x_n) = \rho_n x_n$ for all $n \in \mathbb{N}$, we clearly obtain that $\|D\| = \|(\rho_n)_{n\in\mathbb{N}}\|_\infty$.

(ii) Fix arbitrary elements $x, y \in X$. By hypothesis, $\sum_{n=1}^{\infty} \rho_n J(x_n)(x)x_n$ is convergent, thus Lemma 3.5 assures that

$$J\left(\sum_{n=1}^{\infty} \rho_n J(x_n)(x)x_n\right) = w^* \sum_{n=1}^{\infty} J\left(\rho_n J(x_n)(x)x_n\right).$$

Then

$$\begin{aligned} J(D(x))(y) &= J\left(\sum_{n=1}^{\infty} \rho_n J(x_n)(x)x_n\right)(y) \\ &= \sum_{n=1}^{\infty} J\left(\rho_n J(x_n)(x)x_n\right)(y) \text{ by Lemma 3.5} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \rho_n \overline{J(x_n)(x)} J(x_n)(y) \text{ by Remark 1.1} \\
&= \sum_{n=1}^{\infty} \rho_n J(x_n)(y) J(x)(x_n) \text{ by Remark 3.4} \\
&= J(x)\left(\sum_{n=1}^{\infty} \rho_n J(x_n)(y) x_n\right) \\
&= J(x)(D(y)).
\end{aligned}
$$

(iii) Fix an arbitrary $x \in X$. Then

$$
\begin{aligned}
J(x)(D(x)) &= J(x)\left(\sum_{n=1}^{\infty} \rho_n J(x_n)(x) x_n\right) \\
&= \sum_{n=1}^{\infty} \rho_n J(x_n)(x) J(x)(x_n) \\
&= \sum_{n=1}^{\infty} \rho_n |J(x_n)(x)|^2 \text{ by Remark 3.4} \\
&\geq 0.
\end{aligned}
$$

(iv) We will prove that $D = C^2$, where $C(x) = \sum_{n=1}^{\infty} \sqrt{\rho_n} J(x_n)(x) x_n$ for all $x \in X$. Indeed, for every $x \in X$,

$$
\begin{aligned}
C(C(x)) &= \sum_{n=1}^{\infty} \sqrt{\rho_n} J(x_n)(C(x)) x_n \\
&= \sum_{n=1}^{\infty} \sqrt{\rho_n} \sum_{m=1}^{\infty} \sqrt{\rho_m} J(x_m)(x) J(x_n)(x_m) x_n \\
&= \sum_{n=1}^{\infty} \rho_n J(x_n)(x) x_n \text{ by orthogonality} \\
&= D(x).
\end{aligned}
$$

Finally, notice that C is well defined in view of Proposition 3.2, and C is self-adjoint by (ii).

(v) Fix an arbitrary $x \in X$. Then

$$
\begin{aligned}
J(x)(D(x)) &= J(x)\left(\sum_{n=1}^{\infty} \rho_n J(x_n)(x) x_n\right) = \sum_{n=1}^{\infty} \rho_n J(x_n)(x) J(x)(x_n) \\
&= \sum_{n=1}^{\infty} \rho_n |J(x_n)(x)|^2 \text{ by Remark 3.4} \\
&\leq \sum_{n=1}^{\infty} |J(x_n)(x)|^2 \leq \|x\|^2 \text{ by Lemma 3.7} \\
&= J(x)(x).
\end{aligned}
$$

(vi) Fix an arbitrary $x \in X$. On the one hand, note that $D^2(x) = D(D(x)) = \sum_{n=1}^{\infty} \rho_n J(x_n)(D(x))x_n$, so

$$J(x)\left(D^2(x)\right) = \sum_{n=1}^{\infty} \rho_n J(x_n)(D(x)) J(x)(x_n)$$
$$= \sum_{n=1}^{\infty} \rho_n \sum_{k=1}^{\infty} \rho_k J(x_k)(x) J(x_n)(x_k) J(x)(x_n).$$

On the other hand, since D is self-adjoint, we have that $J(x)\left(D^2(x)\right) = J(D(x))(D(x)) = \|D(x)\|^2$, so $J(x)\left(D^2(x)\right)$ is real and positive. Finally,

$$\begin{aligned}
J(x)\left(D^2(x)\right) &= \Re\left(J(x)\left(D^2(x)\right)\right) = \sum_{n=1}^{\infty} \rho_n \sum_{k=1}^{\infty} \rho_k \Re\left(J(x_k)(x) J(x_n)(x_k) J(x)(x_n)\right) \\
&\leq \sum_{n=1}^{\infty} \rho_n \sum_{k=1}^{\infty} \rho_k \left|J(x_k)(x) J(x_n)(x_k) J(x)(x_n)\right| \\
&= \sum_{n=1}^{\infty} \rho_n |J(x)(x_n)| \sum_{k=1}^{\infty} \rho_k \left|J(x_k)(x)\right| \left|J(x_n)(x_k)\right| \\
&\leq \sum_{n=1}^{\infty} \rho_n |J(x)(x_n)| \sum_{k=1}^{\infty} \rho_k |J(x_k)(x)| \\
&= \left(\sum_{n=1}^{\infty} \rho_n |J(x)(x_n)|\right)\left(\sum_{k=1}^{\infty} \rho_k |J(x_k)(x)|\right) \\
&= \left(\sum_{n=1}^{\infty} \rho_n |J(x)(x_n)|\right)^2 = \left(\sum_{n=1}^{\infty} \sqrt{\rho_n}\sqrt{\rho_n} |J(x)(x_n)|\right)^2 \\
&\leq \left(\sum_{n=1}^{\infty} \rho_n\right)\left(\sum_{n=1}^{\infty} \rho_n |J(x)(x_n)|^2\right) \text{ by Hölder's inequality} \\
&\leq J(x)(D(x)) \text{ by (iii).}
\end{aligned}$$

(vii) By (iii), $J(e_k)(D(e_k)) = \sum_{n=1}^{\infty} \rho_n |J(e_k)(x_n)|^2$ for every $k \in \mathbb{N}$. Fix an arbitrary $l \in \mathbb{N}$. Keeping in mind Lemma 3.7,

$$\sum_{k=1}^{l} |J(e_k)(x_n)|^2 \leq \sum_{k=1}^{\infty} |J(e_k)(x_n)|^2 \leq \|x_n\|^2$$

for every $n \in \mathbb{N}$. Then

$$\sum_{k=1}^{l} J(e_k)(D(e_k)) = \sum_{k=1}^{l}\sum_{n=1}^{\infty} \rho_n |J(e_k)(x_n)|^2 = \sum_{n=1}^{\infty} \rho_n \sum_{k=1}^{l} |J(e_k)(x_n)|^2$$
$$\leq \sum_{n=1}^{\infty} \rho_n \|x_n\|^2 = \|(\rho_n)_{n\in\mathbb{N}}\|_1 .$$

The arbitrariness of $l \in \mathbb{N}$ assures that

$$\operatorname{tr}_S(D) := \sum_{k=1}^{\infty} J(e_k)(D(e_k)) \leq \|(\rho_n)_{n\in\mathbb{N}}\|_1 .$$

(viii) Following Corollary 3.8, since X is smooth and $(e_n)_{n\in\mathbb{N}} \subseteq \mathsf{S}_X$ is a binormalized Schauder basis of L^2-summand vectors, the basic sequence of coordinate functionals is precisely $(J(e_n))_{n\in\mathbb{N}} \subseteq \mathsf{S}_{X^*}$, thus, for every $x \in X$, $x = \sum_{n=1}^{\infty} J(e_n)(x)e_n$ and $\|x\|^2 = \sum_{n=1}^{\infty} |J(e_n)(x)e_n|^2$. In particular, we have that $1 = \|x_n\|^2 = \sum_{k=1}^{\infty} |J(e_k)(x_n)|^2$ for every $n \in \mathbb{N}$. According to (iii), $J(e_k)(D(e_k)) = \sum_{n=1}^{\infty} \rho_n |J(e_k)(x_n)|^2$ for every $k \in \mathbb{N}$. Next, if we consider the double series $\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \rho_n |J(e_k)(x_n)|^2$, then we have that for every $k \in \mathbb{N}$, the series $\sum_{n=1}^{\infty} \rho_n |J(e_k)(x_n)|^2$ is absolutely convergent to $J(e_k)(D(e_k))$, and for every $n \in \mathbb{N}$, the series $\sum_{k=1}^{\infty} \rho_n |J(e_k)(x_n)|^2$ is absolutely convergent as well to $\rho_n \|x_n\|^2$. Therefore, we can call on Lemma 3.9 to conclude that

$$\begin{aligned} \sum_{k=1}^{\infty} J(e_k)(D(e_k)) &= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \rho_n |J(e_k)(x_n)|^2 = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \rho_n |J(e_k)(x_n)|^2 \\ &= \sum_{n=1}^{\infty} \rho_n \|x_n\|^2 = \|(\rho_n)_{n\in\mathbb{N}}\|_1 . \end{aligned}$$

(ix) For every $n \in \mathbb{N}$, $D(x_n) = \rho_n x_n$, thus $\rho_n \in \sigma_p(D)$ for all $n \in \mathbb{N}$. Next, if $\lambda \in \sigma_p(D) \setminus \{0\}$, then there exists $x \in X \setminus \{0\}$ such that $D(x) = \lambda x$. Then $\sum_{n=1}^{\infty} \rho_n J(x_n)(x)x_n = \lambda x$, meaning that $\rho_n J(x_n)(x) = \lambda J(x_n)(x)$ for all $n \in \mathbb{N}$ by orthogonality. Since $x \neq 0$ and $\lambda \neq 0$, there must exist $n_0 \in \mathbb{N}$ for which $J(x_{n_0})(x) \neq 0$, which implies that $\rho_{n_0} = \lambda$. Suppose now that $(x_n)_{n\in\mathbb{N}}$ is a binormalized Schauder basis. If $0 \in \sigma_p(D)$, then wen can find $x \in \ker(D) \setminus \{0\}$. Note that $x = \sum_{n=1}^{\infty} J(x_n)(x)x_n$, so there exists $n_0 \in \mathbb{N}$ such that $J(x_{n_0})(x) \neq 0$. The equation $D(x) = 0$ means that $\sum_{n=1}^{\infty} \rho_n J(x_n)(x)x_n = 0$, so in particular $\rho_{n_0} J(x_{n_0})(x) = 0$, yielding $\rho_{n_0} = 0$.

(x) Fix arbitrary $m_1, \ldots, m_k \in \mathbb{N}$ such that $\left|\rho_{m_n}\right| = \left\|(\rho_n)_{n\in\mathbb{N}}\right\|_{\infty} = \|D\|$ for all $n \in \{1, \ldots, k\}$. Take any $x \in \operatorname{span}\left(\left\{x_{m_1}, \ldots, x_{m_k}\right\}\right)$. Note that $x = \sum_{n=1}^{k} J(x_{m_n})(x)x_{m_n}$, hence

$$D(x) = \sum_{n=1}^{k} \rho_{m_n} J(x_{m_n})(x)x_{m_n} = \sum_{n=1}^{k} J(x_{m_n})(D(x))x_{m_n}.$$

By applying Lemma 3.7,

$$\|D(x)\|^2 = \sum_{n=1}^{k} |J(x_{m_n})(D(x))|^2 = \sum_{n=1}^{k} \left|\rho_{m_n}\right|^2 \left|J\left(x_{m_n}\right)(x)\right|^2 = \|D\|^2 \|x\|^2.$$

As a consequence, $x \in \operatorname{suppv}(D)$. This shows that

$$\operatorname{span}\left(\left\{x_m : |\rho_m| = \left\|(\rho_n)_{n\in\mathbb{N}}\right\|_{\infty}\right\}\right) \subseteq \operatorname{suppv}(D).$$

Since $\operatorname{suppv}(D)$ is trivially closed, we have that

$$\overline{\operatorname{span}}\left(\left\{x_m : |\rho_m| = \left\|(\rho_n)_{n\in\mathbb{N}}\right\|_{\infty}\right\}\right) \subseteq \operatorname{suppv}(D).$$

Next, fix an arbitrary $x \in \operatorname{suppv}(D)$. Then

$$\|D\|^2 \|x\|^2 = \|D(x)\|^2 = \sum_{n=1}^{\infty} |\rho_n|^2 |J(x_n)(x)|^2 \text{ by (i)}$$

$$\leq \left\|(\rho_n)_{n\in\mathbb{N}}\right\|_\infty^2 \sum_{n=1}^{\infty} |J(x_n)(x)|^2 \leq \left\|(\rho_n)_{n\in\mathbb{N}}\right\|_\infty^2 \|x\|^2 \text{ by Lemma 3.7}$$
$$= \|D\|^2 \|x\|^2 \text{ by (i).}$$

Therefore, if there exists $m \in \mathbb{N}$ for which $J(x_m)(x) \neq 0$, then $|\rho_m| = \left\|(\rho_n)_{n\in\mathbb{N}}\right\|_\infty$. Finally, if $(x_n)_{n\in\mathbb{N}}$ is a binormalized Schauder basis and $x \in X$ satisfies that $|\rho_m| = \left\|(\rho_n)_{n\in\mathbb{N}}\right\|_\infty$ whenever $J(x_m)(x) \neq 0$, then

$$x = \sum_{n=1}^{\infty} J(x_n)(x)x_n \in \overline{\operatorname{span}}\left(\left\{x_m : |\rho_m| = \left\|(\rho_n)_{n\in\mathbb{N}}\right\|_\infty\right\}\right).$$

□

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