Nash blowups of toric varieties in prime characteristic

Daniel Duarte[1] · Jack Jeffries[2] · Luis Núñez-Betancourt[3]

Abstract

We initiate the study of the resolution of singularities properties of Nash blowups over fields of prime characteristic. We prove that the iteration of normalized Nash blowups desingularizes normal toric surfaces. We also introduce a prime characteristic version of the logarithmic Jacobian ideal of a toric variety and prove that its blowup coincides with the Nash blowup of the variety. As a consequence, the Nash blowup of a, not necessarily normal, toric variety of arbitrary dimension in prime characteristic can be described combinatorially.

Introduction

The Nash blowup of an algebraic variety is a modification that replaces singular points by limits of tangent spaces. It has been proposed to solve singularities by iterating this blowup [16, 20]. This question has been extensively studied [1, 2, 5, 6, 8–11, 13, 14, 16, 19, 21].

Until recently, the resolution properties of Nash blowups have been studied only over fields of characteristic zero. This is due to a well-known example given by A. Nobile that shows that the Nash blowup could be trivial in prime characteristic [16]. It was recently shown that Nash blowups behave as expected in prime characteristic after adding the condition of normality [7]. Hence, the original question regarding Nash blowups and resolution of singularities can

✉ Daniel Duarte
adduarte@matmor.unam.mx

Jack Jeffries
jack.jeffries@unl.edu

Luis Núñez-Betancourt
luisnub@cimat.mx

1 Centro de Ciencias Matemáticas, UNAM, Campus Morelia, Morelia, MICH, Mexico

2 University of Nebraska-Lincoln, Lincoln, USA

3 Centro de Investigación en Matemáticas, Guanajuato, GTO, Mexico

be reconsidered in arbitrary characteristic by iterating normalized Nash blowups. We stress that the condition of normality is frequently assumed for many results also in characteristic zero. For instance, M. Spivakovsky showed that the iteration of normalized Nash blowups gives a resolution of singularities for complex surfaces [21].

In this paper we initiate the study of the resolution properties of normalized Nash blowups in prime characteristic. We study this question in the context of toric varieties.

Our first goal in this paper is to describe combinatorially the Nash blowup of a not necessarily normal toric variety of arbitrary dimension over fields of prime characteristic. In characteristic zero, the key ingredient for such a description is given by the so-called logarithmic Jacobian ideal, which was originally introduced by González-Sprinberg [8]. This is a monomial ideal determined by linear relations on the generators of the semigroup of the toric variety. González-Sprinberg showed that the Nash blowup of a normal toric variety is isomorphic to the blowup of its logarithmic Jacobian ideal. This result was later revisited and generalized by several authors. For instance, M. Lejeune-Jalabert and A. Reguera gave a new proof of this result [15]. In addition, P. González and B. Teissier extended this result for not necessarily normal toric varieties [11].

It is worth mentioning that the isomorphism between Nash blowups of toric varieties and the blowup of their logarithmic Jacobian ideal does not hold over fields of prime characteristic (see Example 1.5). In this paper we define a positive characteristic version of the logarithmic Jacobian ideal (see Definition 1.6) that allows us to extend the known result in characteristic zero to prime characteristic.

Theorem A (Theorem 1.9) *The Nash blowup of a toric variety over a field of characteristic $p > 0$ is isomorphic to the blowup of its logarithmic Jacobian ideal modulo p.*

Combining Theorem A with previous work by Gónzalez-Teissier [11], one obtains a combinatorial description of the Nash blowup in prime characteristic. We specialize this description to dimension two to obtain our second main theorem.

Theorem B (Theorem 2.5) *The iteration of normalized Nash blowups desingularizes normal toric surfaces over fields of prime characteristic.*

G. González-Sprinberg proved the characteristic-zero version of Theorem B [8]. An important step in his proof consists in showing that the normalized Nash blowup of a normal toric surface is determined by a subdivision of the cone defining the surface. The key step to prove Theorem B is to show that this subdivision is actually the same in prime characteristic. As a consequence, it is equivalent to solve normal toric surfaces iterating normalized Nash blowups in zero or prime characteristic. Once this is shown, the rest of the proof of Theorem B follows the work of González-Sprinberg.

As mentioned before, the combinatorics of normalized Nash blowups of toric surfaces are independent of the characteristic. One may ask whether this is true for higher dimensions. We conclude this paper by exhibiting an example of a toric variety of dimension three whose Nash blowup behaves differently in characteristic zero and two. In particular, the normalized Nash blowup of this variety is nonsingular in characteristic zero but singular in characteristic two. A further iteration of normalized Nash blowup gives a nonsingular variety in this case.

Convention 0.1 Throughout this manuscript $\mathbb{K}$ denotes an algebraically closed field.

1 Nash blowups of toric varieties

We are interested in studying the Nash blowup of a toric variety over fields of prime characteristic. We first recall some classical results of the characteristic zero case.

Definition 1.1 Let $\mathbb{K}$ be an algebraically closed field of arbitrary characteristic. Let $X \subseteq \mathbb{K}^n$ be an equidimensional algebraic variety of dimension d. Consider the Gauss map:

$$G : X \setminus \mathrm{Sing}(X) \to \mathrm{Gr}(d, n)$$
$$x \mapsto T_x X,$$

where $\mathrm{Gr}(d, n)$ is the Grassmanian of d-dimensional vector spaces in $\mathbb{K}^n$, and $T_x X$ is the tangent space to X at x. Denote by X^* the Zariski closure of the graph of G. Call ν the restriction to X^* of the projection of $X \times \mathrm{Gr}(d, n)$ to X. The pair (X^*, ν) is called the Nash blowup of X.

Definition 1.2 ([4, 22]) Let $\Gamma \subseteq \mathbb{Z}^d$ be a semigroup generated by $\{\gamma_1, \dots, \gamma_n\}$. Consider the $\mathbb{K}$-algebra homomorphism $\pi_\Gamma : \mathbb{K}[x_1, \dots, x_n] \to \mathbb{K}[t_1^\pm, \dots, t_d^\pm]$, $x_i \mapsto t^{\gamma_i}$. Let $I_\Gamma = \ker \pi_\Gamma$. The variety $X_\Gamma = \mathbf{V}(I_\Gamma) \subseteq \mathbb{K}^n$ is called the toric variety defined by Γ. We denote as $\mathbb{K}[t^\Gamma]$ the coordinate ring of X_Γ.

The first step towards a combinatorial description of Nash blowups of toric varieties in characteristic zero is given by the so-called logarithmic Jacobian ideal. This ideal was originally introduced by G. González-Sprinberg [8, Section 2], and later revisited by several authors [3, 11, 15].

Definition 1.3 Suppose that $\mathrm{char}(\mathbb{K}) = 0$. Let $\Gamma = \langle \gamma_1, \dots, \gamma_n \rangle_\mathbb{N} \subseteq \mathbb{Z}^d$ be a semigroup such that $\langle \gamma_1, \dots, \gamma_n \rangle_\mathbb{Z} = \mathbb{Z}^d$. Consider the following ideal:

$$\mathcal{J}_0 = \langle t^{\gamma_{i_1} + \cdots + \gamma_{i_d}} \mid \det(\gamma_{i_1} \cdots \gamma_{i_d}) \neq 0, \ 1 \leq i_1 < \cdots < i_d \leq n \rangle \subseteq \mathbb{K}[t^\Gamma].$$

The ideal $\mathcal{J}_0$ is called the *logarithmic Jacobian ideal of X_Γ*.

Theorem 1.4 ([8, 11, 15]) *Suppose that* $\mathrm{char}(\mathbb{K}) = 0$. *The Nash blowup of X_Γ is isomorphic to the blowup of its logarithmic Jacobian ideal.*

The previous theorem is false over fields of prime characteristic.

Example 1.5 Suppose that $\mathrm{char}(\mathbb{K}) = 2$, $\Gamma = \langle 2, 3 \rangle_\mathbb{N}$, and $X_\Gamma = \mathbf{V}(x^3 - y^2)$. Then $\mathrm{Bl}_{\langle t^2, t^3 \rangle} X_\Gamma$ is nonsingular but $X_\Gamma^* \cong X_\Gamma$ [16, Example 1].

Our first goal is to give the positive characteristic version of Theorem 1.4.

Definition 1.6 Suppose that $\mathrm{char}(\mathbb{K}) = p > 0$. Let $\Gamma = \langle \gamma_1, \dots, \gamma_n \rangle_\mathbb{N} \subseteq \mathbb{Z}^d$ be a semigroup such that $\langle \gamma_1, \dots, \gamma_n \rangle_\mathbb{Z} = \mathbb{Z}^d$. Consider the following ideal:

$$\mathcal{J}_p = \langle t^{\gamma_{i_1} + \cdots + \gamma_{i_d}} \mid \det(\gamma_{i_1} \cdots \gamma_{i_d}) \neq 0 \bmod p, \ 1 \leq i_1 < \cdots < i_d \leq n \rangle \subseteq \mathbb{K}[t^\Gamma].$$

The ideal $\mathcal{J}_p$ is called the *logarithmic Jacobian ideal modulo p of X_Γ*.

Example 1.7 Let $\Gamma = \langle 2, 3 \rangle_\mathbb{N} \subseteq \mathbb{N}$. Then $\mathcal{J}_2 = \langle t^3 \rangle$, $\mathcal{J}_3 = \langle t^2 \rangle$, and $\mathcal{J}_p = \langle t^2, t^3 \rangle$, for $p = 0$ and $p \geq 5$.

We now present a property of short exact sequence of matrices. In fact, there is a more general version of this property in the context of realizable matroids, which is known as Gale duality [18, Theorem 2.2.8]. We give the proof of this property for the sake of completeness.

Lemma 1.8 *Let*

$$0 \longrightarrow \mathbb{K}^c \xrightarrow{B} \mathbb{K}^n \xrightarrow{A} \mathbb{K}^d \longrightarrow 0 .$$

be a short exact sequence of vector spaces over $\mathbb{K}$. Let $K \subseteq [n] = \{1, \ldots, n\}$, $|K| = c$. Denote as B_K the matrix formed by the rows of B corresponding to K. Similarly, denote as $A_{[n]\setminus K}$ the matrix formed by the columns of A corresponding to $[n] \setminus K$. Then

$$\det B_K \neq 0 \Leftrightarrow \det A_{[n]\setminus K} \neq 0.$$

Proof Without loss of generality we assume $K = \{d + 1, \ldots, n\}$. Denote $D = [n] \setminus K$.

Assume that $\det(A_D) \neq 0$. Let $u \in \mathbb{K}^c$. Since multiplication by B^T is surjective, there exists $\begin{pmatrix} w_0 \\ v_0 \end{pmatrix}$ such that $B^T \begin{pmatrix} w_0 \\ v_0 \end{pmatrix} = B_D^T w_0 + B_K^T v_0 = u$. Since $\det(A_D^T) = \det(A_D) \neq 0$, there exists $z \in \mathbb{K}^d$ such that $A_D^T z = w_0$. Let $v = v_0 - A_K^T z$. Thus, $\begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} w_0 \\ v_0 \end{pmatrix} - \begin{pmatrix} A_D^T z \\ A_K^T z \end{pmatrix} = \begin{pmatrix} w_0 \\ v_0 \end{pmatrix} - A^T z$. Hence,

$$B_K^T v = 0 + B_K^T v = B^T \begin{pmatrix} 0 \\ v \end{pmatrix} = B^T \begin{pmatrix} w_0 \\ v_0 \end{pmatrix} - B^T A^T z = B^T \begin{pmatrix} w_0 \\ v_0 \end{pmatrix} = u.$$

Hence, the linear transformation associated to B_K^T is surjective, and so, $\det(B_K) = \det(B_K^T) \neq 0$.

The other implication follows as in the previous paragraph working on the dual exact sequence.   □

Theorem 1.9 *Suppose that $\mathrm{char}(\mathbb{K}) = p > 0$. The Nash blowup of X_Γ is isomorphic to the blowup of its logarithmic Jacobian ideal modulo p.*

Proof The proof follows the arguments given by González Pérez and Teissier [11, Proposition 60], combined with the previous lemma.

Assume that $\{\gamma_1, \ldots, \gamma_n\}$ is a minimal generating set of $\Gamma \subseteq \mathbb{Z}^d$ and $\langle \gamma_1, \ldots, \gamma_n \rangle_\mathbb{Z} = \mathbb{Z}^d$. Let $A = (\gamma_1 \cdots \gamma_n)_{d \times n}$ and $c = n - d$. Since $\mathbb{K}$ is algebraically closed, there exist $x^{\alpha_1} - x^{\beta_1}, \ldots, x^{\alpha_c} - x^{\beta_c} \in I_\Gamma$ such that $\alpha_1 - \beta_1, \ldots, \alpha_c - \beta_c$ define a basis of the kernel of A. Let $B = (b_1 \cdots b_c)_{n \times c}$, where $b_i = \alpha_i - \beta_i$. Then we have an exact sequence,

$$0 \longrightarrow \mathbb{Z}^c \xrightarrow{B} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^d \longrightarrow 0 .$$

Since the kernel of A is a saturated sublattice of $\mathbb{Z}^n$, one can extend $b_1, \ldots, b_c$ to a basis of $\mathbb{Z}^n$. We denote as $\bar{A}$ and $\bar{B}$ the matrices obtained by taking the entries modulo p. Hence, we obtain an exact sequence of $\mathbb{Z}_p$-vector spaces,

$$0 \longrightarrow \mathbb{Z}_p^c \xrightarrow{\bar{B}} \mathbb{Z}_p^n \xrightarrow{\bar{A}} \mathbb{Z}_p^d \longrightarrow 0 . \tag{1}$$

Take $D_0 \subseteq \{1, \ldots, n\}$ such that $|D_0| = d$ and $\det(\bar{A}_{D_0}) \neq 0$. Letting $K_0 = [n] \setminus D_0$, we get $\det(\bar{B}_{K_0}) \neq 0$ by Lemma 1.8. Denote as J_{K_0} the matrix formed by the columns corresponding to K_0 of the Jacobian matrix $\mathrm{Jac}(x^{\alpha_i} - x^{\beta_i})_{1 \le i \le c}$.

We have that $\prod_{k \in K_0} x_k \det J_{K_0} \equiv \prod_{i=1}^{c} x^{\alpha_i} \det \bar{B}_{K_0} \mod I_\Gamma$ [8, Section 2.2, Lemme 2], [11, Proposition 60]. We point out that this congruence is usually proved over fields of characteristic zero; however, the same proof holds in arbitrary characteristic. Since I_Γ is a prime ideal and $\{\gamma_1, \ldots, \gamma_n\}$ is a minimal generating set of Γ, we obtain $\det(J_{K_0}) \neq 0$ mod I_Γ. Hence, $\mathrm{Jac}(x^{\alpha_i} - x^{\beta_i})_{1 \leq i \leq c}$ has maximal rank equal to c. It follows that the blowup of X_Γ along the ideal $\langle \det J_K \mid K \subseteq [n], |K| = c \rangle$ is isomorphic to the Nash blowup of X_Γ [16, Theorem 1].

Let $K \subseteq [n], |K| = c$. As before, $\prod_{k \in K} x_k \det J_K \equiv \prod_{i=1}^{c} x^{\alpha_i} \det \bar{B}_K \mod I_\Gamma$. Multiply this congruence by the monomial $\prod_{i \in [n] \setminus K} x_i$ to obtain:

$$(x_1 \cdots x_n) \det J_K \equiv \left(\prod_{i \in [n] \setminus K} x_i \right) \prod_{i=1}^{c} x^{\alpha_i} \det \bar{B}_K \mod I_\Gamma. \qquad (2)$$

Since multiplying an ideal by a principal ideal gives isomorphic blowups, Congruence (2) implies:

$$
\begin{aligned}
X_\Gamma^* &\cong \mathrm{Bl}_{\langle \det J_K \mid K \subseteq [n], |K| = c \rangle} X_\Gamma \\
&\cong \mathrm{Bl}_{\langle x_1 \cdots x_n \rangle \langle \det J_K \mid K \subseteq [n], |K| = c \rangle} X_\Gamma \\
&\cong \mathrm{Bl}_{\langle (\prod_{i \in [n] \setminus K} x_i)(\prod_{i=1}^{c} x^{\alpha_i}) \det \bar{B}_K \mid K \subseteq [n], |K| = c \rangle} X_\Gamma \\
&\cong \mathrm{Bl}_{\langle (\prod_{i \in [n] \setminus K} x_i) \det \bar{B}_K \mid K \subseteq [n], |K| = c \rangle} X_\Gamma.
\end{aligned}
$$

Using the isomorphism $\mathbb{K}[x_1, \ldots, x_n]/I_\Gamma \cong \mathbb{K}[t^\Gamma]$ and Lemma 1.8 applied to the exact sequence (1), the ideal

$$\left\langle \left(\prod_{i \in [n] \setminus K} x_i \right) \det \bar{B}_K \; \middle| \; \begin{matrix} K \subseteq [n] \\ |K| = c \end{matrix} \right\rangle = \left\langle \prod_{i \in [n] \setminus K} x_i \; \middle| \; \det \bar{B}_K \neq 0, \; \begin{matrix} K \subseteq [n] \\ |K| = c \end{matrix} \right\rangle$$

corresponds to the ideal

$$\langle t^{\gamma_{i_1} + \cdots + \gamma_{i_d}} \mid \det(\gamma_{i_1} \cdots \gamma_{i_d}) \neq 0 \mod p, \; 1 \leq i_1 < \cdots < i_d \leq n \rangle = \mathcal{J}_p.$$

In conclusion, $X_\Gamma^* \cong \mathrm{Bl}_{\mathcal{J}_p} X_\Gamma$. $\qquad \square$

Using Theorem 1.9, a combinatorial description of the Nash blowup of a toric variety can be obtained with the framework developed by Gónzalez-Teissier for the blowup of a toric variety along any monomial ideal [11, Section 2.6].

Remark 1.10 As mentioned in the introduction, the characteristic zero analogue of Theorem 1.9 was studied by several authors. In particular, there is a version using the language of Minkowski sums [1, Theorem 2.9]. Even though the characteristic zero assumption is not explicitly stated in that work [1, Theorem 2.9], it is implicitly used in the proof.

2 Resolution of normal toric surfaces by iterated normalized Nash blowups

In this section we prove that normalized Nash blowups solve the singularities of normal toric surfaces over fields of prime characteristic.

Notation 2.1 Throughout this section, $\sigma \subseteq \mathbb{R}^2$ denotes a nonregular strongly convex rational polyhedral cone of dimension 2, and $\Gamma = \check{\sigma} \cap \mathbb{Z}^2$, where $\check{\sigma} \subseteq \mathbb{R}^2$ is the dual cone of σ. Let $\Theta \subseteq \mathbb{R}^2$ be the convex hull of $\Gamma \setminus \{(0, 0)\}$. Let $\{\gamma_1, \ldots, \gamma_n\} \subseteq \mathbb{Z}^2$ be the points lying on the compact edges of the boundary polygon $\partial \Theta$. We order $\{\gamma_1, \ldots, \gamma_n\}$ according to the counterclockwise order.

We note that $n \geq 3$ since σ is nonregular. We now recall the following description of the minimal generating set of Γ.

Proposition 2.2 ([17, Proposition 1.21]) *In the context of Notation 2.1, $\{\gamma_1, \ldots, \gamma_n\}$ is the minimal set of generators of Γ.*

We use Theorem 1.9 to prove that iterated normalized Nash blowups solve normal toric surfaces in prime characteristic. We show that the involved combinatorics are independent of the characteristic. The result thus follows from the classical work of G. González-Sprinberg [8].

Lemma 2.3 *Let $\alpha, \beta, \gamma \in \mathbb{Z}_{\geq 0}^2$ be pairwise linearly independent vectors, in counterclockwise orientation. For a subset $S \subseteq \mathbb{R}^2$, denote as $\mathrm{Conv}(S)$ the convex hull of S.*

(1) *If $\beta \in \mathrm{Conv}(\{\alpha, \gamma\})$, then $\alpha + \gamma \in \mathrm{Conv}(\{\alpha + \beta, \beta + \gamma\})$.*
(2) *If $\beta \notin \mathrm{Conv}(\{\alpha, \gamma\}) + \mathbb{R}_{\geq 0}(\alpha, \gamma)$, then*

$$\alpha + \gamma \in \mathrm{Conv}(\{\alpha + \beta, \beta + \gamma\}) + \mathbb{R}_{\geq 0}(\alpha, \gamma).$$

Proof The first statement of the lemma follows from direct computation.

We now prove (2). There exist unique $r, s \in \mathbb{R}$ such that $\beta = r\alpha + s\gamma$. We claim that $r, s > 0$ and $0 < r + s < 1$. By the orientation, we have that β is in the cone spanned by α and γ. Thus, $r, s \geq 0$ and $r = 0$ or $s = 0$ contradicts linear independence. If $r + s \geq 1$, let $r = r' + r''$ and $s = s' + s''$ with $r' + s' = 1$ and $r', s', r'', s'' \geq 0$. Then

$$\beta = (r'\alpha + s'\gamma) + (r''\alpha + s''\gamma) \in \mathrm{Conv}(\{\alpha, \gamma\}) + \mathbb{R}_{\geq 0}(\alpha, \gamma),$$

which contradicts the hypothesis on β. This justifies the claim. Hence, $0 < r, s < 1$ and $0 < r + s < 1$.

We have $\alpha + \beta = (1 + r)\alpha + s\gamma$ and $\beta + \gamma = r\alpha + (s + 1)\gamma$. Set $u = 1 - r$ and $v = 1 - s$. Then $0 < u, v < 1$ and $1 < u + v$. Write $u = u' + u''$ and $v = v' + v''$ with $u' + v' = 1$ and $u', v', u'', v'' \geq 0$. Thus,

$$\begin{aligned}
\alpha + \gamma &= (\alpha + \gamma - \beta) + \beta \\
&= (u\alpha + v\gamma) + (u' + v')\beta \\
&= u'(\alpha + \beta) + v'(\beta + \gamma) + u''\alpha + v''\gamma \\
&\in \mathrm{Conv}(\{\alpha + \beta, \beta + \gamma\}) + \mathbb{R}_{\geq 0}(\alpha, \gamma).
\end{aligned}$$

$\square$

Proposition 2.4 *In the setting of Notation 2.1, let $p = \mathrm{char}(\mathbb{K})$, not necessarily positive. Let $\mathcal{N}_\sigma(\mathcal{J}_p)$ denote the convex hull of*

$$\{(\gamma_i + \gamma_j) + \check{\sigma} \mid \det(\gamma_i\ \gamma_j) \neq 0 \bmod p\}_{1 \leq i < j \leq n}.$$

Then the vertices of $\mathcal{N}_\sigma(\mathcal{J}_p)$ are contained in the set

$$\{\gamma_\ell + \gamma_{\ell+1} \mid \ell \in \{1, \ldots, n - 1\}\}.$$

In particular, $\mathcal{N}_\sigma(\mathcal{J}_p) = \mathcal{N}_\sigma(\mathcal{J}_0)$, for any $p > 0$.

Proof As before, $\{\gamma_1, \ldots, \gamma_n\}$ denotes the minimal generating set of Γ, ordered counterclockwise. Recall that $n \geq 3$. Up to a change of coordinates, we can assume that $\Gamma \subseteq \mathbb{Z}_{\geq 0}^2$. Let $1 \leq i < j < k \leq n$. By Proposition 2.2, we know that $\gamma_i, \gamma_j, \gamma_k$, satisfy the first and second conditions in Lemma 2.3. By the lemma,

$$\gamma_i + \gamma_k \in \text{Conv}\left(\{\gamma_i + \gamma_j, \gamma_j + \gamma_k\}\right) \subseteq \mathcal{N}_\sigma(\mathcal{J}_p),$$

or

$$\gamma_i + \gamma_k \in \text{Conv}\left(\{\gamma_i + \gamma_j, \gamma_j + \gamma_k\}\right) + \mathbb{R}_{\geq 0}(\gamma_i, \gamma_k) \subseteq \mathcal{N}_\sigma(\mathcal{J}_p).$$

Hence, $\gamma_i + \gamma_k$ is either on the facet determined by $\gamma_i + \gamma_j$ and $\gamma_j + \gamma_k$, or it is in the interior of $\mathcal{N}_\sigma(\mathcal{J}_p)$. In particular, it is not a vertex of $\mathcal{N}_\sigma(\mathcal{J}_p)$. We conclude that the vertices of $\mathcal{N}_\sigma(\mathcal{J}_p)$ must be of the form $\gamma_\ell + \gamma_{\ell+1}$ for some $\ell \in \{1, \ldots, n-1\}$.

The last part of the proposition follows from the fact $\det(\gamma_\ell \ \gamma_{l+1}) = 1$ for all $\ell \in \{1, \ldots, n-1\}$. $\qquad\square$

A similar statement to Proposition 2.4 was already known [1, Proposition 4.11].

Theorem 2.5 *In the setting of Notation 2.1, suppose that $\text{char}(\mathbb{K}) = p > 0$. The iteration of Nash blowups followed by normalization solves the singularities of normal toric surfaces.*

Proof Recall the following classical theorem due to G. González-Sprinberg: if $\text{char}(\mathbb{K}) = 0$, normalized Nash blowups solves normal toric surfaces [8, Section 2.3, Théorème]. Equivalently, by Theorem 1.4, normalized blowups of the logarithmic Jacobian ideal $\mathcal{J}_0$ solves normal toric surfaces.

The normalized blowup of the logarithmic Jacobian ideal $\mathcal{J}_0$ is determined by the Newton polygon $\mathcal{N}_\sigma(\mathcal{J}_0)$, which, in turn, induces a subdivision of σ [11, Section 2.6 and Remark 26]. Hence, the iteration of normalized blowups of $\mathcal{J}_0$ gives place to a regular subdivision of σ.

Now suppose $\text{char}(\mathbb{K}) = p > 0$. By Proposition 2.4, $\mathcal{N}_\sigma(\mathcal{J}_p) = \mathcal{N}_\sigma(\mathcal{J}_0)$. In particular, the subdivision of σ induced by $\mathcal{N}_\sigma(\mathcal{J}_p)$ is the same as the one induced by $\mathcal{N}_\sigma(\mathcal{J}_0)$. Hence, the iteration of normalized blowups of $\mathcal{J}_p$ gives place to a regular subdivision of σ. By Theorem 1.9, we conclude that the iteration of normalized Nash blowups solves X_Γ. $\qquad\square$

3 Higher dimensional toric varieties

In view of the results of previous sections, one may wonder whether the combinatorics of the Nash blowup of toric varieties are independent of the characteristic in general. In other words, me may ask whether the last statement of Proposition 2.4 also holds for higher dimensional toric varieties. We conclude this paper by showing that this is not the case already in dimension three.

Let $\sigma \subseteq \mathbb{R}^3$ be a cone such that $\check{\sigma} = \langle (1,0,0), (0,1,0), (1,1,2) \rangle_{\mathbb{R}_{\geq 0}} \subseteq \mathbb{R}^3$. Using Macaulay2 [12], we obtain that $\Gamma = \check{\sigma} \cap \mathbb{Z}^3$ is minimally generated by $\gamma_1 = (1,0,0)$, $\gamma_2 = (0,1,0)$, $\gamma_3 = (1,1,1)$, and $\gamma_4 = (1,1,2)$. Hence, the logarithmic Jacobian ideals in characteristic zero and two are, respectively, $\mathcal{J}_0 = \langle t^{(2,2,1)}, t^{(2,3,3)}, t^{(3,2,3)}, t^{(2,2,2)} \rangle$, $\mathcal{J}_2 = \langle t^{(2,2,1)}, t^{(2,3,3)}, t^{(3,2,3)} \rangle$.

Let $\mathcal{N}_\sigma(\mathcal{J}_p) = \text{Conv}\{\gamma_i + \gamma_j + \gamma_k + \check{\sigma} \mid \det(\gamma_i \ \gamma_j \ \gamma_k) \neq 0 \mod p\}_{1 \leq i < j < k \leq 4}$. We claim that $\mathcal{N}_\sigma(\mathcal{J}_0) \neq \mathcal{N}_\sigma(\mathcal{J}_2)$. We obtain $\{x - 2 = 0\}$, $\{y - 2 = 0\}$, and $\{x + y - z - 2 = 0\}$ are supporting hyperplanes of $\mathcal{N}_\sigma(\mathcal{J}_0)$ from a direct computation. The intersection of these three

planes is $\{(2, 2, 2)\}$. Hence, $(2, 2, 2)$ is a vertex of $\mathcal{N}_\sigma(\mathcal{J}_0)$. Finally, notice that $(2, 2, 2) \notin \mathcal{N}_\sigma(\mathcal{J}_2)$.

Now we study the normalized Nash blowup of X_Γ. In characteristic zero, the normalization of X_Γ^* is nonsingular [1, Section 6, Table 5].

To compute X_Γ^* in characteristic two, we compute the blowup of $\mathcal{J}_2$ following the combinatorial description for the blowup of a toric variety along any monomial ideal [11, Section 2.6]. The polyhedron $\mathcal{N}_\sigma(\mathcal{J}_2)$ has three vertices: $v_1 = (2, 2, 1)$, $v_2 = (3, 2, 3)$, $v_3 = (2, 3, 3)$. These vertices give place to the affine charts of X_Γ^*. Each of these affine charts are also toric varieties, determined by the following semigroups:

$$\begin{aligned}
\Gamma_1 &= \Gamma + \langle v_2 - v_1, v_3 - v_1 \rangle_\mathbb{N} \\
&= \langle (1, 0, 0), (0, 1, 0), (1, 1, 1), (1, 0, 2), (0, 1, 2) \rangle_\mathbb{N}, \\
\Gamma_2 &= \Gamma + \langle v_1 - v_2, v_3 - v_2 \rangle_\mathbb{N} \\
&= \langle (1, 0, 0), (1, 1, 1), (1, 1, 2), (-1, 0, -2), (-1, 1, 0) \rangle_\mathbb{N}, \\
\Gamma_3 &= \Gamma + \langle v_1 - v_3, v_2 - v_3 \rangle_\mathbb{N} \\
&= \langle (0, 1, 0), (1, 1, 1), (1, 1, 2), (0, -1, -2), (1, -1, 0) \rangle_\mathbb{N}.
\end{aligned}$$

To compute the normalization of X_Γ^* we need to find the saturation of these semigroups. Using Macaulay2 we obtain the following minimal set of generators for the saturation of each Γ_i, denoted as $\overline{\Gamma_i}$:

$$\begin{aligned}
\overline{\Gamma_1} &= \langle (1, 0, 0), (0, 1, 0), (1, 0, 2), (0, 1, 2), (1, 0, 1), (0, 1, 1) \rangle_\mathbb{N}, \\
\overline{\Gamma_2} &= \langle (1, 0, 0), (-1, 1, 0), (0, 0, -1), (-1, 0, -2), (1, 1, 2), (0, 1, 1) \rangle_\mathbb{N}, \\
\overline{\Gamma_3} &= \langle (1, -1, 0), (0, 1, 0), (0, 0, -1), (0, -1, -2), (1, 1, 2), (1, 0, 1) \rangle_\mathbb{N}.
\end{aligned}$$

In particular, the normalization of X_Γ^* is singular. Repeating the entire algorithm for each $\overline{\Gamma_i}$, the resulting saturated semigroups can all be generated by three elements. We conclude that by iterating twice the normalized Nash blowup in characteristic two, we obtain a resolution of X_Γ.

We note also that the corresponding fans for the normalized blowups of X_Γ with respect to $\mathcal{J}_0$ and $\mathcal{J}_2$ have different rays; in particular, neither blowup factors through the other via a small morphism.

References

1. Atanasov, A., Lopez, C., Perry, A., Proudfoot, N., Thaddeus, M.: Resolving toric varieties with Nash blow-ups. Exp. Math. **20**(3), 288–303 (2011)

2. Chachapoyas, N.: Invariantes de variedades determinantais. Thesis, ICMC-USP (2014)
3. Chávez-Martínez, E., Duarte, D., Giles Flores, A.: A higher-order tangent map and a conjecture on the higher Nash blowup of curves. Math. Z. **297**, 1767–1791 (2021)
4. Cox, D., Little, J., Schenck, H.: Toric varieties, graduate studies in mathematics, vol. 124. AMS, (2011)
5. Duarte, D.: Nash modification on toric surfaces. Rev. de la Real Acad. de Cienc. Exactas Físicas y Na. Serie A Math. **108**(1), 153–171 (2014)
6. Duarte, D., Green Tripp, D.: Nash modification on toric curves. In: Greuel, G.-M., Narváez Macarro, L., Xambó-Descamps, S. (eds.) Singularities Algebraic Geometry Commutative Algebra and Related Topics, pp. 191–202. Springer Nature, Switzerland AG (2018). https://doi.org/10.1007/978-3-319-96827-8-8
7. Duarte, D., Núñez-Betancourt, L.: Nash blowups in positive characteristic. Rev. Math. Iberoam. Electron. Publ. (2021). https://doi.org/10.4171/RMI/1278
8. Gonzalez-Sprinberg, G.: Eventails en dimension 2 et transformé de Nash, pp. 1–68. Publ. de l'E.N.S, Paris (1977)
9. Gonzalez-Sprinberg, G.: Résolution de Nash des points doubles rationnels. Ann. Inst. Fourier Grenoble **32**(2), 111–178 (1982)
10. Gonzalez-Sprinberg, G.: On Nash blow-up of orbifolds. Adv. Stud. Pure Math. 56 (2009), Singularities-Niigata-Toyama 133-149 (2007)
11. González Perez, P.D., Teissier, B.: Toric geometry and the Semple-Nash modification. Rev. de la Real Acad. de Cienc. Exactas Físicas y Na. Serie A Math. **108**(1), 1–48 (2014)
12. Grayson, D., Stillman, M.: Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/
13. Grigoriev, D., Milman, P.: Nash resolution for binomial varieties as Euclidean division. A priori termination bound, polynomial complexity in essential dimension 2. Adv Math **231**, 3389–3428 (2012)
14. Hironaka, H.: On Nash blowing-up. Arithmetic and geometry II, Progr. Math., vol. 36, pp. 103-111. Birkhauser Boston, Mass, (1983)
15. Lejeune-Jalabert, M., Reguera, A.: The Denef-Loeser series for toric surfaces singularities. In: Proceedings of the International Conference on Algebraic Geometry and Singularities (Spanish) (Sevilla, 2001). Rev. Mat. Iberoamericana **19**, 581-612 (2003)
16. Nobile, A.: Some properties of the Nash blowing-up. Pac. J. Math. **60**, 297–305 (1975)
17. Oda, T.: Convex bodies and algebraic geometry, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 15. Springer-Verlag, Berlin (1988)
18. Oxley, J.: Matroid theory, Oxford graduate texts in mathematics, 2nd edn, vol. 21, (2011)
19. Rebassoo, V.: Desingularisation properties of the Nash blowing-up process. University of Washington, Thesis (1977)
20. Semple, J.G.: Some investigations in the geometry of curve and surface elements. Proc. Lond. Math. Soc. **4**(3), 24–49 (1954)
21. Spivakovsky, M.: Sandwiched singularities and desingularisation of surfaces by normalized Nash transformations. Ann. Math. **131**(3), 411–491 (1990)
22. Sturmfels, B.: Gröbner Bases and Convex Polytopes, University Lecture Series, vol. 8. AMS, Providence, RI (1996)