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Halfspace type theorems for self-shrinkers in arbitrary codimension

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Abstract
In this paper, we generalize some halfspace type theorems for self-shrinkers of codimension 1 to the case of arbitrary codimension.

Keywords Halfspace type theorem · Divergence type theorem · Self-shrinkers

Mathematics Subject Classification Primary 53C21; Secondary 35J60

1 Introduction

The halfspace theorem says that *"There is no non-planar, complete, minimal surface properly immersed in a halfspace of* $\mathbb{R}^3$*."* The theorem is due to Hoffman and Meeks. In fact they proved a stronger version, the strong halfspace theorem, *"Two disjoint complete properly immersed minimal surfaces in* $\mathbb{R}^3$ are planes" (see [14]).

The halfspace theorem is essentially a three-dimensional one. In $\mathbb{R}^n$, $n > 3$, the halfspace theorem is false because there are minimal Catenoids with bounded height.

Many generalizations of the theorem have been made by several authors, see [8, 9, 19, 20, 22, 23] and references therein.

The first halfspace theorem for self-shrinker in codimension 1 was proved in [21] based on the weighted parabolicity of self-shrinkers. A similar result in a more general setting was proved recently in [15].

Theorem 1 [Theorem 3 in [21]; Theorem 1.1 in [2]] *Let P be a hyperplane passing through the origin. The only properly immersed self-shrinker contained in one of the closed halfspace determined by P is $\Sigma = P$.*

In contrast with the case of minimal surfaces, the halfspace theorem for self-shrinkers holds true in any dimension. Moreover, one can consider a type of halfspace theorems for self-shrinker containing inside or outside a hypercylinder.

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In 2016, Cavalcante and Espinar [2] showed some halfspace type theorems for self-shrinkers of codimension 1 including Theorem 1 with a different proof.

Theorem 2 [Theorem 1.2 in [2]] *The only complete self-shrinker properly immersed in a closed cylinder* $\overline{B^{k+1}(R)} \times \mathbb{R}^{n-k} \subset \mathbb{R}^{n+1}$, *for some* $k \in \{1, 2, \dots, n\}$ *and radius* R, $R \leq \sqrt{2k}$, *is the cylinder* $S^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$.

Theorem 3 [Theorem 1.3 in [2]] *The only complete self-shrinker properly immersed in an exterior closed cylinder* $\overline{E^{k+1}(R)} \times \mathbb{R}^{n-k} \subset \mathbb{R}^{n+1}$, *for some* $k \in \{1, 2, \dots, n\}$ *and radius* R, $R \geq \sqrt{2k}$, *is the cylinder* $S^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$. *Here* $E^{k+1}(R) = R^{k+1} - \overline{B^{k+1}(R)}$.

In 2018, Vieira and Zhou [26] proved similar results, where spheres or balls center at the origin are replaced by ones with arbitrary centers and suitable radii. Recently, Impera, Pigola and Rimoldi [18] recovered Cavalcante and Espinar's results with short proofs by using potential theoretic arguments.

The aim of this paper is to generalize the above hasflpace type results to the case of arbitrary codimension. The first step in our approach is somewhat similar to the one in [18] for codimension 1 but the use of maximal principle for weighted superharmonic functions together with the weighted parabolicity of self-shrinkers is replaced by an application of a divergence type theorem (Theorem 6). In fact our proofs recovered some key formulas originally proved by Colding and Minicozzi [7] for codimension 1 self-shrinkers and extended for higher codimension by Arezzo and Sun [1].

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2 Preliminaries

In this paper, we use the following notations:

1. $B^k(a, R)$, the k-ball with center a and radius R;
2. $E^k(a, R) = \mathbb{R}^k - \overline{B^k(a, R)}$, the complement of $\overline{B^k(a, R)}$;
3. $S^k(a, R)$, the k-sphere with center a and radius R.

When the center of spheres or balls is the origin we simply write $B^k(R)$, $E^k(R)$, $S^k(R)$.

2.1 Self-shrinkers

An n-dimensional submanifold Σ immersed in $\mathbb{R}^m$, $m > n$, is called a self-shrinker, if

$$\mathbf{H} = -\frac{1}{2} X^N, \tag{1}$$

where $\mathbf{H}$ is the mean curvature vector of Σ, X is the position vector, and X^N denotes the normal part of X.

Self-shrinkers are self-similar solutions to the mean curvature flow and play an important role in the study of its singularities. For more information about self-shrinkers as well as singularities, we refer the readers to [5, 6, 16, 17].

A complete manifold Σ^n in $\mathbb{R}^m$ is said to have polynomial volume growth if there exist constants C and d such that for all $R \geq 1$, there holds

$$\mathrm{Vol}(B^m(R) \cap \Sigma) \leq C R^d. \tag{2}$$

In [10], Ding-Xin proved that:

Theorem 4 [Theorem 1.1 in [10]] *A complete non-compact properly immersed self-shrinker Σ^n in $\mathbb{R}^m$, $m > n$, has Euclidean volume growth at most, i.e.*

$$\mathrm{Vol}(B^m(R) \cap \Sigma) \leq C R^n$$

for $R \geq 1$.

This result was previously proved by Cheng and Zhou [4] in the codimension one case.

2.2 Some typical examples

It is not hard to verify all of the followings are n-dimensional complete self-shrinkers in $\mathbb{R}^m$.

1. An n-plane passing through the origin.
2. $S^n(\sqrt{2n}) \subset \mathbb{R}^{n+1}$.
3. The cylinder $S^k(\sqrt{2k}) \times \mathbb{R}^{n-k} \subset \mathbb{R}^{n+1}, 0 < k < n$.
4. $S^{n_1}(\sqrt{2n_1}) \times S^{n_2}(\sqrt{2n_2}) \times \ldots \times S^{n_k}(\sqrt{2n_k}) \subset \mathbb{R}^{n+1}$, $n_1 + n_2 + \ldots + n_k = n$.
5. $S^{n_1}(\sqrt{2n_1}) \times S^{n_2}(\sqrt{2n_2}) \times \ldots \times S^{n_k}(\sqrt{2n_k}) \times \mathbb{R}^p \subset \mathbb{R}^{n+1}$, $p \geq 1$ and $n_1 + n_2 + \ldots + n_k + p = n$.
6. n-dimensional complete minimal submanifolds of the sphere $S^{m-1}(\sqrt{2n})$ (see Theorem 4.1 in [1] or subsection 1.4 in [25]).

For some more well-known results about complete self-shrinkers, we refer the readers to [7, 17, 21] for the case of codimension 1 and [3, 24] for the case of arbitrary codimension.

2.3 Some basic formulas

In this subsection, we calculate the surface divergence of some vector fields that will be used in the proofs of the main results. The calculations are straightforward, but for the sake of completeness we present them here.

Let $e_1, e_2, \ldots, e_m$ be the coordinate vector fields for $\mathbb{R}^m$, Σ^n be a complete self-shrinker in $\mathbb{R}^m$, $\{E_1, E_2, \ldots, E_n\}$ be an orthonormal basis for $T_X\Sigma$, $X = \sum_{i=1}^m x_i e_i$ be the position vector field and $u = \sum_{i=1}^{k+1} x_i e_i$, $k \leq m-1$. We have the following lemma.

Lemma 5 *1.*

$$\mathrm{div}_\Sigma X^T = n - \frac{1}{2}|X^N|^2. \tag{3}$$

2.

$$\mathrm{div}_\Sigma e_l^T = -\frac{1}{2}\langle X, e_l^N\rangle,\ l = 1, 2, \ldots, m. \tag{4}$$

3.

$$\mathrm{div}_\Sigma x_l e_l^T = |e_l^T|^2 - \frac{1}{2}x_l\langle X, e_l^N\rangle,\ l = 1, 2, \ldots, m. \tag{5}$$

4.

$$\mathrm{div}_\Sigma u^T = (k+1) - \frac{1}{2}|u^N|^2 - \sum_{i=1}^{k+1}|e_i^N|^2. \tag{6}$$

5.

$$\operatorname{div}_\Sigma \frac{1}{|u|} u^T = \frac{1}{|u|} \left[k - \frac{1}{2}|u^N|^2 - \sum_{i=1}^{k+1} |e_i^N|^2 + \frac{|u^N|^2}{|u|^2} \right]. \tag{7}$$

Proof We use the summation convention.

1. We have

$$\operatorname{div}_\Sigma X = n,$$

and

$$\begin{aligned} \operatorname{div}_\Sigma X^N &= \langle E_i, \nabla_{E_i} X^N \rangle = \nabla_{E_i} \langle E_i, X^N \rangle - \langle \nabla_{E_i} E_i, X^N \rangle \\ &= \nabla_{E_i}(0) - \langle (\nabla_{E_i} E_i)^N, X \rangle = -\langle \mathbf{H}, X \rangle = \frac{1}{2}|X^N|^2. \end{aligned}$$

Therefore,

$$\operatorname{div}_\Sigma X^T = n - \frac{1}{2}|X^N|^2.$$

2.

$$\begin{aligned} \operatorname{div}_\Sigma e_l^T &= \operatorname{div}_\Sigma e_l - \operatorname{div}_\Sigma e_l^N = 0 - \langle E_i, \nabla_{E_i} e_l^N \rangle \\ &= \langle \nabla_{E_i} E_i, e_l^N \rangle = \langle (\nabla_{E_i} E_i)^N, e_l \rangle = \langle \mathbf{H}, e_l \rangle \\ &= -\frac{1}{2} \langle X, e_l^N \rangle. \end{aligned}$$

3.

$$\begin{aligned} \operatorname{div}_\Sigma x_l e_l^T &= \operatorname{div}_\Sigma x_l e_l - \operatorname{div}_\Sigma x_l e_l^N = |e_l^T|^2 - \langle E_i, \nabla_{E_i} x_l e_k^N \rangle \\ &= |e_l^T|^2 + \langle (\nabla_{E_i} E_i)^N, x_l e_l \rangle = |e_l^T|^2 + \langle \mathbf{H}, x_l e_l \rangle \\ &= |e_l^T|^2 - \frac{1}{2} x_l \langle X, e_l^N \rangle. \end{aligned}$$

4. For $v \in T_p\Sigma$,

$$\nabla_v u = \pi_1(v) = \langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2 + \ldots + \langle v, e_{k+1} \rangle e_{k+1}.$$

We have

$$\begin{aligned} \operatorname{div}_\Sigma(u) &= \langle E_i, \nabla_{E_i} u \rangle = \sum_{j=1}^{k+1} \sum_{i=1}^{n} \langle E_i, e_j \rangle^2 \\ &= \sum_{j=1}^{k+1} |e_j^T|^2 = (k+1) - \sum_{j=1}^{k+1} |e_j^N|^2, \end{aligned}$$

and

$$\begin{aligned} \operatorname{div}_\Sigma u^N &= \langle E_i, \nabla_{E_i} u^N \rangle = \nabla_{E_i} \langle E_i, u^N \rangle - \langle \nabla_{E_i} E_i, u^N \rangle \\ &= \nabla_{E_i}(0) - \langle (\nabla_{E_i} E_i)^N, u \rangle = -\langle \mathbf{H}, u \rangle = \frac{1}{2}|u^N|^2. \end{aligned}$$

Therefore,

$$\mathrm{div}_\Sigma\, u^T = (k+1) - \frac{1}{2}|u^N|^2 - \sum_{i=1}^{k+1} |e_i^N|^2.$$

5.

$$\begin{aligned}\mathrm{div}_\Sigma\, \frac{1}{|u|}u^T &= \left\langle \nabla_\Sigma \frac{1}{|u|}, u^T \right\rangle + \frac{1}{|u|}\,\mathrm{div}\, u^T \\ &= -\frac{|u^T|^2}{|u|^3} + \frac{1}{|u|}[(k+1) - \frac{1}{2}|u^N|^2 - \sum_{i=1}^{k+1}|e_i^N|^2] \\ &= \frac{1}{|u|}\left[k - \frac{1}{2}|u^N|^2 - \sum_{i=1}^{k+1}|e_i^N|^2 + \frac{|u^N|^2}{|u|^2}\right].\end{aligned}$$

□

3 A divergence type theorem

In this section, Σ is assumed to be an n-dimensional complete (without boundary) self-shrinker properly immersed in $\mathbb{R}^m$, $m > n$.

The condition of polynomial volume growth is essential for using an integral formula that is similar to the generalized divergence theorem for compact manifolds. We have the following theorem.

Theorem 6 *Let F be a smooth tangent vector field on Σ. Assume that there exist positive constants C and d such that $|\,\mathrm{div}_\Sigma\, F(X)| \le C|X|^d$. Then*

$$\int_\Sigma \mathrm{div}_\Sigma (e^{-\frac{|X|^2}{4}} F(X)) dV = 0. \tag{8}$$

Proof Suppose that Σ is inside a ball, since it is proper it must be compact and the theorem holds true by divergence theorem. Now suppose that Σ is not inside any ball, i.e. $\partial(B_R \cap \Sigma) \neq \emptyset$ when R is large enough. Since F is tangent to Σ, the generalized divergence theorem for $e^{-\frac{|X|^2}{4}} F$ yields

$$\int_{B_R\cap\Sigma} \mathrm{div}_\Sigma (e^{-\frac{|X|^2}{4}} F(X)) dV = e^{-\frac{R^2}{4}} \int_{\partial(B_R\cap\Sigma)} \langle F(X), \nu\rangle\, dA.$$

Taking the limit when $R \to \infty$, the theorem is proved because

$$\begin{aligned}\lim_{R\to\infty} e^{-\frac{R^2}{4}} \left| \int_{\partial(B_R\cap\Sigma)} \langle F(X), \nu\rangle\, dA \right| &= \lim_{R\to\infty} e^{-\frac{R^2}{4}} \left| \int_{B_R\cap\Sigma} \mathrm{div}_\Sigma\, F(X) dV \right| \\ &\le \lim_{R\to\infty} e^{-\frac{R^2}{4}} C R^d \int_{B_R\cap\Sigma} dV \\ &\le \lim_{R\to\infty} e^{-\frac{R^2}{4}} C_1 C R^{d+n} = 0.\end{aligned}$$

The first and the second inequalities hold true because of the assumption and Theorem 4, respectively. □

Applying Theorem 6 with suitable choices of tangent vector fields F, we obtain the main results of the paper.

3.1 Halfspace type result w.r.t. hyperplanes

The following theorem says that Σ intersects every hyperplane passing through the origin.

Theorem 7 *Let P be a hyperplane passing through the origin. If Σ lies in a closed halfspace determined by P, then $\Sigma \subset P$.*

Proof Without loss of generality, we can suppose that P is the hyperplane $x_m = 0$ and Σ is in the closed halfspace $\{(x_1, x_2, \ldots, x_m) : x_m \geq 0\}$.

By (4),

$$\begin{aligned}\mathrm{div}_\Sigma(e^{-\frac{|X|^2}{4}} e_m^T) &= e^{-\frac{|X|^2}{4}} \mathrm{div}_\Sigma\, e_m^T - e^{-\frac{|X|^2}{4}} \frac{1}{2}\langle X, e_m^T\rangle \\ &= -\frac{1}{2} e^{-\frac{|X|^2}{4}} \left[\langle X, e_m^N\rangle + \langle X, e_m^T\rangle\right] \\ &= -\frac{1}{2} e^{-\frac{|X|^2}{4}} x_m.\end{aligned}$$

Then Theorem 6 applying for $F = e_m^T$ yields (see [7] for the case of codimension 1, also see [1])

$$\int_\Sigma e^{-\frac{|X|^2}{4}} x_m dV = 0. \tag{9}$$

Therefore, $x_m = 0$, i.e. $\Sigma \subset P$.

□

Remark 8 If $n = m - 1$, then $\Sigma = P$ ([21], Theorem 3; [2], Theorem 1.1).

Corollary 9 *If there exist $m - n$ orthonormal vectors $v_1, v_2, \ldots, v_{m-n}$ such that for $i = 1, 2, \ldots, m - n$, $\langle X, v_i\rangle$ does not change sign, then Σ is an n-plane passing through the origin.*

Proof Without loss of generality, we can assume that $v_i = e_{n+i}$ if $\langle X, v_i\rangle \geq 0$ and $v_i = -e_{n+i}$ if $\langle X, v_i\rangle \leq 0$. The assumption guarantees that Σ is in the closed halfspace $\{(x_1, x_2, \ldots, x_m) : x_{n+i} \geq 0, i = 1, 2, \ldots, m - n\}$. The proof is then followed by applying Theorem 7 in turn for $v_1, v_2, \ldots, v_{m-n}$. □

Based on the Bernstein result for self-shrinkers of codimension 1, "*An entire graphic self-shrinker must be a hyperplane passing through the origin*" (see [11, 13, 27]), and with the same argument as in the proof of Corollary 9, we have the following.

Corollary 10 [A Bernstein type theorem] *Let $F : \mathbb{R}^n \to \mathbb{R}^{m-n}$, $F(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_{m-n}(\mathbf{x}))$ be a smooth function and $\Sigma = \{(\mathbf{x}, F(\mathbf{x})) : \mathbf{x} \in \mathbb{R}^n\}$ be its graph. If there exist at least $(m - n - 1)$ functions f_i that do not change sign, then Σ is an n-plane passing through the origin.*

3.2 Self-shrinkers inside or outside a ball

The following theorem says that a complete properly immersed self-shrinker Σ^n and $S^{m-1}(\sqrt{2n})$ must be intersected.

Theorem 11 *If $\Sigma \subset \overline{E^m(\sqrt{2n})}$ or $\Sigma \subset \overline{B^m(\sqrt{2n})}$, then Σ is compact and $\Sigma \subset S^{m-1}(\sqrt{2n})$, i.e. Σ is a minimal submanifold of $S^{m-1}(\sqrt{2n})$. Moreover, if $n = m-1$, then $\Sigma = S^n(\sqrt{2n})$.*

Proof By (3),

$$\begin{aligned}\operatorname{div}_\Sigma(e^{-\frac{|X|^2}{4}}X^T) &= e^{-\frac{|X|^2}{4}}\operatorname{div}_\Sigma X^T - e^{-\frac{|X|^2}{4}}\left\langle\frac{1}{2}X, X^T\right\rangle\\ &= e^{-\frac{|X|^2}{4}}\left(n-\frac{1}{2}|X^N|^2\right) - e^{-\frac{|X|^2}{4}}\frac{1}{2}|X^T|^2\\ &= e^{-\frac{|X|^2}{4}}\left(n-\frac{1}{2}|X|^2\right).\end{aligned}$$

Applying Theorem 6 with $F = X^T$ (see [7] for the case of codimension 1, also see [1]),

$$\int_\Sigma e^{-\frac{|X|^2}{4}}\left(n-\frac{1}{2}|X|^2\right)dV = 0. \tag{10}$$

If $\Sigma \subset \overline{E^m(\sqrt{2n})}$ ($\Sigma \subset \overline{B^m(\sqrt{2n})}$), then $2n - |X|^2 \le 0$ ($2n - |X|^2 \ge 0$). By (10), it follows that $2n - |X|^2 = 0$, i.e. $\Sigma \subset S^{m-1}(\sqrt{2n})$. Since Σ is proper, it must be compact.

The case of $n = m-1$ is obvious. □

The following theorem can be seen as an arbitrary codimension version of Theorem 1 in [26]. Here the proof is also applied to the case of self-shrinkers that are outside of spheres.

Theorem 12
1. *Any complete self-shrinker Σ^n properly immersed in $\mathbb{R}^m$, $m > n$, intersects all members of the collection $\mathcal{C}$ given by*

$$\mathcal{C} := \{S^{m-1}(a, \sqrt{2n+|a|^2}) : a \text{ is a vector in } \mathbb{R}^m\}.$$

2. *If the Σ lies in $\overline{B^m(a, \sqrt{2n+|a|^2})}$ or in $\mathbb{R}^m - B^m(a, \sqrt{2n+|a|^2})$ then $\Sigma \subset S^{m-1}(a, \sqrt{2n+|a|^2})$. Moreover, if $n = m-1$, then Σ is the sphere $S^n(\sqrt{2n})$.*

Proof From (9), it follows that

$$\int_\Sigma e^{-\frac{|X|^2}{4}}\langle X, a\rangle dV = 0. \tag{11}$$

Therefore, (10) and (11) yields

$$\int_\Sigma e^{-\frac{|X|^2}{4}}(|X-a|^2 - (2n+|a|^2))dV = 0. \tag{12}$$

The theorem is proved easily by some arguments as in the proof of Theorem 11. Note that, for codimension 1 case, the sphere $S^n(a, \sqrt{2n+|a|^2})$ is a self-shrinker if and only if $a = 0$. □

Remark 13 Theorem 5.1 in [12] shows another version of Theorem 11, where self-shrinkers are assumed to be parabolic instead of proper. And a different proof of Theorem 11, stated in terms of λ-self-shrinkers, was also done in [12] (Theorem 6.3).

3.3 Halfspace type results w. r. t. cylinders

Theorem 14 *[Self-shrinker inside a hypercylinder] Let* $k \in \{m-n, m-n+1, \ldots, m-2\}$, $p = m-k-1$ *and* $R = \sqrt{2(n-p)}$. *If* Σ *is inside the closed cylinder* $\overline{B^{k+1}(R)} \times \mathbb{R}^p$, *then* $\Sigma \subset S^k(R) \times \mathbb{R}^p$.

Proof By (5)

$$\begin{aligned}\operatorname{div}_\Sigma\left(e^{-\frac{|X|^2}{4}} x_i e_i^T\right) &= e^{-\frac{|X|^2}{4}}\left[\operatorname{div}_\Sigma\left(x_i e_i^T\right) - \frac{1}{2}\left\langle X, x_i e_i^T\right\rangle\right] \\ &= e^{-\frac{|X|^2}{4}}\left[|e_i^T|^2 - \frac{1}{2}x_i\left\langle X, e_i^N\right\rangle - \frac{1}{2}x_i\langle X, e_i^T\rangle\right] \\ &= e^{-\frac{|X|^2}{4}}\left[|e_i^T|^2 - \frac{1}{2}x_i^2\right].\end{aligned}$$

Applying Theorem 6 with $F = x_i e_i^T$, we have (see [7] for the case of codimension 1, also see [1])

$$\int_\Sigma e^{-\frac{|X|^2}{4}} x_i^2 dV = 2\int_\Sigma e^{-\frac{|X|^2}{4}} |e_i^T|^2 dV. \tag{13}$$

Let $\{e_1, e_2, \ldots, e_m\}$ be the standard basis in $\mathbb{R}^m$, where $\{e_1, e_2, \ldots, e_{k+1}\} \subset \mathbb{R}^{k+1}$ and $\{e_{k+2}, e_{k+3}, \ldots, e_m\} \subset \mathbb{R}^p$. Denote $X = (u, v)$, where $u \in \mathbb{R}^{k+1}$, $v \in \mathbb{R}^p$.

By (10) and (13), we get

$$\begin{aligned}\int_\Sigma e^{-\frac{|X|^2}{4}}\left[|X|^2 - 2n - \sum_{i=k+2}^m x_i^2\right] dV &= \int_\Sigma e^{-\frac{|X|^2}{4}}[|u|^2 - 2n] dV \\ &= -2\int_\Sigma e^{-\frac{|X|^2}{4}} \sum_{i=k+2}^m |e_i^T|^2 dV.\end{aligned}$$

Since $|e_i^T|^2 = 1 - |e_i^N|^2$, it follows that

$$\int_\Sigma e^{-\frac{|X|^2}{4}}\left[|u|^2 - R^2\right] dV = 2\int_\Sigma e^{-\frac{|X|^2}{4}} \sum_{i=k+2}^m |e_i^N|^2 dV \geq 0.$$

The assumption that Σ is inside the closed cylinder $\overline{B^{k+1}(R)} \times \mathbb{R}^p$, means

$$|u|^2 - R^2 \leq 0.$$

Therefore,

$$|u|^2 - R^2 = 0,$$

i.e. $\Sigma \subset S^k(R) \times \mathbb{R}^p$. □

Remark 15 1. We see in the above proof that $e_i^N = 0$, i.e. $e_i = e_i^T$, $i = k+2, \ldots, m$. Therefore, $\Sigma = \Gamma \times \mathbb{R}^p$, where $\Gamma \subset S^k$ is an $(n-p)$-dimensional self-shrinker, i.e. an $(n-p)$-dimensional minimal submanifold of S^k.
2. If $n = m-1$, then $\Sigma = S^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$ ([2], Theorem 1.2).

Theorem 16 [Self-shrinker outside a hypercylinder] *Let* $k \in \{1, 2, ..., n\}$. *If* Σ *is contained in* $\overline{E^{k+1}(\sqrt{2k})} \times \mathbb{R}^{m-k-1}$, *then* $\Sigma \subset S^k(\sqrt{2k}) \times \mathbb{R}^{m-k-1}$.

Proof Let $u = \sum_{i=1}^{k+1} x_i e_i$. By (7)

$$\mathrm{div}_\Sigma \left(e^{-\frac{|X|^2}{4}} \frac{1}{|u|} u^T \right) = \left[e^{-\frac{|X|^2}{4}} \mathrm{div}_\Sigma \left(\frac{1}{|u|} u^T \right) - \frac{1}{2} \left\langle X, \frac{u^T}{|u|} \right\rangle \right]$$
$$= e^{-\frac{|X|^2}{4}} \frac{1}{|u|} \left[k - \frac{1}{2}|u|^2 - \sum_{i=1}^{k+1} |e_i^N|^2 + \frac{|u^N|^2}{|u|^2} \right].$$

It is not hard to check that

$$\sum_{i=1}^{k+1} |e_i^N|^2 \geq \frac{|u^N|^2}{|u|^2}.$$

Indeed, we have

$$|u^N|^2 = \left| \sum_{i=1}^{k+1} x_i e_i^N \right|^2 = \sum_{i=1}^{k+1} x_i^2 |e_i^N|^2 + 2 \sum_{i \neq j} x_i x_j \langle e_i^N, e_j^N \rangle$$
$$\leq \sum_{i=1}^{k+1} x_i^2 |e_i^N|^2 + \sum_{i \neq j} x_i^2 |e_j^N|^2$$
$$\leq \left(\sum_{i=1}^{k+1} x_i^2 \right) \left(\sum_{i=1}^{k+1} |e_i^N|^2 \right) = |u|^2 \left(\sum_{i=1}^{k+1} |e_i^N|^2 \right).$$

Applying Theorem 6 with $F = \frac{1}{|u|} u^T$,

$$\int_\Sigma e^{-\frac{|X|^2}{4}} \frac{1}{|u|} \left(2k - |u|^2 \right) dV \geq 0. \tag{14}$$

But the assumption that Σ is in $\overline{E^{k+1}(\sqrt{2k})} \times \mathbb{R}^{m-k-1}$ means

$$|u|^2 - 2k \geq 0$$

Therefore, $|u|^2 - 2k = 0$, i.e. $\Sigma \subset S^k(\sqrt{2k}) \times \mathbb{R}^{m-k-1}$. □

Remark 17 If $n = m - 1$, then $\Sigma = S^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$ ([2], Theorem 1.3).

With the same arguments as in the proof of Theorem 12, we have the following theorem (see Colollary 1, [26] for the case of codimension 1).

Theorem 18 1. *If the self-shrinker Σ^n lies inside the closed cylinder*

$$\overline{B^{k+1}(a, \sqrt{2(n-p) + |a|^2})} \times \mathbb{R}^p,$$

where $a \in \mathbb{R}^{k+1}$, then $\Sigma \subset S^k(a, \sqrt{2(n-p) + |a|^2}) \times \mathbb{R}^p$. Moreover, if $n = m - 1$, then $\Sigma = S^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$.

2. *The self-shrinker cannot lie outside the closed cylinder*

$$\overline{B^{k+1}(a, \sqrt{2(k+1) + |a|^2})} \times \mathbb{R}^p,$$

for any vector a in $\mathbb{R}^{k+1}$.

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References

1. Arezzo, C., Sun, J.: Self-shrinkers for the mean curvature flow in arbitrary codimension. Math. Z. **274**(3–4), 993–1027 (2013)
2. Cavalcante, M.P., Espinar, J.M.: Halfspace type theorems for self-shrinkers. Bull. Lond. Math. Soc. **48**(2), 242–250 (2016)
3. Cheng, Q.-M., Peng, Y.: Complete self-shrinkers of the mean curvature flow. Calc. Var. Partial Differ. Equ. **52**(3–4), 497–506 (2015)
4. Cheng, X., Zhou, D.: Volume estimate about shrinkers. Proc. Am. Math. Soc. **141**(2), 687–696 (2013)
5. Colding, T.H., Minicozzi, W.P., II.: A Course in Minimal Surfaces, Graduate Studies in Mathematics, vol. 121. American Mathematical Society, Providence (2011)
6. Colding, T.H., Minicozzi, W.P., II.: Smooth compactness of self-shrinkers. Comment. Math. Helv. **87**(2), 463–475 (2012)
7. Colding, T.H., Minicozzi, W.P.: Generic mean curvature flow I generic singularities. Ann. Math. **175**(2), 755–833 (2012)
8. Daniel, B., Hauswirth, L.: Half-space theorem, embedded minimal annuli and minimal graphs in the Heisenberg group. Proc. Lond. Math. Soc. **98**(2), 445–470 (2009)
9. Daniel, B., Meeks, W.H., III., Rosenberg, H.: Half-space theorems for minimal surfaces in Nil_3 and Sol_3,. J. Differ. Geom. **88**(1), 41–59 (2011)
10. Ding, Q., Xin, Y.L.: Volume growth, eigenvalue and compactness for self-shrinkers. Asian J. Math. **17**(3), 443–456 (2013)
11. Ecker, K., Huisken, G.: Mean curvature evolution of entire graphs. Ann. Math. **130**, 453–471 (1989)
12. Gimeno, V., Palmer, V.: Parabolicity, Brownian exit time and properness of solitons of the direct and inverse mean curvature flow. J. Geom. Anal. **31**(1), 579–618 (2021)
13. Hieu, D.T.: A weighted volume estimate and its application to Bernstein type theorems in Gauss space Colloq. Math. **159**(1), 25–28 (2020)
14. Hoffman, D., Meeks, W.H., III.: The strong halfspace theorem for minimal surfaces. Invent. Math. **101**(2), 373–377 (1990)
15. Hurtado, A., Palmer, V., Rosales, C.: Parabolicity criteria and characterization results for submanifolds of bounded mean curvature in model manifolds with weights. Nonlinear Anal. **192**, 111681 (2020)
16. Huisken, G.: Asymptotic behavior for singularities of the mean curvature flow. J. Differ. Geom. **31**(1), 285–299 (1990)
17. Huisken, G.: Local and global behaviour of hypersurfaces moving by mean curvature, In: Greene, R., Yau, S.T. (eds.), Differential Geometry: Partial Differential Equations on Manifolds, Los Angeles, Proceedings of Symposium on Pure Mathematics, vol. 54, Amer. Math. Soc, Providence (1990)
18. Impera, D., Pigola, S., Rimoldi, M.: The Frankel property for self-shrinkers from the viewpoint of elliptic PDE's. J. Reine Angew. Math. **773**, 1–20 (2021)
19. Mazet, L.: A general halfspace theorem for constant mean curvature surfaces. Am. J. Math. **135**, 801–834 (2013)
20. Nelli, B., Sa Earp, R.: A halfspace theorem for mean curvature $H = 1/2$ surfaces in $H^2 \times \mathbb{R}$. J. Math. Anal. Appl. **365**(1), 167–170 (2010)
21. Pigola, S., Rimoldi, M.: Complete self-shrinkers confined into some regions of the space. Ann. Global Anal. Geom. **45**, 47–65 (2014)
22. Rodriguez, L., Rosenberg, H.: Half-space theorems for mean curvature one surfaces in hyperbolic space. Proc. Am. Math. Soc. **126**(9), 2755–2762 (1998)
23. Rosenberg, H., Schultze, F., Spruck, J.: The halfspace property and entire minimal graphs in $M \times \mathbb{R}$,. J. Differ. Geom. **95**, 321–336 (2013)
24. Smoczyk, K.: Self-shrinkers of the mean curvature flow in arbitrary codimension. Int. Math. Res. Not. **48**, 2983–3004 (2005)
25. Xin, Y.: Minimal Submanifolds and Related Topics, [MR2035469], 2nd edn. Nankai Tracts in Mathematics, vol. 16, World Scientific Publishing Co. Pte. Ltd., Hackensack (2019)
26. Vieira, M., Zhou, D.: Geometric properties of self-shrinkers in cylinder shrinking Ricci solitons. J. Geom. Anal. **28**(1), 170–189 (2018)
27. Wang, L.: A Bernstein type theorem for self-similar shrinkers. Geom. Dedicata **151**, 297–303 (2011)