Collectanea Mathematica (2024) 75:409–423
https://doi.org/10.1007/s13348-023-00392-x

# The normalized depth function of squarefree powers

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Received: 27 September 2022 / Accepted: 8 January 2023 / Published online: 26 January 2023


## Abstract
The depth of squarefree powers of a squarefree monomial ideal is introduced. Let $I$ be a squarefree monomial ideal of the polynomial ring $S = K[x_1, \dots, x_n]$. The $k$-th squarefree power $I^{[k]}$ of $I$ is the ideal of $S$ generated by those squarefree monomials $u_1 \cdots u_k$ with each $u_i \in G(I)$, where $G(I)$ is the unique minimal system of monomial generators of $I$. Let $d_k$ denote the minimum degree of monomials belonging to $G(I^{[k]})$. One has $\operatorname{depth}(S/I^{[k]}) \geq d_k - 1$. Setting $g_I(k) = \operatorname{depth}(S/I^{[k]}) - (d_k - 1)$, one calls $g_I(k)$ the normalized depth function of $I$. The computational experience strongly invites us to propose the conjecture that the normalized depth function is nonincreasing. In the present paper, especially the normalized depth function of the edge ideal of a finite simple graph is deeply studied.

**Keywords** Normalized depth function · Squarefree powers · Matchings · Edge ideals

**Mathematics Subject Classification** 13C15 · 05E40 · 05C70

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## 1 Introduction

Let $K$ be a field, and let $S = K[x_1, \dots, x_n]$ be the polynomial ring in $n$ indeterminates over $K$. The depth function of a homogeneous ideal $I$ is the integer valued function $f_I(k) = \operatorname{depth}(S/I^k)$. While it is known by Brodmann [3] that $f_I(k)$ is constant for all $k \gg 0$, the initial behaviour of the depth function is not so easy to understand. In [10] it was conjectured that any bounded convergent function $\mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ could be the depth function of a suitable ideal. This conjecture has been proved several years later by H.T. Hà, H. Nguyen, N. Trung and T. Trung in [9, Theorem 4.1].

For a longer time it was expected that the depth function of a squarefree monomial ideal is nonincreasing. Francisco, Hà and Van Tuyl [7] showed that this expected behaviour for the powers of unmixed height 2 squarefree monomial ideals would be a consequence of a combinatorial statement which says that for every positive integer $k$ and every $k$-critical (i.e., critically $k$-chromatic) graph, there is a set of vertices whose replication produces a $(k+1)$-critical graph. However in 2014, Kaiser, Stehlík and Škrekovski [16] gave a counterexample to this and constructed an example of a squarefree monomial ideal $I \subset S$ with $\operatorname{depth}(S/I^3) = 0$ but $\operatorname{depth}(S/I^4) = 4$. It is still open whether the depth function of the edge ideal of a graph is nonincreasing.

In the present paper, we study squarefree monomial ideals and their squarefree powers. Several algebraic properties of such powers have been studied in [2, 6] and [5]. Let $I \subset S$ be a squarefree monomial ideal. The uniquely determined minimal set of generators of $I$ is denoted by $G(I)$. We denote by $I^{[k]}$ the $k$th *squarefree power* of $I$. The generators of $I^{[k]}$ are the products $u_1 \cdots u_k$ with $u_i \in G(I)$, which form a squarefree monomial. Thus $u_1 \cdots u_k \in G(I^{[k]})$ if and only if $u_1, \dots, u_k$ is a regular sequence.

A case of special interest is the squarefree powers of the edge ideal of a graph. Let $G$ be a finite simple graph on $[n] = \{1, \dots, n\}$, and let as before $S = K[x_1, \dots, x_n]$ be the polynomial ring in $n$ variables over a field $K$. The *edge ideal* of $G$ is the squarefree monomial ideal $I(G)$ of $S$, which is generated by those $x_i x_j$ for which $\{i, j\}$ is an edge of $G$. Now, given an integer $k > 0$, the $k$th squarefree power of $I(G)$ is the squarefree monomial ideal $I(G)^{[k]}$ of $S$ which is generated by the squarefree monomials

$$x_{i_1} x_{j_1} x_{i_2} x_{j_2} \cdots x_{i_k} x_{j_k},$$

where each $\{i_q, j_q\}$ is an edge of $G$ and where $\{i_q, j_q\} \cap \{i_r, j_r\} = \emptyset$ for $q \neq r$. It follows that the minimal set of generators of $I(G)^{[k]}$ are in bijection to the vertex sets of $k$-matchings of $G$. Recall that a set $M$ of edges of $G$ is called a *matching*, if no distinct two edges of $G$ have a common vertex. The matching $M$ is called a *$k$-matching*, if $|M| = k$. The set of all matchings forms a simplicial complex, the so-called matching complex of $G$. The *matching number* $\nu(G)$ of $G$ is the maximum cardinality of a matching of $G$. We have $I(G)^{[k]} \neq 0$ if and only if $k > \nu(G)$.

Let again $I \subset S$ be an arbitrary squarefree monomial ideal. We denote by $\nu(I)$ the maximum length of a monomial regular sequence in $I$. Thus if $G$ is a graph, then $\nu(I(G)) = \nu(G)$. We are interested in the depth of the squarefree powers of $I$. Let $d_k = \min\{\deg u \, u \in G(I^{[k]})\}$. We also set $d = d_1$. Note that $d_{k+1} \geq d_k + d \geq (k+1)d$ for all $k < \nu(I)$. Our first result is Proposition 1.1, where it is shown that $\operatorname{depth}(S/I^{[k]}) \geq d_k - 1$ for $k = 1, \dots, \nu(I)$. In particular, in the important special case that $I$ is generated in the single degree $d$, we have $\operatorname{depth}(S/I^{[k]}) \geq dk - 1$ for $k = 1, \dots, \nu(I)$. Thus, in contrast to ordinary powers, the depth function of squarefree powers tends to be nondecreasing. The picture changes, if we consider the function

$$g_I(k) = \text{depth}(S/I^{[k]}) - (d_k - 1)$$

for $k = 1, \dots, \nu(I)$. We call $g_I(k)$ the *normalized depth function* of $I$. In all our results and the examples we considered, $g_I(k)$ is a nonincreasing function. This fact and also Corollary 3.5 leads us to the following

**Conjecture** *Let $I$ be a squarefree monomial ideal. Then $g_I(k)$ is a nonincreasing function.*

We say that the squarefree powers of $I$ have *minimum depth* if $g_I(k) = 0$ for all $k = 1, \dots, \nu(I)$. In Sect. 2, we give examples of ideals whose squarefree powers have minimum depth. Among them are the edge ideals of complete graphs and complete bipartite graphs. More generally, any squarefree Veronese ideal as well as any matroidal ideal has minimum depth, see Examples 1.2 and Theorem 1.6.

In the following two sections, we focus on edge ideals and give criteria for minimum depth. One of the main results of this paper is

**Corollary** *Let $G$ be a graph with no isolated vertices and matching number $\nu(G)$. Then the following statements are equivalent*:

(1) *$G^c$ is disconnected.*
(2) $g_{I(G)}(1) = 0$.
(3) $g_{I(G)}(k) = 0$ *for all* $1 \leq k \leq \nu(G)$.

For the proof of this result, the concept of well-ordered facet covers, due to Erey and Faridi [4], is used to give a non-vanishingness condition for Betti numbers of squarefree powers.

A subset $D$ of vertices of a graph $G$ is called a *dominating set* if every vertex of $G$ which is not in $D$ is adjacent to some vertex in $D$. A complete subgraph $K_m$ of $G$ is called a *dominating clique* if $V(K_m)$ is a dominating set. This notion of dominating cliques provides a sufficient condition for having minimum depth for a given squarefree power $k$. Indeed, we have

**Theorem** *Let $G$ be a graph and let $2 \leq k \leq \nu(G)$. If $G$ has a dominating clique $K_{2k-1}$, then $g_{I(G)}(k) = 0$.*

We call a $k$-matching $M$ a *dominating $k$-matching* if $V(M)$ is a dominating set. For edge ideals with the property that $I^{[k]}$ has linear quotients, we have a criterion for minimum depth, see Proposition 3.3. As corollaries we obtain

**Corollary** *Let $G$ be a graph with no isolated vertices and $k$ be an integer with $1 \leq k \leq \nu(G)$ where $I(G)^{[k]}$ has linear quotients. If $g_{I(G)}(k) = 0$, then $G$ has a dominating $k$-matching. In particular, the statement holds for any cochordal graph $G$ with no isolated vertices.*

**Corollary** *Let $G$ be a graph with no isolated vertices. Then $g_{I(G)}(\nu(G)) = 0$.*

As a final application for our minimum depth criterion, we discuss the depth of squarefree powers of edge ideals of *multiple whiskered complete graphs* which are obtained by attaching at least one whisker to each vertex of a complete graph $K_s$ with $s \geq 2$. We denote such a graph by $G = H(a_1, \dots, a_s)$ where $a_i \geq 0$ is the number of whiskers attached to the vertex $i$ of $K_s$. For this family of graphs minimum depth is achieved in the second half of the interval $[1, \nu(G)]$. The precise result is the following.

**Theorem** *Let $G = H(a_1, \dots, a_s)$ with $a_i \geq 1$ for all $i = 1, \dots, s$ and let $k = 1, \dots, \nu(G)$. Then we have*:

(a) $\nu(G) = s$;
(b) *G is cochordal*;
(c) *The following statements are equivalent*:

  (1) $g_{I(G)}(k) = 0$.
  (2) $\lfloor s/2 \rfloor + 1 \leq k \leq s$.

Without any doubt, one of the most challenging open questions is to find all possible normalized depth functions. Our conjecture implies that if $g_I(k) = 0$, then $g_I(k + 1) = 0$ for all $k < \nu(I)$. Therefore, as a partial answer to the above question, it would be nice to solve the following

**Problem** For given integers $1 \leq s < m$, find a finite simple graph $G$ with $\nu(G) = m$ and

(1) $g_{I(G)}(k) > 0$ for $k = 1, \dots, s$;
(2) $g_{I(G)}(k) = 0$ for $k = s + 1, \dots, m$.

Throughout the paper, unless otherwise stated, $S = K[x_1, \dots, x_n]$ is the polynomial ring in $n$ variables over a field $K$, all graphs and simplicial complexes have $n$ vertices corresponding to the $n$ variables of $S$.

## 2 A lower bound for the depth of squarefree powers

In this section, we consider squarefree powers of any squarefree monomial ideal and provide a lower bound for their depth. Besides discussing several examples, we also show that the normalized depth function of a matroidal ideal is zero.

For the lower bound of the depth of squarefree powers we have the following result.

**Proposition 1.1** *Let $I \subset S$ be a squarefree monomial ideal. Then*

(a) $I^{[k]} = 0$ *if and only if* $k > \nu(I)$;
(b) $g_I(k) \geq 0$ *for all* $k = 1, \dots, \nu(I)$.

***Proof*** (a) Let $m = \nu(I)$ and $v_1, \dots, v_m$ be a maximal regular sequence of monomials in $I$. Then for each $i$, there exists $u_i \in G(I)$ which divides $v_i$, and hence $u_1, \dots, u_m$ is again a maximal regular sequence of monomials in $I$. In particular, $\gcd(u_i, u_j) = 1$ for all $i \neq j$. It follows that $u_1 \cdots u_m$ is squarefree. This implies that $I^{[k]} \neq 0$ for any $k \leq \nu(I)$. Any product of generators with more than $m$ many factors cannot be squarefree, since these factors cannot form a regular sequence. This shows that $I^{[k]} = 0$ for $k > m$.

(b) Let $k = 1, \dots, \nu(I)$. By using the Auslander-Buchsbaum formula, it suffices to show that $\operatorname{proj\,dim}(S/I^{[k]}) \le n - d_k + 1$ where $d_k = \min\{\deg u\, u \in G(I^{[k]})\}$. We observe that for any $i > 0$ for which $\beta_{i,j}(S/I^{[k]}) \ne 0$, we have $j \ge d_k + i - 1$.

It follows from Hochster's formula that $n \ge j$ for any $j$ such that $\beta_{i,j}(S/I^{[k]}) \ne 0$. Therefore, $n \ge d_k + i - 1$, as desired. □

Let $I$ be a squarefree monomial ideal, and let $k$ be an integer with $1 \le k \le \nu(I)$. We say that $S/I^{[k]}$ has *minimum depth* (or simply say that $I^{[k]}$ has *minimum depth* when the polynomial ring is clear from the context) if $g_I(k) = 0$.

***Example 1.2*** (a) Let $\mathfrak{m}$ be the graded maximal ideal of $S$, and let $I = \mathfrak{m}^{[d]}$ for some $d \le n$. By [10, Corollary 3.4], we have $\operatorname{depth}(S/I) = d - 1$. Since $I^{[k]} = \mathfrak{m}^{[kd]}$, it follows that $\operatorname{depth}(S/I^{[k]}) = dk - 1$ for $dk \le n$.

(b) Consider the polynomial rings $S_1 = K[x_1, \dots, x_n]$ and $S_2 = K[y_1, \dots, y_m]$, and let $I \subset S_1$ and $J \subset S_2$ be graded ideals. Moreover, let $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$. Then [15, Corollary 3.2] implies that $\operatorname{depth}(S/IJ) = \operatorname{depth}(S_1/I) + \operatorname{depth}(S_2/J) + 1$.

In the given situation, assume that $I$ is a monomial ideal generated in degree $d_1$ and $J$ is a monomial ideal generated in degree $d_2$. Then $IJ$ is generated in degree $d = d_1 + d_2$. We have $(IJ)^{[k]} = I^{[k]}J^{[k]}$, since $I$ and $J$ are ideals in different sets of variables. Therefore, $\operatorname{depth}(S/(IJ)^{[k]}) = \operatorname{depth}(S_1/I^{[k]}) + \operatorname{depth}(S_2/J^{[k]}) + 1$. By Proposition 1.1 we have $\operatorname{depth}(S/(IJ)^{[k]}) \ge dk - 1$, $\operatorname{depth}(S_1/I^{[k]}) \ge d_1k - 1$ and $\operatorname{depth}(S_2/J^{[k]}) \ge d_2k - 1$. Thus we see that $S/(IJ)^{[k]}$ has minimum depth if and only if both $S_1/I^{[k]}$ and $S_2/J^{[k]}$ have minimum depth.

(c) Let $I$ be the edge ideal of a complete bipartite graph with the vertex set partition $[m] \cup [n]$. Then $I = (x_1, \dots, x_m)(y_1, \dots, y_n)$, and we may apply (a) and (b) to see that $I^{[k]}$ has minimum depth for all $1 \le k \le \nu(I)$.

(d) Let $G$ be the graph which is a 3-cycle with two whiskers at each vertex, and let $I$ be its edge ideal. It can be checked that $\operatorname{depth}(S/I) = 5$ and $\operatorname{depth}(S/I^{[2]}) = 3$. This shows that if $I \subset S$ is a squarefree monomial ideal generated in a single degree, then $\operatorname{depth}(S/I^{[k]})$ is not necessarily an increasing function of $k$. This example, and a related family of graphs will be studied in Sect. 3 in more details.

By the *squarefree part* of a monomial ideal $J$, we mean the ideal generated by squarefree generators of $J$. It is clear that for any $k$, the squarefree part of $J^k$ coincides with $J^{[k]}$.

Let $I \subset S$ be a squarefree monomial ideal generated in degree $d$, and suppose that $I^{[k]} = \mathfrak{m}^{[dk]}$ for some $k$ with $dk \le n$. Then not only $S/I^{[k]}$ has minimum depth, but we also have $I^{[\ell]} = \mathfrak{m}^{[d\ell]}$ for all $\ell \ge k$ (which then implies that $S/I^{[\ell]}$ has minimum depth for $\ell \ge k$ with $\ell k \le n$). This follows from Example 1.2 (a) and the next slightly more general result.

**Proposition 1.3** *Let $I \subset J \subset S$ be squarefree monomial ideals, and suppose that $I^{[k]} = J^{[k]}$ for some $k$. Then $I^{[\ell]} = J^{[\ell]}$ for all $\ell \ge k$.*

***Proof*** It suffices to show that $I^{[k+1]} = J^{[k+1]}$. For any two monomial ideals $L$ and $M$ we define the squarefree product, denoted by $L * M$, as the squarefree part of $LM$. Since $I^{[k]} \subseteq I^{[k-1]} * J \subseteq J^{[k]} = I^{[k]}$, it follows that $I^{[k]} = I^{[k-1]} * J$. Then

$$I^{[k+1]} = I * I^{[k]} = I * (I^{[k-1]} * J) = (I * I^{[k-1]}) * J = I^{[k]} * J = J^{[k]} * J = J^{[k+1]}.$$

□

***Example 1.4*** For $n \geq 4$, let $P_n$ be the path graph with the vertex set $[n] = \{1, \dots, n\}$ and edges $\{i, i+1\}$ for $i = 1, \dots, n-1$. Let $G = P_n^c$ be the complementary graph of $P_n$ with the edge ideal $I$. By Corollary 2.6, $\operatorname{depth}(S/I) > 1$. In fact, we will show that $\operatorname{depth}(S/I) = 2$. It suffices to show that $\operatorname{proj\,dim}(S/I) \geq n-2$. Indeed, $\{1, \dots, n-2\}$ is a minimal vertex cover of $G$. By a well-known theorem of Terai [18], $\operatorname{proj\,dim}(S/I)$ is equal to the regularity of the Alexander dual of $I$. Since the regularity of the dual ideal cannot be less than the maximum degree of its minimal generators, the assertion follows.

We claim that $\operatorname{depth}(S/I^{[k]}) = 2k - 1$ for $2 \leq k \leq \nu(G)$. Indeed, if $a$, $b$, $c$, $d$ are pairwise distinct vertices of $P_n$, then we may assume that $\{a, b\}$ and $\{c, d\}$ are non-edges of $P_n$. Then $\{a, b\}$ and $\{c, d\}$ are edges of $G$. This implies that $I^{[2]} = \mathfrak{m}^{[4]}$. Then $I^{[k]} = \mathfrak{m}^{[2k]}$ for all $k \geq 2$ follows from Proposition 1.3. Then, Example 1.2(a) yields the desired conclusion.

Let $I \subset S$ be a monomial ideal with linear quotients. In other words, the elements of $G(I)$ can be ordered as $u_1, \dots, u_s$ such that for all $i = 2, \dots, s$, the colon ideal $(u_1, \dots, u_{i-1}) : u_i$ is generated by variables. Let $r_i$ be the minimum number of variables generating this colon ideal. By [11, Corollary 8.2.2] one has

$$\operatorname{depth}(S/I) = n - \max\{r_2, \dots, r_s\} - 1. \tag{1}$$

By the *support* of a monomial $u$, denoted by $\operatorname{supp}(u)$, we mean the set of all $i$'s where $x_i$ divides $u$, and we put $\operatorname{supp}(I) = \cup_{u \in G(I)} \operatorname{supp}(u)$.

In the sequel we will use the following lemma.

**Lemma 1.5** *Let $I$ be a squarefree monomial ideal with $G(I) = \{u_1, \dots, u_s\}$, and let $J = (u_1, \dots, u_{s-1}) : u_s$. If $v \in G(J)$, then $vu_s$ is squarefree. In particular, $v$ is squarefree and* $\operatorname{supp}(v) \cap \operatorname{supp}(u_s) = \emptyset$.

***Proof*** Since $v \in J$, it follows that $u_j | vu_s$ for some $j = 1, \dots, s-1$. It follows that $u_j$ divides $w = \prod_{i \in \operatorname{supp}(vu_s)} x_i$ which is a squarefree monomial, because $u_j$ is squarefree. Since $u_s$ is squarefree, we have $w = v' u_s$ where $v'$ is a squarefree monomial with $v' | v$ and $\operatorname{supp}(v') \cap \operatorname{supp}(u_s) = \emptyset$. Therefore, $v' \in J$. Since $v \in G(J)$, it follows that $v' = v$. □

Now we use (1) to show the following theorem for matroidal ideals (i.e. squarefree polymatroidal ideals). See, for example, [11, Section 12.6] for more properties of polymatroidal ideals.

**Theorem 1.6** *Let $I \subset S = K[x_1, \dots, x_n]$ be a matroidal ideal with* $\operatorname{supp}(I) = [n]$. *Then all squarefree powers of $I$ have minimum depth.*

The proof of Theorem 1.6 will follow immediately from the next two results.

**Proposition 1.7** *Let $I \subset S = K[x_1, \dots, x_n]$ be a matroidal ideal with* $\operatorname{supp}(I) = [n]$. *Suppose that $I$ is generated in degree $d$. Then* $\operatorname{depth}(S/I) = d - 1$.

***Proof*** By [12, Lemma 1.3] we know that $I$ has linear quotients. Let $u_1, \dots, u_s$ be a linear quotients order for the elements of $G(I)$. By Lemma 1.5, we have $r_i \leq n - d$ for all $i$. Let $A = \{i \, i \notin \operatorname{supp}(u_s)\}$. Then $|A| = n - d$. Let $i \in A$. Then there exists $u_j$ such that $x_i | u_j$.

By the exchange property of matroidal ideals, there exists $k$ such that $x_k|u_s$, $x_k \nmid u_j$ and $x_i(u_s/x_k) \in I$. This implies that $x_i \in (u_1, \dots, u_{s-1}) : u_s$. Hence, $r_s = n - d$ and (1) implies that $\operatorname{depth}(S/I) = n - (n-d) - 1 = d - 1$, as desired. □

**Proposition 1.8** *The squarefree part of a polymatroidal ideal is a matroidal ideal.*

***Proof*** Let $I$ be a polymatroidal ideal with $G(I) = \{u_1, \dots, u_s\}$ and let $I'$ be the squarefree part of $I$ with $G(I') = \{v_1, \dots, v_r\}$. Let $v_i, v_j, x_k$ satisfy $x_k|v_i, x_k \nmid v_j$. Since $I$ is polymatroidal, there is $x_\ell$ with $x_\ell \nmid v_i, x_\ell|v_j$ for which $x_\ell(v_i/x_k) \in G(I)$. Since $x_\ell(v_i/x_k)$ is squarefree, it follows that $x_\ell(v_i/x_k) \in G(I')$. Thus $I'$ is a matroidal ideal, as desired. □

***Proof*** *(Theorem 1.6)* By [10, Theorem 12.6.3], $I^k$ is a polymatroidal ideal generated in degree $kd$. Proposition 1.8 implies that $I^{[k]}$ is a matroidal ideal generated in degree $kd$. Thus, Proposition 1.7 completes the proof. □

## 3 Minimum depth for squarefree powers of edge ideals

In this section, we give a characterization of squarefree powers of edge ideals which have minimum depth with respect to reduced homologies of a certain simplicial complex. In particular, we give a more explicit classification of all edge ideals which have minimum depth. We show that all squarefree powers of such ideals have minimum depth as well. Moreover, we give a sufficient condition for squarefree powers of edge ideals in terms of the so-called dominating cliques to have minimum depth.

Recall that a *simplicial complex* $\Delta$ on a finite vertex set $V(\Delta)$ is a set of subsets of $V(\Delta)$ such that $\{v\} \in \Delta$ for every $v \in V(\Delta)$ and if $F \in \Delta$, then $G \in \Delta$ for every $G \subseteq F$. An element $F \in \Delta$ is a *face* of $\Delta$ and a *facet* is a face of $\Delta$ which is maximal with respect to inclusion. The set of all facets of $\Delta$ is denoted by $\operatorname{Facets}(\Delta)$. If $\operatorname{Facets}(\Delta) = \{F_1, \dots, F_q\}$, then we write $\Delta = \langle F_1, \dots, F_q \rangle$.

Now, we define a simplicial complex related to matchings of a graph. Let $G$ be a graph and $k = 1, \dots, \nu(G)$. Then, we define

$$\Gamma_k(G) = \{F \subseteq V(G) : V(M) \nsubseteq F \text{ for any } k\text{-matching } M \text{ of } G\}.$$

It is easily seen that $\Gamma_k(G)$ is a simplicial complex on $V(G)$ whose Stanley-Reisner ideal is $I(G)^{[k]}$. In particular, $\Gamma_1(G)$ is the well-known independence complex of $G$ (or equivalently the clique complex of the complementary graph $G^c$ of $G$) whose Stanley–Reisner ideal is the edge ideal of $G$.

We know from Proposition 1.1 that for any graph $G$, the depth of $S/I(G)^{[k]}$ is at least $2k-1$. In the following proposition, we give an equivalent condition for attaining this lower bound.

**Proposition 2.1** *Let $G$ be a graph and $k = 1, \dots, \nu(G)$. Then the following statements are equivalent*:

(1) $g_{I(G)}(k) = 0$.
(2) $\tilde{H}_{2k-2}(\Gamma_k(G);K) \neq 0$.

***Proof*** By the Auslander-Buchsbaum formula, we know that (i) is equivalent to proj $\dim(S/I(G)^{[k]}) = n - 2k + 1$ which holds if and only if $\beta_{n-2k+1,n}(S/I(G)^{[k]}) \neq 0$, since $I(G)^{[k]}$ is generated in degree $2k$. By Hochster's formula, we know that

$$\beta_{n-2k+1,n}(S/I(G)^{[k]}) = \dim_K \tilde{H}_{2k-2}(\Gamma_k(G);K).$$

Therefore, $\beta_{n-2k+1,n}(S/I(G)^{[k]}) \neq 0$ if and only if $\tilde{H}_{2k-2}(\Gamma_k(G);K) \neq 0$, and hence the desired result follows. □

The next corollary shows that an edge ideal has minimum depth if and only if the complementary graph of $G$ is disconnected.

**Corollary 2.2** *Let $G$ be a graph. Then* $\operatorname{depth}(S/I(G)) = 1$ *if and only if $G^c$ is disconnected.*

***Proof*** Applying Proposition 2.1 for $k = 1$, we have $\operatorname{depth}(S/I(G)) = 1$ if and only if $\tilde{H}_0(\Gamma_1(G);K) \neq 0$ which is equivalent to $\Gamma_1(G)$ being disconnected. The latter is also equivalent to $G^c$ being disconnected, since $\Gamma_1(G)$ is the clique complex of $G^c$. Thus, we get the desired conclusion. □

Next, we recall some definitions about simplicial complexes and their facet ideals and we fix some notation which will be used in the rest of the section.

A *subcollection* of a simplicial complex $\Delta$ is a simplicial complex $\Gamma$ such that every facet of $\Gamma$ is also a facet of $\Delta$. If $A \subseteq V(\Delta)$, then the *induced subcollection* $\Delta_A$ is the simplicial complex $\langle F \in \text{Facets}(\Delta) \mid F \subseteq A \rangle$.

A set $D \subseteq \text{Facets}(\Delta)$ is called a *facet cover* of $\Delta$ if every vertex $v$ of $\Delta$ belongs to some $F$ in $D$. A facet cover is called *minimal* if no proper subset of it is a facet cover of $\Delta$.

Let $\Delta$ be a simplicial complex on the vertices $x_1, \dots, x_n$. Recall that the *facet ideal* of $\Delta$, denoted by $\mathcal{F}(\Delta)$, is the squarefree monomial ideal

$$\mathcal{F}(\Delta) = (x_{i_1} \cdots x_{i_k} \mid \{x_{i_1}, \dots, x_{i_k}\} \in \text{Facets}(\Delta)) \subset S.$$

If $u$ is a squarefree monomial, then $\Delta_u$ denotes the subcollection $\Delta_U$ where $U$ is the set of variables which divide $u$.

Erey and Faridi in [4] introduced the concept of well-ordered facet cover to give a non-vanishingness condition for Betti numbers of facet ideals (see [4, Definition 3.1]). Well-ordered facet covers generalize the concept of strongly disjoint bouquets which was introduced by Kimura [17].

A sequence $F_1, \dots, F_k$ of facets of a simplicial complex $\Delta$ is called a *well-ordered facet cover* if $\{F_1, \dots, F_k\}$ is a minimal facet cover of $\Delta$ and for every facet $H \notin \{F_1, \dots, F_k\}$ of $\Delta$ there exists $i \leq k-1$ such that $F_i \subseteq H \cup F_{i+1} \cup F_{i+2} \cup \cdots \cup F_k$.

Existence of well-ordered facet covers yields non-zero Betti numbers as follows:

**Theorem 2.3** [4, *Corollary* 3.4] *Let $\Delta$ be a simplicial complex and let $u$ be a squarefree monomial. If $\Delta_u$ has a well-ordered facet cover of cardinality $i$, then $\beta_{i,u}(S/\mathcal{F}(\Delta)) \neq 0$.*

We need the following technical lemma which will be useful in the sequel.

**Lemma 2.4** *Let $G$ be a graph with no isolated vertices and $v(G) \geq 2$. Suppose that $G$ is not a complete bipartite graph and $G^c$ is disconnected, and let $2 \leq k \leq v(G)$. Then there exists* *two induced subgraphs $G_1$ and $G_2$ on disjoint sets of vertices, a $(k-1)$-matching $M$ and a vertex $v$ of $G$ such that*

(1) $V(G_1) \cup V(G_2) = V(G)$,
(2) $\{x_1, x_2\} \in E(G)$ *for all* $x_1 \in V(G_1)$ *and* $x_2 \in V(G_2)$,
(3) $v \notin e$ *for every* $e \in M$,
(4) $M \cap E(G_1) \neq \emptyset$ *and*
(5) $v \in V(G_2)$.

***Proof*** Since $G^c$ is disconnected, there exist two induced subgraphs $G_1$ and $G_2$ of $G$, on disjoint sets of vertices, satisfying conditions (1) and (2). By the symmetry, it suffices to consider the following cases:

Case 1: Suppose that there exists a matching $\{e_1, \ldots, e_k\}$ of $G$ such that $e_1 \in E(G_1)$. Let $v$ be a vertex of $G_2$. If there exists $t \in \{2, \ldots, k-1\}$ with $v \in e_t$, then let $M = \{e_1, \ldots, e_k\} \setminus \{e_t\}$. Otherwise, let $M = \{e_1, \ldots, e_{k-1}\}$. In both cases, $v$ and $M$ fulfill the desired conditions.

Case 2: Suppose that no $k$-matching of $G$ has an edge contained in $G_1$ or $G_2$. Let $\{e_1, \ldots, e_k\}$ be a matching of $G$ with $e_i = \{a_i, b_i\}$ such that $a_i \in V(G_1)$ and $b_i \in V(G_2)$. Without loss of generality, it is enough to consider the following cases:

Case 2.1: Suppose that there is an edge $e$ of $G$ such that $e \subseteq \{a_1, \ldots, a_k\}$. We may assume that $e = \{a_{k-1}, a_k\}$. Then $v = b_k$ and $M = \{e_1, \ldots, e_{k-2}, e\}$ satisfy the required conditions.

Case 2.2: Suppose that both $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$ are independent sets of $G$. Since $G$ is not a complete bipartite graph, we may assume that there exists an edge $f = \{f_1, f_2\}$ such that $f \in E(G_1)$. Since $A$ is independent, we know that at least one vertex of $f$, say $f_1$, is not in $A$. Without loss of generality, we assume that $e_i \cap f = \emptyset$ for all $i = 2, \ldots, k$. Observe that for $v = b_k$ and $M = \{f, e_2, \ldots, e_{k-1}\}$, the required conditions are satisfied. □

In the next theorem, we give a sufficient condition for existence of a well-ordered facet cover of certain cardinality for the simplicial complex whose facet ideal is $I(G)^{[k]}$ where $G$ is assumed to have certain properties.

**Theorem 2.5** *Let $G$ be a graph with no isolated vertices and $\nu(G) \geq 2$, which is not a complete bipartite graph. Suppose that $G^c$ is disconnected and for any $2 \leq k \leq \nu(G)$, let $\Delta$ be the simplicial complex with facet ideal $I(G)^{[k]}$. Then $\Delta$ has a well-ordered facet cover of cardinality $n - 2k + 1$.*

***Proof*** Let $2 \leq k \leq \nu(G)$ and let $G_1, G_2, M$ and $v$ be as in Lemma 2.4. Suppose that $M = \{e_1, \ldots, e_{k-1}\}$ such that $e_1 \in E(G_1)$. Let $e_1 = \{y, z\}$. We set $U = e_1 \cup \cdots \cup e_{k-1} \cup \{v\}$. Let $V(G_1) \backslash U = \{x_1, \ldots, x_i\}$ and $V(G_2) \backslash U = \{x_{i+1}, \ldots, x_{n-2k+1}\}$. We define

$$F_j = e_1 \cup \cdots \cup e_{k-1} \cup \{v, x_j\}$$

for each $j = 1, \ldots, n - 2k + 1$.

If $x_j \in V(G_1) \setminus U$, then $F_j$ is a facet of $\Delta$ corresponding to the $k$-matching $\{\{v, x_j\}\} \cup M$. On the other hand, if $x_j \in V(G_2) \setminus U$, then $F_j$ is a facet corresponding to the $k$-matching $\{\{v, y\}, \{z, x_j\}, e_2, \ldots, e_{k-1}\}$.

We claim that $F_1, \dots, F_{n-2k+1}$ is a well-ordered facet cover of $\Delta$. Since every vertex of $\Delta$ belongs to some $F_j$, these facets indeed form a cover. Also, since for every $j = 1, \dots, n-2k+1$ we have $x_j \in F_t$ if and only if $j = t$, it follows that this cover is minimal. To prove the "well-ordered" property, let $H$ be a facet of $\Delta$ such that $H \notin \{F_1, \dots, F_{n-2k+1}\}$. Observe that $H \nsubseteq U$, since $H$ has $2k$ vertices whereas $U$ has $2k-1$. On the other hand, since $H \notin \{F_1, \dots, F_{n-2k+1}\}$, there exists at least two indices $\ell < m$ such that $\{x_\ell, x_m\} \subseteq H$. Then the proof follows from the inclusion $F_\ell \subseteq H \cup F_{\ell+1} \cup \dots \cup F_{n-2k+1}$. □

Next, we show that having minimum depth for the edge ideal itself implies the same for all squarefree powers and vice versa.

**Corollary 2.6** *Let G be a graph with no isolated vertices and matching number $\nu(G)$. Then the following statements are equivalent*:

(1) *$G^c$ is disconnected.*
(2) *$g_{I(G)}(1) = 0$.*
(3) *$g_{I(G)}(k) = 0$ for all $1 \le k \le \nu(G)$.*

***Proof*** Equivalence of (1) and (2) was already proved in Corollary 2.2. It is enough to show that (*i*) implies (*iii*). Let $2 \le k \le \nu(G)$. If $G$ is a complete bipartite graph, then the result follows from Example 1.2 (c). Otherwise, let $\Delta$ be the simplicial complex whose facet ideal is $I(G)^{[k]}$. Then by Theorem 2.5, $\Delta$ has a well-ordered facet cover of cardinality $n-2k+1$. Therefore, Theorem 2.3 implies $\beta_{n-2k+1,n}(S/I(G)^{[k]}) \neq 0$. Thus, $\operatorname{proj\,dim}(S/I(G)^{[k]}) \ge n-2k+1$, and hence the desired result follows from the Auslander-Buchsbaum formula. □

Let $G$ be a graph. A subset $D$ of $V(G)$ is called a *dominating set* if every vertex $v \in V(G) - D$ is adjacent to a vertex in $D$. A complete subgraph $K_m$ of $G$ is called a *dominating clique* if $V(K_m)$ is a dominating set.

The final result of this section applies the notion of dominating cliques to provide a sufficient condition for having minimum depth for a given squarefree power $k$.

**Theorem 2.7** *Let G be a graph and let $2 \le k \le \nu(G)$. If G has a dominating clique $K_{2k-1}$, then $g_{I(G)}(k) = 0$.*

***Proof*** Let $K_{2k-1}$ be a dominating clique with $V(K_{2k-1}) = \{x_1, \dots, x_{2k-1}\}$. Let $V(G) - V(K_{2k-1}) = \{x_{2k}, x_{2k+1}, \dots, x_n\}$. For every $j = 2k, \dots, n$, we set $F_j = \{x_1, \dots, x_{2k-1}\} \cup \{x_j\}$. Let $\Delta$ be the simplicial complex with facet ideal $\mathcal{F}(\Delta) = I(G)^{[k]}$. Then, each $F_j$ is a facet of $\Delta$. Indeed, without loss of generality, if $x_j$ is adjacent to $x_{2k-1}$, then $\{\{x_1, x_2\}, \dots, \{x_{2k-3}, x_{2k-2}\}, \{x_j, x_{2k-1}\}\}$ is a $k$-matching of $G$. It is clear that $\{F_{2k}, \dots, F_n\}$ is a minimal facet cover of $\Delta$. As in the proof of Theorem 2.5 one can show that $F_{2k}, \dots, F_n$ is a well-ordered facet cover. The proof then follows from Theorem 2.3 and the Auslander-Buchsbaum formula. □

## 4 Depth of squarefree powers with linear quotients

In this section, we provide a criterion for squarefree powers of edge ideals with linear quotients to have minimum depth. Applying that criterion, we discuss the depth of squarefree powers of the edge ideal of a class of chordal graphs which are obtained by adding some whiskers to a complete graph. Indeed, we determine when the depth of such ideals is minimum.

In the next lemma, we show that having linear quotients is inherited by the squarefree part.

**Lemma 3.1** *Let $I$ be a monomial ideal with linear quotients. Then the squarefree part of $I$ has also linear quotients.*

***Proof*** Let $G(I) = \{u_1, \dots, u_s\}$, and assume that $u_1, \dots, u_s$ is a linear quotients ordering. Let $I'$ be the squarefree part of $I$ and $G(I') = \{u_{i_1}, \dots, u_{i_t}\}$ with $1 \le i_1 < \cdots < i_t \le s$. Let $A_j = (u_{i_1}, \dots, u_{i_{j-1}}) : u_{i_j}$. We show that $A_j$ is generated by variables. Let $v \in G(A_j)$. Then $v \in (u_1, \dots, u_{i_j - 1}) : u_{i_j}$. Hence there exists $t$ such that $x_t | v$ and $u_k | x_t u_{i_j}$ for some $k \le i_j - 1$. Lemma 1.5 implies that $x_t u_{i_j}$ is squarefree. This implies that $u_k$ is squarefree, and hence $k \in \{i_1, \dots, i_{j-1}\}$. Therefore, $x_t \in A_j$ which implies that $v = x_t$, since $v \in G(A_j)$ and $x_t | v$. Thus $A_j$ is generated by variables, as desired. □

Recall that a *cochordal* graph is a graph whose complementary graph is chordal, i.e. has no induced cycle of length greater than 3. The next result follows by combining the well-known Fröberg's Theorem [F], Lemma 3.1, [11, Theorem 10.1.9] and [11, Theorem 10.2.5].

**Corollary 3.2** *Let $G$ be a cochordal graph. Then $I(G)^{[k]}$ has linear quotients for any $k = 1, \dots, \nu(G)$.*

In Corollary 2.2, we provided an explicit combinatorial characterization of edge ideals with the minimum depth. In the next proposition, we provide a combinatorial criterion for all squarefree powers of edge ideals with linear quotients admitting minimum depth.

We call a $k$-matching $M$ a *dominating k-matching* if $V(M)$ is a dominating set, i.e. any vertex $v \in V(G) - V(M)$ is adjacent to a vertex in $V(M)$.

**Proposition 3.3** *Let $G$ be a graph with no isolated vertices and $1 \le k \le \nu(G)$. If $I(G)^{[k]}$ has linear quotients with respect to the ordering $u_1, \dots, u_s$ of its minimal generators, then the following statements are equivalent:*

(1) $g_{I(G)}(k) = 0$.
(2) *There exists a dominating k-matching $M$ and some $i = 2, \dots, s$ which satisfy the following conditions:*
    (a) *$V(M) = \mathrm{supp}(u_i)$, and*
    (b) *for any $t \in V(G) - V(M)$, there exists a k-matching $M'$ with $V(M') = \mathrm{supp}(u_j)$ for some $j = 1, \dots, i-1$ such that $V(M') \subseteq V(M) \cup \{t\}$.*

In particular, if $G$ is a cochordal graph, then statements (1) and (2) are equivalent.
***Proof*** Suppose that $V(G)=[n]$. Following notation of Sect. 2, for any $i=2,\dots,r$ let $r_i$ be the number of variables in $(u_1,\dots,u_{i-1}) : u_i$. According to (1), $\operatorname{depth}(S/I(G)^{[k]})=2k-1$ if and only if $r_i=n-2k$ for some $i=2,\dots,r$. This is the case if and only if $(u_1,\dots,u_{i-1}) : u_i$ is generated by $n-2k$ variables, namely

$$(u_1,\dots,u_{i-1}) : u_i=(x_t : t\in V(G)-\operatorname{supp}(u_i)),$$

since $|\operatorname{supp}(u_i)|=2k$.

In other words, for any $t\in V(G)-\operatorname{supp}(u_i)$, there exists $j=1,\dots,i-1$ such that $u_j|x_tu_i$, or equivalently $\operatorname{supp}(u_j)\subseteq\{t\}\cup\operatorname{supp}(u_i)$.

Let $M$ be a $k$-matching with $V(M)=\operatorname{supp}(u_i)$ and $M'$ be a $k$-matching with $V(M')=\operatorname{supp}(u_j)$. The inclusion $V(M')\subseteq V(M)\cup\{t\}$ for any $t\in V(G)-V(M)$ implies that $t$ is adjacent to some vertices of $M$, and hence $M$ is a dominating $k$-matching of $G$. Thus, the statements (1) and (2) are equivalent, as desired.

In particular, if $G$ is cochordal, then the result follows from Corollary 3.2. □

As it was mentioned at the end of the proof of Proposition 3.3, the condition that $M$ is a “dominating" matching follows from other conditions. Therefore, we can drop the word “dominating" from the statement of the proposition. However, to emphasize on this combinatorial condition, we keep it in the statement, especially that this provides us a nice necessary condition for having minimum depth. Indeed, as an immediate consequence of Proposition 3.3, we get the following necessary condition to have minimum depth.

**Corollary 3.4** *Let $G$ be a graph with no isolated vertices and $k$ be an integer with $1\le k\le\nu(G)$ where $I(G)^{[k]}$ has linear quotients. If $g_{I(G)}(k)=0$, then $G$ has a dominating $k$-matching. In particular, the statement holds for any cochordal graph $G$ with no isolated vertices.*

As another consequence of Proposition 3.3, we show that the highest non-zero squarefree power of the edge ideal of any graph has the minimum depth.

**Corollary 3.5** *Let $G$ be a graph with no isolated vertices. Then $g_{I(G)}(\nu(G))=0$.*

***Proof*** Let $k=\nu(G)$. It was proved in [2, Theorem 5.1] that $I(G)^{[k]}$ has linear quotients with respect to lexicographic order on the generators where the vertices can have any fixed labelling. Let $u_1,\dots,u_s$ be a linear quotients order on the minimal monomial generators of $I(G)^{[k]}$. Let $M$ be a $k$-matching with $V(M)=\operatorname{supp}(u_s)$. It suffices to show that $M$ is a $k$-matching which satisfies condition (2) of Proposition 3.3. Let $v\in V(G)-V(M)$. Since $G$ has no isolated vertices, $v$ is adjacent to at least one vertex of $G$, say $w$. Observe that if $w\notin V(M)$, then $M$ together with the edge $\{v,w\}$ is a matching of $G$ of size greater than $k$, which is a contradiction. Therefore, we may assume that $w\in e$ for some $e\in M$. Then it suffices to put $M'=(M-\{e\})\cup\{e'\}$ with $e'=\{v,w\}$, which completes the proof. □

The corollary above does not generalize to squarefree monomial ideals. Indeed, there are examples of squarefree monomial ideals whose highest non-zero squarefree powers do not have minimum depth. For example, the monomial ideal

$$I=(x_1x_3x_5,x_2x_4x_6,x_5x_7x_9,x_4x_6x_8,x_4x_7x_{10},x_9x_{10}x_{11},x_5x_8x_{11})$$

in $S = K[x_1, \dots, x_{11}]$ satisfies $\nu(I) = 3$ with $g_I(3) = 1 \neq 0$.

As an application of Proposition 3.3, we discuss the depth of squarefree powers of edge ideals of *multiple whiskered complete graphs* which are obtained by attaching some whiskers to each vertex of a complete graph $K_s$ with $s \geq 2$. We denote such a graph by $H(a_1, \dots, a_s)$ where $a_i \geq 0$ is the number of whiskers attached to the vertex $i$ of $K_s$. Here the vertex set of $K_s$ is assumed to be $[s] = \{1, \dots, s\}$. The case $G = H(a_1, \dots, a_s)$ with $a_1 = \cdots = a_s = s - 1$ came up in [8] where edge ideals of minimum projective dimension were considered. In the same article, besides other results, it was shown that $\operatorname{proj\,dim}(S/I(G)) = s - 2$ which means that $\operatorname{depth}(S/I(G)) = s^2 - 2s + 2$ is not minimum.

**Theorem 3.6** *Let $G = H(a_1, \dots, a_s)$ with $a_i \geq 1$ for all $i = 1, \dots, s$ and let $k = 1, \dots, \nu(G)$. Then we have*:

(a) $\nu(G) = s$;
(b) *$G$ is cochordal*;
(c) *The following statements are equivalent*:

  (1) $g_{I(G)}(k) = 0$;
  (2) $\lfloor s/2 \rfloor + 1 \leq k \leq s$.

***Proof*** (a) and (b) can be easily proved. We prove (c). By Corollary 3.2, $I(G)^{[k]}$ has linear quotients. Let $u_1, \dots, u_r$ be a linear quotients ordering for the minimal generators of $I(G)^{[k]}$. We consider two cases:

Case 1: Suppose that $s = 2k$. To prove the equivalence of (1) and (2), we must show that $g_{I(G)}(k) \neq 0$. To this end, we apply the criterion in Proposition 3.3. Assume on the contrary that there is a dominating $k$-matching which satisfies condition (2) of Proposition 3.3. Let $u_m$ be the generator corresponding to such matching. Since $s = 2k$, we have $u_m = x_1 \dots x_s$. Let $W$ be the set of all leaves of $G$. By assumption, for every $v \in W$, there exists $j < m$ such that $u_j | u_m x_v$. For each $v \in W$, let $u_{\bar{v}}$ be the smallest generator in the linear quotients ordering whose support contains $v$ but no other leaves. In other words, we define

$$\bar{v} = \min\{t : \operatorname{supp}(u_t) \cap W = \{v\}\}. \tag{2}$$

Note that we have $\bar{v} < m$ for every leaf $v \in W$ by the initial assumption on $u_m$. We set $\alpha = \max\{\bar{v} : v \in W\}$. The support of $u_\alpha$ has exactly one leaf, say $w$. Without loss of generality, we assume that $w$ is adjacent to 1. Since $u_\alpha$ is of degree $2k$, there is exactly one vertex of $K_s$ that is missing in the support of $u_\alpha$, say $j \neq 1$. Let $z$ be a leaf adjacent to $j$. By definition of $\alpha$, we have $\bar{z} < \alpha < m$. Observe that $u_{\bar{z}} / \gcd(u_{\bar{z}}, u_\alpha) = x_j x_z$. Now we consider the ideal

$$J = (u_1, \dots, u_{\alpha-1}) : u_\alpha.$$

Since $J$ is generated by variables, either $x_j$ or $x_z$ must be a generator of $J$. Since $z$ is a leaf adjacent to $j$, any minimal generator of $I(G)^{[k]}$ which is divisible by $x_z$ is also divisible by $x_j$. On the other hand, since $x_j$ does not divide $u_\alpha$, we see that $x_z$ cannot be a generator of $J$. Therefore, $x_j$ is a generator of $J$.

Then there exists $\beta < \alpha$ such that $u_\beta / \gcd(u_\beta, u_\alpha) = x_j$. If the support of $u_\beta$ has no leaves, then $\beta = m$ which is a contradiction as $\beta < \alpha < m$. On the other hand, since the support of $u_\beta$ cannot contain multiple leaves, $w$ must be the only leaf in it. In that case, $\beta < \alpha = \bar{w}$ contradicts the definition (2) of $\bar{w}$.

Case 2: Suppose that $s \neq 2k$. Let $W_i = \{v_1^{(i)}, v_2^{(i)}, \dots, v_{a_i}^{(i)}\}$ denote the set of leaves attached to the vertex $i$ of $K_s$.

(1) $\Rightarrow$ (2): If either $s$ is odd with $1 \leq k \leq \lfloor s/2 \rfloor$ or $s$ is even with $1 \leq k < s/2$, then for each $k$-matching $M$ of $G$ one has $[s] \nsubseteq V(M)$. Since each $a_i \geq 1$, it follows that $M$ can not be a dominating $k$-matching of $G$. Hence, $\operatorname{depth}(S/I(G)^{[k]}) > 2k-1$, by Corollary 3.4.

(2) $\Rightarrow$ (1): If $k = s$, then the result follows from Corollary 3.5. Assume that $\lfloor s/2 \rfloor + 1 \leq k < s$. Then there is a $k$-matching $N$ of $G$ with $[s] \subsetneq V(N)$. Let $M_1, \dots, M_q$ be those $k$-matchings of $G$ on distinct sets of vertices with $[s] \subsetneq V(M_t)$ for each $t$.

We may assume that $V(M_i) = \operatorname{supp}(u_{\ell_i})$ with $1 \leq \ell_1 < \cdots < \ell_q \leq r$. Letting $M = M_q$, we claim that $M$ is a $k$-matching which satisfies condition (2) of Proposition 3.3. To this end, let $v \in V(G) - V(M)$, say $v = v_j^{(i)}$. Now we consider the following two cases to conclude the proof.

First suppose that $a_i \geq 2$ and $\{i, v_{j'}^{(i)}\} \in M$ where $j \neq j'$. Then

$$M' = (M - \{\{i, v_{j'}^{(i)}\}\}) \cup \{\{i, v_j^{(i)}\}\}$$

is a $k$-matching of $G$ and $M' = M_{q'}$ with $q' < q$ and $V(M') \subseteq V(M) \cup \{v\}$.

Next suppose that $\{i, v_{j'}^{(i)}\} \notin M$ for all $1 \leq j' \leq a_i$. Then there is some $i' \neq i$ with $1 \leq i' \leq s$ such that $\{i, i'\} \in M$. Also, there is some $i'' \neq i, i'$ with $1 \leq i'' \leq s$ such that $\{i'', v_{j''}^{(i'')}\} \in M$ for some $j''$. Then

$$M'' = (M - \{\{i, i'\}, \{i'', v_{j''}^{(i'')}\}\}) \cup \{\{i', i''\}, \{i, v_j^{(i)}\}\}$$

is a $k$-matching of $G$ and $M'' = M_{q''}$ with $q'' < q$ and $V(M'') \subseteq V(M) \cup \{v\}$. □

Besides the given characterization in Theorem 3.6, it would be also interesting to find the exact values of the normalized depth function for $1 \leq k \leq \lfloor s/2 \rfloor$. In the next example, we give a few computed cases.

***Example 3.7*** Our computations with CoCoA [1] shows the following for the squarefree powers of edge ideals of multiple whiskered complete graphs $G$ with $a_i = 1$ for all $i = 1, \dots, s$ which do not have minimum depth:

(1) If $s = 4$, then $g_{I(G)}(1) = 3$ and $g_{I(G)}(2) = 1$.
(2) If $s = 5$, then $g_{I(G)}(1) = 4$ and $g_{I(G)}(2) = 2$.
(3) If $s = 6$, then $g_{I(G)}(1) = 5$, $g_{I(G)}(2) = 3$ and $g_{I(G)}(3) = 1$.

Finally, we would like to remark that in the proof of Theorem 3.6, we used the fact that $a_i \geq 1$, for all $i = 1, \dots, s$. If we also allow some $a_i$'s to be equal to zero, then, using Theorem 2.7, we can guarantee for a specific squarefree power to have minimum depth. More precisely, let $G$ be a multiple whiskered graph with $a_i = 0$ for at least one $i$ and let $k \geq 2$. If either $s = 2k$ or $s = 2k-1$, then $G$ has a dominating $K_{2k-1}$ clique. Therefore, by Theorem 2.7, it follows that $S/I(G)^{[k]}$ has minimum depth.

**Acknowledgements** Jürgen Herzog and Sara Saeedi Madani was supported by TÜBİTAK (2221-Fellowships for Visiting Scientists and Scientists on Sabbatical Leave) to visit Nursel Erey at Gebze Technical University. Takayuki Hibi was partially supported by JSPS KAKENHI 19H00637. Sara Saeedi Madani was in part supported by a grant from IPM (No. 1401130112). We thank the referee for his/her careful reading and useful suggestions.

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