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# Orlicz spaces associated to a quasi-Banach function space: applications to vector measures and interpolation

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**Abstract**
The Orlicz spaces $X^{\Phi}$ associated to a quasi-Banach function space $X$ are defined by replacing the role of the space $L^1$ by $X$ in the classical construction of Orlicz spaces. Given a vector measure $m$, we can apply this construction to the spaces $L^1_w(m)$, $L^1(m)$ and $L^1(\|m\|)$ of integrable functions (in the weak, strong and Choquet sense, respectively) in order to obtain the known Orlicz spaces $L^{\Phi}_w(m)$ and $L^{\Phi}(m)$ and the new ones $L^{\Phi}(\|m\|)$. Therefore, we are providing a framework where dealing with different kind of Orlicz spaces in a unified way. Some applications to complex interpolation are also given.

**Keywords** Orlicz spaces · Quasi-Banach function spaces · Vector measures · Complex interpolation

**Mathematics Subject Classification** 46E30 · 46G10

## 1 Introduction

The Banach lattice $L^1(m)$ of integrable functions with respect to a vector measure $m$ (defined on a $\sigma$-algebra of sets and with values in a Banach space) has been systematically studied during the last 30 years and it has proved to be a efficient tool to describe the optimal domain of operators between Banach function spaces (see [18] and the references

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therein). The Orlicz spaces $L^{\Phi}(m)$ and $L^{\Phi}_{w}(m)$ associated to $m$ were introduced in [8] and they have recently shown in [5] their utility in order to characterize compactness in $L^1(m)$.

On the other hand, the quasi-Banach lattice $L^1(\|m\|)$ of integrable functions (in the Choquet sense) with respect to the semivariation of $m$ was introduced in [9]. Some properties of this space and their corresponding $L^p(\|m\|)$ with $p > 1$ have been obtained, but in order to achieve compactness results in $L^1(\|m\|)$ we would need to dispose of certain Orlicz spaces related to $L^1(\|m\|)$.

In [10] some generalized Orlicz spaces $X_{\Phi}$ have been obtained by replacing the role of the space $L^1$ by a Banach function space $X$ in the classical construction of Orlicz spaces. Moreover, the spaces $X$ they consider are allways supposed to possess the $\sigma$-Fatou property. However, these Orlicz spaces do not cover our situation since:

- the space $L^1(\|m\|)$ is only a quasi-Banach function space, and
- in most of the time $L^1(m)$ lacks the $\sigma$-Fatou property.

Thus, the purpose of this work is to provide a construction of certain Orlicz spaces $X^{\Phi}$ valid for the case of $X$ being an arbitrary quasi-Banach function space (in general without the $\sigma$-Fatou property), with the underlying idea that it can be applied simultaneously to the spaces $L^1(\|m\|)$ and $L^1(m)$ among others. In a subsequent paper [6] we shall employ these Orlicz spaces $L^1(\|m\|)^{\Phi}$ and their main properties here derived in order to study compactness in $L^1(\|m\|)$.

The organization of the paper goes as follows: Section 2 contains the preliminaries which we will need later. Section 3 contains a discussion of completeness in the quasi-normed context without any additional hypothesis on $\sigma$-Fatou property. Section 4 is devoted to introduce the Orlicz spaces $X^{\Phi}$ associated to a quasi-Banach function space $X$ and obtain their main properties. In Sect. 5, we show that the construction of the previous section allows to capture the Orlicz spaces associated to a vector measure and we take advantage of its generality to introduce the Orlicz spaces associated to its semivariation. Finally, in Sect. 6 we present some applications of this theory to compute their complex interpolation spaces.

## 2 Preliminaries

Throughout this paper, we shall always assume that $\Omega$ is a nonempty set, $\Sigma$ is a $\sigma$-algebra of subsets of $\Omega$, $\mu$ is a finite positive measure defined on $\Sigma$ and $L^0(\mu)$ is the space of ($\mu$-a.e. equivalence classes of) measurable functions $f : \Omega \to \mathbb{R}$ equipped with the topology of convergence in measure.

Recall that a *quasi-normed space* is any real vector space $X$ equipped with a *quasi-norm*, that is, a function $\|\cdot\|_X : X \to [0, \infty)$ which satisfies the following axioms:

(Q1) $\|x\|_X = 0$ if and only if $x = 0$.
(Q2) $\|\alpha x\|_X = |\alpha|\|x\|_X$, for $\alpha \in \mathbb{R}$ and $x \in X$.
(Q3) There exists $K \geq 1$ such that $\|x_1 + x_2\|_X \leq K\big(\|x_1\|_X + \|x_2\|_X\big)$, for all $x_1, x_2 \in X$.

The constant $K$ in (Q3) is called *a quasi-triangle constant* of $X$. Of course if we can take $K = 1$, then $\|\cdot\|_X$ is a norm and $X$ is a normed space. A *quasi-normed function space* over $\mu$ is any quasi-normed space $X$ satisfying the following properties:

(a) $X$ is an ideal in $L^0(\mu)$ and a quasi-normed lattice with respect to the $\mu$-a.e. order, that is, if $f \in L^0(\mu)$, $g \in X$ and $|f| \leq |g|$ $\mu$-a.e., then $f \in X$ and $\|f\|_X \leq \|g\|_X$.
(b) The characteristic function of $\Omega$, $\chi_\Omega$, belongs to $X$.

If, in addition, the quasi-norm $\|\cdot\|_X$ happens to be a norm, then $X$ is called a *normed function space*. Note that, with this definition, any quasi-normed function space over $\mu$ is continuously embedded into $L^0(\mu)$, as it is proved in [18, Proposition 2.2].

***Remark 1*** Many of the results that we will present in this paper are true if we assume that the measure space $(\Omega, \Sigma, \mu)$ is $\sigma$-finite. In this case, the previous condition (b) must be replaced by

(b') The characteristic functions $\chi_A$ belong to $X$ for all $A \in \Sigma$ such that $\mu(A) < \infty$.

Nevertheless we prefer to present the results in the finite case for clarity and simplicity in the proofs.

We say that a quasi-normed function space $X$ has the *$\sigma$-Fatou property* if for any positive increasing sequence $(f_n)_n$ in $X$ with $\sup_n \|f_n\|_X < \infty$ and converging pointwise $\mu$-a.e. to a function $f$, then $f \in X$ and $\|f\|_X = \sup_n \|f_n\|_X$. And a quasi-normed function space $X$ is said to be *$\sigma$-order continuous* if for any positive increasing sequence $(f_n)_n$ in $X$ converging pointwise $\mu$-a.e. to a function $f \in X$, then $\|f - f_n\|_X \to 0$.

A complete quasi-normed function space is called a *quasi-Banach function space* (briefly q-B.f.s.). If, in addition, the quasi-norm happens to be a norm, then $X$ is called a *Banach function space* (briefly B.f.s.). It is known that if a quasi-normed function space has the $\sigma$-Fatou property, then it is complete and hence a q-B.f.s. (see [18, Proposition 2.35]) and that inclusions between q-B.f.s. are automatically continuous (see [18, Lemma 2.7]).

Given a countably additive vector measure $m : \Sigma \to Y$ with values in a real Banach space $Y$, there are several ways of constructing $q$-B.f.s. of integrable functions. Let us recall them briefly. The *semivariation* of $m$ is the finite subadditive set function defined on $\Sigma$ by

$$\|m\|(A) := \sup \left\{ |\langle m, y^* \rangle|(A) : y^* \in B_{Y^*} \right\},$$

where $|\langle m, y^* \rangle|$ denotes the variation of the scalar measure $\langle m, y^* \rangle : \Sigma \to \mathbb{R}$ given by $\langle m, y^* \rangle(A) := \langle m(A), y^* \rangle$ for all $A \in \Sigma$, and $B_{Y^*}$ is the unit ball of $Y^*$, the dual of $Y$. A set $A \in \Sigma$ is called *m-null* if $\|m\|(A) = 0$. A measure $\mu := |\langle m, y^* \rangle|$, where $y^* \in B_{Y^*}$, that is equivalent to $m$ (in the sense that $\|m\|(A) \to 0$ if and only if $\mu(A) \to 0$) is called a *Rybakov control measure* for $m$. Such a measure always exists (see [7, Theorem 2, p.268]). Let $L^0(m)$ be the space of ($m$-a.e. equivalence classes of) measurable functions $f : \Omega \to \mathbb{R}$. Thus, $L^0(m)$ and $L^0(\mu)$ are just the same whenever $\mu$ is a Rybakov control measure for $m$.

A measurable function $f : \Omega \to \mathbb{R}$ is called *weakly integrable* (with respect to $m$) if $f$ is integrable with respect to $|\langle m, y^* \rangle|$ for all $y^* \in Y^*$. A weakly integrable function $f$ is said to be *integrable* (with respect to $m$) if, for each $A \in \Sigma$ there exists an element (necessarily unique) $\int_A f \, dm \in Y$, satisfying

$$\left\langle \int_A f\,dm, y^* \right\rangle = \int_A f\,d\langle m, y^* \rangle, \quad y^* \in Y^*.$$

Given a measurable function $f : \Omega \to \mathbb{R}$, we shall also consider its distribution function (with respect to the semivariation of the vector measure $m$)

$$\|m\|_f : t \in [0,\infty) \to \|m\|_f(t) := \|m\|\big(\big[|f| > t\big]\big) \in [0,\infty),$$

where $\big[|f| > t\big] := \{w \in \Omega : |f(w)| > t\}$. This distribution function is bounded, non-increasing and right-continuous.

Let $L^1_w(m)$ be the space of all ($m$-a.e. equivalence classes of) weakly integrable functions, $L^1(m)$ the space of all ($m$-a.e equivalence classes of) integrable functions and $L^1(\|m\|)$ the space of all ($m$-a.e. equivalence classes of) measurable functions $f$ such that its distribution function $\|m\|_f$ is Lebesgue integrable in $(0,\infty)$. Letting $\mu$ be any Rybakov control measure for $m$, we have that $L^1_w(m)$ becomes a B.f.s. over $\mu$ with the $\sigma$-Fatou property when endowed with the norm

$$\|f\|_{L^1_w(m)} := \sup\left\{ \int_\Omega |f|\,d|\langle m, y^*\rangle| : y^* \in B_{Y^*} \right\}.$$

Moreover, $L^1(m)$ is a closed $\sigma$-order continuous ideal of $L^1_w(m)$. In fact, it is the closure of $\mathscr{S}(\Sigma)$, the space of simple functions supported on $\Sigma$. Thus, $L^1(m)$ is a $\sigma$-order continuous B.f.s. over $\mu$ endowed with same norm (see [18, Theorem 3.7] and [18, p.138])). It is worth noting that space $L^1(m)$ does not generally have the $\sigma$-Fatou property. In fact, if $L^1(m) \neq L^1_w(m)$, then $L^1(m)$ does not have the $\sigma$-Fatou property. See [2] for details.

On the other hand, $L^1(\|m\|)$ equipped with the quasi-norm

$$\|f\|_{L^1(\|m\|)} := \int_0^\infty \|m\|_f(t)\,dt.$$

is a q-B.f.s. over $\mu$ with the $\sigma$-Fatou property (see [4, Proposition 3.1]) and it is also $\sigma$-order continuous (see [4, Proposition 3.6]). We will denote by $L^\infty(m)$ the B.f.s. of all ($m$-a.e. equivalence classes of) essentially bounded functions equipped with the essential sup-norm.

## 3 Completeness of quasi-normed lattices

In this section we present several characterizations of completeness which will be needed later. We begin by recalling one of them valid for general quasi-normed spaces (see [10, Theorem 1.1]).

**Theorem 1** *Let X be a quasi-normed space with a quasi-triangle constant K. The following conditions are equivalent*:

(i) *X is complete.*

(ii) *For every sequence* $(x_n)_n \subseteq X$ *such that* $\sum_{n=1}^{\infty} K^n \|x_n\|_X < \infty$ *we have* $\sum_{n=1}^{\infty} x_n \in X$. *In this case, the inequality* $\left\|\sum_{n=1}^{\infty} x_n\right\|_X \leq K \sum_{n=1}^{\infty} K^n \|x_n\|_X$ *holds.*

The next result is a version of Amemiya's Theorem ( [10, Theorem 2, p.290]) for quasi-normed lattices.

**Theorem 2** *Let X be a quasi-normed lattice. The following conditions are equivalent*:

(i) *X is complete.*
(ii) *For any positive increasing Cauchy sequence* $(x_n)_n$ *in X there exists* $\sup_n x_n \in X$.

***Proof*** (i) ⇒ (ii) is evident because the limit of increasing convergent sequences in a quasi-normed lattice is always its supremum.

(ii) ⇒ (i) Let $(x_n)_n$ be a positive increasing Cauchy sequence in $X$. It is sufficient to prove that $(x_n)_n$ is convergent in $X$ for $X$ being complete (see, for example [1, Theorem 16.1]). By hypothesis, there exists $x := \sup_n x_n \in X$. We have to prove that $(x_n)_n$ converges to $x$ and for this it is enough the convergence of a subsequence of $(x_n)_n$. So, let us take a subsequence of $(x_n)_n$, that we still denote by $(x_n)_n$, such that $\|x_{n+1} - x_n\|_X \leq \frac{1}{K^n n^3}$, for all $n \in \mathbb{N}$ where $K$ is a quasi-triangle constant of $X$. Thus, the sequence $y_n := \sum_{i=1}^{n} i(x_{i+1} - x_i)$ is positive, increasing and Cauchy. Indeed, given $m > n$, we have

$$\|y_m - y_n\|_X \leq \sum_{i=n+1}^{m} iK^{i-n}\|x_{i+1} - x_i\|_X \leq \frac{1}{K^n} \sum_{i=n+1}^{m} \frac{1}{i^2}.$$

Applying (ii) again, we deduce that there exists $y := \sup_n y_n \in X$. Moreover, given $n \in \mathbb{N}$, we have

$$n(x - x_n) = n\left(\sup_{m>n} x_{m+1} - x_n\right) = n \sup_{m>n}(x_{m+1} - x_n) = n \sup_{m>n} \sum_{i=n}^{m}(x_{i+1} - x_i) \leq \sup_{m>n} y_n = y.$$

Therefore, $0 \leq x - x_n \leq \frac{1}{n} y$ and hence $\|x - x_n\|_X \leq \frac{1}{n}\|y\|_X \to 0$. □

Applying Theorem 2 to the sequence of partial sums of a given sequence, we see that completeness in quasi-normed lattices can still be characterized by a Riesz-Fischer type property.

**Corollary 1** *Let X be a quasi-normed lattice with a quasi-triangle constant K. The following conditions are equivalent:*

(i) *X is complete.*

(ii) *For every positive sequence* $(x_n)_n \subseteq X$ *such that* $\sum_{n=1}^{\infty} K^n \|x_n\|_X < \infty$ *there exists* $\sup_n \sum_{i=1}^{n} x_i \in X$.

## 4 Orlicz spaces $X^{\Phi}$

In this section we introduce the Orlicz spaces $X^{\Phi}$ associated to a quasi-Banach function space $X$ and a Young function $\Phi$ and obtain their main properties.

Recall that a *Young function* is any function $\Phi : [0,\infty) \to [0,\infty)$ which is strictly increasing, continuous, convex, $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$. A Young function $\Phi$ satisfies the following useful inequalities (which we shall use without explicit mention) for all $t \geq 0$:

$$\begin{cases} \Phi(\alpha t) \leq \alpha\, \Phi(t) & \text{if } \ 0 \leq \alpha \leq 1, \\ \Phi(\alpha t) \geq \alpha\, \Phi(t) & \text{if } \ \alpha \geq 1. \end{cases}$$

In particular, from the second of the previous inequalities it follows that for all $t_0 > 0$ there exists $C > 0$ such that $\Phi(t) \geq Ct$ for all $t \geq t_0$. For a given $t_0 > 0$, just take $C := \frac{\Phi(t_0)}{t_0} > 0$ and observe that $\Phi(t) = \Phi\left(t_0 \frac{t}{t_0}\right) \geq \frac{t}{t_0}\Phi(t_0) = Ct$ for all $t \geq t_0$.

Moreover, it is easy to prove using the convexity of $\Phi$ that

$$\Phi\left(\sum_{n=1}^{N} t_n\right) \leq \sum_{n=1}^{N} \frac{1}{2^n \alpha^n} \Phi(2^n \alpha^n t_n) \tag{1}$$

for all $N \in \mathbb{N}, \alpha \geq 1$ and $t_1, \ldots, t_N \geq 0$.

A Young function $\Phi$ has the $\Delta_2$*-property*, written $\Phi \in \Delta_2$, if there exists a constant $C > 1$ such that $\Phi(2t) \leq C\Phi(t)$ for all $t \geq 0$. Equivalently, $\Phi \in \Delta_2$ if for any $c > 1$ there exists $C > 1$ such that $\Phi(ct) \leq C\Phi(t)$, for all $t \geq 0$.

**Definition 1** Let $\Phi$ be a Young function. Given a quasi-normed function space $X$ over $\mu$, the corresponding (generalized) *Orlicz class* $\widetilde{X}^{\Phi}$ is defined as the following set of ($\mu$-a.e. equivalence classes of) measurable functions:

$$\widetilde{X}^{\Phi} := \left\{ f \in L^0(\mu) : \Phi(|f|) \in X \right\}.$$

**Proposition 1** *Let* $\Phi$ *be a Young function and* $X$ *be a quasi-normed function space over* $\mu$*. Then,* $\widetilde{X}^{\Phi}$ *is a solid convex set in* $L^0(\mu)$*. Moreover,* $\widetilde{X}^{\Phi} \subseteq X$*.*

***Proof*** Let $f, g \in \widetilde{X}^{\Phi}$ and $0 \leq \alpha \leq 1$. According to the convexity and monotonicity properties of $\Phi$ we have $\Phi(|\alpha f + (1-\alpha)g|) \leq \alpha\Phi(|f|) + (1-\alpha)\Phi(|g|) \in X$. The ideal property of $X$ yields $\Phi(|\alpha f + (1-\alpha)g|) \in X$ which means that $\alpha f + (1-\alpha)g \in \widetilde{X}^{\Phi}$ and proves the convexity of $\widetilde{X}^{\Phi}$. Clearly, $\widetilde{X}^{\Phi}$ is solid, since $|h| \leq |f|$ implies that $\Phi(|h|) \leq \Phi(|f|) \in X$, for any $h \in L^0(\mu)$. Moreover, since $\Phi$ is a convex function, there exists $C > 0$ such that $\Phi(t) \geq Ct$, for all $t > 1$. Thus, for all $f \in \widetilde{X}^{\Phi}$,

$$|f| = |f|\chi_{[|f|>1]} + |f|\chi_{[|f|\le 1]} \le \frac{1}{C}\Phi\Big(|f|\chi_{[|f|>1]}\Big) + \chi_\Omega \le \frac{1}{C}\Phi(|f|) + \chi_\Omega \in X,$$

which gives $f \in X$. □

**Definition 2** Let $\Phi$ be a Young function. Given a quasi-normed function space $X$ over $\mu$, the corresponding (generalized) *Orlicz space* $X^\Phi$ is defined as the following set of ($\mu$-a.e. equivalence classes of) measurable functions:

$$X^\Phi := \left\{ f \in L^0(\mu) : \exists\, c > 0 : \frac{|f|}{c} \in \widetilde{X}^\Phi \right\}.$$

**Proposition 2** *Let $\Phi$ be a Young function and $X$ be a quasi-normed function space over $\mu$. Then, $X^\Phi$ is a linear space, an ideal in $L^0(\mu)$ and $\widetilde{X}^\Phi \subseteq X^\Phi \subseteq X$.*

***Proof*** Let $f, g \in X^\Phi$ and $\alpha \in \mathbb{R}$. Then, there exist $c_1, c_2 > 0$ such that $\frac{|f|}{c_1}, \frac{|g|}{c_2} \in \widetilde{X}^\Phi$. Setting $c := \max\{c_1, c_2\}$ and using the convexity of $\widetilde{X}^\Phi$ we have

$$\frac{|f+g|}{2c} \le \frac{1}{2}\frac{|f|}{c} + \frac{1}{2}\frac{|g|}{c} \le \frac{1}{2}\frac{|f|}{c_1} + \frac{1}{2}\frac{|g|}{c_2} \in \widetilde{X}^\Phi$$

and hence $\frac{|f+g|}{2c} \in \widetilde{X}^\Phi$ since $\widetilde{X}^\Phi$ is solid, which proves that $f + g \in X^\Phi$. Note that this also implies that $nf \in X^\Phi$ for any $n \in \mathbb{N}$. Taking $n_0 \in \mathbb{N}$ such that $|\alpha| \le n_0$, it follows that there exists $c_0 > 0$ such that $\frac{|\alpha f|}{c_0} \le \frac{n_0|f|}{c_0} \in \widetilde{X}^\Phi$, which yields $\frac{|\alpha f|}{c_0} \in \widetilde{X}^\Phi$ and so $\alpha f \in X^\Phi$.

It is evident that $\widetilde{X}^\Phi \subseteq X^\Phi$ and $X^\Phi$ inherits the ideal property from $\widetilde{X}^\Phi$, since $|h| \le |f|$ implies that $\frac{|h|}{c_1} \le \frac{|f|}{c_1} \in \widetilde{X}^\Phi$ for any $h \in L^0(\mu)$. Moreover, taking into account Proposition 1, we have $\frac{|f|}{c_1} \in \widetilde{X}^\Phi \subseteq X$ and so $f \in X$ which proves that $X^\Phi \subseteq X$. □

**Definition 3** Let $\Phi$ be a Young function and $X$ be a quasi-normed function space over $\mu$. Given $f \in X^\Phi$, we define

$$\|f\|_{X^\Phi} := \inf\left\{ k > 0 : \frac{|f|}{k} \in \widetilde{X}^\Phi \text{ with } \left\|\Phi\left(\frac{|f|}{k}\right)\right\|_X \le 1 \right\}.$$

The functional $\|\cdot\|_{X^\Phi}$ in $X^\Phi$ is called the Luxemburg quasi-norm.

**Proposition 3** *Let $\Phi$ be a Young function and $X$ be a quasi-normed function space (respectively, normed function space) over $\mu$. Then, $\|\cdot\|_{X^\Phi}$ is a quasi-norm (respectively, norm) in $X^\Phi$. Moreover, $X^\Phi$ equipped with the Luxemburg quasi-norm, is a quasi-normed (respectively, normed) function space over $\mu$.*

***Proof*** First, note that $\|\cdot\|_{X^\Phi} : X^\Phi \to [0,\infty)$. Given $f \in X^\Phi$, there exists $c > 0$ such that $\Phi\left(\frac{|f|}{c}\right) \in X$. Let $M := \left\|\Phi\left(\frac{|f|}{c}\right)\right\|_X < \infty$. On the one hand, if $M \le 1$ then

$\|f\|_{X^\Phi} \leq c < \infty$. On the other hand, if $M > 1$ then $\Phi\left(\frac{|f|}{Mc}\right) \leq \frac{1}{M}\Phi\left(\frac{|f|}{c}\right) \in X$ and so $\left\|\Phi\left(\frac{|f|}{Mc}\right)\right\|_X \leq \frac{1}{M}\left\|\Phi\left(\frac{|f|}{c}\right)\right\| = 1$, which implies that $\|f\|_{X^\Phi} \leq Mc < \infty$.

If $f = 0$, then $\left\|\Phi\left(\frac{|f|}{c}\right)\right\|_X = 0 \leq 1$ for all $c > 0$ and so $\|f\|_{X^\Phi} = 0$. Now, suppose that $\|f\|_{X^\Phi} = 0$ and that $\mu\big([f \neq 0]\big) > 0$, that is, $\left\|\Phi\left(\frac{|f|}{c}\right)\right\|_X \leq 1$ for all $c > 0$ and there exist $\varepsilon > 0$ and $A \in \Sigma$ such that $\mu(A) > 0$ and $|f|\chi_A \geq \varepsilon\chi_A$. Given $c > 0$, we have $\Phi\left(\frac{\varepsilon}{c}\right)\chi_A \leq \Phi\left(\frac{|f|\chi_A}{c}\right) \leq \Phi\left(\frac{|f|}{c}\right)$. Therefore,

$$\left\|\Phi\left(\frac{|f|}{c}\right)\right\|_X \geq \left\|\Phi\left(\frac{\varepsilon}{c}\right)\chi_A\right\|_X = \Phi\left(\frac{\varepsilon}{c}\right)\|\chi_A\|_X$$

and keeping in mind that $\lim_{t\to\infty} \Phi(t) = \infty$, we can take $c > 0$ such that

$$\Phi\left(\frac{\varepsilon}{c}\right)\|\chi_A\|_X > 1$$

which yields a contradiction.

On the other hand, given $f \in X^\Phi$ and $\lambda \in \mathbb{R}$, it is clear that

$$\begin{aligned}\|\lambda f\|_{X^\Phi} &= \inf\left\{k > 0 : \left\|\Phi\left(\frac{|\lambda f|}{k}\right)\right\|_X \leq 1\right\} = \inf\left\{k > 0 : \left\|\Phi\left(\frac{|f|}{\frac{k}{|\lambda|}}\right)\right\|_X \leq 1\right\} \\ &= |\lambda| \inf\left\{\frac{k}{|\lambda|} > 0 : \left\|\Phi\left(\frac{|f|}{\frac{k}{|\lambda|}}\right)\right\|_X \leq 1\right\} = |\lambda|\|f\|_{X^\Phi}.\end{aligned}$$

Now, let $f, g \in X^\Phi$ and take $K \geq 1$ as in (Q3). Given $a, b > 0$ such that $\left\|\Phi\left(\frac{|f|}{a}\right)\right\|_X \leq 1$ and $\left\|\Phi\left(\frac{|g|}{b}\right)\right\|_X \leq 1$, we have

$$\begin{aligned}\Phi\left(\frac{|f+g|}{K(a+b)}\right) &\leq \frac{1}{K}\,\Phi\left(\frac{|f+g|}{a+b}\right) \leq \frac{1}{K}\,\Phi\left(\frac{a}{(a+b)}\frac{|f|}{a} + \frac{b}{(a+b)}\frac{|g|}{b}\right) \\ &\leq \frac{1}{K}\frac{a}{(a+b)}\Phi\left(\frac{|f|}{a}\right) + \frac{1}{K}\frac{b}{(a+b)}\Phi\left(\frac{|g|}{b}\right).\end{aligned}$$

Hence, $\left\|\Phi\left(\frac{|f+g|}{K(a+b)}\right)\right\|_X \leq \frac{a}{(a+b)}\left\|\Phi\left(\frac{|f|}{a}\right)\right\|_X + \frac{b}{(a+b)}\left\|\Phi\left(\frac{|g|}{b}\right)\right\|_X \leq 1$ which implies that $\|f+g\|_{X^\Phi} \leq K(a+b)$. By the arbitrariness of $a$ and $b$ we deduce that $\|f+g\|_{X^\Phi} \leq K(\|f\|_{X^\Phi} + \|g\|_{X^\Phi})$.

Thus, we have proved that $\|\cdot\|_{X^\Phi}$ is a quasi-norm in $X^\Phi$ with the same quasi-triangle constant as the one of the quasi-norm of $X$. Moreover, we have already proved that $X^\Phi$ equipped with the Luxemburg quasi-norm is a quasi-normed space and an ideal in $L^0(\mu)$. It is also clear that the Luxemburg quasi-norm is a lattice quasi-norm: $|f| \leq |g|$ implies that

$\Phi\left(\frac{|f|}{k}\right) \le \Phi\left(\frac{|g|}{k}\right)$ for all $k > 0$ and this guarantees that $\|f\|_{X^\Phi} \le \|g\|_{X^\Phi}$. In addition, $\chi_\Omega \in X^\Phi$, since $\Phi\left(\frac{|\chi_\Omega|}{c}\right) = \Phi\left(\frac{1}{c}\right)\chi_\Omega \in X$, for all $c > 0$, and hence $X^\Phi$ is in fact a quasi-normed function space. □

***Remark 2*** The inclusion of $X^\Phi \subseteq X$ is continuous provided $X$ and $X^\Phi$ be q-B.f.s. We will see in Theorem 3 that the completeness is transferred from $X$ to $X^\Phi$.

Once we have checked that $X^\Phi$ is quasi-normed function space, it is immediate that $L^\infty(\mu)$ is contained in $X^\Phi$ and this inclusion is continuous with norm $\|\chi_\Omega\|_{X^\Phi}$. The next result establishes the relation between the norm of this inclusion and the norm $\|\chi_\Omega\|_X$ of the continuous inclusion of $L^\infty(\mu)$ into $X$.

**Lemma 1** *Let $\Phi$ be a Young function and $X$ be a quasi-normed function space over $\mu$.*

(i) *For all $A \in \Sigma$ with $\mu(A) > 0$,* $\|\chi_A\|_{X^\Phi} = \dfrac{1}{\Phi^{-1}\left(\frac{1}{\|\chi_A\|_X}\right)}$.

(ii) *For all $f \in L^\infty(\mu)$,* $\|f\|_{X^\Phi} \le \dfrac{\|f\|_{L^\infty(\mu)}}{\Phi^{-1}\left(\frac{1}{\|\chi_\Omega\|_X}\right)}$.

***Proof*** (i) Write $\alpha := \dfrac{1}{\Phi^{-1}\left(\frac{1}{\|\chi_A\|_X}\right)}$. On the one hand,

$$\left\|\Phi\left(\frac{|\chi_A|}{\alpha}\right)\right\|_X = \Phi\left(\frac{1}{\alpha}\right)\|\chi_A\|_X = \Phi\left(\Phi^{-1}\left(\frac{1}{\|\chi_A\|_X}\right)\right)\|\chi_A\|_X = 1,$$

and so $\|\chi_A\|_{X^\Phi} \le \alpha$. On the other hand, given $k > 0$ such that $\frac{\chi_A}{k} \in \widetilde{X}^\Phi$ with $\left\|\Phi\left(\frac{\chi_A}{k}\right)\right\|_X \le 1$, we have $\Phi\left(\frac{1}{k}\right)\|\chi_A\|_X \le 1$, that is, $\Phi\left(\frac{1}{k}\right) \le \frac{1}{\|\chi_A\|_X}$ or, equivalently, $\frac{1}{k} \le \Phi^{-1}\left(\frac{1}{\|\chi_A\|_X}\right)$, which finally leads to $\alpha \le k$ and so $\alpha \le \|\chi_A\|_{X^\Phi}$.

(ii) Since $|f| \le \|f\|_{L^\infty(\mu)}\chi_\Omega$, for any $f \in L^\infty(\mu)$, we have $\|f\|_{X^\Phi} \le \|f\|_{L^\infty(\mu)}\|\chi_\Omega\|_{X^\Phi}$ and the result follows applying (i) to $\chi_\Omega$. □

The following two results explore the close relationship between the quantities $\|f\|_{X^\Phi}$ and $\|\Phi(|f|)\|_X$. This entails interesting consequences on boundedness in $X^\Phi$, allowing us to obtain a sufficient condition and a necessary condition for it.

**Lemma 2** *Let $\Phi$ be a Young function, $X$ be a quasi-normed function space over $\mu$ and $H \subset L^0(\mu)$.*

(i) *If $f \in \widetilde{X}^\Phi$, then* $\|f\|_{X^\Phi} \le \max\{1, \|\Phi(|f|)\|_X\}$.

(ii) *If $\{\Phi(|h|) : h \in H\}$ is bounded in $X$, then $H$ is bounded in $X^\Phi$.*

***Proof*** (i) On the one hand, $\|\Phi(|f|)\|_X \le 1$ directly implies that

$$\|f\|_{X^\Phi} \leq 1 = \max\{1, \|\Phi(|f|)\|_X\}.$$

On the other hand, if $\|\Phi(|f|)\|_X \geq 1$, then $\Phi\left(\frac{|f|}{\|\Phi(|f|)\|_X}\right) \leq \frac{1}{\|\Phi(|f|)\|_X}\Phi(|f|) \in X$ and hence $\Phi\left(\frac{|f|}{\|\Phi(|f|)\|_X}\right) \in X$ with $\left\|\Phi\left(\frac{|f|}{\|\Phi(|f|)\|_X}\right)\right\|_X \leq 1$. This also leads to $\|f\|_{X^\Phi} \leq \|\Phi(|f|)\|_X = \max\{1, \|\Phi(|f|)\|_X\}$.

(ii) If $\|\Phi(|h|)\|_X \leq M < \infty$, for all $h \in H$, according to (i) we have that $\|h\|_{X^\Phi} \leq \max\{1, \|\Phi(|h|)\|_X\} \leq \max\{1, M\} < \infty$, for all $h \in H$. □

**Lemma 3** *Let $\Phi$ be a Young function, $X$ be a quasi-normed function space over $\mu$ and $f \in X^\Phi$.*

(i) *If $\|f\|_{X^\Phi} < 1$, then $f \in \widetilde{X}^\Phi$ with $\|\Phi(|f|)\|_X \leq \|f\|_{X^\Phi}$.*
(ii) *If $\|f\|_{X^\Phi} > 1$ and $f \in \widetilde{X}^\Phi$, then $\|\Phi(|f|)\|_X \geq \|f\|_{X^\Phi}$.*
(iii) *If $H \subseteq X^\Phi$ is bounded, then there exists a Young function $\Psi$ such that the set $\{\Psi(|h|) : h \in H\}$ is bounded in $X$.*

***Proof*** (i) Given $0 < k < 1$ such that $\frac{|f|}{k} \in \widetilde{X}^\Phi$ with $\left\|\Phi\left(\frac{|f|}{k}\right)\right\|_X \leq 1$, we have

$$\Phi(|f|) = \Phi\left(k\frac{|f|}{k}\right) \leq k\,\Phi\left(\frac{|f|}{k}\right) \in X.$$

Therefore, $\Phi(|f|) \in X$ with $\|\Phi(|f|)\|_X \leq k\left\|\Phi\left(\frac{|f|}{k}\right)\right\|_X \leq k$ and keeping in mind that $\|f\|_{X^\Phi} < 1$, we obtain

$$\|\Phi(|f|)\|_X \leq \inf\left\{0 < k < 1 \;:\; \frac{|f|}{k} \in \widetilde{X}^\Phi \text{ with } \left\|\Phi\left(\frac{|f|}{k}\right)\right\|_X \leq 1\right\} = \|f\|_{X^\Phi}.$$

(ii) Let $0 < \varepsilon < \|f\|_{X^\Phi} - 1$ and observe that $\left\|\Phi\left(\frac{|f|}{\|f\|_{X^\Phi} - \varepsilon}\right)\right\|_X > 1$. Thus,

$$\begin{aligned}\|\Phi(|f|)\|_X &= \left\|\Phi\left((\|f\|_{X^\Phi} - \varepsilon)\frac{|f|}{\|f\|_{X^\Phi} - \varepsilon}\right)\right\|_X \\ &\geq (\|f\|_{X^\Phi} - \varepsilon)\left\|\Phi\left(\frac{|f|}{\|f\|_{X^\Phi} - \varepsilon}\right)\right\|_X \geq \|f\|_{X^\Phi} - \varepsilon,\end{aligned}$$

and letting $\varepsilon \to 0$, it follows that $\|\Phi(|f|)\|_X \geq \|f\|_{X^\Phi}$.

(iii) Take $M > 0$ such that $\|h\|_{X^\Phi} < M$, for all $h \in H$. Since $\left\|\frac{h}{M}\right\|_{X^\Phi} < 1$, for all $h \in H$, (i) guarantees that $\Phi\left(\frac{|h|}{M}\right) \in X$ with $\left\|\Phi\left(\frac{|h|}{M}\right)\right\|_X \leq \left\|\frac{h}{M}\right\|_{X^\Phi} < 1$, for all $h \in H$. Defining $\Psi(t) := \Phi\left(\frac{t}{M}\right)$, for all $t \geq 0$, we produce a Young function such that $\{\Psi(|h|) : h \in H\}$ is bounded in $X$. □

We are now in a position to establish the remarkable fact that Orlicz spaces $X^{\Phi}$ are always complete for any q-B.f.s. $X$. It is worth pointing out that standard proofs in the Banach setting require the $\sigma$-Fatou property of $X$ to obtain the $\sigma$-Fatou property of $X^{\Phi}$ (see the next Theorem 4) and as a byproduct, the completeness of this last space. However, as we have said before, there are many complete spaces without the $\sigma$-Fatou property, to which it is not possible to apply Theorem 4. Herein lies the importance of the result that we will show next about completeness of $X^{\Phi}$.

**Theorem 3** *Let $\Phi$ a Young function and $X$ be a q-B.f.s. over $\mu$. Then, $X^{\Phi}$ is complete (and hence it is a q-B.f.s. over $\mu$).*

***Proof*** Let $(h_n)_n$ be a positive increasing Cauchy sequence in $X^{\Phi}$ and take $K \geq 1$ as in (Q3). Then, we can choose a subsequence of $(h_n)_n$, that we denote by $(f_n)_n$, such that $\|f_{n+1} - f_n\|_{X^{\Phi}} < \frac{1}{2^{2n}K^{2n}}$, for all $n \in \mathbb{N}$. Thus,

$$\left\|2^nK^n(f_{n+1} - f_n)\right\|_{X^{\Phi}} < \frac{1}{2^nK^n} < 1$$

for all $n \in \mathbb{N}$, and by Lemma 3 it follows that

$$\left\|\Phi\left(2^nK^n\left(f_{n+1} - f_n\right)\right)\right\|_X \leq \left\|2^nK^n\left(f_{n+1} - f_n\right)\right\|_{X^{\Phi}} < \frac{1}{2^nK^n}, \quad n \in \mathbb{N},$$

which proves that $\sum_{n=1}^{\infty} K^n \left\|\Phi\left(2^nK^n\left(f_{n+1} - f_n\right)\right)\right\|_X \leq \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$. The completeness of $X$ ensures that the function $f := \sum_{n=1}^{\infty} \Phi\left(2^nK^n\left(f_{n+1} - f_n\right)\right) \in X$, by Theorem 1. Note that $f \in L^0(\mu)$ and the convergence of that series is also $\mu$-a.e, since $X$ is continuously included in $L^0(\mu)$. Given $N \in \mathbb{N}$, let $g_N := \sum_{n=1}^{N}(f_{n+1} - f_n)$ and denote by $g := \sup_N g_N$ pointwise $\mu$-a.e. Applying (1) with $\alpha := K$, it follows that for all $N \in \mathbb{N}$,

$$\begin{aligned}\Phi(g_N) = \Phi\left(\sum_{n=1}^{N}(f_{n+1} - f_n)\right) &\leq \sum_{n=1}^{N} \frac{1}{2^nK^n}\Phi\left(2^nK^n\left(f_{n+1} - f_n\right)\right) \\ &\leq \sum_{n=1}^{N} \Phi(2^nK^n(f_{n+1} - f_n)) \leq f\end{aligned}$$

Therefore, $0 \leq g_N \leq \Phi^{-1}(f) \in L^0(\mu)$ for all $N \in \mathbb{N}$ and so $g \in L^0(\mu)$ with $0 \leq g \leq \Phi^{-1}(f) \in X^{\Phi}$, which guarantees that $g \in X^{\Phi}$. But

$$f_{N+1} = \sum_{n=1}^{N}(f_{n+1} - f_n) + f_1 = g_N + f_1$$

for all $N \in \mathbb{N}$ and so there also exists $\sup_n f_n = g + f_1 \in X^{\Phi}$. Since $(f_n)_n$ is a subsequence of the original increasing sequence $(h_n)_n$, the supremum of the whole sequence must exists and be the same as the supremum of the subsequence. By applying Amemiya's Theorem 2 we conclude that $X^{\Phi}$ is complete. □

If the q-B.f.s. $X$ has the $\sigma$-Fatou property, then we can improve a little more our knowledge about $X^{\Phi}$ as the following proposition makes evident.

**Theorem 4** *Let $\Phi$ be a Young function and $X$ be a q-B.f.s. over $\mu$ with the $\sigma$-Fatou property.*

(i) *If* $0 \neq f \in X^{\Phi}$ *then* $\frac{|f|}{\|f\|_{X^{\Phi}}} \in \widetilde{X}^{\Phi}$ *with* $\left\| \Phi\left(\frac{|f|}{\|f\|_{X^{\Phi}}}\right)\right\|_X \leq 1$.

(ii) *If* $f \in X^{\Phi}$ *with* $\|f\|_{X^{\Phi}} \leq 1$ *then* $f \in \widetilde{X}^{\Phi}$ *with* $\|\Phi(|f|)\|_X \leq \|f\|_{X^{\Phi}}$.

(iii) $X^{\Phi}$ *also has the $\sigma$-Fatou property.*

***Proof*** (i) Take a sequence $(k_n)_n$ such that $k_n \downarrow \|f\|_{X^{\Phi}}$ and $\left\|\Phi\left(\frac{|f|}{k_n}\right)\right\|_X \leq 1$, for all $n \in \mathbb{N}$. Then, $\frac{|f|}{k_n} \uparrow \frac{|f|}{\|f\|_{X^{\Phi}}}$ and so $\Phi\left(\frac{|f|}{k_n}\right) \uparrow \Phi\left(\frac{|f|}{\|f\|_{X^{\Phi}}}\right)$, since $\Phi$ is continuous and increasing. The $\sigma$-Fatou property of $X$ guarantees that $\Phi\left(\frac{|f|}{\|f\|_{X^{\Phi}}}\right) \in X$ and

$$\left\|\Phi\left(\frac{|f|}{\|f\|_{X^{\Phi}}}\right)\right\|_X = \sup_n \left\|\Phi\left(\frac{|f|}{k_n}\right)\right\|_X \leq 1.$$

(ii) According to (i) and the inequality

$$\Phi(|f|) = \Phi\left(\|f\|_{X^{\Phi}} \frac{|f|}{\|f\|_{X^{\Phi}}}\right) \leq \|f\|_{X^{\Phi}}\, \Phi\left(\frac{|f|}{\|f\|_{X^{\Phi}}}\right)$$

we deduce that $\Phi(|f|) \in X$ and $\|\Phi(|f|)\|_X \leq \|f\|_{X^{\Phi}} \left\|\Phi\left(\frac{|f|}{\|f\|_{X^{\Phi}}}\right)\right\|_X \leq \|f\|_{X^{\Phi}}$.

(iii) Let $(f_n)_n$ in $X^{\Phi}$ with $0 \leq f_n \uparrow f$ $\mu$-a.e. and $M := \sup_n \|f_n\|_{X^{\Phi}} < \infty$. Then, $\Phi\left(\frac{f_n}{M}\right) \uparrow \Phi\left(\frac{f}{M}\right)$ $\mu$-a.e. and $\left\|\frac{f_n}{M}\right\|_{X^{\Phi}} \leq 1$ for all $n \in \mathbb{N}$. Applying (ii), we deduce that $\Phi\left(\frac{f_n}{M}\right) \in X$ with $\left\|\Phi\left(\frac{f_n}{M}\right)\right\|_X \leq 1$ for all $n \in \mathbb{N}$ and using the $\sigma$-Fatou property of $X$, it follows that $\Phi\left(\frac{f}{M}\right) \in X$ with $\left\|\Phi\left(\frac{f}{M}\right)\right\|_X = \sup_n \left\|\Phi\left(\frac{f_n}{M}\right)\right\|_X \leq 1$. This implies that $f \in X^{\Phi}$ with $\|f\|_{X^{\Phi}} \leq M$ and we also have $M \leq \|f\|_{X^{\Phi}}$, since $f_n \leq f \in X^{\Phi}$. Thus, $\|f\|_{X^{\Phi}} = M$, which proves that $X^{\Phi}$ has the $\sigma$-Fatou property. □

The relation between the Orlicz class and its corresponding Orlicz space is greatly simplified when the Young function has the $\Delta_2$-property. In addition, this has far-reaching consequences on convergence in $X^{\Phi}$ as we state in the next result.

**Theorem 5** *Let $X$ be a quasi-normed function space over $\mu$ and $\Phi \in \Delta_2$.*

(i) *The Orlicz space and the Orlicz class coincide*: $X^{\Phi} = \widetilde{X}^{\Phi}$.

(ii) $\|f_n\|_{X^{\Phi}} \to 0$ *if and only if* $\|\Phi(|f_n|)\|_X \to 0$, *for all* $(f_n)_n \subseteq X^{\Phi}$.

(iii) *If $X$ is $\sigma$-order continuous, then $X^{\Phi}$ is also $\sigma$-order continuous.*

***Proof*** (i) Given $f \in X^{\Phi}$, there exists $c > 0$ such that $\Phi\left(\frac{|f|}{c}\right) \in X$. If $c \leq 1$, then

$$\Phi(|f|) = \Phi\left(c\,\frac{|f|}{c}\right) \leq c\,\Phi\left(\frac{|f|}{c}\right) \in X,$$

and if $c > 1$, then there exist $C > 1$ such that $\Phi(ct) \leq C\Phi(t)$ for all $t \geq 0$ by the $\Delta_2$-property of $\Phi$. Therefore, $\Phi(|f|) = \Phi\left(c\,\frac{|f|}{c}\right) \leq C\,\Phi\left(\frac{|f|}{c}\right) \in X$. In any case, it follows that $\Phi(|f|) \in X$, which means that $f \in \widetilde{X}^{\Phi}$.

(ii) If $\|f_n\|_{X^{\Phi}} \to 0$, then $\|\Phi(|f_n|)\|_X \to 0$ as a consequence of Lemma 3 (i). Suppose now that $\|f_n\|_{X^{\Phi}}$ does not converges to 0. Then, there exists $\varepsilon > 0$ and a subsequence $(f_{n_k})_k$ of $(f_n)_n$ such that $\|f_{n_k}\|_{X^{\Phi}} > \varepsilon$ for all $k \in \mathbb{N}$. We can assume that $\varepsilon < 1$ and that $(f_{n_k})_k$ is the whole $(f_n)_n$ without loss of generality. Since $\Phi \in \Delta_2$ and $\frac{1}{\varepsilon} > 1$, there exist $C > 1$ such that $\Phi\left(\frac{|f_n|}{\varepsilon}\right) \leq C\Phi(|f_n|)$. By (i), we deduce that $\Phi\left(\frac{|f_n|}{\varepsilon}\right) \in X$ and hence $\left\|\Phi\left(\frac{|f_n|}{\varepsilon}\right)\right\|_X > 1$. Thus, $\|\Phi(|f_n|)\|_X \geq \frac{1}{C}\left\|\Phi\left(\frac{|f_n|}{\varepsilon}\right)\right\|_X > \frac{1}{C} > 0$, which means that $\|\Phi(|f_n|)\|_X$ does not converges to 0.

(iii) Let $(f_n)_n$ and $f$ in $X^{\Phi}$ such that $0 \leq f_n \uparrow f$ $\mu$-a.e. Then, $\Phi\big(f - f_n\big) \downarrow 0$ $\mu$-a.e. Since $X$ is $\sigma$-order continuous, it follows that $\|\Phi\big(f - f_n\big)\|_X \to 0$ and by (ii) this implies that $\|f - f_n\|_{X^{\Phi}} \to 0$, which gives the $\sigma$-order continuity of $X^{\Phi}$. □

## 5 Application: Orlicz spaces associated to a vector measure

First of all observe that classical Orlicz spaces $L^{\Phi}(\mu)$ with respect to a positive finite measure $\mu$ are obtained applying the construction $X^{\Phi}$ of section 4 to the B.f.s. $X = L^1(\mu)$, that is, $L^{\Phi}(\mu) = L^1(\mu)^{\Phi}$ equipped with the norm $\|\cdot\|_{L^{\Phi}(\mu)} := \|\cdot\|_{L^1(\mu)^{\Phi}}$. Using these classical Orlicz spaces, the Orlicz spaces $L^{\Phi}_w(m)$ and $L^{\Phi}(m)$ with respect to a vector measure $m : \Sigma \to Y$ were introduced in [8] in the following way:

$$L^{\Phi}_w(m) := \big\{f \in L^0(m) : f \in L^{\Phi}(|\langle m, y^*\rangle|),\ \forall\, y^* \in Y^*\big\},$$

equipped with the norm

$$\|f\|_{L^{\Phi}_w(m)} := \sup\left\{\|f\|_{L^{\Phi}(|\langle m,y^*\rangle|)} : y^* \in B_{Y^*}\right\},$$

and $L^{\Phi}(m)$ is the closure of simple functions $\mathscr{S}(\Sigma)$ in $L^{\Phi}_w(m)$. The next result establishes that these Orlicz spaces $L^{\Phi}_w(m)$ and $L^{\Phi}(m)$ can be obtained as generalized Orlicz spaces $X^{\Phi}$ by taking $X$ to be $L^1_w(m)$ and $L^1(m)$, respectively.

**Proposition 4** *Let $\Phi$ be a Young function and $m : \Sigma \to Y$ a vector measure.*

(i) $L^{\Phi}_w(m) = L^1_w(m)^{\Phi}$ *and* $\|f\|_{L^{\Phi}_w(m)} = \|f\|_{L^1_w(m)^{\Phi}}$, *for all* $f \in L^{\Phi}_w(m)$.
(ii) $L^{\Phi}(m) \subseteq L^1(m)^{\Phi}$ *and if* $\Phi \in \Delta_2$, *then* $L^{\Phi}(m) = L^1(m)^{\Phi}$.

***Proof*** (i) Suppose that $f \in L^1_w(m)^\Phi$ and let $k > 0$ such that $\Phi\left(\frac{|f|}{k}\right) \in L^1_w(m)$ with $\left\|\Phi\left(\frac{|f|}{k}\right)\right\|_{L^1_w(m)} \leq 1$. Given $y^* \in B_{Y^*}$ we have $\Phi\left(\frac{|f|}{k}\right) \in L^1(|\langle m, y^*\rangle|)$ with

$$\left\|\Phi\left(\frac{|f|}{k}\right)\right\|_{L^1(|\langle m,y^*\rangle|)} \leq \left\|\Phi\left(\frac{|f|}{k}\right)\right\|_{L^1_w(m)} \leq 1.$$

This implies that $f \in L^\Phi(|\langle m, y^*\rangle|)$ with $\|f\|_{L^\Phi(|\langle m,y^*\rangle|)} \leq k$. Hence, $f \in L^\Phi_w(m)$ with $\|f\|_{L^\Phi_w(m)} \leq \|f\|_{L^1_w(m)^\Phi}$.

Reciprocally, suppose now that $f \in L^\Phi_w(m)$, write $M := \|f\|_{L^\Phi_w(m)}$ and let $y^* \in B_{Y^*}$. Since $f \in L^\Phi(|\langle m, y^*\rangle|)$ and $\|f\|_{L^\Phi(|\langle m,y^*\rangle|)} \leq M$, we have that $\frac{f}{M} \in L^\Phi(|\langle m, y^*\rangle|)$ with $\left\|\frac{f}{M}\right\|_{L^\Phi(|\langle m,y^*\rangle|)} \leq 1$. Applying Theorem 4 (ii) to the space $X = L^1(|\langle m, y^*\rangle|)$, it follows that $\Phi\left(\frac{|f|}{M}\right) \in L^1(|\langle m, y^*\rangle|)$ with $\left\|\Phi\left(\frac{|f|}{M}\right)\right\|_{L^1(|\langle m,y^*\rangle|)} \leq \left\|\frac{f}{M}\right\|_{L^\Phi(|\langle m,y^*\rangle|)} \leq 1$. Then, the arbitrariness of $y^* \in B_{Y^*}$ guarantees that $\Phi\left(\frac{|f|}{M}\right) \in L^1_w(m)$ with $\left\|\Phi\left(\frac{|f|}{M}\right)\right\|_{L^1_w(m)} \leq 1$ and hence $f \in L^1_w(m)^\Phi$ with $\|f\|_{L^1_w(m)^\Phi} \leq M$.

(ii) Since $L^1(m)^\Phi$ is a B.f.s., simple functions $\mathscr{S}(\Sigma) \subseteq L^1(m)^\Phi$ and $L^1(m)^\Phi$ is a closed subspace of $L^1_w(m)^\Phi$. Thus, taking in account (i), we deduce that $L^\Phi(m) \subseteq L^1(m)^\Phi$. If in addition $\Phi \in \Delta_2$, we have

$$L^1(m)^\Phi = \{f \in L^0(m) : \Phi(|f|) \in L^1(m)\} = L^\Phi(m),$$

where the first equality is due to Theorem 5 (i) applied to $X = L^1(m)$ and the second one can be found in [8, Proposition 4.4]. □

The Orlicz spaces $L^\Phi(m)$ have been recently employed in [5] to locate the compact subsets of $L^1(m)$. Motivated by the idea of studying compactness in $L^1(\|m\|)$ (see [6] for details), we introduce the Orlicz spaces $L^\Phi(\|m\|)$ as the Orlicz spaces $X^\Phi$ associated to the q-B.f.s. $X = L^1(\|m\|)$. For further reference, we collect together all the information that our general theory provide about these new Orlicz spaces.

**Definition 4** Let $\Phi$ be a Young function and $m : \Sigma \to Y$ a vector measure. We define the *Orlicz spaces associated to the semivariation* of $m$ as $L^\Phi(\|m\|) := L^1(\|m\|)^\Phi$ equipped with $\|f\|_{L^\Phi(\|m\|)} := \|f\|_{L^1(\|m\|)^\Phi}$, for all $f \in L^\Phi(\|m\|)$.

**Corollary 2** *Let $\Phi$ be a Young function, $m : \Sigma \to Y$ a vector measure and $\mu$ any Rybakov control measure for $m$. Then,*

(i) *$L^\Phi(\|m\|)$ is a q-B.f.s. over $\mu$ with the $\sigma$-Fatou property.*
(ii) *If $\Phi \in \Delta_2$, then $L^\Phi(\|m\|)$ is $\sigma$-order continuous.*
(iii) *$L^\Phi(\|m\|) \subseteq L^1(\|m\|)$ with continuous inclusion.*

***Proof*** Apply Theorems 3, 4 and 5 to the q-B.f.s $X = L^1(\|m\|)$. See also Proposition 2 and Remark 2. □

**Corollary 3** *Let $\Phi$ be a Young function, $m : \Sigma \to Y$ a vector measure, $f \in L^{\Phi}(\|m\|)$ and $H \subseteq L^0(m)$.*

(i) *If $\Phi(|f|) \in L^1(\|m\|)$, then $\|f\|_{L^{\Phi}(\|m\|)} \leq \max\{1, \|\Phi(|f|)\|_{L^1(\|m\|)}\}$.*
(ii) *If $\|f\|_{L^{\Phi}(\|m\|)} \leq 1$, then $\Phi(|f|) \in L^1(\|m\|)$ and $\|\Phi(|f|)\|_{L^1(\|m\|)} \leq \|f\|_{L^{\Phi}(\|m\|)}$.*
(iii) *If $\|f\|_{L^{\Phi}(\|m\|)} > 1$ and $\Phi(|f|) \in L^1(\|m\|)$, then $\|\Phi(|f|)\|_{L^1(\|m\|)} \geq \|f\|_{L^{\Phi}(\|m\|)}$.*
(iv) *If $\{\Phi(|h|) : h \in H\}$ is bounded in $L^1(\|m\|)$, then $H$ is bounded in $L^{\Phi}(\|m\|)$.*
(v) *If $H$ is bounded in $L^{\Phi}(\|m\|)$, then there exists a Young function $\Psi$ such that $\{\Psi(|h|) : h \in H\}$ is bounded in $L^1(\|m\|)$.*

***Proof*** Particularize Lemmas 2 and 3 to $X = L^1(\|m\|)$. Note that, in fact, we can use (ii) of Theorem 4. □

**Corollary 4** *Let $\Phi \in \Delta_2$, $m : \Sigma \to Y$ a vector measure and $(f_n)_n \subseteq L^{\Phi}(\|m\|)$.*

(i) $L^{\Phi}(\|m\|) = \{f \in L^0(m) : \Phi(|f|) \in L^1(\|m\|)\}$.
(ii) $\|f_n\|_{L^{\Phi}(\|m\|)} \to 0$ *if and only if* $\|\Phi(|f_n|)\|_{L^1(\|m\|)} \to 0$.

***Proof*** Apply Theorem 5 to the space $X = L^1(\|m\|)$. □

## 6 Application: interpolation of Orlicz spaces

In this section all the q-B.f.s. will be supposed to be complex. This means that $L^0(\mu)$ will be assumed to be in fact the space of all ($\mu$-a.e. equivalence classes of) $\mathbb{C}$-valued measurable functions on $\Omega$. Recall that a complex q-B.f.s. $X$ over $\mu$ is the *complexification* of the real q-B.f.s. $X_{\mathbb{R}} := X \cap L^0_{\mathbb{R}}(\mu)$, where $L^0_{\mathbb{R}}(\mu)$ is the space of all ($\mu$-a.e. equivalence classes of) $\mathbb{R}$-valued measurable functions on $\Omega$ (see [18, p.24] for more details) and this allows to extend all the real q-B.f.s. defined above to complex q-B.f.s. following a standard argument.

The complex method of interpolation, $[X_0, X_1]_\theta$ with $0 < \theta < 1$, for pairs $(X_0, X_1)$ of quasi-Banach spaces was introduced in [10] as a natural extension of Calderón's original definition for Banach spaces. It relies on a theory of analytic functions with values in quasi-Banach spaces which was developed in [10] and [10]. It is important to note that there is no analogue of the Maximum Modulus Principle for general quasi-Banach spaces, but there is a wide subclass of quasi-Banach spaces called *analytically convex* (A-convex) in which that principle does hold. For a q-B.f.s. $X$ it can be proved that analytical convexity is equivalent to *lattice convexity* (L-convexity), i.e., there exists $0 < \varepsilon < 1$ so that if $f \in X$ and $0 \leq f_i \leq f$, $i = 1, \ldots, n$, satisfy $\dfrac{f_1 + \cdots + f_n}{n} \geq (1 - \varepsilon)f$, then $\max_{1 \leq i \leq n} \|f_i\|_X \geq \varepsilon \|f\|_X$ (see [10, Theorem 4.4]). This is also equivalent to $X$ be $s$-convex for some $s > 0$ (see [10, Theorem 2.2]). We recall that $X$ is called $s$-convex if there exists $C \geq 1$ such that

$$\left\| \left( \sum_{k=1}^{n} |f_k|^s \right)^{\frac{1}{s}} \right\|_X \leq C \left( \sum_{k=1}^{n} \|f_k\|_X^s \right)^{\frac{1}{s}}$$

for all $n \in \mathbb{N}$ and $f_1, \dots, f_n \in X$. Observe that, $X$ is $s$-convex if and only if its $s$-th power $X_{[s]}$ is 1-convex, where the *s-th power* $X_{[s]}$ of a q-B.f.s. $X$ over $\mu$ (for any $0 < s < \infty$) is the q-B.f.s. $X_{[s]} := \left\{ f \in L^0(\mu) \, : \, |f|^{\frac{1}{s}} \in X \right\}$ equipped with the quasi-norm $\|f\|_{X_{[s]}} = \left\| |f|^{\frac{1}{s}} \right\|_X^s$, for all $f \in X_{[s]}$ (see [18, Proposition 2.22]).

The following result provides a condition under which the L-convexity of $X$ can be transferred to its Orlicz space $X^\Phi$. When $X$ possesses the $\sigma$-Fatou property, this can be derived from [10, Proposition 3.3], but we make apparent that this property can be dropped. Recall that a function $\psi$ on the semiaxis $[0, \infty)$ is said to be *quasiconcave* if $\psi(0) = 0$, $\psi(t)$ is positive and increasing for $t > 0$ and $\dfrac{\psi(t)}{t}$ is decreasing for $t > 0$. Observe that a quasi-concave function $\psi$ satisfies the following inequalities for all $t \geq 0$:

$$\begin{cases} \psi(\alpha t) \geq \alpha \, \psi(t) & \text{if} \quad 0 \leq \alpha \leq 1, \\ \psi(\alpha t) \leq \alpha \, \psi(t) & \text{if} \quad \alpha \geq 1. \end{cases}$$

**Theorem 6** *If $X$ is an L-convex q-B.f.s. and $\Phi \in \Delta_2$, then $X^\Phi$ is L-convex.*

***Proof*** Since $\Phi \in \Delta_2$, there exists $s > 1$ such that $\Phi(2t) \leq s\Phi(t)$ for all $t \geq 0$. From the inequality

$$t\Phi'(t) \leq \int_t^{2t} \Phi'(u)\, du \leq \int_0^{2t} \Phi'(u)\, du = \Phi(2t) \leq s\Phi(t), \; t > 0$$

it is easy to check that $\dfrac{\Phi(t)}{t^s}$ is decreasing and then $\dfrac{\Phi\left(t^{\frac{1}{s}}\right)}{t}$ so is. Therefore, the function $\psi(t) := \Phi\left(t^{\frac{1}{s}}\right)$ is quasiconcave. Take $0 < \delta < 1$ such that $(1-\delta)^s = 1 - \varepsilon$, where $\varepsilon$ is the constant from the L-convexity of $X$. Let $f \in X^\Phi$ and $0 \leq f_i \leq f$, $i = 1, \dots, n$ satisfying $\dfrac{f_1 + \cdots + f_n}{n} \geq (1-\delta)f$. We can also assume that $\|f\|_{X^\Phi} = 1$ without loss of generality. Note that this implies $\|\Phi(f)\|_X \geq 1$. If we suppose, on the contrary, that $\|\Phi(f)\|_X < 1$ and we take $0 < k < 1$ such that $\|\Phi(f)\|_X < k^s < 1$, then

$$\left\| \Phi\left(\frac{f}{k}\right) \right\|_X = \left\| \psi\left(\frac{f^s}{k^s}\right) \right\|_X \leq \frac{1}{k^s} \|\psi(f^s)\|_X = \frac{1}{k^s} \|\Phi(f)\|_X < 1,$$

and therefore $\|f\|_{X^\Phi} < k < 1$. Moreover, we have $0 \leq \Phi(f_i) \leq \Phi(f) \in X$ and

$$\begin{aligned} \frac{\Phi(f_1) + \cdots + \Phi(f_n)}{n} &\geq \Phi\left(\frac{f_1 + \cdots + f_n}{n}\right) \geq \Phi((1-\delta)f) \\ &\geq (1-\delta)^s \psi(f^s) = (1-\delta)^s \Phi(f) = (1-\varepsilon)\Phi(f). \end{aligned}$$

Thus, the L-convexity of $X$ implies that $\max\limits_{1 \leq i \leq n} \|\Phi(f_i)\|_X \geq \varepsilon \|\Phi(f)\|_X \geq \varepsilon$ and hence $\max\limits_{1 \leq i \leq n} \|f_i\|_{X^\Phi} \geq \varepsilon > \delta$ by (i) of Lemma 3. □

The *Calderón product* $X_0^{1-\theta}X_1^{\theta}$ of two q-B.f.s. $X_0$ and $X_1$ over $\mu$ is the q-B.f.s. of all functions $f \in L^0(\mu)$ such that there exist $f_0 \in B_{X_0}$, $f_1 \in B_{X_1}$ and $\lambda > 0$ for which

$$|f(w)| \leq \lambda |f_0(w)|^{1-\theta} |f_1(w)|^{\theta}, \quad w \in \Omega \ (\mu\text{-a.e. }) \tag{2}$$

endowed with the quasi-norm $\|f\|_{X_0^{1-\theta}X_1^{\theta}} = \inf \lambda$, where the infimum is taken over all $\lambda$ satisfying (2). The complex method gives the result predicted by the Calderón product for nice pairs of q-B.f.s. (see [10, Theorem 3.4]).

**Theorem 7** *Let $\Omega$ be a Polish space and let $\mu$ be a finite Borel measure on $\Omega$. Let $X_0$, $X_1$ be a pair of $\sigma$-order continuous L-convex q-B.f.s. over $\mu$. Then $X_0 + X_1$ is L-convex and $[X_0, X_1]_\theta = X_0^{1-\theta}X_1^{\theta}$ with equivalence of quasi-norms.*

On the other hand, it is easy to compute the Calderón product of two Orlicz spaces associated to the same q-B.f.s:

**Proposition 5** *Let X be a q-B.f.s. over $\mu$, $\Phi_0, \Phi_1$ Young functions, $0 < \theta < 1$ and $\Phi$ such that $\Phi^{-1} := (\Phi_0^{-1})^{1-\theta}(\Phi_1^{-1})^{\theta}$. Then $\left(X^{\Phi_0}\right)^{1-\theta}\left(X^{\Phi_1}\right)^{\theta} = X^{\Phi}$.*

***Proof*** Given $f \in X^{\Phi}$, there exists $c > 0$ such that $h := \Phi\left(\frac{|f|}{c}\right) \in X$ and hence $f_0 := \Phi_0^{-1}(h) \in X^{\Phi_0}$ and $f_1 := \Phi_1^{-1}(h) \in X^{\Phi_1}$. Taking $\alpha := \max\{\|f_0\|_{X^{\Phi_0}}, \|f_1\|_{X^{\Phi_1}}\}$, it follows that

$$|f| = c\,\Phi^{-1}(h) \ = \ c\,(\Phi_0^{-1}(h))^{1-\theta}(\Phi_1^{-1}(h))^{\theta} \ = \ c|f_0|^{1-\theta}|f_1|^{\theta} \leq \quad c\alpha\left(\frac{f_0}{\alpha}\right)^{1-\theta}\left(\frac{f_1}{\alpha}\right)^{\theta},$$

which yields $f \in \left(X^{\Phi_0}\right)^{1-\theta}\left(X^{\Phi_1}\right)^{\theta}$.

Conversely, if $f \in \left(X^{\Phi_0}\right)^{1-\theta}\left(X^{\Phi_1}\right)^{\theta}$, then there exist $\lambda > 0$, $f_0 \in X^{\Phi_0}$ and $f_1 \in X^{\Phi_1}$ such that $|f| \leq \lambda |f_0|^{1-\theta}|f_1|^{\theta}$. This implies the existence of $c > 0$ such that $h_0 := \Phi_0\left(\frac{|f_0|}{c}\right) \in X$ and $h_1 := \Phi_1\left(\frac{|f_1|}{c}\right) \in X$. Thus, taking $h := h_0 + h_1 \in X$, we deduce that

$$\begin{aligned} |f| &\leq \lambda |f_0|^{1-\theta}|f_1|^{\theta} = \lambda c\left(\frac{|f_0|}{c}\right)^{1-\theta}\left(\frac{|f_1|}{c}\right)^{\theta} = \lambda c(\Phi_0^{-1}(h_0))^{1-\theta}(\Phi_1^{-1}(h_1))^{\theta} \\ &\leq \lambda c(\Phi_0^{-1}(h))^{1-\theta}(\Phi_1^{-1}(h))^{\theta} = \lambda \Phi^{-1}(h) \in X^{\Phi}, \end{aligned}$$

and hence $f \in X^{\Phi}$. □

Combining the three previous results, we obtain conditions under which the complex method applied to Orlicz spaces associated to a q-B.f.s. over $\mu$ keeps on producing an Orlicz space associated to the same q-B.f.s.

**Corollary 5** *Let $\Omega$ be a Polish space and let $\mu$ be a finite Borel measure on $\Omega$. Let X be an L-convex, $\sigma$-order continuous q-B.f.s. over $\mu$, $\Phi_0, \Phi_1 \in \Delta_2$, $0 < \theta < 1$ and $\Phi$ such that $\Phi^{-1} := (\Phi_0^{-1})^{1-\theta}(\Phi_1^{-1})^{\theta}$. Then, $[X^{\Phi_0}, X^{\Phi_1}]_\theta = X^{\Phi}$.*

***Proof*** According to Theorems 5 and 6, the hypotheses guarantee that $X^{\Phi_0}$ and $X^{\Phi_1}$ are $L$-convex, $\sigma$-order continuous q-B.f.s. Therefore, the result follows by applying Theorem 7 and Proposition 5. □

Let us denote $L^s(\|m\|) := L^1(\|m\|)_{\left[\frac{1}{s}\right]}$, for $0 < s < \infty$ and $m : \Sigma \to Y$ a vector measure. In [4, Proposition 4.1] we proved that if $s > 1$, then $L^s(\|m\|)$ is $r$-convex for every $r < s$. In fact, this is true for all $0 < s < \infty$ because if $0 < s \leq 1$ and $r < s$, then $\frac{s}{r} > 1$ and hence $L^{\frac{s}{r}}(\|m\|)$ is 1-convex, that is $L^s(\|m\|)_{[r]}$ is 1-convex, which is equivalent to $L^s(\|m\|)$ be $r$-convex. This means that $L^s(\|m\|)$ is $L$-convex for all $0 < s < \infty$. In particular, $L^1(\|m\|)$ is $L$-convex and we can apply Corollary 5 to it.

**Corollary 6** *Let $\Omega$ be a Polish space and let $\mu$ be a Borel measure which is a Rybakov control measure for $m$. Let $\Phi_0, \Phi_1 \in \Delta_2$, $0 < \theta < 1$ and $\Phi$ such that $\Phi^{-1} := (\Phi_0^{-1})^{1-\theta}(\Phi_1^{-1})^{\theta}$. Then, $[L^{\Phi_0}(\|m\|), L^{\Phi_1}(\|m\|)]_\theta = L^{\Phi}(\|m\|)$.*

For a similar result about complex interpolation of Orlicz type spaces $L^{\Phi}(m)$ and $L^{\Phi}_w(m)$ see [3, Corollary 4.2 and Theorem 4.5].

Note that, for $p > 1$, $\frac{1}{p}$-th powers are an special case of Orlicz spaces, since $X_{\left[\frac{1}{p}\right]} = X^{\Phi_{[p]}}$, where $\Phi_{[p]}(t) = t^p$. If we particularize the previous Corollary to these powers, then we obtain the interpolation result below for $L^p(\|m\|)$ spaces. In fact, this result is valid for all $0 < p_0, p_1 < \infty$ due to the fact that the Calderón product *commutes* with powers for all indices.

**Corollary 7** *Let $\Omega$ be a Polish space and let $\mu$ be a Borel measure which is a Rybakov control measure for $m$. Let $0 < \theta < 1$ and $0 < p_0, p_1 < \infty$. Then $[L^{p_0}(\|m\|), L^{p_1}(\|m\|)]_\theta = L^p(\|m\|)$, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.*

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