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Integration-by-parts characterizations of Gaussian processes

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Abstract
The Malliavin integration-by-parts formula is a key ingredient to develop stochastic analysis on the Wiener space. In this article we show that a suitable integration-by-parts formula also characterizes a wide class of Gaussian processes, the so-called Gaussian Fredholm processes.

Keywords Gaussian processes · Malliavin calculus · Stein's lemma

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1 Introduction

It is well-known that the law of a standard normal random variable X is fully characterized by the Stein's equation (also known as the integration-by-parts formula)

$$\mathbb{E}\left[f'(X)\right] = \mathbb{E}\left[Xf(X)\right]. \tag{1.1}$$

More exactly, X follows the standard normal distribution if and only if for any function $f: \mathbb{R} \to \mathbb{R}$ that is integrable with respect to the standard Gaussian measure on $\mathbb{R}$, the relation (1.1) holds true. The formula (1.1) can be extended to finite-dimensional Gaussian vectors and it can be also expressed in terms of the Malliavin calculus in various ways (see e.g. Hsu [5] or Nourdin and Peccati [7]).

Our purpose is to prove an integration-by-parts formula that characterizes (centered) Gaussian stochastic processes. The framework is to view the stochastic processes as random paths on $\mathscr{L}^2 = \mathscr{L}^2([0,1])$, and to show that the law $\mathbb{P} = \mathbb{P}^X$ of the co-ordinate process X satisfies a certain integration-by-parts formula depending on a covariance function R if and only if under $\mathbb{P}$ it is a centered Gaussian process with the covariance function R. The line of attack

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is to use the Fredholm representation of $\mathscr{L}^2$-valued Gaussian processes provided in [12] and [13].

On related research we mention Barbour [2], Coutin and Decreusefond [3], Kuo and Lee [6], Shih [11], and Sun and Guo [15]. In particular, we note that Theorem 3.1 of Shih [11] characterizes Gaussian measures on Banach spaces via the following integration-by-parts formula (the formulation uses the machinery of abstract Wiener spaces $(i, \mathscr{R}, \mathscr{B})$, where $i : \mathscr{R} \to \mathscr{B}$ is the canonical embedding and $\mathscr{R}$ is the Cameron–Martin space of an $\mathscr{B}$-valued Gaussian random variable): Let X be a $\mathscr{B}$-valued random variable. Then $\mathbb{P}$ is Gaussian if and only if

$$\mathbb{E}\left[\langle X, \mathrm{D}f(X)\rangle_{\mathscr{B},\mathscr{B}^*}\right] = \mathbb{E}\left[\mathrm{Tr}_{\mathscr{R}}\mathrm{D}^2 f(X)\right]$$

for all $f : \mathscr{B} \to \mathbb{R}$ such that $\mathrm{D}^2 f(X)$ is trace-class on $\mathscr{R}$, where D denotes the Gross derivative. In Sect. 5, we will discuss the connection between our integration-by-parts formula (4.6) and results in Shih [11]. In particular, we will show that if one works on the smaller space of continuous functions on [0, 1] vanishing at the origin, then our formula and Shih's results are different. On the other hand, our approach does not use the Cameron–Martin space and the abstract Wiener space structure. Instead, we work solely on the fixed path-space $\mathscr{L}^2 = \mathscr{L}^2([0, 1])$ and this makes somehow our characterization of Gaussian measures simpler.

The rest of the paper is organized as follows. In Sect. 2 we recall some preliminaries including operators on $\mathscr{L}^2$ spaces and associated Gaussian Fredholm processes. We also define "pathwise" Malliavin derivative that is crucial for our results. Indeed, as the underlying processes in Sect. 4 are not Gaussian *a priori*, the Malliavin derivative cannot be defined in a traditional sense using Gaussian spaces. We formulate and prove our integration-by-parts characterizations in Sect. 4. We also provide several examples including characterizations for Brownian motion and Brownian bridges. We end the paper with a short discussion in Sect. 5 on the links between our findings and the characterization of Gaussian processes in Shih [11].

2 Preliminaries

We recall some necessary preliminaries in order to prove our results. We begin by considering kernels and operators in $\mathscr{L}^2 = \mathscr{L}^2([0, 1])$ which are then connected with $\mathscr{L}^2$-valued centered Gaussian processes via the so-called Fredholm representation. For details of the facts and constructions related to Fredholm processes, we refer to [12–14].

2.1 Some operators and kernels

Let $\mathbf{1}_A$ denote the indicator of a set A. We use the short-hand $\mathbf{1}_t = \mathbf{1}_{[0,t)}$, which is related with the interpretation $\int_a^b = \int_{[a,b)}$. Recall that $\mathscr{L}^2 = \mathscr{L}^2([0, 1])$. We use the identification $\mathscr{L}^2 \times \mathscr{L}^2 = \mathscr{L}^2([0, 1]^2)$. We denote by $\mathscr{E} \subset \mathscr{L}^2$ the set of right-continuous step-functions.

Definition 2.1 (*Associated Operator*) For a kernel $T \in \mathscr{L}^2 \times \mathscr{L}^2$ its *associated operator* $\mathrm{T} : \mathscr{L}^2 \to \mathscr{L}^2$ is defined as

$$\mathrm{T}f(t) = \int_0^1 f(s)T(t, s)\,\mathrm{d}s.$$

We note that the associated operator $\mathrm{T}\colon \mathscr{L}^2 \to \mathscr{L}^2$ is bounded. Indeed, by the Cauchy–Schwarz inequality

$$\begin{aligned}\|\mathrm{T}f\|_{\mathscr{L}^2}^2 &= \int_0^1 \left(\int_0^1 f(s)T(t,s)\,\mathrm{d}s\right)^2 \mathrm{d}t \\ &\le \int_0^1 \left(\int_0^1 f(s)^2\,\mathrm{d}s\right)\left(\int_0^1 T(t,s)^2\,\mathrm{d}s\right)\mathrm{d}t \\ &= \int_0^1 f(s)^2\,\mathrm{d}s \int_0^1\int_0^1 T(t,s)^2\,\mathrm{d}s\mathrm{d}t \\ &= \|f\|_{\mathscr{L}^2}^2 \|T\|_{\mathscr{L}^2\times\mathscr{L}^2}^2.\end{aligned}$$

Example 2.1 For the indicator kernel $I(t,s) = \mathbf{1}_t(s)$ the associated operator is just the definite integral

$$\mathrm{I}f(t) = \int_0^t f(s)\,\mathrm{d}s.$$

Definition 2.2 (*Associated Adjoint Operator*) For a kernel $T \in \mathscr{L}^2 \times \mathscr{L}^2$ its *associated adjoint operator* T^* is defined by extending linearly the relations

$$\mathrm{T}^*\mathbf{1}_t(s) = T(t,s), \quad t \in [0,1].$$

The domain of T^* is the Hilbert space $\mathrm{dom}(\mathrm{T}^*)$ that is generated by the indicators $\mathbf{1}_t$, $t \in [0,1]$, and closed under the inner product

$$\langle \mathbf{1}_t, \mathbf{1}_s\rangle_{\mathrm{dom}(\mathrm{T}^*)} = \int_0^1 T(t,u)T(s,u)\,\mathrm{d}u.$$

We note that $\mathrm{dom}(\mathrm{T}^*)$ may not be a function space in general. Also, we note that, by construction, T^* is an isometry to a subspace of $\mathscr{L}^2$ and

$$\langle f, g\rangle_{\mathrm{dom}(\mathrm{T}^*)} = \langle \mathrm{T}^*f, \mathrm{T}^*g\rangle_{\mathscr{L}^2}.$$

The following result gives more understanding on the operator T^* by considering the case where the kernel T is continuous and of bounded variation in its first argument, and $T(0,\cdot) \equiv 0$. In relation to Fredholm Gaussian processes X, the condition $T(0,\cdot) \equiv 0$ simply means that $X_0 \equiv 0$, if T is the Fredholm kernel of X.

Lemma 2.1 *If the kernel T is left-continuous and of bounded variation in its first argument and $T(0,\cdot) \equiv 0$, then*

$$\mathrm{T}^*f(t) = \int_0^1 f(s)\,T(\mathrm{d}s, t), \tag{2.1}$$

for all $f \in \mathscr{E}$, and T^* is the adjoint of T in the sense that

$$\int_0^1 \mathrm{T}^*f(t)\,g(t)\,\mathrm{d}t = \int_0^1 f(t)\,\mathrm{T}g(\mathrm{d}t). \tag{2.2}$$

for all $f \in \mathscr{E}$ and $g \in \mathscr{L}^2$.

Proof For (2.1) it is enough to show that its right-hand-side is $T(t,s)$ for $f(s) = \mathbf{1}_t(s)$. But this is straightforward:

$$T(t,s) = T(t-,s) - T(0,s) = \int_0^t T(\mathrm{d}u, s)$$
$$= \int_0^1 \mathbf{1}_t(u)\, T(\mathrm{d}u, s).$$

Let us then show (2.2). By the Fubini theorem

$$\int_0^1 \mathrm{T}^* f(t)\, g(t)\, \mathrm{d}t = \int_0^1 \int_0^1 f(s) T(\mathrm{d}s, t)\, g(t)\, \mathrm{d}t$$
$$= \int_0^1 f(t) \int_0^1 T(\mathrm{d}t, s)\, g(s)\, \mathrm{d}s$$
$$= \int_0^1 f(t)\, \mathrm{T}g(\mathrm{d}t),$$

which proves the claim. □

Example 2.2 For the integral operator I we have

$$\mathrm{I}^* f(t) = \int_0^1 f(s)\, \delta_t(\mathrm{d}s),$$

where δ_t is the unit mass at point t. In other words, I^* is the identity operator. This also provides an example where (2.1) holds provided that $T(\mathrm{d}s, t)$ is understood as a measure.

Remark 2.1 If the kernel T is Volterra type (i.e. $T(t,s) = 0$ if $s > t$) and of bounded variation in its first argument, then the operator T^* coincides with the adjoint operator of Lemma 1 of Alòs at al. [1].

2.2 Gaussian Fredholm processes

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. For concreteness, we assume that $\Omega = \mathscr{L}^2$, $\mathscr{F}$ is the associated Borel σ-field and $\mathbb{P}$ is the probability measure of the co-ordinate process $X_t(\omega) = \omega(t)$. The following result can be found from [13] although here we present the statement in a slightly different form.

Lemma 2.2 (Fredholm Representation) *Suppose $X = (X_t)_{t\in[0,1]}$ is a centered process with covariance R. Then X takes values in $\mathscr{L}^2$ if and only if the covariance operator* R *is trace-class, i.e.,*

$$\int_0^1 R(t,t)\, \mathrm{d}t < \infty. \tag{2.3}$$

In this case the square root K *of the covariance* R *admits a* Fredholm kernel*, i.e.,*

$$R(t,s) = \int_0^1 K(t,u) K(s,u)\, \mathrm{d}u.$$

The kernel $K \in \mathscr{L}^2 \times \mathscr{L}^2$ can be assumed to be positive symmetric, and in this case it is unique. Consequently, we have the Fredholm representation *for Gaussian processes with*

trace-class covariance operator R*:*

$$X_t = \int_0^1 K(t,s)\,\mathrm{d}W_s \tag{2.4}$$

for some Brownian motion.

Remark 2.2 (Construction of Gaussian Fredholm Processes) Lemma 2.2 can be used to construct a Gaussian process having values in $\mathscr{L}^2$. Indeed, let $(e_n)_{n=1}^{\infty}$ be any orthonormal basis on $\mathscr{L}^2$, and let $(\xi_n)_{n=1}^{\infty}$ be i.i.d. standard Gaussian random variables. Then a Gaussian process X with Fredholm kernel K can be constructed using the following $\mathscr{L}^2$-convergent series:

$$X_t = \sum_{n=1}^{\infty} \int_0^1 \mathrm{e}_n(s) K(t,s)\,\mathrm{d}s\ \xi_n.$$

Definition 2.3 (Cameron–Martin Space) The *Cameron–Martin space*, or the *Reproducing kernel Hilbert space*, $\mathscr{R}$ of a centered Gaussian process X with covariance R is the Hilbert space of real-valued functions on $[0,1]$ generated by the functions $R(t,\cdot)$, $t \in [0,1]$, and the inner product

$$\langle R(t,\cdot), R(s,\cdot)\rangle_{\mathscr{R}} = R(t,s).$$

For Gaussian Fredholm processes with representation (2.4) the implicit Definition 2.3 can be made completely concrete. Indeed, in this case we have $\mathscr{R} = \mathrm{K}\mathscr{L}^2$ and the inner product is given by

$$\langle f, g\rangle_{\mathscr{R}} = \int_0^1 \mathrm{K}^{-1} f(t)\,\mathrm{K}^{-1} g(t)\,\mathrm{d}t.$$

Example 2.3 The Brownian motion is a Gaussian Fredholm process with Fredholm kernel $K(t,s) = \mathbf{1}_t(s)$. Consequently, the Cameron–Martin space of a Brownian motion is $\mathrm{I}\mathscr{L}^2$ and the inner product is

$$\langle f, g\rangle_{\mathscr{R}} = \int_0^1 f'(t) g'(t)\,\mathrm{d}t.$$

Definition 2.4 (*Linear Space*) The *linear space* $\mathscr{H}_1$ or the *first chaos* of a centered Gaussian process X is the closed subspace of $\mathscr{L}^2(\Omega, \sigma(X), \mathbb{P})$ generated by the random variables X_u, $u \in [0,1]$.

Definition 2.5 (*Integrand Space*) The *integrand space* $\mathscr{I}$ of a centered Gaussian process X with covariance R is the closure of step-functions $f \in \mathscr{E}$ under the norm induced by the inner product generated by the relation

$$\langle \mathbf{1}_t, \mathbf{1}_s\rangle_{\mathscr{I}} = R(t,s).$$

In general, the Hilbert space $\mathscr{I}$ may contain distributions. Note also that $\mathscr{I} = \mathrm{dom}(\mathrm{K}^*)$.

Suppose the centered Gaussian process $X\colon \Omega \to \mathscr{L}^2$ is infinite-dimensional in the sense that its Cameron–Martin space $\mathscr{R}$ is infinite-dimensional. Then all the spaces $\mathscr{R}$, $\mathscr{H}_1$ and $\mathscr{I}$ are isometric to $\mathscr{L}^2$. For example, $\mathrm{K}\colon \mathscr{L}^2 \to \mathscr{R}$ is an isometry and $\mathrm{K}^*\colon \mathscr{I} \to \mathscr{L}^2$ is an isometry.

Definition 2.6 (*Abstract Wiener Integral*) Let X be a Gaussian Fredholm process with kernel K. Let $f \in \mathscr{I}$. The *abstract Wiener integral*

$$\int_0^1 f(t)\,\mathrm{d}X_t$$

is the image in $\mathscr{H}_1$ under the isometry built from linearly extending the mapping $\mathbf{1}_u \mapsto X_u$.

Finally, we note that the Fredholm representation (2.4) extends into the following *transfer principle* (see [13] for details).

Lemma 2.3 (Transfer Principle) *Let X be a Gaussian Fredholm process with kernel K and associated Brownian motion W. Then for all $f \in \mathscr{I}$*

$$\int_0^1 f(t)\,\mathrm{d}X_t = \int_0^1 \mathrm{K}^* f(t)\,\mathrm{d}W_t,$$

where the left-hand-side is an abstract Wiener integral and the right-hand-side is the classical Wiener integral.

Finally, we note that by taking $(e_n)_{n=1}^{\infty}$ to be an orthonormal basis of $\mathscr{L}^2$, one can construct orthonormal bases for $\mathscr{R}$, $\mathscr{H}_1$ and $\mathscr{I}$. For example, $(\mathrm{K}e_n)_{n=1}^{\infty}$ is an orthonormal basis on $\mathscr{R}$.

2.3 Classical Malliavin differentiation

We recall briefly the essential elements of Malliavin calculus. For further details, see Nualart [8], Nualart and Nualart [9] and Nourdin and Peccati [7].

2.3.1 Isonormal processes and Wiener-Itô Chaos expansion

Let $\mathscr{H}$ be a real separable Hilbert space. For any $q \in \mathbb{N}$, we denote by $\mathscr{H}^{\otimes q}$ and $\mathscr{H}^{\odot q}$, respectively, the qth tensor power and the qth symmetric tensor power of $\mathscr{H}$. We also set by convention $\mathscr{H}^{\otimes 0} = \mathscr{H}^{\odot 0} = \mathbb{R}$.

Remark 2.3 If $\mathscr{H} = \mathscr{L}^2(A, \mathscr{A}, \mu) = \mathscr{L}^2(\mu)$, where μ is a σ-finite and non-atomic measure on the measurable space $(A, \mathscr{A})$, then $\mathscr{H}^{\otimes q} = \mathscr{L}^2(A^q, \mathscr{A}^q, \mu^q) = L^2(\mu^q)$, and $\mathscr{H}^{\odot q} = \mathscr{L}^2_s(A^q, \mathscr{A}^q, \mu^q) = \mathscr{L}^2_s(\mu^q)$, where $\mathscr{L}^2_s(\mu^q)$ stands for the subspace of $\mathscr{L}^2(\mu^q)$ composed of those functions that are μ^q-almost everywhere symmetric.

We denote by $W = \{W(h)\,;\, h \in \mathscr{H}\}$ the *isonormal Gaussian process* over $\mathscr{H}$. This means that W is a centered Gaussian family, defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with a covariance structure given by the relation $\mathbb{E}\,[W(h)W(g)] = \langle h, g\rangle_{\mathscr{H}}$. We also assume that $\mathscr{F} = \sigma(W)$, that is, $\mathscr{F}$ is generated by W, and use the shorthand notation $\mathscr{L}^2(\Omega) = \mathscr{L}^2(\Omega, \mathscr{F}, \mathbb{P})$.

Remark 2.4 The isonormal Gaussian process can be constructed from a centered Gaussian process as follows. Let X be a centered Gaussian process with covariance R and associated integrand space $\mathscr{I}$. Then $W(h) = \int_0^1 h(t)\,\mathrm{d}X_t$, $h \in \mathscr{I}$, is the isonormal Gaussian process with $\mathscr{H} = \mathscr{I}$. In particular, the Brownian motion corresponds to the isonormal Gaussian process with $\mathscr{H} = \mathscr{L}^2$.

For $q \geq 1$, let $\mathscr{H}_q$ be the *qth chaos* of W, defined as the closed linear subspace of $\mathscr{L}^2(\Omega)$ generated by the family $\{H_q(W(h))\,;\ h \in \mathscr{H}, \|h\|_{\mathscr{H}} = 1\}$, where H_q is the qth Hermite polynomial, defined as

$$H_q(x) = (-1)^q \mathrm{e}^{\frac{x^2}{2}} \frac{\mathrm{d}^q}{\mathrm{d}x^q}\left[\mathrm{e}^{-\frac{x^2}{2}}\right].$$

We write by convention $\mathscr{H}_0 = \mathbb{R}$. The mapping

$$\mathbf{I}_q(h^{\otimes q}) = H_q(W(h))$$

can be extended to a linear isometry between the symmetric tensor product $\mathscr{H}^{\odot q}$ (equipped with the modified norm $\sqrt{q!}\|\cdot\|_{\mathscr{H}^{\otimes q}}$) and the qth Wiener chaos $\mathscr{H}_q$. For $q = 0$, we write by convention $\mathbf{I}_0(c) = c$, $c \in \mathbb{R}$.

It is well-known that $\mathscr{L}^2(\Omega)$ can be decomposed into the infinite orthogonal sum of the spaces $\mathscr{H}_q$: any square-integrable random variable $F \in \mathscr{L}^2(\Omega)$ admits the following *Wiener-Itô chaotic expansion*

$$F = \sum_{q=0}^{\infty} \mathbf{J}_q(F),$$

where the series converges in $\mathscr{L}^2(\Omega)$ and $\mathbf{J}_q$ is the orthogonal projection operator on the qth chaos $\mathscr{H}_q$.

2.3.2 Malliavin operators

We briefly introduce some basic elements of the Malliavin calculus with respect to the isonormal Gaussian process W.

Let $\mathscr{S}$ be the set of all cylindrical random variables of the form

$$F = g\left(W(\phi_1), \ldots, W(\phi_n)\right),$$

where $n \geq 1$, $g\colon \mathbb{R}^n \to \mathbb{R}$ is an infinitely differentiable function such that it and all its partial derivatives have at most polynomial growth, and $\phi_i \in \mathscr{H}$, $i = 1, \ldots, n$. The *Malliavin derivative* of F with respect to W is the element of $\mathscr{L}^2(\Omega; \mathscr{H})$ defined as

$$\mathbf{D}F \;=\; \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}\left(W(\phi_1), \ldots, W(\phi_n)\right)\phi_i.$$

In particular, $\mathbf{D}W(h) = h$ for every $h \in \mathscr{H}$. By iteration, one can define the mth order derivative $\mathbf{D}^m F$, which is an element of $\mathscr{L}^2(\Omega; \mathscr{H}^{\odot m})$ for every $m \geq 2$. For $m \geq 1$ and $p \geq 1$, let $\mathbb{D}^{m,p}$ denote the closure of $\mathscr{S}$ with respect to the norm $\|\cdot\|_{m,p}$, defined by the relation

$$\|F\|_{m,p}^p \;=\; \mathbb{E}\left[|F|^p\right] + \sum_{i=1}^{m} \mathbb{E}\left[\|\mathbf{D}^i F\|_{\mathscr{H}^{\otimes i}}^p\right].$$

By Proposition 1.2.1 of Nualart [8] and the following discussion there, the (iterative) Malliavin derivatives $\mathbf{D}^m$ are closable, and can thus be extended to the spaces $\mathbb{D}^{m,p}$ for any $p \geq 1$.

The Malliavin derivative $\mathbf{D}$ obeys the following chain rule: If $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable with bounded partial derivatives and if $F = (F_1, \ldots, F_n)$ is a vector of elements in $\mathbb{D}^{1,2}$, then $\varphi(F) \in \mathbb{D}^{1,2}$ and

$$\mathbf{D}\varphi(F) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(F)\mathbf{D}F_i.$$

The operator L, defined as

$$\mathrm{L} = \sum_{q=0}^{\infty} -q J_q,$$

is the *infinitesimal generator of the Ornstein-Uhlenbeck semigroup*. The domain of L is

$$\mathrm{dom}(\mathrm{L}) = \left\{ F \in L^2(\Omega) : \sum_{q=1}^{\infty} q^2 \left\| J_q F \right\|^2_{\mathscr{L}^2(\Omega)} < \infty \right\} = \mathbb{D}^{2,2}.$$

For $F \in \mathscr{L}^2(\Omega)$, we define

$$\mathrm{L}^{-1} F = \sum_{q=1}^{\infty} -\frac{1}{q} J_q(F).$$

The operator L^{-1} is called the *pseudo-inverse* of L. Indeed, for any $F \in \mathscr{L}^2(\Omega)$, we have $\mathrm{L}^{-1}F \in \mathrm{domL} = \mathbb{D}^{2,2}$, and

$$\mathrm{L}\mathrm{L}^{-1} F = F - \mathbb{E}[F].$$

The following integration-by-parts formula can be found e.g. in Nourdin and Peccati [7], Theorem 2.9.1.

Proposition 2.1 *Suppose that $F \in \mathbb{D}^{1,2}$ and $G \in L^2(\Omega)$. Then, $\mathrm{L}^{-1}G \in \mathbb{D}^{2,2}$ and*

$$\mathbb{E}[FG] = \mathbb{E}[F]\mathbb{E}[G] + \mathbb{E}\left[\langle \mathbf{D}F, -\mathbf{D}\mathrm{L}^{-1}G\rangle_{\mathscr{H}}\right].$$

3 Pathwise differentiation

We introduce several (classical) pathwise derivatives as well as a pathwise version of the Malliavin differentiation in the path space $\mathscr{L}^2$ without a priori assuming an isonormal Gaussian structure as explained in Sect. 2.3.

Definition 3.1 Let $f : \mathscr{L}^2 \to \mathbb{C}$.

(a) (Fréchet derivative) The *Fréchet derivative* of f at point $x \in \mathscr{L}^2$ is the element $\nabla f(x) \in \mathscr{L}^2$ such that for all $y \in \mathscr{L}^2$

$$\lim_{\|y\|_{\mathscr{L}^2} \to 0} \frac{\left| f(x+y) - f(x) - \langle \nabla f(x), y\rangle_{\mathscr{L}^2} \right|}{\|y\|_{\mathscr{L}^2}} = 0.$$

(b) (Gâteaux derivative) The *Gâteaux derivative* of f at point $x \in \mathscr{L}^2$ to direction $y \in \mathscr{L}^2$ is

$$\nabla_y f(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon y) - f(x)}{\varepsilon}.$$

(c) (Pathwise Malliavin derivative) Let $\mathscr{C}_p^\infty(\mathbb{R}^n)$ denote the space of all polynomially bounded functions with polynomially bounded partial derivatives of all orders. Consider functionals $f : \mathscr{L}^2 \to \mathbb{C}$ of the form

$$f(x) = g(z_1, \ldots, z_n),$$

where $n \in \mathbb{N}$ and $g \in \mathscr{C}_p^\infty(\mathbb{R}^n)$, and

$$z_k = \int_0^1 e_k(t)\, \mathrm{d}x(t) \tag{3.1}$$

for some elementary functions $e_k \in \mathscr{E}$. For such f we write $f \in \mathscr{S}$. We call the elements of class $\mathscr{S}$ the *smooth* functionals. The *pathwise Malliavin derivative* of such $f \in \mathscr{S}$ is

$$\mathrm{D}_t f(x) = \sum_{k=1}^n \frac{\partial}{\partial z_k} g(z_1, \ldots, z_n)\, e_k(t). \tag{3.2}$$

More generally, by iteration for every $m \in \mathbb{N}$, the pathwise Malliavin derivative of order m is defined as follows: for every $t_1, \ldots, t_m \in [0, 1]$,

$$\mathrm{D}^m_{t_m,\ldots,t_1} f(x) = \sum_{1 \le k_1, \ldots, k_m \le n} \frac{\partial^m}{\partial z_{k_1} \cdots \partial z_{k_m}} g(z_{k_1}, \ldots, z_{k_n}) \left(e_{k_1} \otimes \cdots \otimes e_{k_m}\right)(t_1, \ldots, t_m).$$

Remark 3.1 If f is Fréchet differentiable at point $x \in \mathscr{L}^2$, then the Gâteaux derivative can be written as

$$\nabla_y f(x) = \langle \nabla f(x), y \rangle_{\mathscr{L}^2} = \int_0^1 \nabla_t f(x)\, y(t)\, \mathrm{d}t.$$

Throughout the article the notation ∇ can mean either the Fréchet differential or the Gâteaux derivative, whenever confusion cannot arise.

Remark 3.2 If X is a Gaussian process, then our definition of the pathwise Malliavin derivative coincides with the classical one introduced in Sect. 2.3.2 on the class of smooth functionals. In particular, it does not depend on the representation (3.1). For details in this case, we refer to Nualart [8].

Remark 3.3 (Caution)

(a) In our definition of pathwise Malliavin derivative, it is not necessary to take $\mathscr{L}^2$ as the domain of smooth functionals. In fact, any suitable space of functions can be realized as good integrators with respect to elementary functions. However, the $\mathscr{L}^2$ space can be seen as a convenient reference function space later on, since as we are going to apply our results in a setting where x in (3.1) plays the role of a typical sample path over the interval $[0, 1]$ of a Gaussian process.

(b) The $\mathscr{E}$-valued operator D defined in (3.2) is in fact a linear unbounded operator. It is well known (cf. Nualart [8], Lemma 1.2.1) that the domain of the classical Malliavin derivative can be extended in $\mathscr{L}^2(\Omega)$ fashion, if $\mathbb{P} = \mathbb{P}^X$ is a Gaussian measure. The key part in the extension is the Gaussian integration-by-parts formula

$$\mathbb{E}\left[\langle \mathrm{D}F, h \rangle_{\mathscr{H}}\right] = \mathbb{E}\left[FW(h)\right],$$

$h \in \mathscr{H}$, $h \in \mathscr{S}$, which implies the closability of the Malliavin derivative as an operator $\mathrm{D}\colon \mathscr{L}^2(\Omega) \to \mathscr{L}^2(\Omega; \mathscr{I})$, as shown in Nualart [8], Proposition 1.2.1. However, in the pathwise setting the closability of operator D is not available. On the other hand, surprisingly such requirement is not needed in order to establish our results.

The next lemma relates the pathwise Malliavin derivative to the Fréchet derivative. A similar result can be found in Nualart and Saussereau [10] for the particular case of the fractional Brownian motion.

Lemma 3.1 *Let $f \in \mathscr{S}$ and $y \in \mathscr{L}^2$. Then*

$$\langle \nabla f(x), \mathrm{I}y \rangle_{\mathscr{L}^2} = \langle \mathrm{D}f(x), y \rangle_{\mathscr{L}^2}$$

Proof Straightforward calculations yield, with $e_k \in \mathscr{E}$ and $x \in \mathscr{L}^2$, that

$$\begin{aligned}
&f(x + \mathrm{I}y) \\
&= g\left(\int_0^1 e_1(t)\, \mathrm{d}(x(t) + \mathrm{I}y(t)), \ldots, \int_0^1 e_n(t)\, \mathrm{d}(x(t) + y(t)) \right) \\
&= g\left(\int_0^1 e_1(t)\, \mathrm{d}x(t) + \int_0^1 e_1(t)\, \mathrm{d}\mathrm{I}y(t), \ldots, \int_0^1 e_n(t)\, \mathrm{d}x(t) + \int_0^1 e_n(t)\, \mathrm{d}\mathrm{I}y(t) \right) \\
&= g\left(\int_0^1 e_1(t)\, \mathrm{d}x(t) + \int_0^1 e_1(t) y(t)\, \mathrm{d}t, \ldots, \int_0^1 e_n(t)\, \mathrm{d}x(t) + \int_0^1 e_n(t) y(t)\, \mathrm{d}t \right) \\
&= f\left(z_1 + \langle y_1, y \rangle_{\mathscr{L}^2}, \ldots, z_1 + \langle y_n, y \rangle_{\mathscr{L}^2} \right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\langle \nabla f(x), \mathrm{I}y \rangle_{\mathscr{L}^2} &= \nabla_{\mathrm{I}y} f(x) \\
&= \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon \mathrm{I}y) - f(x)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{f(z_1 + \varepsilon \langle y_1, y \rangle_{\mathscr{L}^2}, \ldots, z_1 + \varepsilon \langle y_n, y \rangle_{\mathscr{L}^2}) - f(z_1, \ldots, z_n)}{\varepsilon} \\
&= \sum_{k=1}^{n} \frac{\partial}{\partial z_k} f(z_1, \ldots, z_n) \langle y_k, y \rangle_{\mathscr{L}^2} \\
&= \langle \mathrm{D}f(x), y \rangle_{\mathscr{L}^2}
\end{aligned}$$

proving the claim. □

We also need the following two lemmas in order to establish our novel integration-by-parts formulas in Sect. 4.

Lemma 3.2 *Let $X = (X_t)_{t \in [0,1]}$ be a centered Gaussian process satisfying (2.3), and let* **D** *denote the associated classical Malliavin derivative. Then for every $f \in \mathscr{S}$ we have $f(X) \in \mathrm{dom}(\mathbf{D})$ and that $\mathrm{D}_t f(X) = \mathbf{D}_t f(X)$ for every $t \in [0, 1]$. More generally, for every $m \in \mathbb{N}$, and $t_1, \ldots, t_m \in [0, 1]$, it holds that, $\mathrm{D}_{t_m, \ldots, t_1} f(X) \in \mathrm{dom}(\mathbf{D})$, and*

$$\mathrm{D}_t \left(\mathrm{D}_{t_m, \ldots, t_1} f(X) \right) = \mathrm{D}_{t, t_m, \ldots, t_1} f(X) = \mathbf{D}_{t, t_m, \ldots, t_1} f(X)$$

for every $t \in [0, 1]$.

Proof By Lemma 2.2 the paths of the process X belong to $\mathscr{L}^2$. Moreover, $\mathscr{E} \subset \mathscr{I}$, where $\mathscr{I}$ stands for the associated integrand space of Gaussian process X. Furthermore, for every $e \in \mathscr{E}$, the pathwise integral $\int_0^1 e(t)\mathrm{d}X_t$ coincides with the abstract Wiener integral of Definition 2.6. Thus, the claim follows. □

Lemma 3.3 *Let $f \in \mathscr{S}$, $K \in \mathscr{L}^2 \times \mathscr{L}^2$, and let X have paths in $\mathscr{L}^2$. Then $f(X)$ is twice pathwise Malliavin differentiable and the mapping $\mathrm{D}^2_{t,\cdot} f(X)$ belongs to* $\mathrm{dom}(\mathrm{K}^*)$.

Proof Second order pathwise Malliavin differentiability is obvious. Indeed, we have

$$\mathrm{D}^2_{t,\cdot} f(X) = \sum_{1\le k,l\le n} \frac{\partial^2}{\partial z_k \partial z_l} g\left(\int_0^1 e_1(t)\mathrm{d}X_t, \ldots, \int_0^1 e_n(t)\mathrm{d}X_t\right)(e_k \otimes e_l)(t,\cdot).$$

This also shows that $\mathrm{D}^2_{t,\cdot} f(X) \in \mathrm{dom}(\mathrm{K}^*)$, since $\mathscr{E} \subset \mathrm{dom}(\mathrm{K}^*)$. □

4 Integration-by-parts characterization of Gaussian processes

We begin with the following stronger formulation of the integration-by-parts characterization.

Theorem 4.1 (General Gaussian Processes, Strong Version) *The co-ordinate process $X\colon \Omega \to \mathscr{L}^2$ is centered Gaussian with Fredholm kernel $K \in \mathscr{L}^2 \times \mathscr{L}^2$ if and only if*

$$\mathbb{E}\left[X_t \mathrm{D}_t f(X)\right] = \mathbb{E}\left[\int_0^1 K(t,s)\mathrm{K}^*\left[\mathrm{D}^2_{t,\cdot} f(X)\right](s)\,\mathrm{d}s\right] \tag{4.1}$$

for all $t \in [0,1]$ and $f \in \mathscr{S}$.

Remark 4.1 If $X_0 = 0$ and the kernel K is left-continuous and of bounded variation in its first argument, then we can reformulate (4.1) as

$$\mathbb{E}\left[X_t \mathrm{D}_t f(X)\right] = \mathbb{E}\left[\int_0^1\int_0^1 K(t,s)\mathrm{D}^2_{t,u} f(X)\, K(\mathrm{d}u,s)\,\mathrm{d}s\right].$$

Proof of Theorem 4.1 *"If" part:* Suppose the co-ordinate process $X\colon \Omega \to \mathscr{L}^2$ satisfies (4.1). We begin by considering the covariance function of X, which will justify the use of the Fubini theorem later and make a tedious variance calculations unnecessary. For this, take $f(X) = \frac{1}{2}X_u^2$ for some $u \in [0,1]$. Then $f \in \mathscr{S}$. We have $\mathrm{D}_t f(X) = X_u \mathbf{1}_u(t)$ and $\mathrm{D}_{t,s} f(X) = \mathbf{1}_u(s)\mathbf{1}_u(t)$. Consequently, (4.1) yields

$$\begin{aligned}\mathbb{E}\left[X_t X_u\right]\mathbf{1}_u(t) &= \mathbb{E}\left[\int_0^1 K(t,s)\mathrm{K}^*\left[\mathbf{1}_u\right](s)\,\mathrm{d}s\right]\mathbf{1}_u(t)\\ &= \int_0^1 K(t,s)K(u,s)\,\mathrm{d}s\,\mathbf{1}_u(t)\\ &= R(t,u)\mathbf{1}_u(t).\end{aligned} \tag{4.2}$$

This shows that X has the covariance function R given by the Fredholm kernel K. In particular, we have

$$\mathbb{E}\left[X_t^2\right] = \int_0^1 K(t,s)^2\,\mathrm{d}s,$$

and since $K \in \mathscr{L}^2 \times \mathscr{L}^2$, we have $\int_0^1 \mathbb{E}\left[X_t^2\right] \mathrm{d}t < \infty$ which justifies the use of the Fubini theorem in the rest of the proof. Next we are going to show that any finite linear combination

$$Z = \sum_{k=1}^{n} a_k \left(X_{t_k} - X_{t_{k-1}}\right) = \int_0^1 e(t)\, \mathrm{d}X_t$$

with $e = \sum_k^n a_k \mathbf{1}_{(t_{k-1}, t_k]} \in \mathscr{E}$ is a Gaussian random variable. Now, note that for every θ the complex-valued exponential functional $e^{\mathrm{i}\theta Z} = \cos(\theta Z) + \mathrm{i}\sin(\theta Z)$ belongs to $\mathscr{S}$, meaning that the real and imaginary parts both belong to $\mathscr{S}$. Let φ be the characteristic function of Z. Then

$$\begin{aligned} \mathrm{D}_t \mathrm{e}^{\mathrm{i}\theta Z} &= \mathrm{i}\theta e(t)\, \mathrm{e}^{\mathrm{i}\theta Z}, \\ \mathrm{D}^2_{t,s} \mathrm{e}^{\mathrm{i}\theta Z} &= -\theta^2 e(t) e(s)\, \mathrm{e}^{\mathrm{i}\theta Z}. \end{aligned}$$

Hence $\mathbb{E}\left[X_t \mathrm{D}_t \mathrm{e}^{\mathrm{i}\theta Z}\right] = \mathrm{i}\theta e(t) \mathbb{E}\left[X_t \mathrm{e}^{\mathrm{i}\theta Z}\right]$. Also, by a direct application of Fubini theorem

$$\begin{aligned} &\mathbb{E}\left[\int_0^1 K(t,s) \mathrm{K}^*\left[\mathrm{D}^2_{t,\cdot}\, \mathrm{e}^{\mathrm{i}\theta Z}\right](s)\, \mathrm{d}s\right] \\ &= -\mathbb{E}\left[\int_0^1 K(t,s) \mathrm{K}^*\left[\theta^2 e(t) e(\cdot) \mathrm{e}^{\mathrm{i}\theta Z}\right](s)\, \mathrm{d}s\right] \\ &= -\theta^2 e(t) \mathbb{E}\left[\int_0^1 K(t,s) \mathrm{K}^*\left[e(\cdot) \mathrm{e}^{\mathrm{i}\theta Z}\right](s)\, \mathrm{d}s\right] \\ &= -\theta^2 e(t) \int_0^1 K(t,s) e^*(s)\, \mathrm{d}s\ \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\theta Z}\right], \end{aligned}$$

where we have denoted $e^* = \mathrm{K}^* e$. Consequently, the integration-by-parts formula (4.1) yields

$$\mathrm{i}\,\mathbb{E}\left[X_t \mathrm{e}^{i\theta Z}\right] = -\theta \int_0^1 K(t,s) e^*(s)\, \mathrm{d}s\ \varphi(\theta). \tag{4.3}$$

By Fubini theorem justified by the covariance computation (4.2), we also have

$$\varphi'(\theta) = \mathbb{E}\left[\mathrm{i}Z\, \mathrm{e}^{\mathrm{i}\theta Z}\right].$$

Thus we obtain by several application of (4.3) that $\varphi'(\theta) = -c\theta\, \varphi(\theta)$, where we have denoted

$$c = \int_0^1 \left(\sum_{k=1}^{n} a_k \left(K(t_k, s) - K(t_{k-1}, s)\right) e^*(s)\right) \mathrm{d}s < \infty.$$

This implies that $\varphi(\theta) = \mathrm{e}^{-\frac{1}{2}c\theta^2}$, and since φ is a characteristic function, $c > 0$. Consequently, Z is a centered Gaussian random variable with variance c.

"Only if" part: Since the co-ordinate process $X\colon \Omega \to \mathscr{L}^2$ is Gaussian, we have the full power of Malliavin calculus at our disposal. In particular, we can use Proposition 2.1 with $F = \mathrm{D}_t f(X)$ and $G = X_t$. Since $\mathbb{E}[X_t] = 0$, we obtain

$$\mathbb{E}\left[X_t \mathrm{D}_t f(X)\right] = \mathbb{E}\left[\langle \mathrm{D}^2_{t,\cdot} f(X), -\mathrm{DL}^{-1} X_t \rangle_{\mathscr{I}}\right]$$

But $-\mathrm{DL}^{-1}X_t = \mathbf{1}_t$ and K^* is an isometry between $\mathscr{I}$ and $\mathscr{L}^2$. Therefore, by noticing that $\mathrm{K}^*\mathbf{1}_t(s) = K(t,s)$, we obtain

$$\begin{aligned}\mathbb{E}\left[X_t \mathrm{D}_t f(X)\right] &= \mathbb{E}\left[\langle \mathrm{D}^2_{t,\cdot} f(X), \mathbf{1}_t\rangle_{\mathscr{I}}\right] \\ &= \mathbb{E}\left[\langle \mathrm{K}^*\mathrm{D}^2_{t,\cdot} f(X), \mathrm{K}^*\mathbf{1}_t\rangle_{\mathscr{L}^2}\right] \\ &= \mathbb{E}\left[\int_0^1 \mathrm{K}^*\left[\mathrm{D}^2_{t,\cdot} f(X)\right](s)\mathrm{K}^*\mathbf{1}_t(s)\,\mathrm{d}s\right] \\ &= \mathbb{E}\left[\int_0^1 \mathrm{K}^*\left[\mathrm{D}^2_{t,\cdot} f(X)\right](s)K(t,s)\,\mathrm{d}s\right]\end{aligned}$$

showing the claim. □

Remark 4.2 It is classical that a random variable $X \approx \mathscr{N}(0,\sigma^2)$ if and only if its characteristic function φ_X satisfies $\varphi_X'(\theta) \approx -\sigma^2\theta\varphi_X(\theta)$. The latter is equivalent to

$$\mathbb{E}\left[Xe^{\mathrm{i}\theta X}\right] \approx \mathrm{i}\,\sigma^2\,\theta\,\mathbb{E}\left[e^{\mathrm{i}\theta X}\right]. \tag{4.4}$$

Hence, as a direct consequence, relation (4.4) is also equivalent to the fact that for a given diffusive (satisfying the chain rule) gradient operator D on the space of random variables, it holds that

$$\mathbb{E}\left[XDe^{\mathrm{i}\theta X}\right] \approx \sigma^2\,\mathbb{E}\left[D^2e^{\mathrm{i}\theta X}\right].$$

For example, in the setting of Theorem 4.1 and for a random variable X_t, by considering the functional $f(X) = e^{\mathrm{i}\theta X_t}$, one can easily infer that

$$\mathbb{E}\left[X_t e^{\mathrm{i}\theta X_t}\right] = \mathrm{i}\,\theta\int_0^1 K(t,s)^2\,\mathrm{d}s\,\mathbb{E}\left[e^{\mathrm{i}\theta X_t}\right].$$

This implies that $X_t \sim \mathscr{N}(0,\sigma^2)$ with $\sigma^2 = \int_0^1 K(t,s)^2\mathrm{d}s$. Indeed, Theorem 4.1 is a functional version of the aforementioned considerations in order to capture the Gaussian structure of X as a process.

If we have additional information on the co-ordinate process, then we can obtain a weaker integration-by-parts characterization. This is the topic of the next theorem.

Theorem 4.2 (General Gaussian Processes, Weak Version) *Let $K \in \mathscr{L}^2 \times \mathscr{L}^2$ be a square integrable kernel. Assume that the co-ordinate process $X\colon \Omega \to \mathscr{L}^2$ satisfies $X \in \mathscr{L}^2(\mathrm{d}t \otimes \mathbb{P})$, i.e.*

$$\int_0^1 \mathbb{E}\left[X_t^2\right]\mathrm{d}t < \infty. \tag{4.5}$$

Then X is centered Gaussian with the Fredholm kernel K if and only if

$$\mathbb{E}\left[\int_0^1 X_t \mathrm{D}_t f(X)\,\mathrm{d}t\right] = \mathbb{E}\left[\int_0^1\int_0^1 K(t,s)\mathrm{K}^*\left[\mathrm{D}^2_{t,\cdot} f(X)\right](s)\,\mathrm{d}s\mathrm{d}t\right] \tag{4.6}$$

for all $f \in \mathscr{S}$.

Before proving Theorem 4.2, let us consider its similarities and differences to the integration-by-parts characterization of finite-dimensional Gaussian vectors.

Remark 4.3 A random vector $X = (X_1, \ldots, X_d)$ is centered Gaussian with covariance matrix R if and only if

$$\mathbb{E}\left[\sum_{i=1}^{d} X_i \frac{\partial}{\partial x_i} f(X)\right] = \mathbb{E}\left[\sum_{i=1}^{d}\sum_{j=1}^{d} \mathrm{R}_{ij} \frac{\partial^2}{\partial x_i \partial x_j} f(X)\right]$$

for all smooth $f : \mathbb{R}^d \to \mathbb{R}$ such that the expectations above exist. Thus, by a simple analogy, one would guess (wrongly!) that a process is Gaussian if and only if

$$\mathbb{E}\left[\int_0^1 X_t \mathrm{D}_t f(X)\, \mathrm{d}t\right] = \mathbb{E}\left[\int_0^1\int_0^1 R(t,s)\mathrm{D}^2_{t,s} f(X)\, \mathrm{d}s\mathrm{d}t\right].$$

This formula is not, however, true even for the Brownian motion. It seems that there is no integration-by-parts formula in terms of the covariance directly and the simplest formula one can obtain is (4.6) that is given in terms of the Fredholm kernel.

Proof of Theorem 4.2 By (4.5) and the Fubini theorem, the weak integration-by-parts formula (4.6) follows from the strong integration-by-parts formula (4.1) by integrating with respect to t over the interval $[0, 1]$.

Conversely, suppose formula (4.6) holds for all $f \in \mathscr{S}$. Let $u \in (0, 1]$ be chosen arbitrary and take $f \in \mathscr{S}$ such that $f(X)$ depends on X only through its path up to time u. Then, by definition of pathwise Malliavin derivative, we infer that $\mathrm{D}_t f(X) = 0$ for all $t > u$. Consequently, using the Fubini theorem again, (4.6) becomes

$$\int_0^u \mathbb{E}\left[X_t \mathrm{D}_t f(X)\right] \mathrm{d}t = \int_0^u \mathbb{E}\left[\int_0^1 K(t,s)\mathrm{K}^*\left[\mathrm{D}^2_{t,\cdot} f(X)\right](s)\, \mathrm{d}s\right] \mathrm{d}t.$$

Since the latter identity holds for arbitrary $u \in (0, 1]$, a direct application of fundamental Theorem of calculus ensures that the formula (4.1) takes place for Lebesgue almost every t, and for every $f \in \mathscr{S}$. Finally, we note that under assumption (4.5) the functions $t \in [0, 1] \mapsto \mathbb{E}\left[X_t \mathrm{D}_t f(X)\right]$ and

$$t \in [0, 1] \mapsto \mathbb{E}\left[\int_0^1 K(t,s)\mathrm{K}^*\left[\mathrm{D}^2_{t,\cdot} f(X)\right](s)\, \mathrm{d}s\right]$$

belong to $\mathscr{L}^1(\mathrm{d}t)$. Now the rest of the proof follows similar lines as the proof of Theorem 4.1. □

Corollary 4.1 (Brownian Motion) *The co-ordinate process W satisfying assumption (4.5) is the Brownian motion if and only if*

$$\mathbb{E}\left[\int_0^1 W_t\, \mathrm{D}_t f(W)\, \mathrm{d}t\right] = \mathbb{E}\left[\int_0^1\int_0^t \mathrm{D}^2_{t,s} f(W)\, \mathrm{d}s\mathrm{d}t\right] \tag{4.7}$$

for all $f \in \mathscr{S}$.

Proof The Brownian motion is a Gaussian Fredholm process with kernel $I(t, s) = \mathbf{1}_t(s)$. The claim follows from this by noticing that $I(\mathrm{d}u, s) = \delta_s(\mathrm{d}u)$, (see Example 2.2), where δ_s is the unit mass at s. □

Corollary 4.2 (Gaussian Martingales) *The co-ordinate process M satisfying assumption (4.5) is a Gaussian martingale with bracket $\langle M\rangle$ if and only if*

$$\mathbb{E}\left[\int_0^1 M_t\, \mathrm{D}_t f(M)\, \mathrm{d}t\right] = \mathbb{E}\left[\int_0^1\int_0^t \mathrm{D}^2_{t,s} f(M)\, \mathrm{d}\langle M\rangle_s\, \mathrm{d}t\right]$$

for all $f \in \mathscr{S}$.

Proof By using a time-change, we observe that Gaussian martingales are Gaussian Fredholm processes with kernel $K(t,s) = I(\langle M\rangle_t, s)$. Consequently, $K(\mathrm{d}u, s) = \delta_{\langle M\rangle_s^{-1}}(\mathrm{d}u)$. Therefore,

$$\begin{aligned}\int_0^1 K(t,s)\mathrm{K}^*\left[\mathrm{D}^2_{t,\cdot} f(M)\right](s)\,\mathrm{d}s &= \int_0^{\langle M\rangle_t}\int_0^1 \mathrm{D}^2_{t,u} f(M)\,\delta_{\langle M\rangle_s^{-1}}(\mathrm{d}u)\mathrm{d}s \\ &= \int_0^{\langle M\rangle_t} \mathrm{D}^2_{t,\langle M\rangle_s^{-1}} f(M)\,\mathrm{d}s,\end{aligned}$$

from which the claim follows by making a change-of-variables. □

Corollary 4.3 (Brownian Bridge) *The co-ordinate process B satisfying assumption* (4.5) *is the Brownian bridge if and only if*

$$\mathbb{E}\left[\int_0^1 B_t \mathrm{D}_t f(B)\,\mathrm{d}t\right] = \mathbb{E}\left[\int_0^1\int_0^1\int_0^1 [\mathbf{1}_t(s) - t]\,\mathrm{D}^2_{t,u} f(B)\,[\delta_s(u) - \mathrm{d}u]\,\mathrm{d}s\mathrm{d}t\right] \quad (4.8)$$

for all $f \in \mathscr{S}$.

Proof The integration-by-parts formula (4.8) follows from the orthogonal representation $B_t = W_t - tW_1$ of the Brownian bridge. Indeed, we have

$$\begin{aligned}K(t,s) &= \mathbf{1}_t(s) - t,\\ K(\mathrm{d}u, s) &= \delta_s(u) - \mathrm{d}u.\end{aligned}$$

□

Remark 4.4 The Brownian bridge also admits the so-called canonical representation

$$B_t = \int_0^t \frac{1-t}{1-s}\,\mathrm{d}W_t,$$

Consequently, we have

$$\begin{aligned}K(t,s) &= \frac{1-t}{1-s}\mathbf{1}_t(s),\\ K(\mathrm{d}u, s) &= \frac{1}{1-s}\Big[(1-u)\delta_s(\mathrm{d}u) + \mathbf{1}_u(s)\mathrm{d}u\Big].\end{aligned}$$

It follows that an equivalent formulation for the integration-by-parts formula (4.8) is

$$\begin{aligned}&\mathbb{E}\left[\int_0^1 B_t \mathrm{D}_t f(B)\,\mathrm{d}t\right]\\ &= \mathbb{E}\left[\int_0^1\int_0^1\int_0^1 \frac{1-t}{1-s}\mathbf{1}_t(s)\mathrm{D}^2_{t,u} f(B)\left[\frac{1-u}{1-s}\delta_s(\mathrm{d}u) + \frac{\mathbf{1}_u(s)}{1-s}\mathrm{d}u\right]\mathrm{d}s\mathrm{d}t\right]\\ &= \mathbb{E}\left[\int_0^1\int_0^t \frac{1-t}{(1-s)^2}\int_0^1 \mathrm{D}^2_{t,u} f(B)\,[(1-u)\delta_s(\mathrm{d}u) + \mathbf{1}_u(s)\mathrm{d}u]\,\mathrm{d}s\mathrm{d}t\right]\\ &= \mathbb{E}\left[\int_0^1\int_0^t \frac{1-t}{(1-s)^2}\left[(1-s)\mathrm{D}^2_{t,s} f(B) + \int_0^s \mathrm{D}^2_{t,u} f(B)\,\mathrm{d}u\right]\mathrm{d}s\mathrm{d}t\right].\end{aligned}$$

Remark 4.5 Corollary 4.3 can be further extended to generalized bridges with respect to a general class of Gaussian processes by using the representation results of [14].

5 Connection with the abstract Wiener space approach of [11]

In this part, we will discuss the link between our results and the integration-by-parts formula of Shih [11]. For simplicity, we consider only the case of Brownian motion.

Let us denote by $\mathscr{B}$ the space $\mathscr{C}_0([0,1])$ of continuous functions on $[0,1]$, vanishing at zero. Let $X = W = (W_t)_{t\in[0,1]}$ be the standard Brownian motion. Then $\mathrm{K} = \mathrm{I}$ is just the integral operator and I^* is the identity operator (see Example 2.2). The integrand space is $\mathscr{I} = \mathscr{L}^2$ and the Cameron–Martin space is $\mathscr{R} = \mathrm{I}\mathscr{L}^2$. It is well-known that $\mathrm{I}\colon \mathscr{I} \to \mathscr{R} \subset \mathscr{B}$ embeds $\mathscr{I}$ densely into $\mathscr{B}$. Consequently, $(\mathrm{I}, \mathscr{R}, \mathscr{B})$ is an abstract Wiener space in the sense of Gross [4].

Now, the pathwise Malliavin derivative introduced in Sect. 3 (which coincides with the standard Malliavin derivative, see Lemma 3.2) satisfies

$$\langle \mathrm{D}f(x), y\rangle_{\mathscr{L}^2} = \nabla_{\mathrm{I}y} f(x)$$

for every $f \in \mathscr{S}, x, y \in \mathscr{L}^2$. It can be shown that $\nabla_{\mathrm{I}y} f(x)$ coincides with the Gross $\mathscr{R}$-derivative of $f(x)$ at $\mathrm{I}y$, see [11], p. 1241 or [14].

In [11], the following characterization of Gaussian measures on $\mathscr{B}$ was obtained: if X is a $\mathscr{B}$-valued random variable, then $\mathbb{P}$ is a Gaussian measure if and only if

$$\mathbb{E}\left[\langle X, \mathrm{D}f(X)\rangle_{\mathscr{B},\mathscr{B}^*}\right] = \mathbb{E}\left[\mathrm{Tr}_{\mathscr{R}}\mathrm{D}^2 f(X)\right] \tag{5.1}$$

for all $f : \mathscr{B} \to \mathbb{R}$ such that $\mathrm{D}^2 f(X)$ is trace-class on $\mathscr{R}$. Here the notation $\langle\cdot,\cdot,\rangle_{\mathscr{B},\mathscr{B}^*}$ means the usual dual pairing and $\mathrm{Tr}_{\mathscr{R}}\mathrm{D}^2 f(X)$ is the trace of the Malliavin derivative D^2 (also called the Gross Laplacian).

Let us discuss the connection between our result in Corollary 4.1 and the above formula (5.1). We will formally compute the left-hand side of (5.1). Let $\dot{W}$ be the so-called white noise, which is formally defined as a Gaussian process with covariance $\mathbb{E}[\dot{W}_t \dot{W}_s] = \delta(t-s)$. Recall that for every $g \in \mathscr{L}^2$, integrals of the form $\int_0^1 g(s)\dot{W}_s \mathrm{d}s$ are well-defined centered Gaussian random variables. Also recall the formula that links the dual pairing $\mathscr{B} - \mathscr{B}^*$ (recall that $\mathscr{B}^*$ is the space of signed measures) to the scalar product in $\mathscr{L}^2$ (see e.g. [4], p. 1241):

$$\langle \mathrm{I}x, h\rangle_{\mathscr{B},\mathscr{B}^*} = \langle x, \mathrm{I}'h\rangle_{\mathscr{L}^2}, \tag{5.2}$$

for any $x \in \mathscr{L}^2$ and $h \in \mathscr{B}^*$, where I' is the injection from $\mathscr{B}^*$ into $(\mathscr{L}^2)^* \simeq \mathscr{L}^2$ given by (see e.g. [16], Chapter 1)

$$\mathrm{I}'h(t) = \int_t^1 h(\mathrm{d}s).$$

Using (5.2), the left-hand side of (5.1) can be expressed as follows: by setting $x = \dot{W}$ and $h(\mathrm{d}u) = \mathrm{d}\mathrm{D}_u f(W)$, we obtain

$$\begin{aligned}
\mathbb{E}\left[\langle W, \mathrm{D}f(W)\rangle_{\mathscr{B},\mathscr{B}^*}\right] &= \mathbb{E}\left[\int_0^1 \dot{W}_t \left(\int_t^1 h(\mathrm{d}u)\right)\mathrm{d}t\right] \\
&= \mathbb{E}\left[\int_0^1 W_u\, h(\mathrm{d}u)\right] \\
&= \mathbb{E}\left[\int_0^1 W_u\, \mathrm{d}\mathrm{D}_u f(W)\right],
\end{aligned}$$

which does not coincide with the left-hand side of (4.7). Therefore, our formula in Corollary 4.1 is different from the Shih's formula (5.1).

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