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Mean Lipschitz spaces and a generalized Hilbert operator

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Abstract If μ is a positive Borel measure on the interval $[0, 1)$ we let $\mathcal{H}_\mu$ be the Hankel matrix $\mathcal{H}_\mu = (\mu_{n,k})_{n,k\geq 0}$ with entries $\mu_{n,k} = \mu_{n+k}$, where, for $n = 0, 1, 2, \ldots, \mu_n$ denotes the moment of order n of μ. This matrix induces formally the operator

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\mu_{n,k}a_k\right)z^n$$

on the space of all analytic functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$, in the unit disc $\mathbb{D}$. This is a natural generalization of the classical Hilbert operator. In this paper we study the action of the operators $\mathcal{H}_\mu$ on mean Lipschitz spaces of analytic functions.

Keywords Hankel matrix · Generalized Hilbert operator · Mean Lipschitz spaces · Carleson measures

Mathematics Subject Classification 30H10 · 47B35

1 Introduction and main results

Let $\mathbb{D}$ be the unit disc in the complex plane $\mathbb{C}$, and let $\mathcal{H}ol(\mathbb{D})$ denote the space of all analytic functions in $\mathbb{D}$. For $0 < r < 1$ and $f \in \mathcal{H}ol(\mathbb{D})$, we set

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$$M_p(r,f) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f(re^{i\theta})\right|^p d\theta\right)^{1/p}, \quad 0<p<\infty,$$
$$M_\infty(r,f) = \max_{|z|=r}|f(z)|.$$

For $0<p\le\infty$ the Hardy space H^p consists of those functions f, analytic in $\mathbb{D}$, for which

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(r,f) < \infty.$$

We refer to [9] for the theory of Hardy spaces.

The space $BMOA$ consists of those functions $f \in H^1$ whose boundary values have bounded mean oscillation on $\partial\mathbb{D}$. The Bloch space $\mathcal{B}$ consists of all analytic functions f in $\mathbb{D}$ with bounded invariant derivative:

$$f\in\mathcal{B} \quad\Leftrightarrow\quad \|f\|_{\mathcal{B}} \stackrel{\text{def}}{=} |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)\,|f'(z)| < \infty\,.$$

We mention [1,13,23] as excellent references for these spaces. Let us recall that $BMOA \subsetneq \mathcal{B}$.

If μ is a finite positive Borel measure on $[0,1)$ and $n=0,1,2,\dots$, we let μ_n denote the moment of order n of μ, that is, $\mu_n = \int_{[0,1)} t^n\,d\mu(t)$, and we let $\mathcal{H}_\mu$ be the Hankel matrix $(\mu_{n,k})_{n,k\ge0}$ with entries $\mu_{n,k}=\mu_{n+k}$. The matrix $\mathcal{H}_\mu$ induces formally an operator, also denoted $\mathcal{H}_\mu$, on spaces of analytic functions in the following way: if $f(z)=\sum_{k=0}^\infty a_k z^k \in \mathcal{H}ol(\mathbb{D})$ we define

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^\infty\left(\sum_{k=0}^\infty \mu_{n,k}a_k\right)z^n,$$

whenever the right hand side makes sense and defines an analytic function in $\mathbb{D}$.

If μ is the Lebesgue measure on $[0,1)$ the matrix $\mathcal{H}_\mu$ reduces to the classical Hilbert matrix $\mathcal{H} = \left((n+k+1)^{-1}\right)_{n,k\ge0}$, which induces the classical Hilbert operator $\mathcal{H}$. The Hilbert operator is known to be well defined on H^1 and bounded from H^p into itself, if $1<p<\infty$, but not if $p=1$ or $p=\infty$ [8].

The question of describing the measures μ for which the operator $\mathcal{H}_\mu$ is well defined and bounded on distinct spaces of analytic functions has been studied in a good number of papers (see [2,7,10,15,16,19–21]). The measures in question are Carleson-type measures.

If $I\subset\partial\mathbb{D}$ is an interval, $|I|$ will denote the length of I. The *Carleson square* $S(I)$ is defined as $S(I)=\{re^{it} : e^{it}\in I,\quad 1-\frac{|I|}{2\pi}\le r<1\}$.

If $s>0$ and μ is a positive Borel measure on $\mathbb{D}$, we shall say that μ is an s-Carleson measure if there exists a positive constant C such that

$$\mu\left(S(I)\right) \le C|I|^s, \quad \text{for any interval } I\subset\partial\mathbb{D}.$$

A 1-Carleson measure will be simply called a Carleson measure.

If μ is a positive Borel measure on $\mathbb{D}$, $0\le\alpha<\infty$, and $0<s<\infty$ we say that μ is an α-logarithmic s-Carleson measure [22] if there exists a positive constant C such that

$$\frac{\mu\left(S(I)\right)\left(\log\frac{2\pi}{|I|}\right)^\alpha}{|I|^s} \le C, \quad \text{for any interval } I\subset\partial\mathbb{D}.$$

A positive Borel measure μ on $[0,1)$ can be seen as a Borel measure on $\mathbb{D}$ by identifying it with the measure $\tilde{\mu}$ defined by

$$\tilde{\mu}(A) = \mu\left(A\cap[0,1)\right), \quad \text{for any Borel subset } A \text{ of } \mathbb{D}.$$

In this way a positive Borel measure μ on $[0, 1)$ is an s-Carleson measure if and only if there exists a positive constant C such that

$$\mu([t, 1)) \leq C(1-t)^s, \quad 0 \leq t < 1.$$

We have a similar statement for α-logarithmic s-Carleson measures.

Widom [21, Theorem 3. 1] (see also [20, Theorem 3] and [19, p. 42, Theorem 7. 2]) proved that $\mathcal{H}_\mu$ is a bounded operator from H^2 into itself if and only μ is a Carleson measure. Galanopoulos and Peláez [10] studied the operators $\mathcal{H}_\mu$ acting on H^1. The action of $\mathcal{H}_\mu$ on the Hardy spaces H^p, $0 < p \leq \infty$, has been studied in [7,15,16]. The papers [15] and [16] study also the operators $\mathcal{H}_\mu$ acting on distinct subspaces of the Bloch space, including $BMOA$, Besov spaces, and the Q_s-spaces.

In this paper we shall study the operators $\mathcal{H}_\mu$ acting on mean Lipschitz spaces of analytic functions.

If $f \in \mathcal{H}ol(\mathbb{D})$ has a non-tangential limit $f(e^{i\theta})$ at almost every $e^{i\theta} \in \partial\mathbb{D}$ and $\delta > 0$, we define

$$\omega_p(\delta, f) = \sup_{0<|t|\leq\delta} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f(e^{i(\theta+t)}) - f(e^{i\theta}) \right|^p d\theta \right)^{1/p}, \quad \text{if } 1 \leq p < \infty,$$

$$\omega_\infty(\delta, f) = \sup_{0<|t|\leq\delta} \left(\operatorname*{ess.sup}_{\theta\in[-\pi,\pi]} |f(e^{i(\theta+t)}) - f(e^{i\theta})| \right).$$

Then $\omega_p(\cdot, f)$ is the integral modulus of continuity of order p of the boundary values $f(e^{i\theta})$ of f.

Given $1 \leq p \leq \infty$ and $0 < \alpha \leq 1$, the mean Lipschitz space Λ^p_α consists of those functions $f \in \mathcal{H}ol(\mathbb{D})$ having a non-tangential limit almost everywhere for which $\omega_p(\delta, f) = O(\delta^\alpha)$, as $\delta \to 0$. If $p = \infty$ we write Λ_α instead of Λ^∞_α. This is the usual Lipschitz space of order α.

A classical result of Hardy and Littlewood [17] (see also [9, Chapter 5]) asserts that for $1 \leq p \leq \infty$ and $0 < \alpha \leq 1$, we have that $\Lambda^p_\alpha \subset H^p$ and

$$\Lambda^p_\alpha = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : M_p(r, f') = \mathrm{O}\left(\frac{1}{(1-r)^{1-\alpha}} \right) \right\}. \tag{1}$$

It is known that if $1 < p < \infty$ and $\alpha > \frac{1}{p}$ then each $f \in \Lambda^p_\alpha$ is bounded and has a continuous extension to the closed unit disc ([6], p. 88). This is not true for $\alpha = \frac{1}{p}$, because the function $f(z) = \log(1-z)$ belongs to $\Lambda^p_{1/p}$ for all $p \in (1, \infty)$. By a theorem of Hardy and Littlewood [9, Theorem 5.9] and of [6, Theorem 2.5] we have

$$\Lambda^p_{1/p} \subset \Lambda^q_{1/q} \subset BMOA \quad 1 \leq p < q < \infty.$$

The inclusion $\Lambda^p_{1/p} \subset BMOA$, $1 \leq p < \infty$ was proved to be sharp in a very strong sense in [3,11,12] using the following generalization of the spaces Λ^p_α which occurs frequently in the literature. Let $\omega : [0, \pi] \to [0, \infty)$ be a continuous and increasing function with $\omega(0) = 0$ and $\omega(t) > 0$ if $t > 0$. Then, for $1 \leq p \leq \infty$, the mean Lipschitz space $\Lambda(p, \omega)$ consists of those functions $f \in H^p$ such that

$$\omega_p(\delta, f) = O(\omega(\delta)), \quad \text{as } \delta \to 0.$$

With this notation we have $\Lambda^p_\alpha = \Lambda(p, \delta^\alpha)$.

The question of finding conditions on ω so that it is possible to obtain results on the spaces $\Lambda(p,\omega)$ analogous to those proved by Hardy and Littlewood for the spaces Λ^p_α has been studied by several authors (see [4–6]). We shall say that ω satisfies the Dini condition or that ω is a Dini-weight if there exists a positive constant C such that

$$\int_0^\delta \frac{\omega(t)}{t}\,dt \le C\omega(\delta), \quad 0<\delta<1.$$

We shall say that ω satisfies the condition b_1 or that $\omega \in b_1$ if there exists a positive constant C such that

$$\int_\delta^\pi \frac{\omega(t)}{t^2}\,dt \le C\frac{\omega(\delta)}{\delta}, \quad 0<\delta<1.$$

In order to simplify our notation, let $\mathcal{AW}$ denote the family of all functions $\omega : [0,\pi] \to [0,\infty)$ which satisfy the following conditions:

(i) ω is continuous and increasing in $[0,\pi]$.
(ii) $\omega(0)=0$ and $\omega(t)>0$ if $t>0$.
(iii) ω is a Dini-weight.
(iv) ω satisfies the condition b_1.

The elements of $\mathcal{AW}$ will be called admissible weights. Characterizations and examples of admissible weights can be found in [4,5].

Blasco and de Souza extended the above mentioned result of Hardy and Littlewood showing in [4, Th. 2.1] that if $\omega \in \mathcal{AW}$ then,

$$\Lambda(p,\omega) = \left\{ f \text{ analytic in } \mathbb{D} : M_p(r,f') = \mathrm{O}\left(\frac{\omega(1-r)}{1-r}\right), \text{ as } r\to 1\right\}.$$

In [3,11,12] it is proved that if $1 \le p < \infty$ and ω is an admissible weight such that

$$\frac{\omega(\delta)}{\delta^{1/p}} \to \infty, \text{ as } \delta \to 0,$$

then there exists a function $f \in \Lambda(p,\omega)$ which is a not a normal function (see [1] for the definition). Since any Bloch function is normal, if follows that for such admissible weights ω one has that $\Lambda(p,\omega) \not\subset \mathcal{B}$.

One of the main results in [16] is the following one.

Theorem A [16] *Let μ be a positive Borel measure on* $[0,1)$ *and let X be a Banach space of analytic functions in $\mathbb{D}$ with $\Lambda^2_{1/2} \subset X \subset \mathcal{B}$. Then the following conditions are equivalent.*

(i) *The operator $\mathcal{H}_\mu$ is well defined in X and, furthermore, it is a bounded operator from X into the Bloch space $\mathcal{B}$.*
(ii) *The operator $\mathcal{H}_\mu$ is well defined in X and, furthermore, it is a bounded operator from X into $\Lambda^2_{1/2}$.*
(iii) *The measure μ is a* 1*-logarithmic* 1*-Carleson measure.*
(iv) $\int_{[0,1)} t^n \log\frac{1}{1-t}\,d\mu(t) = \mathrm{O}\left(\frac{1}{n}\right)$.

A key ingredient in the proof of Theorem A is the fact that for any space X with $\Lambda^2_{1/2} \subset X \subset \mathcal{B}$ the functions $f \in X$ of the form $f(z)=\sum_{n=0}^\infty a_n z^n$ whose sequence of Taylor coefficients $\{a_n\}$ is a decreasing sequence of non-negative numbers are the same. Indeed, for such a function f and such a space X we have that $f \in X \Leftrightarrow a_n = \mathrm{O}\left(\frac{1}{n}\right)$. This result remains true if we substitute $\Lambda^2_{1/2}$ by $\Lambda^p_{1/p}$ for any $p>1$. That is, the following result holds:

Lemma 1 *Suppose that* $1 < p < \infty$ *and let* $f \in \mathcal{H}ol(\mathbb{D})$ *be of the form* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *with* $\{a_n\}_{n=0}^{\infty}$ *being a decreasing sequence of nonnegative numbers. If* X *is a subspace of* $\mathcal{H}ol(\mathbb{D})$ *with* $\Lambda^p_{1/p} \subset X \subset \mathcal{B}$, *then*

$$f \in X \quad \Leftrightarrow \quad a_n = \mathrm{O}\left(\frac{1}{n}\right).$$

Lemma 1 is a consequence of the following one which will be proved in Sect. 2.

Lemma 2 *Let* $1 < p < \infty$, $\omega \in \mathcal{AW}$ *and let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *with* $\{a_n\}_{n=0}^{\infty}$ *being a decreasing sequence of nonnegative numbers. Then*

$$f \in \Lambda(p, \omega) \quad \Leftrightarrow \quad a_n = \mathrm{O}\left(\frac{\omega(1/n)}{n^{1-1/p}}\right). \tag{2}$$

Using Lemma 1 and following the proof of Theorem A in [16], we obtain

Theorem 1 *Suppose that* $1 < p < \infty$. *Let* μ *be a positive Borel measure on* $[0, 1)$ *and let* X *be a Banach space of analytic functions in* $\mathbb{D}$ *with* $\Lambda^p_{1/p} \subset X \subset \mathcal{B}$. *Then the following conditions are equivalent.*

(i) *The operator* $\mathcal{H}_\mu$ *is well defined in* X *and, furthermore, it is a bounded operator from* X *into the Bloch space* $\mathcal{B}$.
(ii) *The operator* $\mathcal{H}_\mu$ *is well defined in* X *and, furthermore, it is a bounded operator from* X *into* $\Lambda^p_{1/p}$.
(iii) *The measure* μ *is a* 1-*logarithmic* 1-*Carleson measure.*
(iv) $\int_{[0,1)} t^n \log \frac{1}{1-t} d\mu(t) = \mathrm{O}\left(\frac{1}{n}\right)$.

As an immediate consequence of Theorem 1 we obtain the following result.

Corollary 1 *Let* μ *be a positive Borel measure on* $[0, 1)$ *and* $1 < p < \infty$. *Then the operator* $\mathcal{H}_\mu$ *is well defined in* $\Lambda^p_{1/p}$ *and, furthermore, it is a bounded operator from* $\Lambda^p_{1/p}$ *into itself if and only if* μ *is a* 1-*logarithmic* 1-*Carleson measure.*

Let us turn our attention now to the spaces $\Lambda(p, \omega)$ with $\frac{\omega(\delta)}{\delta^{1/p}} \nearrow \infty$, $\delta \searrow 0$ which, as noted before, are not included in the Bloch space. We have the following result which shows that the situation is different from the one covered in Theorem 1.

Theorem 2 *Let* $1 < p < \infty$, $\omega \in \mathcal{AW}$ *with* $\frac{\omega(\delta)}{\delta^{1/p}} \nearrow \infty$ *when* $\delta \searrow 0$. *The following conditions are equivalent:*

(i) *The operator* $\mathcal{H}_\mu$ *is well defined in* $\Lambda(p, \omega)$ *and, furthermore, it is a bounded operator from* $\Lambda(p, \omega)$ *into itself.*
(ii) *The measure* μ *is a Carleson measure.*

The proofs of Lemma 2 and Theorem 2 will be presented in Sect. 2. We close this section noticing that, as usual, we shall be using the convention that $C = C(p, \alpha, q, \beta, \dots)$ will denote a positive constant which depends only upon the displayed parameters $p, \alpha, q, \beta \dots$ (which sometimes will be omitted) but not necessarily the same at different occurrences. Moreover, for two real-valued functions E_1, E_2 we write $E_1 \lesssim E_2$, or $E_1 \gtrsim E_2$, if there exists a positive constant C independent of the arguments such that $E_1 \leq CE_2$, respectively $E_1 \geq CE_2$. If we have $E_1 \lesssim E_2$ and $E_1 \gtrsim E_2$ simultaneously then we say that E_1 and E_2 are equivalent and we write $E_1 \asymp E_2$.

2 Proofs of the main results

We start recalling that for a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ analytic in $\mathbb{D}$, the polynomials $\Delta_j f$ are defined as follows:

$$\Delta_j f(z) = \sum_{k=2^j}^{2^{j+1}-1} a_k z^k, \quad \text{for } j \geq 1,$$
$$\Delta_0 f(z) = a_0 + a_1 z.$$

The proof of Lemma 2 is based in the following result of Girela and González [14, Theorem 2].

Theorem B *Let $1 < p < \infty$ and let ω be an admissible weight. If $f \in \mathcal{H}ol(\mathbb{D})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then*

$$f \in \Lambda(p, \omega) \;\Leftrightarrow\; \|\Delta_N f\|_{H^p} = O\left(\omega\left(\frac{1}{2^N}\right)\right).$$

Proof of Lemma 2 By Lemma A of [18], since $a_n \searrow 0$, we have

$$\|\Delta_N f\|_{H^p} \asymp a_{2^N} 2^{N(1-1/p)}, \quad N \geq 1.$$

So by Theorem B we have that

$$f \in \Lambda(p, \omega) \Leftrightarrow a_{2^N} \lesssim \frac{\omega\left(1/2^N\right)}{2^{N(1-1/p)}}, \quad N \geq 1.$$

This easily implies (2). □

Lemma 3 *Suppose that $1 < p < \infty$. Let ν be a positive Borel measure on $[0, 1)$, and let $\omega \in \mathcal{AW}$ satisfying that $x^{-1/p}\omega(x) \nearrow \infty$, as $x \searrow 0$. Then following conditions are equivalent:*

(i) $\nu_n \lesssim \frac{\omega(1/n)}{n^{1-1/p}}$, $n \geq 2$.
(ii) $\nu([b, 1)) \lesssim (1-b)^{1-1/p}\omega(1-b)$, $b \in [0, 1)$.

Proof Suppose (i). Then we have that

$$\begin{aligned}
1 &\gtrsim \frac{n^{1-1/p}\,\nu_n}{\omega(1/n)} = \frac{n^{1-1/p}}{\omega(1/n)} \int_{[0,1)} t^n \, d\nu(t) \geq \frac{n^{1-1/p}}{\omega(1/n)} \int_{[1-1/n,1)} t^n \, d\nu(t) \\
&\geq \frac{n^{1-1/p}}{\omega(1/n)} \nu([1-1/n, 1)) \left(1 - \frac{1}{n}\right)^n \\
&\geq \frac{n^{1-1/p}}{\omega(1/n)} \nu([1-1/n, 1)) \inf_{m \geq 2} \left(1 - \frac{1}{m}\right)^m \\
&\gtrsim \frac{n^{1-1/p}}{\omega(1/n)} \nu([1-1/n, 1)).
\end{aligned}$$

So $\nu([1-1/n, 1)) \lesssim \frac{\omega(1/n)}{n^{1-1/p}}$ for $n \geq 2$.

Let now $b \in [1/2, 1)$. There exists $n \geq 2$ such that $1 - \frac{1}{n} \leq b < 1 - \frac{1}{n+1}$ so using the above we have that

$$\nu([b, 1)) \leq \nu([1-1/n, 1)) \lesssim \frac{\omega(1/n)}{n^{1-1/p}}.$$

This, and the facts that $\omega(1/n)n^{1/p} \le \omega(1/(n+1))(n+1)^{1/p}$ and that the weight ω increases give (ii).

Suppose now (ii). Then

$$\begin{aligned}\nu_n &= \int_{[0,1)} t^n \, d\nu(t) = n \int_0^1 \nu([t,1))t^{n-1}\, dt \\ &\lesssim n \int_0^1 (1-t)^{1-1/p}\omega(1-t)t^{n-1}\, dt \\ &= n \int_0^{1-\frac{1}{n}} + \int_{1-\frac{1}{n}}^1 \left((1-t)^{1-1/p}\omega(1-t)t^{n-1}\right) dt.\end{aligned}$$

The first integral can be estimated bearing in mind that $(1-t)^{-1/p}\omega(1-t) \nearrow \infty$ when $t \nearrow 1$ as follows

$$\begin{aligned} & n \int_0^{1-\frac{1}{n}} (1-t)^{1-1/p}\omega(1-t)t^{n-1}\, dt \\ & \le n^{1+1/p}\omega(1/n) \int_0^{1-\frac{1}{n}} (1-t)t^{n-1}\, dt \\ & = n^{1+1/p}\omega(1/n) \left(1-\frac{1}{n}\right)^n \left(\frac{1}{n} - \frac{n-1}{n(n+1)}\right) \\ & \lesssim \frac{\omega(1/n)}{n^{1-1/p}}.\end{aligned}$$

To estimate of the second integral we use that $(1-t)^{1-1/p}\omega(1-t) \searrow 0$ when $t \nearrow 1$ to obtain

$$\begin{aligned} & n \int_{1-\frac{1}{n}}^1 (1-t)^{1-1/p}\omega(1-t)t^{n-1}\, dt \\ & \le n^{1/p}\omega(1/n) \int_{1-\frac{1}{n}}^1 t^{n-1}\, dt \\ & = \frac{\omega(1/n)}{n^{1-1/p}} \left(1 - \left(1-\frac{1}{n}\right)^n\right) \\ & \lesssim \frac{\omega(1/n)}{n^{1-1/p}}.\end{aligned}$$

Then (i) follows. □

Proof of Theorem 2 (i) ⇒ (ii) Suppose that $\mathcal{H}_\mu : \Lambda(p,\omega) \to \Lambda(p,\omega)$ is bounded. By Lemma 2 we have that the function f defined by $f(z) = \sum_{n=1}^\infty \frac{\omega(1/n)}{n^{1-1/p}} z^n$ belongs to the space $\Lambda(p,\omega)$ so, by the hypothesis, $\mathcal{H}_\mu(f)$ belongs also to $\Lambda(p,\omega)$. Now

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^\infty \left(\sum_{k=1}^\infty \frac{\omega(1/k)}{k^{1-1/p}}\, \mu_{n+k}\right) z^n.$$

Notice that $\sum_{k=1}^\infty \frac{\omega(1/k)}{k^{1-1/p}}\, \mu_{n+k} \searrow 0,\ n \to \infty$, so using again Lemma 2 it holds that

$$\sum_{k=1}^\infty \frac{\omega(1/k)}{k^{1-1/p}}\, \mu_{n+k} = \int_{[0,1)} t^n \sum_{k=1}^\infty \frac{\omega(1/k)}{k^{1-1/p}}\, t^k \, d\mu(t) \lesssim \frac{\omega(1/n)}{n^{1-1/p}},$$

that is, the moments of the measure ν defined by

$$d\nu(t) = \sum_{k=1}^{\infty} \frac{\omega(1/k)}{k^{1-1/p}} t^k \, d\mu(t)$$

satisfy that

$$\nu_n \lesssim \frac{\omega(1/n)}{n^{1-1/p}},$$

so by Lemma 3 we have that $\nu([b, 1)) \lesssim (1-b)^{1-1/p}\omega(1-b)$, $b \in [0, 1)$.

According to the definition of the measure

$$\begin{aligned}(1-b)^{1-1/p}\omega(1-b) \gtrsim \nu([b, 1)) &= \int_{[b,1)} d\nu(t) \\ &= \int_{[b,1)} \sum_{k=1}^{\infty} \frac{\omega(1/k)}{k^{1-1/p}} t^k \, d\mu(t) \\ &\geq \mu\,([b, 1)) \sum_{k=1}^{\infty} \frac{\omega(1/k)}{k^{1-1/p}} b^k\end{aligned}$$

and the sum can be estimated as follows

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{\omega(1/k)}{k^{1-1/p}} b^k &\asymp \int_1^{\infty} \frac{\omega(1/x)}{x^{1-1/p}} b^x \, dx \\ &\geq \int_1^{\frac{1}{1-b}} \frac{\omega(1/x)}{x^{1-1/p}} b^x \, dx \\ &\geq (1-b)^{1-1/p}\omega(1-b) b^{\frac{1}{1-b}} \left(\frac{1}{1-b} - 1 \right) \\ &\gtrsim \frac{\omega(1-b)}{(1-b)^{1/p}}.\end{aligned}$$

Finally, putting all together we have that

$$\mu([b, 1)) \lesssim 1 - b$$

so μ is a Carleson measure.

(ii) $\Rightarrow$ (i) To prove this implication we need to use the integral operator I_μ considered in [7,10,15,16] which is closely related to the operator $\mathcal{H}_\mu$.

If μ is a positive Borel measure on $[0, 1)$ and $f \in \mathcal{H}ol(\mathbb{D})$, we shall write throughout the paper

$$I_\mu(f)(z) = \int_{[0,1)} \frac{f(t)}{1 - tz} \, d\mu(t),$$

whenever the right hand side makes sense and defines an analytic function in $\mathbb{D}$. It turns out that the operators $\mathcal{H}_\mu$ and I_μ are closely related. Indeed, as shown in the just mentioned papers, it turns out that if f is good enough $\mathcal{H}_\mu$ and $I_\mu(f)$ are well defined and coincide.

Suppose that μ is a Carleson measure supported on $[0, 1)$ and let $f \in \Lambda(p, \omega)$. We claim that

$$\int_{[0,1)} \frac{|f(t)|}{|1 - tz|} \, d\mu(t) < \infty. \tag{3}$$

Indeed, using Lemma 3 of [14] we have that

$$f \in \Lambda(p,\omega) \Rightarrow |f(z)| \lesssim \frac{\omega(1-|z|)}{(1-|z|)^{1/p}}, \quad z \in \mathbb{D}. \tag{4}$$

Then we obtain

$$\begin{aligned}\int_{[0,1)} \frac{|f(t)|}{|1-tz|}\, d\mu(t) &\le \frac{1}{1-|z|} \int_{[0,1)} |f(t)|\, d\mu(t)\\ &\lesssim \frac{1}{1-|z|} \int_{[0,1)} \frac{\omega(1-t)}{(1-t)^{1/p}}\, d\mu(t).\end{aligned}$$

If we choose $r \in [0,1)$ we can split the integral in the intervals $[0,r)$ and $[r,1)$. In the first one, as ω is an increasing weight we have

$$\begin{aligned}\int_{[0,r)} \frac{\omega(1-t)}{(1-t)^{1/p}}\, d\mu(t) &\le \omega(1) \int_{[0,r)} \frac{d\mu(t)}{(1-t)^{1/p}}\\ &\le \omega(1) \int_{[0,1)} \frac{d\mu(t)}{(1-t)^{1/p}}\\ &\lesssim 1,\end{aligned}$$

because μ is a Carleson measure. Using this and the condition $\frac{\omega(\delta)}{\delta^{1/p}} \nearrow \infty$, as $\delta \searrow 0$ we can estimate the other integral as follows

$$\begin{aligned}\int_{[r,1)} \frac{\omega(1-t)}{(1-t)^{1/p}}\, d\mu(t) &\le \frac{\omega(1-r)}{(1-r)^{1/p}} \int_{[r,1)} d\mu(t)\\ &\lesssim \omega(1-r)(1-r)^{1-1/p}\\ &\lesssim 1.\end{aligned}$$

So we have that for $f \in \Lambda(p,\omega)$ and $z \in \mathbb{D}$, (3) holds. This implies that $I_\mu(f)$ is well defined, and, using Fubini's theorem and standard arguments it follows easily that $\mathcal{H}_\mu(f)$ is also well defined and that, furthermore,

$$\mathcal{H}_\mu(f)(z) = I_\mu(f)(z), \quad z \in \mathbb{D}.$$

Now we have,

$$I_\mu(f)'(z) = \int_{[0,1)} \frac{tf(t)}{(1-tz)^2}\, d\mu(t), \quad z \in \mathbb{D},$$

so the mean of order p of $I_\mu(f)'$ has the form

$$M_p\left(r, I_\mu(f)'\right) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\int_{[0,1)} \frac{tf(t)}{(1-tre^{i\theta})^2}\, d\mu(t)\right|^p d\theta\right)^{1/p}.$$

Using again (4), the Minkowski inequality and a classical estimation of integrals we obtain that

$$\begin{aligned}M_p\left(r, I_\mu(f)'\right) &\lesssim \int_{[0,1)} |f(t)| \left(\int_{-\pi}^{\pi} \frac{d\theta}{|1-tre^{i\theta}|^{2p}}\right)^{1/p} d\mu(t)\\ &\lesssim \int_{[0,1)} \frac{|f(t)|}{(1-tr)^{2-1/p}}\, d\mu(t)\\ &\lesssim \int_{[0,1)} \frac{\omega(1-t)}{(1-t)^{1/p}(1-tr)^{2-1/p}}\, d\mu(t).\end{aligned}$$

At this point we split the integrals on the sets $[0, r)$ and $[r, 1)$.

In the first integral we use that $x^{-1/p}\omega(x) \nearrow \infty$, as $x \searrow 0$, and the fact that if μ is a Carleson measure (so that $\mu_n = \int_{[0,1)} t^n \, d\mu(t) \lesssim \frac{1}{n}$) to obtain

$$\begin{aligned}
\int_{[0,r)} \frac{\omega(1-t)}{(1-t)^{1/p}(1-tr)^{2-1/p}} \, d\mu(t) &\le \frac{\omega(1-r)}{(1-r)^{1/p}} \int_{[0,r)} \frac{d\mu(t)}{(1-tr)^{2-1/p}} \\
&\le \frac{\omega(1-r)}{(1-r)^{1/p}} \int_{[0,1)} \frac{d\mu(t)}{(1-tr)^{2-1/p}} \\
&\lesssim \frac{\omega(1-r)}{(1-r)^{1/p}} \sum_{n=1}^{\infty} n^{1-1/p} r^n \int_{[0,1)} t^n \, d\mu(t) \\
&\lesssim \frac{\omega(1-r)}{(1-r)^{1/p}} \sum_{n=1}^{\infty} \frac{r^n}{n^{1/p}} \\
&\lesssim \frac{\omega(1-r)}{(1-r)}.
\end{aligned}$$

In the second integral we use that ω is an increasing weight and the fact that the measure μ being a Carleson measure is equivalent to saying that the measure ν defined by $d\nu(t) = \frac{d\mu(t)}{(1-t)^{1/p}}$ is a $1 - \frac{1}{p}$-Carleson measure so that the moments ν_n of ν satisfy $\nu_n \lesssim \frac{1}{n^{1-\frac{1}{p}}}$. Then we obtain

$$\begin{aligned}
\int_{[r,1)} \frac{\omega(1-t)}{(1-t)^{1/p}(1-tr)^{2-1/p}} \, d\mu(t) &\le \omega(1-r) \int_{[r,1)} \frac{d\nu(t)}{(1-tr)^{2-1/p}} \\
&\le \omega(1-r) \int_{[0,1)} \frac{d\nu(t)}{(1-tr)^{2-1/p}} \\
&\lesssim \omega(1-r) \sum_{n=1}^{\infty} n^{1-1/p} r^n \int_{[0,1)} t^n \, d\nu(t) \\
&\lesssim \omega(1-r) \sum_{n=1}^{\infty} r^n \\
&= \frac{\omega(1-r)}{(1-r)}.
\end{aligned}$$

Therefore $I_\mu(f) \in \Lambda(p, \omega)$ and then the operator I_μ (and hence the operator $\mathcal{H}_\mu$) is bounded from $\Lambda(p, \omega)$ into itself. □

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