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Existence and approximation of solution to stochastic fractional integro-differential equation with impulsive effects

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Abstract In this paper, we study a stochastic fractional integro-differential equation with impulsive effects in separable Hilbert space. Using a finite dimensional subspace, semigroup theory of linear operators and stochastic version of the well-known Banach fixed point theorem is applied to show the existence and uniqueness of an approximate solution. Next, these approximate solutions are shown to form a Cauchy sequence with respect to an appropriate norm, and the limit of this sequence is then a solution of the original problem. Moreover, the convergence of Faedo–Galerkin approximation of solution is shown. In the last, we have given an example to illustrate the applications of the abstract results.

Keywords Analytic semigroup · Banach fixed point theorem · Faedo–Galerkin approximations · Hilbert space · Stochastic fractional integro-differential equation with impulsive effects · Mild solution

Mathematics Subject Classification 34G20 · 34K30 · 34K40 · 34K50 · 35K90

1 Introduction

Stochastic differential equation is an emerging field drawing attention from both theoretical and applied disciplines. In many real world phenomenon, the deterministic models often fluctuate due to random influences or noise, so we have to move from deterministic models to stochastic models. Stochastic differential equation involves randomness into mathematical description of the phenomenon and thus helps to understand more precise description of it, therefore, these differential equations play an important role in option pricing, forecast of

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the growth of population, electromagnetic theory, heat conduction in material with memory etc. The theory of stochastic differential equation can also be successfully applied to various problems outside mathematics for example in chemistry, economics, epidemiology, mechanics, finance and several fields in engineering. For recent works on the existence results of mild solutions for stochastic integro-differential equations see [7,11,12,20,33,36,39]. There have been good literature available on this field see [13,18,29,30].

The fractional differential equations are the generalization of ordinary differential equations to arbitrary non integer orders. The fact that the fractional derivative(integral) is an operator which includes integer order derivative(integral) as special case, is the reason why in present fractional differential equations becomes very popular and many applications are available. Fractional differential equations also provide an appropriate tool for the description of hereditary properties of various materials and processes. In recent years, fractional differential equations have drawn attention to many researchers and the solutions of fractional differential equations in analytical and numerical sense have been finding out. Fractional differential equations have lots of applications in the field of fluid flow, viscoelasticity, control theory of dynamical systems, electrical networks, probability and statistics, dynamical processes in self-similar and porous structures, electrochemistry of corrosion, optics and signal processing, nonlinear oscillations of earthquake, rheology, Bio-sciences etc. For more details on fractional differential equations and applications, we refer to books [21,27,32,35] and papers [8–10,14,25,39,40].

On the other hand, the differential equations involving impulse effects arise naturally in the description of phenomena that are subjected to sudden changes in their states, such as population dynamics, biological systems, optimal control, chemotherapeutic treatment in medicine, mechanical systems with impact, financial systems. In these models, the processes are characterized by the fact that they undergo abrupt changes of state at certain moments of time between intervals of continuous evolution. The presence of impulses implies that the trajectories of the system do not necessarily preserve the basic properties of the non-impulsive dynamical systems. For the study of impulsive differential equations, we refer to books [5,24] and papers [9,14,16,17,20,33,37,38].

The Faedo–Galerkin approximation provide a better tool for numerical approximation of the equation and to study more regular solutions of the equations by imposing higher consistency on the given data. This method may also be appropriate in the variational formulation to find the solutions of the equations under weaker assumptions on the data. Initially, Segal [34], Murakami [26], Heinz and Von Wahl [19] studied about the existence, uniqueness and finite-time blow-up of solutions to the functional Cauchy problem in a separable Hilbert space. Then using the existence results of Heinz and von Wahl [19], Bazley [1,2] showed the uniform convergence of the approximations to solutions of the semilinear wave equation on any closed subinterval. Using the idea of Bazley [1,2], Miletta [28] established the existence of the mild solution and proved convergence results to the functional Cauchy problem by using the Faedo–Galerkin approximations in a separable Hilbert space. Bahuguna and Srivastava [3] extended the results of Miletta [28] and considered the Faedo–Galerkin approximations of the solutions to the functional integro-differential equation. Later on, P. Balasubramaniam [6] studied the Faedo–Galerkin approximate solutions for Stochastic semilinear integro-differential equation in Hilbert Space. Recently, Chadha and Pandey [10] discussed the approximation of the solution for neutral fractional differential equation with nonlocal conditions in an arbitrary separable Hilbert space.

Motivated by the above work, the focus of this investigation is the approximation of mild solutions for the following stochastic fractional integro-differential equation with impulsive effects in separable Hilbert space

$$
\begin{aligned}
{}^{c}\mathbf{D}^{\rho}[u(t)+G(t,u_t)]+Au(t) &= F(t,u_t,u[b(u(t),t)]) \\
&\quad + \int_0^t a(t-s)k(s,u_s)d\omega(s),\ \ 0<t\leqslant T<\infty, t\neq t_j
\end{aligned}
\tag{1.1}
$$

$$
\Delta u|_{t=t_j} = I_j(u_{t_j}), \quad j=1,2,\ldots,r,\ \ r\in\mathbb{N}, \tag{1.2}
$$

$$
u(t) = \phi(t), \quad t\in[-\tau,0],\ \ \tau>0, \tag{1.3}
$$

where the state $u(\cdot)$ takes values in a separable Hilbert space $(\mathcal{H}, \|\cdot\|, (\cdot))$. ${}^{c}\mathbf{D}^{\rho}$ denotes the Caputo fractional derivative of order ρ, $0<\rho\leqslant 1$. $0<t_1<t_2<\cdots<t_r<T$ are pre fixed numbers. $\Delta u|_{t=t_j}$ denotes the jump of $u(t)$ at $t=t_j$ i.e. $\Delta u|_{t=t_j}=u(t_j^+)-u(t_j^-)$, where $u(t_j^+)$ and $u(t_j^-)$ represent the right and left limits of $u(t)$ at $t=t_j$ respectively. $A: D(A)\subset\mathcal{H}\to\mathcal{H}$ is a closed, positive definite, self adjoint linear operator with dense domain $D(A)$ such that $-A$ is the infinitesimal generator of an analytic semigroup $\{S(t): t\geqslant 0\}$ in $\mathcal{H}$. The map a is such that $a^2\in L^p_{loc}(0,\infty)$ for some $1<p<\infty$. F, G, I, b and k are suitably defined functions satisfying certain conditions to be stated later. u_t denotes the function defined by $u_t(\nu)=u(t+\nu)$ for $\nu\in[-\tau,0]$, here u_t represents the time history of the state from the time $t-\tau$ up to the present time t. Let $(\mathcal{K}, \|\cdot\|, (\cdot))$ be another separable Hilbert space. $\{\omega(t): t\geqslant 0\}$ is a given $\mathcal{K}$-valued Wiener process defined on a complete probability space $(\Omega, \Im, \{\Im_t\}_{t\geqslant 0}, \mathbb{P})$ with a finite trace nuclear covariance operator $Q\geqslant 0$. The initial data u_0 is an $\Im_0-$ adapted random variable independent of the winer process.

This manuscript develop a continuation and generalization of the existing results in the literature in two ways. First, we study the Faedo–Galerkin approximate solution to the impulsive stochastic fractional integro-differential equation with impulsive effects, to the best of our knowledge this problem has not been discussed earlier in the literature and second, the results in the manuscript constitute impulsive effects and stochastic invariant of some existing results, for instance,[4,9,10,22,23], which permit us to introduce the noise as well as impulses in the physical models. Further, in past few years, the integro-differential equation with impulsive effects have emerged as a new area of investigation as it describes a kind of system present in the real world, therefore, the stochastic fractional integro-differential equation with impulsive effects deserves a deep study.

2 Preliminaries and assumptions

Let $(\Omega, \Im, \{\Im_t\}_{t\geqslant 0}, \mathbb{P})$ be a filtered complete probability space such that the filtration $\{\Im_t\}_{t\geqslant 0}$ is a right continuous increasing family and $\Im_0$ contains all $\mathbb{P}$-null sets. $\omega(t)$ is a $\mathcal{K}$-valued Q-Wiener process with respect to $\{\Im_t\}_{t\geqslant 0}$. A $\mathcal{H}$-valued random variable is a $\Im$-measurable function $u(t): \Omega\to\mathcal{H}$ and the collection of random variables $S=\{u(t): \Omega\to\mathcal{H} | t\in[0,T]\}$ is called a stochastic process. We assume that there exists complete orthonormal system e_k, $k\geqslant 1$ in $\mathcal{K}$, a bounded sequence of nonnegative real numbers λ_k such that $Qe_k=\lambda_k e_k$, $k=1,2,\ldots,$ and a sequence $\{\widehat{B}_k\}$ of real valued mutually independent Brownian motions such that

$$
(\omega(t), e)_{\mathcal{K}} = \sum_{k=1}^{\infty}\sqrt{\lambda_k}(e_k,e)\widehat{B}_k(t), \quad e\in\mathcal{K},\ \ t\geqslant 0.
$$

In order to define stochastic integrals with respect to the Q-Wiener process $\omega(t)$, we introduce the subspace $\mathcal{K}_0=Q^{1/2}(\mathcal{K})$ of $\mathcal{K}$ which is endowed with the inner product $(\widetilde{u},\widetilde{v})_{\mathcal{K}_0}=$

$(Q^{-1/2}\widetilde{u}, Q^{-1/2}\widetilde{v})$ is a Hilbert space. We assume that $\Im_t = \Im_t^{\omega}$, where $\Im_t^{\omega}$ is the $\sigma-$algebra generated by $(\omega(s) : 0 \leqslant s \leqslant t)$. Let $\mathcal{L}_2(\Omega, \Im, \mathbb{P}; \mathcal{H}) \equiv \mathcal{L}_2(\Omega; \mathcal{H})$ denote the Banach space of strongly-measurable, square integrable random variables equipped with norm

$$\|u\|_{\mathcal{L}_2(\Omega,\mathcal{H})} = (\mathbb{E}\|u\|^2_{\mathcal{H}})^{1/2},$$

where $\mathbb{E}$ is defined as integration with respect to probability measure $\mathbb{P}$. Let $\mathcal{L}_2^0 = \mathcal{L}_2(\mathcal{K}_0, \mathcal{H})$ denote the space of all Hilbert-Schmidt operators from $\mathcal{K}_0$ to $\mathcal{H}$ with the norm

$$\|\phi\|^2_{\mathcal{L}_2^0} = Tr\left(\left(\phi Q^{1/2}\right)\left(\phi Q^{1/2}\right)^*\right),$$

for $\phi \in \mathcal{L}_2^0$. Clearly, for any bounded linear operators $\phi \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, this norm reduces to

$$\|\phi\|^2_{\mathcal{L}_2^0} = Tr((\phi Q \phi^*)) = \sum_{k=1}^{\infty} \|\sqrt{\lambda_k}\phi e_k\|^2.$$

Since $-A$ the infinitesimal generator of an analytic semigroup $\{S(t) : t \geqslant 0\}$ in $\mathcal{H}$. Therefore there exists constants $C \geqslant 1$ and $\delta \geqslant 0$ such that $\|S(t)\| \leqslant Ce^{\delta t}$, $t \geqslant 0$. Moreover

$$\left\|\frac{d^m}{dt^m}S(t)\right\| \leqslant C_m, \quad t > 0, \ m = 1, 2, \ldots$$

where C_m, $m = 1, 2, \ldots$ are some positive constants. Hence without loss of generality, we might accept $S(t)$ is uniformly bounded by C i.e. $\|S(t)\| \leqslant C$ and $0 \in \rho(-A)$, the resolvent set of $-A$. Then for $0 < \alpha \leqslant 1$, it is possible to define the fractional power A^{α} as a closed linear operator on its domain $D(A^{\alpha})$, being dense in $\mathcal{H}$ and we denote the Banach space $D(A^{\alpha})$ by $\mathcal{H}_{\alpha}$ endowed with the norm

$$\|u\|_{\alpha} = \|A^{\alpha}u\|, \quad u \in D(A^{\alpha}),$$

which is equivalent to the graph norm of A^{α}.

Lemma 2.1 *([31]) Let $-A$ be the infinitesimal generator of an analytic semigroup $\{S(t) : t \geqslant 0\}$ such that $\| S(t) \| \leqslant C$, for $t \geqslant 0$ and $0 \in \rho(-A)$. Then,*

1. *For $\alpha \in (0, 1]$, $D(A^{\alpha})$ is a Hilbert space.*
2. *The operator $A^{\alpha}S(t)$ is bounded for every $t > 0$ and*

$$\|AS(t)\| \leqslant Ct^{-1},$$
$$\|A^{\alpha}S(t)\| \leqslant \frac{C_{\alpha}}{t^{\alpha}}.$$

For more details on fractional power operators see Pazy [31].
Now, we have some basic definitions of fractional calculus.

Definition 2.2 [35] The fractional integral of order ρ for a function $F \in L^1(\mathbb{R}^+)$ is defined by

$$I_t^{\rho}F(t) = \frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1}F(s)ds, \quad t > 0, \quad \rho > 0.$$

Definition 2.3 [21] The Caputo fractional derivative of order ρ for a function $F \in C^{m-1}((0, T); \mathcal{H}) \cap \mathcal{L}^1((0, T); \mathcal{H})$ is defined by

$$^c\mathbf{D}_t^{\rho}F(t) = \frac{1}{\Gamma(m-\rho)}\int_0^t (t-s)^{m-\rho-1}F^m(s)ds,$$

where $m - 1 < \rho < m$, $m = [\rho] + 1$ and $[\rho]$ denotes the integral part of the real number ρ.

Lemma 2.4 *[13], For any $r \geqslant 1$ and for arbitrary $\mathcal{L}_0^2$-valued predictable process $h(\cdot)$,*

$$\sup_{s\in[0,t]} \mathbb{E}\left\| \int_0^s h(l)d\omega(l) \right\|^{2r} \leqslant C_r \left(\int_0^t \mathbb{E}\|h(s)\|^2_{\mathcal{L}_2^0} ds \right)^r, \quad \forall t \in [0, \infty).$$

where $C_r = (r(2r-1))^r$.

Let $C_t^\alpha = PC([-\tau, t], \mathcal{H}_\alpha)$ be the Banach space formed by all strongly measurable $\mathcal{H}_\alpha$-valued stochastic processes on $[-\tau, t]$ such that u is continuous everywhere except for a finite number of points t_j such that u is left continuous at t_j and the right limit $u(t_j^+)$ exists for $j = 1, 2, \ldots, r$, endowed with the supremum norm,

$$\|u\|_{t,\alpha} = \left(\sup_{s\in[-\tau,t]} \mathbb{E}\|A^\alpha u(s)\|^2 \right)^{1/2}.$$

Set $C_t^{\alpha-1} = PC([-\tau, t], \mathcal{H}_{\alpha-1}) = \{u \in C_t^\alpha : \mathbb{E}\|u(x) - u(y)\|^2_{\alpha-1} \leqslant L|x-y|^2, \forall x, y \in [-\tau, t]\}$, where L is a positive constant and $0 < \alpha \leqslant 1$.
In order to prove main results we require the following assumptions:

(H1) $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is a closed, positive definite, self adjoint linear operator with dense domain $D(A)$ such that A has a pure point spectrum $0 < \lambda_0 \leqslant \lambda_1 \leqslant \cdots$ and a corresponding complete orthonormal system $\{\psi_i\}$ so that

$$A\psi_i = \lambda_i \psi_i, \quad (\psi_i, \psi_j) = \delta_{ij},$$

where $(\cdot, \cdot)$ is the inner product in $\mathcal{H}$ and δ_{ij} is the Kronecker delta function i.e. $-A$ is the infinitesimal generator of an analytic semigroup $\{S(t) : t \geqslant 0\}$ in $\mathcal{H}$.

(H2) Let $U_1 \subset \mathrm{Dom}(G)$ is an open subset of $[0, T] \times C_0^{\alpha-1}$ and for each $(t, u) \in U_1$ there is a neighborhood $V_1 \subset U_1$. There exist positive constants $0 < \alpha < \beta < 1$, such that the function $A^\beta G$ is continuous for $(t, u) \in [0, T] \times C_0^{\alpha-1}$ such that

$$\mathbb{E}\|A^\beta G(t, u) - A^\beta G(s, v)\|^2 \leqslant L_G \left\{ |t-s|^2 + \mathbb{E}\|u - v\|^2_{0,\alpha-1} \right\},$$
$$\mathbb{E}\|A^\beta G(t, u)\|^2 \leqslant L_G,$$

where $(t, u), (s, v) \in V_1$ and L_G is a positive constant.

(H3) Let $U_2 \subset \mathrm{Dom}(F)$ is an open subset of $[0, T] \times C_0^\alpha \times \mathcal{H}_{\alpha-1}$ and for each $(t, u, v) \in U_2$ there is a neighborhood $V_2 \subset U_2$. The nonlinear map $F : [0, T] \times C_0^\alpha \times \mathcal{H}_{\alpha-1} \to \mathcal{H}$ satisfies

$$\mathbb{E}\|F(t, u, \widetilde{u}) - F(s, v, \widetilde{v})\|^2 \leqslant L_f \left[|t-s|^{2\gamma_1} + \mathbb{E}\|u - v\|^2_{0,\alpha} + \mathbb{E}\|\widetilde{u} - \widetilde{v}\|^2_{\alpha-1} \right],$$

where $0 < \gamma_1 \leqslant 1$, $(t, u, \widetilde{u}) \in V_2$, $(s, v, \widetilde{v}) \in V_2$ and L_f is a positive constant.

(H4) Let $U_3 \in \mathrm{Dom}(b)$ is an open subset of $\mathcal{H}_\alpha \times [0, T]$ and for each $(u, t) \in U_3$ there is a neighborhood $V_3 \subset U_3$. The map $b : \mathcal{H}_\alpha \times [0, T] \to [0, T]$ satisfies

$$|b(u, t) - b(v, s)|^2 \leqslant L_b \left[\mathbb{E}\|u - v\|^2_\alpha + |t-s|^{2\gamma_2} \right],$$

where $0 < \gamma_2 \leqslant 1$, $(u, t), (v, s) \in V_3$, $b(., 0) = 0$ and L_b is a positive constant.

(H5) The nonlinear map k is defined from $[0, T] \times C_0^\alpha$ into $\mathcal{L}_2^0$ and there exists a nonnegative function $L_k \in L^q_{loc}(0, \infty)$, where $1 < q < \infty$, $(1/p) + (1/q) = 1$, such that

$$\mathbb{E}\|k(t, u) - k(t, v)\|^2_{\mathcal{L}_2^0} \leqslant L_k \mathbb{E}\|u - v\|^2_{0,\alpha},$$

and

$$\mathbb{E}\|k(t,u)\|^2_{\mathcal{L}_2^0} \leqslant L_k.$$

(H6) The functions $I_j : C_0^\alpha \to \mathcal{H}_\alpha, j = 1, 2, \ldots, r$ are continuous and there exists positive constants L_j such that

$$\mathbb{E}\|I_j(u) - I_j(v)\|^2_\alpha \leqslant L_j \mathbb{E}\|u - v\|^2_{0,\alpha},$$

and

$$\mathbb{E}\|I_j(u)\|^2_\alpha \leqslant L_j.$$

Define the function $\widetilde{\phi}$ by

$$\widetilde{\phi}(t) = \begin{cases} \phi(t), & t \in [-\tau, 0]; \\ \phi(0), & t \in [0, T_0]. \end{cases}$$

Definition 2.5 [13] A stochastic process $\{u : [-\tau, T_0] \to \mathcal{H}_\alpha\}, 0 < T_0 \leqslant T$ is called a mild solution for the system (1.1)–(1.3) if

1. $u(t)$ is measurable, $\Im_t$-adapted and has Càdlàg paths on $t \in [-\tau, T_0]$.
2. $u(t) \in C_T^\alpha \cap C_T^{\alpha-1}$ and for every $t > s \geqslant 0$, the function $s \to AQ_\rho(t-s)G(s, u_s)ds$ is integrable such that u satisfies the following stochastic integral equation

$$u(t) = \begin{cases} \widetilde{\phi}(t), & t \in [-\tau, 0]; \\ S_\rho(t)[\widetilde{\phi}(0) + G(0, \widetilde{\phi}(0))] - G(t, u_t) + \int_0^t (t-s)^{\rho-1} Q_\rho(t-s) AG(s, u_s) ds & \\ \quad + \int_0^t (t-s)^{\rho-1} Q_\rho(t-s)[F(s, u_s, u[b(u(s), s)]) + \int_0^s a(s-r)k(r, u_r)d\omega(r)]ds & \\ \quad + \sum\limits_{0<t_j<t} S_\rho(t-t_j) I_j(u_{t_j}), & t \in [0, T_0]. \end{cases}$$

with initial value $\widetilde{\phi}(t) \in \mathcal{L}_2^0(\Omega, \mathcal{H}_\alpha)$ for all $t \in [-\tau, 0]$, where

$$S_\rho(t) = \int_0^\infty \zeta_\rho(\theta) S(t^\rho \theta) d\theta,$$

$$Q_\rho(t) = \rho \int_0^\infty \theta \zeta_\rho(\theta) S(t^\rho \theta) d\theta.$$

Here $\zeta_\rho(\theta) = \frac{1}{\rho}\theta^{1-\frac{1}{\rho}} \times \psi_\rho(\theta^{-\frac{1}{\rho}})$ is a probability density function defined on $(0, \infty)$ i.e., $\zeta_\rho(\theta) \geqslant 0, \int_0^\infty \zeta_\rho(\theta) d\theta = 1$ and

$$\psi_\rho(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-n\rho-1} \frac{\Gamma(n\rho+1)}{n!} \sin(n\pi\rho), 0 < \theta < \infty.$$

For more details see [15,40].

Lemma 2.6 ([40]) *The operators $\{S_\rho(t), t \geqslant 0\}$ and $\{Q_\rho(t), t \geqslant 0\}$ are bounded linear operators such that*

- **(i)** $\|S_\rho(t)z\| \leqslant C\|z\|$, $\|Q_\rho(t)z\| \leqslant \frac{\rho C}{\Gamma(1+\rho)}\|z\|$ *and* $\|A^\alpha Q_\rho(t)z\| \leqslant \frac{\rho C_\alpha \Gamma(2-\alpha) t^{-\rho\alpha}}{\Gamma(1+\rho(1-\alpha))}\|z\|$, *for any* $z \in \mathcal{H}$.
- **(ii)** *The families $\{S_\rho(t) : t \geqslant 0\}$ and $\{Q_\rho(t) : t \geqslant 0\}$ are strongly continuous.*
- **(iii)** *If $S(t)$ is compact, then $S_\rho(t)$ and $Q_\rho(t)$ are compact operators for any $t > 0$.*

3 Existance of approximate solutions

Let $\mathcal{H}_n$ denote the finite dimensional subspace of $\mathcal{H}$ spanned by $\{\psi_0, \psi_1, \ldots, \psi_n\}$ and let $P^n : \mathcal{H} \to \mathcal{H}_n$ be the corresponding projection operator for $n = 0, 1, 2, \ldots$.
For $R > 0$, set

$$\mathcal{B}_R = \left\{u \in C^{\alpha}_{T_0} \cap C^{\alpha-1}_{T_0} : \|u - \widetilde{\phi}\|^2_{T_0,\alpha} \leqslant R\right\}.$$

Assumptions $(H3)$–$(H4)$ and $u \in C^{\alpha}_{T_0}$ imply that $F(s, u_s, u[b(u(s), s)])$ is continuous on $[0, T_0]$. Therefore there exists a positive constant N such that

$$\mathbb{E}\|F(s, u_s, u[b(u(s), s)])\|^2 \leqslant L_f\left[T_0^{2\gamma_1} + R(1 + LL_b) + LL_b T_0^{2\gamma_2}\right] + N_0 = N,$$

where $N_0 = \mathbb{E}\|F(0, \widetilde{\phi}(0), \widetilde{\phi}(0))\|^2$. Choose $0 < T_0 < T$ such that

$$M(R) = N + \|a^2\|_{L^p(0,T_0)}\|L_k\|_{L^q(0,T_0)}, \tag{3.1}$$

$$\vartheta := 2\|A^{\alpha-\beta-1}\|^2 L_G + rC^2\sum_{j=1}^{r} L_j < 1/12, \tag{3.2}$$

$$\mathbb{E}\|(S_\rho(t) - I)A^{\alpha}[\widetilde{\phi}(0) + G_n(0, \widetilde{\phi}(0))]\|^2 + \mathbb{E}\|A^{\alpha-\beta}\|^2 L_G T_0^2(1 + L) \leqslant \frac{R}{12}, \tag{3.3}$$

$$\begin{aligned}&\left(\frac{\rho C_{1+\alpha-\beta}\Gamma(1-(\alpha-\beta))}{\Gamma(1+\rho(\beta-\alpha))}\right)^2 \frac{T_0^{2\rho(\beta-\alpha)-1}}{2\rho(\beta-\alpha)-1}L_G\\ &\quad+\left(\frac{\rho C_{\alpha}\Gamma(2-\alpha)}{\Gamma(1+\rho(1-\alpha))}\right)^2 M(R)\frac{T_0^{2\rho(1-\alpha)-1}}{2\rho(1-\alpha)-1} + rC^2\sum_{j=1}^{r} L_j \leqslant \frac{R}{12},\end{aligned} \tag{3.4}$$

$$\begin{aligned}&5\left(\frac{\rho C_{1+\alpha-\beta}\Gamma(1-\alpha+\beta)}{\Gamma(1+\rho(\beta-\alpha))}\right)^2 \frac{T_0^{2\rho(\beta-\alpha)-1}}{2\rho(\beta-\alpha)-1}L_G\|A^{-1}\|\\ &\quad+5\left(\frac{\rho C_{\alpha}\Gamma(2-\alpha)}{\Gamma(1+\rho(1-\alpha))}\right)^2 \frac{T_0^{2\rho(1-\alpha)-1}}{2\rho(1-\alpha)-1}\\ &\quad(L_f(2+LL_b) + M(R)) \leqslant 1 - \vartheta < 1.\end{aligned} \tag{3.5}$$

We define

$$\begin{gathered}G_n : [0, T_0] \times C^{\alpha-1}_t \to \mathcal{H};\ G_n(t, u_t) = G(t, P^n u_t),\\ F_n : [0, T_0] \times C^{\alpha}_t \times \mathcal{H}_{\alpha-1} \to \mathcal{H};\ F_n(t, u_t, u[b(u(t), t)]) = F(t, P^n u_t, P^n u[(b(P^n u(t), t)]),\\ k_n : [0, T_0] \times C^{\alpha}_t \to \mathcal{L}^0_2; k_n(t, u_t) = k(t, P^n u_t),\end{gathered}$$

and

$$I_{j,n} : C^{\alpha}_0 \to \mathcal{H}_{\alpha}; I_{j,n}(u_t) = I_j(P^n u_t).$$

Now, consider the map $\Phi_n : \mathcal{B}_R \to \mathcal{B}_R$ given by

$$(\Phi_n u)(t) = \begin{cases}\widetilde{\phi}(t), \quad t \in [-\tau, 0];\\ S_\rho(t)[\widetilde{\phi}(0) + G_n(0, \widetilde{\phi}(0))] - G_n(t, u_t) + \int_0^t (t-s)^{\rho-1}Q_\rho(t-s)AG_n(s, u_s)ds\\ \quad + \int_0^t (t-s)^{\rho-1}Q_\rho(t-s)[F_n(s, u_s, u[b(u(s), s)]) + \int_0^s a(s-r)k_n(r, u_r)d\omega(r)]ds\\ \quad + \sum_{0<t_j<t} S_\rho(t-t_j)I_j(u_{t_j}), \quad t \in [0, T_0].\end{cases} \tag{3.6}$$

Theorem 3.1 *Let* $(H1)$–$(H6)$ *hold and* $\widetilde{\phi}(t) \in \mathcal{L}_2^0(\Omega, \mathcal{H}_\alpha)$ *for all* $t \in [-\tau, 0]$. *Then there exists a unique* $u_n \in \mathcal{B}_R$ *such that* $\Phi_n u_n = u_n \; \forall \, n = 0, 1, 2, \ldots$.

Proof First we show that $\Phi_n u \in C_{T_0}^{\alpha} \cap C_{T_0}^{\alpha-1}$. Clearly $\Phi_n : C_{T_0}^{\alpha} \to C_{T_0}^{\alpha}$. Therefore we show that $\Phi_n u \in C_{T_0}^{\alpha-1}$ for any $u \in C_{T_0}^{\alpha-1}$. For $u \in C_{T_0}^{\alpha-1}$ and $0 < t' < t'' < T_0$, using Hölder inequality and inequality $\left(\sum_{i=1}^{n} a_i\right)^m \leqslant n^{m-1} \sum_{i=1}^{n} a_i^m$, where a_i are nonnegative constants, we have

$$
\begin{aligned}
&\mathbb{E}\|(\Phi_n u)(t'') - (\Phi_n u)(t')\|_{\alpha-1}^2 \\
&\leqslant 6\Big\{\|(S_\rho(t'') - S_\rho(t'))(\widetilde{\phi}(0) + G_n(0, \widetilde{\phi}(0)))\|_{\alpha-1}^2 \\
&\quad + \mathbb{E}\|A^{\alpha-\beta-1}\|^2 \|(A^\beta G_n(t'', u_{t''}) - A^\beta G_n(t', u_{t'}))\|^2 \\
&\quad + \int_0^{t'} \|A^{\alpha-\beta}((t''-s)^{\rho-1} Q_\rho(t''-s) \\
&\quad - (t'-s)^{\rho-1} Q_\rho(t'-s))\|^2 \mathbb{E}\|A^\beta G_n(s, u_s)\|^2 ds \\
&\quad + \int_{t'}^{t''} \|A^{\alpha-\beta}(t''-s)^{\rho-1} Q_\rho(t''-s)\|^2 \mathbb{E}\|A^\beta G_n(s, u_s)\|^2 ds \\
&\quad + \int_0^{t'} \|A^{\alpha-1}((t''-s)^{\rho-1} Q_\rho(t''-s) - (t'-s)^{\rho-1} Q_\rho(t'-s))\|^2 \\
&\quad \left[\mathbb{E}\|F_n(s, u_s, u[b(u(s), s)])\|^2 + \int_0^s |a(s-r)|^2 \mathbb{E}\|k_n(r, u_r)\|_{\mathcal{L}_2^0}^2 dr\right] ds \\
&\quad + \int_{t'}^{t''} \|A^{\alpha-1}(t''-s)^{\rho-1} Q_\rho(t''-s)\|^2 \Big[\mathbb{E}\|(F_n(s, u_s, b[h(u(s), s)]))\|^2 \\
&\quad + \int_0^s |a(s-r)|^2 \mathbb{E}\|k_n(r, u_r)\|_{\mathcal{L}_2^0}^2 dr\Big] ds \\
&\quad + r \sum_{j=1}^{r} \mathbb{E}\|(S_\rho(t''-t_j) - S_\rho(t'-t_j)) I_{j,n}(u_{t_j})\|_{\alpha-1}^2\Big\} \\
&\leqslant \sum_{i=1}^{7} J_i.
\end{aligned} \tag{3.7}
$$

For $u \in \mathcal{H}$, we have

$$
[S((t'')^\rho \theta) - S((t')^\rho \theta)]u = \int_{t'}^{t''} \frac{d}{dt} S(t^\rho \theta) u dt = \int_{t'}^{t''} \rho \theta t^{\rho-1} A S(t^\rho \theta) u dt.
$$

Therefore

$$
\begin{aligned}
J_1 &\leqslant 6\left(\int_0^\infty \zeta_\rho(\theta) \mathbb{E}\|S(t'')^\rho \theta\right) - S(t'^\rho \theta)\| \|A^{\alpha-1}(\widetilde{\phi}(0) + G_n(0, \widetilde{\phi}(0)))\| d\theta)^2 \\
&\leqslant 6\left(\int_0^\infty \zeta_\rho(\theta) \left[\int_{t'}^{t''} \mathbb{E}\|\frac{d}{dt} S(t^\rho \theta)\| dt\right] \|(\widetilde{\phi}(0) + G_n(0, \widetilde{\phi}(0)))\|_{\alpha-1} d\theta\right)^2 \\
&\leqslant M_1 (t'' - t')^2,
\end{aligned} \tag{3.8}
$$

where $M_1 = 6C_1^2\mathbb{E}\|(\widetilde{\phi}(0) + G_n(0, \widetilde{\phi}(0)))\|^2_{\alpha-1}$.

$$\begin{aligned}
J_2 &\leqslant 6\|A^{\alpha-1-\beta}\|^2 L_G[|t''-t'|^2 + \|u_{t''} - u_{t'}\|^2_{0,\alpha-1}] \\
&\leqslant 6\|A^{\alpha-1-\beta}\|^2 L_G[|t''-t'|^2 + \|u(t''+\nu) - u(t'+\nu)\|^2_{\alpha-1}] \quad \forall \nu \in [-\tau, 0] \\
&\leqslant 6\|A^{\alpha-1-\beta}\|^2 L_G(1+L)|t''-t'|^2 \\
&\leqslant M_2(t''-t')^2, \qquad (3.9)
\end{aligned}$$

where $M_2 = 6\|A^{\alpha-1-\beta}\|^2 L_G(1+L)$.

$$\begin{aligned}
&6A\left[(t''-s)^{\rho-1}S_\rho(t''-s) - (t'-s)^{\rho-1}S_\rho(t'-s))\right] \\
&\quad = 6\int_0^\infty \zeta_\rho(\theta)\left[\frac{d}{dt}S((t-s)^\rho\theta)|_{t=t''} - \frac{d}{dt}S((t-s)^\rho\theta)|_{t=t'}\right]d\theta \\
&\quad = 6\int_0^\infty \zeta_\rho(\theta)\left[\int_{t'}^{t''}\frac{d^2}{dt^2}S((t-s)^\rho\theta)dt\right]d\theta \\
&\quad \leqslant 6C_2(t''-t'). \qquad (3.10)
\end{aligned}$$

Therefore

$$\begin{aligned}
J_3 \leqslant& 6\int_0^{t'}\left(\int_0^\infty \zeta_\rho(\theta)\left[\frac{d}{dt}S((t-s)^\rho\theta)|_{t=t''} - \frac{d}{dt}S((t-s)^\rho\theta)|_{t=t'}\right]d\theta\right)^2 \\
&\|A^{\alpha-\beta-1}\|^2\mathbb{E}\|A^\beta G_n(s, x_s)\|^2 ds \\
\leqslant& M_3(t''-t')^2, \qquad (3.11)\\
J_4 \leqslant& 6\left(\frac{\rho C_{\alpha-\beta}\Gamma(2-(\alpha-\beta))}{\Gamma(1+\rho(1-\alpha+\beta))}\right)^2\int_{t'}^{t''}(t''-s)^{2\rho(\beta-\alpha+1)-2}\mathbb{E}\|A^\beta G_n(s,u_s)\|^2 ds \\
\leqslant& 6\left(\frac{\rho C_{\alpha-\beta}\Gamma(2-(\alpha-\beta))}{\Gamma(1+\rho(1-\alpha+\beta))}\right)^2 L_G\frac{(t''-t')^{2\rho(\beta-\alpha+1)-1}}{2\rho(\beta-\alpha+1)-1}, \qquad (3.12)\\
J_5 \leqslant& 6\int_0^{t'}\left(\int_0^\infty \zeta_\rho(\theta)\left[\int_{t'}^{t''}\frac{d^2}{dt^2}S((t-s)^\rho\theta)dt\right]d\theta\right)^2\|A^{\alpha-2}\|^2[N + \|a^2\|_{L^p}\|L_k\|_{L^q}]ds \\
\leqslant& M_4(t''-t')^2, \qquad (3.13)\\
J_6 \leqslant& 6\left(\frac{\rho C_{\alpha-1}\Gamma(3-\alpha)}{\Gamma(1+\rho(2-\alpha))}\right)^2\int_{t'}^{t''}(t''-s)^{2\rho(2-\alpha)-2}[N + \|a^2\|_{L^p}\|L_k\|_{L^q}]ds \\
\leqslant& 6\left(\frac{\rho C_{\alpha-1}\Gamma(3-\alpha)}{\Gamma(1+\rho(2-\alpha))}\right)^2 M(R)\frac{(t''-t')^{2\rho(2-\alpha)-1}}{2\rho(2-\alpha)-1}, \qquad (3.14)\\
J_7 \leqslant& M_5(t''-t')^2, \qquad (3.15)
\end{aligned}$$

where $M_3 = 6C_2^2\|A^{\alpha-\beta-1}\|^2 L_G T_0$, $M_4 = 6C_2^2\|A^{\alpha-2}\|^2 M(R)T_0$ and $M_5 = rC_1^2\|A^{-1}\|^2 \sum_{j=1}^r L_j$ are constants. Using (3.7)–(3.15), we have the map $\Phi_n : C_{T_0}^{\alpha-1} \to C_{T_0}^{\alpha-1}$ is well defined. Now we will prove that $\Phi_n : \mathcal{B}_R \to \mathcal{B}_R$ i.e. $\Phi_n \in \mathcal{B}_R$ for any $u \in \mathcal{B}_R$

$$\mathbb{E}\|(\Phi_n u)(t)-\widetilde{\phi}(0)\|_\alpha^2 \leqslant 6\bigg\{\mathbb{E}\|(S_\rho(t)-I)A^\alpha[\widetilde{\phi}(0)+G_n(0,\widetilde{\phi}(0))]\|^2 + \mathbb{E}\|A^{\alpha-\beta}\|^2\|A^\beta G_n(t,u_t)-A^\beta G_n(0,\widetilde{\phi}(0))\|^2 + \left(\frac{\rho C_{1+\alpha-\beta}\Gamma(1-(\alpha-\beta))}{\Gamma(1+\rho(\beta-\alpha))}\right)^2\frac{T_0^{2\rho(\beta-\alpha)-1}}{2\rho(\beta-\alpha)-1}L_G + \left(\frac{\rho C_\alpha\Gamma(2-\alpha)}{\Gamma(1+\rho(1-\alpha))}\right)^2 M(R)\frac{T_0^{2\rho(1-\alpha)-1}}{2\rho(1-\alpha)-1} + rC^2\sum_{j=1}^r L_j\bigg\}.$$

It follows from (3.3) and (3.4) that,

$$\mathbb{E}\|(\Phi_n u)(t)-\widetilde{\phi}(0)\|_\alpha^2 \leqslant R.$$

Taking supremum over $(0,T_0]$ we obtain that Φ_n maps $\mathcal{B}_R$ into $\mathcal{B}_R$.
Now we show that Φ_n is a contraction map. For $u, v\in\mathcal{B}_R$ and $-\tau\leqslant t\leqslant 0$, we have

$$\mathbb{E}\|(\Phi_n u)(t)-(\Phi_n v)(t)\|_\alpha^2 = \mathbb{E}\|\widetilde{\phi}(t)-\widetilde{\phi}(t)\|_\alpha^2.$$

For $t\in(0,T_0]$ and $u, v\in\mathcal{B}_R$, we have

$$\begin{aligned}
&\mathbb{E}\|(\Phi_n u)(t)-(\Phi_n v)(t)\|_\alpha^2\\
&\leqslant 5\bigg\{\|A^{\alpha-\beta}\|^2\mathbb{E}\|A^\beta G_n(t,u_t)-A^\beta G_n(t,v_t)\|^2\\
&\quad+\int_0^t \|(t-s)^{\rho-1}Q_\rho(t-s)A^{1+\alpha-\beta}\|^2 E\|A^\beta G_n(s,u_s)-A^\beta G_n(s,v_s)\|^2 ds\\
&\quad+\int_0^t \|(t-s)^{\rho-1}Q_\rho(t-s)A^\alpha\|^2\mathbb{E}\|F_n(s,u_s,u[b(u(s),s)])\\
&\quad-F_n(s,v_s,v[b(v(s),s)])\|^2 ds+\int_0^t \|(t-s)^{\rho-1}Q_\rho(t-s)A^\alpha\|^2\\
&\quad\left(\int_0^s |a(s-r)|^2\mathbb{E}\|k(r,u_r)-k(r,v_r)\|^2 dr\right)ds\\
&\quad+r\sum_{j=1}^r \|S_q(t-t_k)(I_{j,n}(u_{t_j})-I_{j,n}(v_{t_j}))\|_\alpha^2\bigg\}\\
&\leqslant 5\bigg[\|A^{\alpha-\beta-1}\|^2 L_G+\left(\frac{\rho C_{1+\alpha-\beta}\Gamma(1-\alpha+\beta)}{\Gamma(1+\rho(\beta-\alpha))}\right)^2\frac{T_0^{2\rho(\beta-\alpha)-1}}{2\rho(\beta-\alpha)-1}L_G\|A^{-1}\|\\
&\quad+\left(\frac{\rho C_\alpha\Gamma(2-\alpha)}{\Gamma(1+\rho(1-\alpha))}\right)^2\frac{T_0^{2\rho(1-\alpha)-1}}{2\rho(1-\alpha)-1}(L_f(1+LL_b)+M(R))\\
&\quad+rC^2\sum_{j=1}^r L_j\bigg]\mathbb{E}\|u-v\|_{T_0,\alpha}^2.
\end{aligned}$$

Using (3.5) and taking supremum over $t\in(0,T_0]$, we get

$$\mathbb{E}\|(\Phi_n u)(t)-(\Phi_n v)(t)\|_\alpha^2 < \mathbb{E}\|u-v\|_{T_0,\alpha}^2.$$

Thus Φ_n is a strict contraction on $\mathcal{B}_R$. Therefore by Banach contraction principle, there exists a unique $u_n \in \mathcal{B}_R$ such that $\Phi_n u_n = u_n$ i.e. u_n satisfies the approximate integral equation

$$u_n(t) = \begin{cases} \widetilde{\phi}(t), \quad t \in [-\tau, 0]; \\ S_\rho(t)[\widetilde{\phi}(0) + G_n(0, \widetilde{\phi}(0))] - G_n(t, (u_n)_t) + \int_0^t (t-s)^{\rho-1} Q_\rho(t-s) A G_n(s, (u_n)_s) ds \\ \quad + \int_0^t (t-s)^{\rho-1} Q_\rho(t-s)[F_n(s, (u_n)_s, u_n[b(u_n(s), s)]) \\ \quad + \int_0^s a(s-r) k_n(r, (u_n)_r) d\omega(r)] ds + \sum_{j=1}^r S_\rho(t-t_j) I_{j,n}(u_n)_{t_j}), \quad t \in [0, T_0]. \end{cases} \tag{3.16}$$

□

Lemma 3.2 *Let* $(H1)$–$(H5)$ *hold and* $\widetilde{\phi}(t) \in \mathcal{L}_2^0(\Omega, D(A^\alpha))$ *for all* $t \in [-\tau, 0]$. *Then there exists a constant* N_{t_0}*, independent of* n*, such that*

$$\mathbb{E}\|u_n(t)\|_\mu^2 \leqslant N_{t_0}, \quad 0 \leqslant \mu < 1, \ -\tau \leqslant t \leqslant T.$$

Moreover, if $\widetilde{\phi}(0) \in \mathcal{L}_2^0(\Omega, D(A))$*, there exists a constant* N_0*, independent of* n*, such that*

$$\mathbb{E}\|u_n(t)\|_\mu^2 \leqslant N_0, \quad 0 \leqslant \mu < 1, \ 0 \leqslant t \leqslant T.$$

Proof For $t \in [-\tau, 0]$, applying A^μ on both the sides of (3.16) and taking norm, we have

$$\mathbb{E}\|u_n(t)\|_\mu^2 \leqslant \|\widetilde{\phi}(t)\|_\mu^2 \leqslant \|\widetilde{\phi}\|_{0,\mu}^2.$$

For $t \in [t_0, T]$, on applying A^μ on both the sides of (3.16) and taking norm, we have

$$\begin{aligned} \mathbb{E}\|u_n(t)\|_\mu^2 \leqslant 6\Big\{ & \|A^\mu S_\rho(t)[\widetilde{\phi}(0) + G_n(0, \widetilde{\phi}(0))]\|^2 + \|A^{\mu-\beta}\|^2 \mathbb{E}\|A^\beta G_n(t, (u_n)_t)\|^2 \\ & + \int_0^t \|(t-s)^{\rho-1} Q_\rho(t-s) A^{1+\mu-\beta}\|^2 \mathbb{E}\|A^\beta G_n(s, (u_n)_s)\|^2 ds \\ & + \int_0^t \|(t-s)^{\rho-1} Q_\rho(t-s) A^\mu\|^2 \\ & [\mathbb{E}\|F_n(s, (u_n)_s, u_n[b(u_n(s), s)])\|^2 + \int_0^s |a(s-r)|^2 \mathbb{E}\|k_n(r, (u_n)_r)\|^2 dr] ds \\ & + r \sum_{j=1}^r |S_\rho(t-t_j)|^2 \|A^\mu I_j(u_n)_{t_j}\|^2 \Big\} \\ \leqslant 6\Big\{ & C_\mu^2 t_0^{-2\rho\mu} \|\widetilde{\phi}(0) + G_n(0, \widetilde{\phi}(0))\|^2 + \|A^{\mu-\beta}\|^2 L_G \\ & + \left(\frac{\rho C_{1+\mu-\beta} \Gamma(1-\mu+\beta)}{\Gamma(1+\rho(\beta-\mu))} \right)^2 \frac{T_0^{2\rho(\beta-\mu)-1}}{2\rho(\beta-\mu)-1} L_G \\ & + \left(\frac{\rho C_\mu \Gamma(2-\mu)}{\Gamma(1+\rho(1-\mu))} \right)^2 M(R) \frac{T_0^{2\rho(1-\mu)-1}}{2\rho(1-\mu)-1} + r C^2 \sum_{j=1}^r L_j \Big\} \\ \leqslant N_{t_0}. & \end{aligned}$$

From Theorem 3.1, there exists a unique $u_n \in \mathcal{B}_R$ which satisfies (3.16). Now using Part (a) of Theorem 6.13 in Pazy [31] we have $S(t) : \mathcal{H} \to \mathcal{L}_2(\Omega, D(A^\mu))$ for $t > 0$ and $0 \leqslant \mu < 1$. Also Theorem 2.4 in Pazy [31] implies that if $u \in \mathcal{L}_2(\Omega, D(A))$ then $S(t)u \in \mathcal{L}_2(\Omega, D(A))$.

And Theorem 6.8 in Pazy [31] implies $\mathcal{L}_2(\Omega, D(A)) \subseteq \mathcal{L}_2(\Omega, D(A^\mu))$ for $0 \leqslant \mu < 1$. Thus if $\widetilde{\phi}(t) \in \mathcal{L}_2^0(\Omega, D(A))$ then $\widetilde{\phi}(t) \in \mathcal{L}_2^0(\Omega, D(A^\mu))$ for $0 \leqslant \mu < 1$ and we get,

$$\begin{aligned}\mathbb{E}\|u_n(t)\|_\mu^2 \leqslant & 6\Bigg\{C^2\|\widetilde{\phi}(0) + G_n(0, \widetilde{\phi}(0))\|_\mu^2 + \|A^{\mu-\beta}\|^2 L_G \\ & + \left(\frac{\rho C_{1+\mu-\beta}\Gamma(1-\mu+\beta)}{\Gamma(1+\rho(\beta-\mu))}\right)^2 \frac{T_0^{2\rho(\beta-\mu)-1}}{2\rho(\beta-\mu)-1} L_G \\ & + \left(\frac{\rho C_\mu \Gamma(2-\mu)}{\Gamma(1+\rho(1-\mu))}\right)^2 M(R) \frac{T_0^{2\rho(1-\mu)-1}}{2\rho(1-\mu)-1} + rC^2 \sum_{j=1}^r L_j\Bigg\} \\ \leqslant & N_0.\end{aligned}$$

□

4 Convergence of solutions

Theorem 4.1 *Let $(H1)$–$(H6)$ hold and $\widetilde{\phi}(t) \in \mathcal{L}_2^0(\Omega, D(A^\alpha))$ for all $t \in [-\tau, 0]$. Then the sequence $\{u_n\} \in \mathcal{B}_R$ is a Cauchy sequence and therefore converges to a function $u \in \mathcal{B}_R$ satisfying (3.16).*

Proof For $0 < t_0' < t$, we have

$$\begin{aligned}&\mathbb{E}\|u_n(t) - u_m(t)\|_\alpha^2 \\ &\quad \leqslant 6\Bigg\{C^2\|A^{\alpha-\beta}\|^2 \mathbb{E}\|A^\beta G_n(0, \widetilde{\phi}(0)) - A^\beta G_m(0, \widetilde{\phi}(0)))\|^2 \\ &\qquad + \|A^{\alpha-\beta}\|^2 \mathbb{E}\|A^\beta G_n(t, (u_n)_t - A^\beta G_m(t, (u_m)_t)\|^2 \\ &\qquad + \left(\int_0^{t_0'} + \int_{t_0'}^t\right)\|(t-s)^{\rho-1} Q_\rho(t-s) A^{1+\alpha-\beta}\|^2 \\ &\qquad \times \mathbb{E}\|A^\beta G_n(s, (u_n)_s - A^\beta G_m(s, (u_m)_s)\|^2 ds \\ &\qquad + \left(\int_0^{t_0'} + \int_{t_0'}^t\right)\|(t-s)^{\rho-1} Q_\rho(t-s) A^\alpha\|^2 \Bigg[\mathbb{E}\|F_n(s, (u_n)_s, u_n[b(u_n(s), s)]) \\ &\qquad - F_m(s, (u_m)_s, u_m[b(u_m(s), s)])\|^2 \\ &\qquad + \int_0^s |a(s-r)|^2 \mathbb{E}\|k_n(r, (u_n)_r) - k_m(r, (u_m)_r)\|^2 dr\Bigg] ds \\ &\qquad + r\sum_{j=1}^r |S_\rho(t-t_j)|^2 \mathbb{E}\|I_{j,n}(u_n)_{t_j}) - I_{j,m}(u_m)_{t_j})\|_\alpha^2\Bigg\}. \end{aligned} \tag{4.1}$$

Here

$$\begin{aligned}&C^2\|A^{\alpha-\beta}\|^2 \mathbb{E}\|A^\beta G_n(0, \widetilde{\phi}(0)) - A^\beta G_m(0, \widetilde{\phi}(0)))\|^2 \\ &\quad \leqslant C^2\|A^{\alpha-\beta}\|^2 L_G \mathbb{E}\|(P^n - P^m)\widetilde{\phi}(0)\|_{\alpha-1}^2 \\ &\quad \leqslant C^2\|A^{\alpha-\beta-1}\|^2 L_G \mathbb{E}\|(P^n - P^m)A^\alpha \widetilde{\phi}(0)\|^2,\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\|F_n(s,(u_n)_s,u_n[b(u_n(s),s)])-F_m(s,(u_m)_s,u_m[b(u_m(s),s)])\|^2\\
&\quad\leqslant 2[\mathbb{E}\|F_n(s,(u_n)_s,u_n[b(u_n(s),s)])-F_n(s,(u_m)_s,u_m[b(u_m(s),s)])\|^2\\
&\qquad+\mathbb{E}\|F_n(s,(u_m)_s,u_m[b(u_m(s),s)])-F_m(s,(u_m)_s,u_m[b(u_m(s),s)])\|^2]\\
&\quad\leqslant 2[L_f(1+LL_b)\mathbb{E}\|u_n-u_m\|_{s,\alpha}^2+L_f[\mathbb{E}\|(P^n-P^m)u_m\|_{s,\alpha}^2\\
&\qquad+\|A^{-1}\|^2\mathbb{E}\|(P^n-P^m)u_m[b(u_m(s),s)]\|_\alpha^2].
\end{aligned}$$

Also

$$\mathbb{E}\|(P^n-P^m)u_m\|_{s,\alpha}^2\leqslant\mathbb{E}\|A^{\alpha-\mu}(P^n-P^m)A^\mu u_m\|_s^2\leqslant\frac{1}{\lambda_m^{2(\mu-\alpha)}}\mathbb{E}\|A^\mu u_m\|_s^2.$$

Therefore, we have

$$\begin{aligned}
&\mathbb{E}\|F_n(s,(u_n)_s,u_n[b(u_n(s),s)])-F_m(s,(u_m)_s,u_m[b(u_m(s),s)])\|^2\\
&\quad\leqslant 2L_f(1+LL_b)\mathbb{E}\|u_n-u_m\|_{s,\alpha}^2+2L_f\Bigg[\frac{1}{\lambda_m^{2(\mu-\alpha)}}\mathbb{E}\|A^\mu u_m\|_s^2\\
&\qquad+\frac{\|A^{-1}\|^2}{\lambda_m^{2(\mu-\alpha)}}\mathbb{E}\|A^\mu u_m[b(u_m(s),s)]\|^2\Bigg].
\end{aligned}$$

Similarly

$$\begin{aligned}
&\mathbb{E}\|A^\beta G_n(s,(u_n)_s)-A^\beta G_m(s,(u_m)_s)\|^2\\
&\quad\leqslant 2\mathbb{E}\|A^\beta G_n(s,(u_n)_s-A^\beta G_n(s,(u_m)_s)\|^2+2\mathbb{E}\|A^\beta G_n(s,(u_m)_s-A^\beta G_m(s,(u_m)_s)\|^2\\
&\quad\leqslant 2L_G\|A^{-1}\|^2\left[\mathbb{E}\|u_n-u_m\|_{s,\alpha}^2+\frac{1}{\lambda_m^{2(\mu-\alpha)}}\mathbb{E}\|A^\mu u_m\|_s^2\right],
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\|k_n(s,(u_n)_s)-k_m(s,(u_m)_s)\|^2\\
&\quad\leqslant 2[\mathbb{E}\|k_n(s,(u_n)_s)-k_n(s,(u_m)_s)\|^2+\mathbb{E}\|k_n(s,(u_m)_s-k_m(s,(u_m)_s)\|^2]\\
&\quad\leqslant 2L_k\left[\mathbb{E}\|u_n-u_m\|_{s,\alpha}^2+\frac{1}{\lambda_m^{2(\mu-\alpha)}}\mathbb{E}\|A^\mu u_m\|_s^2\right].
\end{aligned}$$

The first and third integrals of (4.1) can be estimated as

$$\begin{aligned}
&\int_0^{t_0'}\|(t-s)^{\rho-1}Q_\rho(t-s)A^{1+\alpha-\beta}\|^2\mathbb{E}\|A^\beta G_n(s,(u_n)_s-A^\beta G_m(s,(u_m)_s)\|^2ds\\
&\quad\leqslant 2L_G\left(\frac{\rho C_{1+\alpha-\beta}\Gamma(1-\alpha+\beta)}{\Gamma(1+\rho(-\alpha+\beta))}\right)^2(t_0-t_0')^{2\rho(\beta-\alpha)-2}t_0',\\
&\int_0^{t_0'}\|(t-s)^{\rho-1}Q_\rho(t-s)A^\alpha\|^2\Bigg[\mathbb{E}\|F_n(s,(u_n)_s,u_n[b(u_n(s),s)])\\
&\qquad-F_m(s,(u_m)_s,u_m[b(u_m(s),s)])\|^2\\
&\qquad+\int_0^s|a(s-r)|^2\mathbb{E}\|k_n(r,(u_n)_r)-k_m(r,(u_m)_r)\|^2dr\Bigg]ds\\
&\quad\leqslant 2M(R)\left(\frac{\rho C_\alpha\Gamma(2-\alpha)}{\Gamma(1+\rho(1-\alpha))}\right)^2(t_0-t_0')^{2\rho(1-\alpha)-2}t_0'.
\end{aligned}$$

Second and fourth integrals of (4.1) can be estimated as

$$\int_{t_0'}^{t} \|(t-s)^{\rho-1}Q_\rho(t-s)A^{1+\alpha-\beta}\|^2\mathbb{E}\|A^\beta G_n(s,(u_n)_s - A^\beta G_m(s,(u_m)_s)\|^2 ds$$
$$\leqslant 2\left(\frac{\rho C_{1+\alpha-\beta}\Gamma(1-\alpha+\beta)}{\Gamma(1+\rho(-\alpha+\beta))}\right)^2 L_G\|A^{-1}\|^2\left(\frac{N_{t_0}T_0^{2\rho(\beta-\alpha)-1}}{\lambda_m^{2(\mu-\alpha)}(2\rho(\beta-\alpha)-1)}\right.$$
$$\left.+\int_{t_0'}^{t}(t-s)^{2\rho(\beta-\alpha)-2}\mathbb{E}\|u_n-u_m\|_{s,\alpha}^2 ds\right),$$
$$\int_{t_0'}^{t} \|(t-s)^{\rho-1}Q_\rho(t-s)A^{\alpha}\|^2\left[\mathbb{E}\|F_n(s,(u_n)_s,u_n[b(u_n(s),s)])\right.$$
$$-F_m(s,(u_m)_s,u_m[b(u_m(s),s)])\|^2$$
$$\left.+\int_0^s |a(s-r)|^2\mathbb{E}\|k_n(r,(u_n)_r)-k_m(r,(u_m)_r)\|^2 dr\right]ds$$
$$\leqslant 2\left(\frac{\rho C_\alpha\Gamma(2-\alpha)}{\Gamma(1+\rho(1-\alpha))}\right)^2\int_{t_0'}^{t}(t-s)^{2\rho(1-\alpha)-2}\left[L_f(1+LL_b)\mathbb{E}\|u_n-u_m\|_{s,\alpha}^2\right.$$
$$+L_f\left(\frac{1}{\lambda_m^{2(\mu-\alpha)}}\mathbb{E}\|A^\mu u_m\|_s^2+\frac{\|A^{-1}\|^2}{\lambda_m^{2(\mu-\alpha)}}\mathbb{E}\|A^\mu u_m[b(u_m(s),s)]\|^2\right)$$
$$\left.+\int_0^s|a(s-r)|^2L_k(\mathbb{E}\|u_n-u_m\|_{r,\alpha}^2+\frac{1}{\lambda_m^{2(\mu-\alpha)}}\mathbb{E}\|A^\mu u_m\|_r^2)dr\right]ds$$
$$\leqslant 2\left(\frac{\rho C_\alpha\Gamma(2-\alpha)}{\Gamma(1+\rho(1-\alpha))}\right)^2\left[(L_f(1+\|A^{-1}\|^2)+\|a^2\|_{L^p}\|L_k\|_{L^q})\frac{N_{t_0}}{\lambda_m^{2(\mu-\alpha)}}\frac{T_0^{2\rho(1-\alpha)-1}}{2\rho(1-\alpha)-1}\right.$$
$$\left.+(L_f(1+LL_b)+\|a^2\|_{L^p}\|L_k\|_{L^q})\int_{t_0'}^{t}(t-s)^{2\rho(1-\alpha)-2}\mathbb{E}\|u_n-u_m\|_{s,\alpha}^2 ds\right].$$

Thus (4.1) can be estimated as

$$\mathbb{E}\|u_n(t)-u_m(t)\|_\alpha^2$$
$$\leqslant 6C^2\|A^{\alpha-\beta-1}\|^2L_G\mathbb{E}\|(P^n-P^m)A^\alpha\widetilde{\phi}(0)\|^2$$
$$+12\|A^{\alpha-\beta-1}\|^2L_G\left(\mathbb{E}\|u_n-u_m\|_{t,\alpha}^2+\frac{N_{t_0}}{\lambda_m^{2(\mu-\alpha)}}\right)$$
$$+12\left(L_G\left(\frac{\rho C_{1+\alpha-\beta}\Gamma(1-\alpha+\beta)}{\Gamma(1+\rho(-\alpha+\beta))}\right)^2(t_0-t_0')^{2\rho(\beta-\alpha)-2}\right.$$
$$\left.+M(R)\left(\frac{\rho C_\alpha\Gamma(2-\alpha)}{\Gamma(1+\rho(1-\alpha))}\right)^2(t_0-t_0')^{2\rho(1-\alpha)-2}\right)t_0'$$
$$+\left[12\left(\frac{\rho C_{1+\alpha-\beta}\Gamma(1-\alpha+\beta)}{\Gamma(1+\rho(-\alpha+\beta))}\right)^2L_G\|A^{-1}\|^2\frac{T_0^{2\rho(\beta-\alpha)-1}}{(2\rho(\beta-\alpha)-1)}\right.$$
$$+12\left(\frac{\rho C_\alpha\Gamma(2-\alpha)}{\Gamma(1+\rho(1-\alpha))}\right)^2(L_f(1+\|A^{-1}\|^2)$$
$$\left.+\|a^2\|_{L^p}\|L_k\|_{L^q})\frac{T_0^{2\rho(1-\alpha)-1}}{2\rho(1-\alpha)-1}\right]\frac{N_{t_0}}{\lambda_m^{2(\mu-\alpha)}}$$

$$+\int_{t_0'}^{t}\left\{12\left(\frac{\rho C_{1+\alpha-\beta}\Gamma(1-\alpha+\beta)}{\Gamma(1+\rho(-\alpha+\beta))}\right)^2 L_G\|A^{-1}\|^2(t-s)^{2\rho(\beta-\alpha)-2}\right.$$
$$+12\left(\frac{\rho C_\alpha\Gamma(2-\alpha)}{\Gamma(1+\rho(1-\alpha))}\right)^2(L_f(1+LL_b)+\|a^2\|_{L^p}\|L_k\|_{L^q})$$
$$\left.(t-s)^{2\rho(1-\alpha)-2}\right\}\mathbb{E}\|u_n-u_m\|_{s,\alpha}^2 ds+6rC^2\sum_{j=1}^{r}L_j\mathbb{E}\|u_n-u_m\|_{t_j,\alpha}^2$$
$$\leqslant D_1\mathbb{E}\|u_n-u_m\|_{t,\alpha}^2+D_2t_0'+\frac{D_3}{\lambda_m^{2(\mu-\alpha)}}+D_4\int_{t_0'}^{t}[(t-s)^{2\rho(\beta-\alpha)-2}$$
$$+(t-s)^{2\rho(1-\alpha)-2}]\mathbb{E}\|u_n-u_m\|_{s,\alpha}^2 ds+6rC^2\sum_{j=1}^{r}L_j\mathbb{E}\|u_n-u_m\|_{t_j,\alpha}^2, \tag{4.2}$$

where

$$D_1=12\|A^{\alpha-\beta-1}\|^2L_G,$$
$$D_2=12\left(L_G\left(\frac{\rho C_{1+\alpha-\beta}\Gamma(1-\alpha+\beta)}{\Gamma(1+\rho(-\alpha+\beta))}\right)^2(t_0-t_0')^{2\rho(\beta-\alpha)-2}\right.$$
$$\left.+M(R)\left(\frac{\rho C_\alpha\Gamma(2-\alpha)}{\Gamma(1+\rho(1-\alpha))}\right)^2(t_0-t_0')^{2\rho(1-\alpha)-2}\right),$$
$$D_3=6C^2N_{t_0}\|A^{\alpha-\beta-1}\|^2L_G\|\widetilde{\phi}\|_{0,\mu}^2+12N_{t_0}\|A^{\alpha-\beta-1}\|^2L_G$$
$$+12N_{t_0}\left(\frac{\rho C_{1+\alpha-\beta}\Gamma(1-\alpha+\beta)}{\Gamma(1+\rho(-\alpha+\beta))}\right)^2L_G\|A^{-1}\|^2\frac{T_0^{2\rho(\beta-\alpha)-1}}{(2\rho(\beta-\alpha)-1)}$$
$$+12N_{t_0}\left(\frac{\rho C_\alpha\Gamma(2-\alpha)}{\Gamma(1+\rho(1-\alpha))}\right)^2(L_f(1+\|A^{-1}\|^2)+\|a^2\|_{L^p}\|L_k\|_{L^q})\frac{T_0^{2\rho(1-\alpha)-1}}{2\rho(1-\alpha)-1},$$
$$D_4=12\left(\frac{\rho C_{1+\alpha-\beta}\Gamma(1-\alpha+\beta)}{\Gamma(1+\rho(-\alpha+\beta))}\right)^2L_G\|A^{-1}\|^2$$
$$+12\left(\frac{\rho C_\alpha\Gamma(2-\alpha)}{\Gamma(1+\rho(1-\alpha))}\right)^2(L_f(1+LL_b)+\|a^2\|_{L^p}\|L_k\|_{L^q}).$$

Replace t by $t+\nu$ in the above inequality, where $\nu\in[t_0'-t,0]$, we get

$$\mathbb{E}\|u_n(t+\nu)-u_m(t+\nu)\|_\alpha^2\leqslant D_1\mathbb{E}\|u_n-u_m\|_{t,\alpha}^2+D_2t_0'+\frac{D_3}{\lambda_m^{2(\mu-\alpha)}}$$
$$+D_4\int_{t_0'}^{t+\nu}[(t+\nu-s)^{2\rho(\beta-\alpha)-2}$$
$$+(t+\nu-s)^{2\rho(1-\alpha)-2}]\mathbb{E}\|u_n-u_m\|_{s,\alpha}^2ds$$
$$+6rC^2\sum_{j=1}^{r}L_j\mathbb{E}\|u_n-u_m\|_{t_j,\alpha}^2$$

Now put $s - \nu = \gamma$ in the integral term of above inequality and we get

$$\begin{aligned}\mathbb{E}\|u_n(t+\nu) - u_m(t+\nu)\|_\alpha^2 &\leqslant D_1 \mathbb{E}\|u_n - u_m\|_{t,\alpha}^2 + D_2 t_0' + \frac{D_3}{\lambda_m^{2(\mu-\alpha)}} \\ &\quad + D_4 \int_{t_0'-\nu}^{t} [(t-\gamma)^{2\rho(\beta-\alpha)-2} \\ &\quad + (t-\gamma)^{2\rho(1-\alpha)-2}] \mathbb{E}\|u_n - u_m\|_{\gamma+\nu,\alpha}^2 d\gamma \\ &\quad + 6rC^2 \sum_{j=1}^{r} L_j \mathbb{E}\|u_n - u_m\|_{t_j,\alpha}^2 \\ &\leqslant D_1 \mathbb{E}\|u_n - u_m\|_{t,\alpha}^2 + D_2 t_0' + \frac{D_3}{\lambda_m^{2(\mu-\alpha)}} \\ &\quad + D_4 \int_{t_0'}^{t} [(t-\gamma)^{2\rho(\beta-\alpha)-2} \\ &\quad + (t-\gamma)^{2\rho(1-\alpha)-2}] \mathbb{E}\|u_n - u_m\|_{\gamma,\alpha}^2 d\gamma \\ &\quad + 6rC^2 \sum_{j=1}^{r} L_j \mathbb{E}\|u_n - u_m\|_{t_j,\alpha}^2 .\end{aligned}$$

Thus,

$$\begin{aligned}\sup_{t_0'-t \leqslant \nu \leqslant 0} \mathbb{E}\|u_n(t+\nu) - u_m(t+\nu)\|_\alpha^2 &\leqslant D_1 \mathbb{E}\|u_n - u_m\|_{t,\alpha}^2 + D_2 t_0' + \frac{D_3}{\lambda_m^{2(\mu-\alpha)}} \\ &\quad + D_4 \int_{t_0'}^{t} [(t-\gamma)^{2\rho(\beta-\alpha)-2} \\ &\quad + (t-\gamma)^{2\rho(1-\alpha)-2}] \mathbb{E}\|u_n - u_m\|_{\gamma,\alpha}^2 d\gamma \\ &\quad + 6rC^2 \sum_{j=1}^{r} L_j \mathbb{E}\|u_n - u_m\|_{t_j,\alpha}^2 . \end{aligned} \tag{4.3}$$

Since $u_n(t+\nu) = \widetilde{\phi}(t+\nu)$ for $t+\nu \leqslant 0$ and $n \geqslant n_0$. Therefore, we have

$$\begin{aligned}\sup_{-\tau-t \leqslant \nu \leqslant 0} \mathbb{E}\|u_n(t+\nu) - u_m(t+\nu)\|_\alpha^2 &\leqslant \sup_{0 \leqslant \nu+t \leqslant t_0'} \mathbb{E}\|u_n(t+\nu) - u_m(t+\nu)\|_\alpha^2 \\ &\quad + \sup_{t_0'-t \leqslant \nu \leqslant 0} \mathbb{E}\|u_n(t+\nu) - u_m(t+\nu)\|_\alpha^2 . \end{aligned} \tag{4.4}$$

For $t \in (0, t_0']$, we have from (4.2)

$$\begin{aligned}\mathbb{E}\|u_n(t+\nu) - u_m(t+\nu)\|_\alpha^2 &\leqslant D_1 \mathbb{E}\|u_n - u_m\|_{t,\alpha}^2 + D_2 t_0' + \frac{D_5}{\lambda_m^{2(\mu-\alpha)}} \\ &\quad + 6rC^2 \sum_{j=1}^{r} L_j \mathbb{E}\|u_n - u_m\|_{t_j,\alpha}^2 , \end{aligned} \tag{4.5}$$

where $D_5 = 6C^2 N_{t_0} \|A^{\alpha-\beta-1}\|^2 L_G \|\widetilde{\phi}\|_{0,\mu}^2 + 12N_{t_0}\|A^{\alpha-\beta-1}\|^2 L_G$. Using (4.3) and (4.5) in (4.4), we get

$$\sup_{-\tau \leqslant t+\nu \leqslant t} \mathbb{E}\|u_n(t+\nu) - u_m(t+\nu)\|_\alpha^2 \leqslant 2D_1 \mathbb{E}\|u_n - u_m\|_{t,\alpha}^2 + 2D_2 t_0' + \frac{D_3 + D_5}{\lambda_m^{2(\mu-\alpha)}} + D_4 \int_{t_0'}^t [(t-\gamma)^{2\rho(\beta-\alpha)-2} + (t-\gamma)^{2\rho(1-\alpha)-2}]\mathbb{E}\|u_n - u_m\|_{\gamma,\alpha}^2 d\gamma + 12rC^2 \sum_{j=1}^r L_j \mathbb{E}\|u_n - u_m\|_{t_j,\alpha}^2.$$

Since $2D_1 + 12rC^2 \sum_{j=1}^r L_j < 1$, we have

$$\mathbb{E}\|u_n(t) - u_m(t)\|_\alpha^2 \leqslant \frac{1}{\left(1 - 2D_1 + 12rC^2 \sum_{j=1}^r L_j\right)} \left[2D_2 t_0' + \frac{D_3 + D_5}{\lambda_m^{2(\mu-\alpha)}} + D_4 \int_{t_0'}^t [(t-\gamma)^{2\rho(\beta-\alpha)-2} + (t-\gamma)^{2\rho(1-\alpha)-2}]\mathbb{E}\|u_n - u_m\|_{\gamma,\alpha}^2 d\gamma\right].$$

From Lemma 5.6.7 in Pazy [31], there exists a constant M such that

$$\mathbb{E}\|u_n(t) - u_m(t)\|_\alpha^2 \leqslant \frac{1}{\left(1 - 2D_1 + 12rC^2 \sum_{j=1}^r L_j\right)} \left[2D_2 t_0' + \frac{D_3 + D_5}{\lambda_m^{2(\mu-\alpha)}}\right] M.$$

Because t_0' is arbitrary and letting $m \to \infty$, the right hand side may be made as small as desired by taking t_0' sufficiently small and we get the required result. □

Theorem 4.2 *Let* $(H1)$–$(H6)$ *hold and* $\widetilde{\phi}(0) \in \mathcal{L}_2(\Omega, D(A^\alpha))$ *for all* $t \in [-\tau, 0]$. *Then there exists a unique function* $u_n \in \mathcal{B}_R$ *and unique* $u \in \mathcal{B}_R$ *satisfying*

$$u_n(t) = \begin{cases} \widetilde{\phi}(t), \quad t \in [-\tau, 0]; \\ S_\rho(t)[\widetilde{\phi}(0) + G_n(0, \widetilde{\phi}(0))] - G_n(t, (u_n)_t) + \int_0^t (t-s)^{\rho-1} Q_\rho(t-s) A G_n(s, (u_n)_s) ds \\ \quad + \int_0^t (t-s)^{\rho-1} Q_\rho(t-s)[F_n(s, (u_n)_s, u_n[b(u_n(s), s)]) \\ \quad + \int_0^s a(s-r) k_n(r, (u_n)_r) d\omega(r)] ds + \sum_{j=1}^r S_\rho(t-t_j) I_{j,n}(u_n)_{t_j}), \quad t \in [0, T_0]. \end{cases} \tag{4.6}$$

and

$$u(t) = \begin{cases} \widetilde{\phi}(t), \quad t \in [-\tau, 0]; \\ S_\rho(t)[\widetilde{\phi}(0) + G(0, \widetilde{\phi}(0))] - G(t, u_t) + \int_0^t (t-s)^{\rho-1} Q_\rho(t-s) A G(s, u_s) ds \\ \quad + \int_0^t (t-s)^{\rho-1} Q_\rho(t-s)[F(s, u_s, u[b(u(s), s)]) \\ \quad + \int_0^s a(s-r) k(r, u_r) d\omega(r)] ds + \sum_{j=1}^r S_\rho(t-t_j) I_{j,n} u_{t_j}), \quad t \in [0, T_0]. \end{cases} \tag{4.7}$$

such that $u_n \to u$ *in* $\mathcal{B}_R$ *as* $n \to \infty$.

Proof Let $\widetilde{\phi}(0) \in \mathcal{L}_2(\Omega, D(A^{\alpha}))$ for all $t \in [-\tau, 0]$. Since for $t \in (0, T]$, there exists $u_n \in \mathcal{B}_R$ such that $A^{\alpha}u_n(t) \to A^{\alpha}u(t) \in \mathcal{B}_R$ as $n \to \infty$ and $u(t) = u_n(t) = \widetilde{\phi}(t)$ for all $t \in [-\tau, 0]$. Also for $t \in [-\tau, T]$, we have $A^{\alpha}u_n(t) \to A^{\alpha}u(t)$ in $\mathcal{L}_2(\Omega, \mathcal{H})$ as $n \to \infty$. Furthermore, since for each $u_n \in \mathcal{B}_R$, we have $u \in \mathcal{B}_R$ and for any $0 < t_0 \leqslant T$,

$$\lim_{n\to\infty} \sup_{t_0\leqslant t\leqslant T} \mathbb{E}\|u_n(t) - u(t)\|_{\alpha}^2 = 0.$$

Also, we have

$$\begin{aligned}
&\sup_{s\in[t_0,T]} \mathbb{E}\|F_n(s, (u_n)_s, u_n[b(u_n(s), s)]) - F(s, u_s, u[b(u(s), s)])\|^2 \\
&\quad\leqslant 2[L_f(1 + LL_b)\mathbb{E}\|u_n - u\|_{s,\alpha}^2 + L_f(\mathbb{E}\|(P^n - I)u_s\|_{\alpha}^2 \\
&\qquad + \|A^{-1}\|^2\mathbb{E}\|(P^n - I)u[b(u(s), s)]\|_{\alpha}^2)] \to 0,
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{s\in[t_0,T]} \mathbb{E}\|A^{\beta}G_n(s, (u_n)_s) - A^{\beta}G(s, u_s)\|^2 \\
&\quad\leqslant 2L_G\|A^{-1}\|^2[\mathbb{E}\|u_n - u\|_{s,\alpha}^2 + \mathbb{E}\|(P^n - I)u_s\|_{\alpha}^2] \to 0,
\end{aligned}$$

and

$$\sup_{s\in[t_0,T]} \mathbb{E}\|k_n(s, (u_n)_s - k(s, u_s)\|^2 \leqslant 2L_k[\mathbb{E}\|u_n - u\|_{s,\alpha}^2 + \mathbb{E}\|(P^n - I)u_s\|_{\alpha}^2] \to 0,$$

as $n \to \infty$. For $t_0' \in (0, t)$, rewrite Eq. (3.16) as

$$\begin{aligned}
u_n(t) =& S_{\rho}(t)[\widetilde{\phi}(0) + G_n(0, \widetilde{\phi}(0))] - G_n(t, (u_n)_t) \\
&+ \left(\int_0^{t_0'} + \int_{t_0'}^{t}\right)(t - s)^{\rho-1}Q_{\rho}(t - s)AG_n(s, (u_n)_s)ds \\
&+ \left(\int_0^{t_0'} + \int_{t_0'}^{t}\right)(t - s)^{\rho-1}Q_{\rho}(t - s)[F_n(s, (u_n)_s, u_n[b(u_n(s), s)]) \\
&+ \int_0^{s} a(s - r)k_n(r, (u_n)_r)d\omega(r)]ds + \sum_{j=1}^{r} S_{\rho}(t - t_j)I_{j,n}(u_n)_{t_j}.
\end{aligned} \tag{4.8}$$

The first and third integrals of (4.8) can be estimated as

$$\begin{aligned}
&\left\|\int_0^{t_0'}(t - s)^{\rho-1}A^{1-\beta}Q_{\rho}(t - s)A^{\beta}G_n(s, (u_n)_s)ds\right\|^2 \leqslant L_G\left(\frac{\rho C_{1-\beta}\Gamma(1 + \beta)}{\Gamma(1 + \rho\beta)}\right)^2 T_0^{2(\rho\beta-1)}t_0', \\
&\left\|\int_0^{t_0'}(t - s)^{\rho-1}Q_{\rho}(t - s)[F_n(s, (u_n)_s, u_n[b(u_n(s), s)]) + \int_0^{s} a(s - r)k_n(r, (u_n)_r)d\omega(r)]ds\right\| \\
&\leqslant M(R)\left(\frac{\rho C}{\Gamma(1 + \rho)}\right)^2 T_0^{2\rho-2}t_0'.
\end{aligned}$$

Thus we conclude that

$$\begin{aligned}
&\Bigg\|u_n(t) - S_{\rho}(t)[\widetilde{\phi}(0) + G_n(0, \widetilde{\phi}(0))] + G_n(t, (u_n)_t) - \int_{t_0'}^{t}(t - s)^{\rho-1}Q_{\rho}(t - s)AG_n(s, (u_n)_s)ds \\
&- \int_{t_0'}^{t}(t - s)^{\rho-1}Q_{\rho}(t - s)[F_n(s, (u_n)_s, u_n[b(u_n(s), s)]) + \int_0^{s} a(s - r)k_n(r, (u_n)_r)d\omega(r)]ds \\
&- \sum_{j=1}^{r} S_{\rho}(t - t_j)I_{j,n}(u_n)_{t_j}\Bigg\| \leqslant \left[L_G\left(\frac{\rho C_{1-\beta}\Gamma(1 + \beta)}{\Gamma(1 + \rho\beta)}\right)^2 T_0^{2(\rho\beta-1)} + M(R)\left(\frac{\rho C}{\Gamma(1 + \rho)}\right)^2 T_0^{2\rho-2}\right]t_0'.
\end{aligned}$$

Letting $n \to \infty$ in the above inequality, we get

$$\Bigg\| u(t) - S_\rho(t)[\widetilde{\phi}(0) + G(0, \widetilde{\phi}(0))] + G(t, u_t) - \int_{t_0'}^{t} (t-s)^{\rho-1} Q_\rho(t-s) AG(s, u_s) ds$$
$$- \int_{t_0'}^{t} (t-s)^{\rho-1} Q_\rho(t-s)[F(s, u_s, u[b(u(s), s)]) + \int_0^s a(s-r)k(r, u_r) d\omega(r)] ds$$
$$- \sum_{j=1}^{r} S_\rho(t-t_j) I_j(u_{t_j}) \Bigg\| \leqslant \left[L_G \left(\frac{\rho C_{1-\beta} \Gamma(1+\beta)}{\Gamma(1+\rho\beta)} \right)^2 T_0^{2(\rho\beta-1)} + M(R) \left(\frac{\rho C}{\Gamma(1+\rho)} \right)^2 T_0^{2\rho-2} \right] t_0'.$$

Since t_0' is arbitrary and hence we conclude that $u(t)$ satisfies Eq. (4.7). □

5 Faedo–Galerkin approximations

For any $0 < T_0 < T$, we have a unique u satisfying the integral equation

$$u(t) = \begin{cases} \widetilde{\phi}(t), \quad t \in [-\tau, 0]; \\ S_\rho(t)[\widetilde{\phi}(0) + G(0, \widetilde{\phi}(0))] - G(t, u_t) + \int_0^t (t-s)^{\rho-1} Q_\rho(t-s) AG(s, u_s) ds \\ + \int_0^t (t-s)^{\rho-1} Q_\rho(t-s)[F(s, u_s, u[b(u(s), s)]) \\ + \int_0^s a(s-r)k(r, u_r) d\omega(r)] ds + \sum_{j=1}^{r} S_\rho(t-t_j) I_{j,n} u_{t_j}, \quad t \in [0, T_0]. \end{cases} \tag{5.1}$$

Also we have a unique solution u_n of the approximate integral equation

$$u_n(t) = \begin{cases} \widetilde{\phi}(t), \quad t \in [-\tau, 0]; \\ S_\rho(t)[\widetilde{\phi}(0) + G(0, P^n \widetilde{\phi}(0))] - G(t, P^n (u_n)_t) \\ \quad + \int_0^t (t-s)^{\rho-1} Q_\rho(t-s) AG(s, P^n (u_n)_s) ds \\ + \int_0^t (t-s)^{\rho-1} Q_\rho(t-s)[F(s, P^n (u_n)_s, P^n u_n[b(P^n u_n(s), s)]) \\ + \int_0^s a(s-r)k(r, P^n (u_n)_r) d\omega(r)] ds + \sum_{j=1}^{r} S_\rho(t-t_j) I_{j,n} (u_n)_{t_j}, \quad t \in [0, T_0]. \end{cases} \tag{5.2}$$

Now, if we project (5.2) onto H_n, we get the Faedo–Galerkin approximations $\widehat{u}_n(t) = P^n u_n(t)$ satisfying

$$\widehat{u}_n(t) = \begin{cases} P^n \widetilde{\phi}(t), \quad t \in [-\tau, 0]; \\ S_\rho(t) P^n [\widetilde{\phi}(0) + G(0, P^n \widetilde{\phi}(0))] - P^n G(t, (\widehat{u}_n)_t) \\ \quad + \int_0^t (t-s)^{\rho-1} Q_\rho(t-s) A P^n G(s, (\widehat{u}_n)_s) ds \\ \quad + \int_0^t (t-s)^{\rho-1} Q_\rho(t-s)[P^n F(s, (\widehat{u}_n)_s, \widehat{u}_n[b(\widehat{u}_n(s), s)]) \\ \quad + \int_0^s a(s-r) P^n k(r, (\widehat{u}_n)_t) d\omega(r)] ds \\ \quad + \sum_{j=1}^{r} S_\rho(t-t_j) P^n I_j(\widehat{u}_n)_{t_j}), \quad t \in [0, T_0]. \end{cases} \tag{5.3}$$

Solutions u of (5.1) and $\widehat{u}_n$ of (5.3) have the representations

$$u(t) = \sum_{j=0}^{\infty} \sigma_j(t) u_j, \qquad \sigma_j(t) = (u(t), u_j), \quad j = 0, 1, \ldots;$$

$$\widehat{u}_n(t) = \sum_{j=0}^{n} \sigma_j^n(t) u_j, \qquad \sigma_j^n(t) = (\widehat{u}_n(t), u_j), \quad j = 0, 1, \ldots; \tag{5.4}$$

as a consequence of Theorems 3.1 and 4.1, we have the following results.

Corollary 5.1 *Let* $(H1)$–$(H6)$ *hold. Then*

(a) *if* $\widetilde{\phi}(t) \in \mathcal{L}_2^0(\Omega, D(A^\alpha)$ *for all* $t \in [-\tau, 0]$, *then*

$$\lim_{n\to\infty} \sup_{\{n \geqslant m, -\tau \leqslant t \leqslant T_0\}} \mathbb{E}\|A^\alpha[\widehat{u}_n(t) - \widehat{u}_m(t)]\|^2 = 0.$$

(b) *if* $\widetilde{\phi}(t) \in \mathcal{L}_2^0(\Omega, D(A)$ *for all* $t \in [-\tau, 0]$, *then*

$$\lim_{n\to\infty} \sup_{\{n \geqslant m, -\tau \leqslant t \leqslant T_0\}} \mathbb{E}\|A^\alpha[\widehat{u}_n(t) - \widehat{u}_m(t)]\|^2 = 0.$$

Theorem 5.2 *Let* $(H1)$–$(H6)$ *hold and let* $\widetilde{\phi}(t) \in \mathcal{L}_2^0(\Omega, D(A^\alpha)$ *for all* $t \in [-\tau, 0]$. *Then, there exist a unique* $\widehat{u}_n \in \mathcal{B}_R$ *satisfying*

$$\widehat{u}_n(t) = \begin{cases} P^n \widetilde{\phi}(t), \quad t \in [-\tau, 0]; \\ S_\rho(t) P^n[\widetilde{\phi}(0) + G(0, P^n \widetilde{\phi}(0))] - P^n G(t, (\widehat{u}_n)_t) \\ \quad + \int_0^t (t-s)^{\rho-1} Q_\rho(t-s) A P^n G(s, (\widehat{u}_n)_s) ds \\ \quad + \int_0^t (t-s)^{\rho-1} Q_\rho(t-s) [P^n F(s, (\widehat{u}_n)_s, \\ \quad \widehat{u}_n[b(\widehat{u}_n(s), s)]) + \int_0^s a(s-r) P^n k(r, (\widehat{u}_n)_t) d\omega(r)] ds \\ \quad + \sum_{j=1}^r S_\rho(t - t_j) P^n I_j (\widehat{u}_n)_{t_j}), \quad t \in [0, T_0]. \end{cases} \tag{5.5}$$

and $u \in \mathcal{B}_R$ *satisfying*

$$u(t) = \begin{cases} \widetilde{\phi}(t), \quad t \in [-\tau, 0]; \\ S_\rho(t)[\widetilde{\phi}(0) + G(0, \widetilde{\phi}(0))] - G(t, u_t) + \int_0^t (t-s)^{\rho-1} Q_\rho(t-s) A G(s, u_s) ds \\ + \int_0^t (t-s)^{\rho-1} Q_\rho(t-s)[F(s, u_s, u[b(u(s), s)]) \\ + \int_0^s a(s-r) k(r, u_r) d\omega(r)] ds + \sum_{j=1}^{r} S_\rho(t - t_j) I_{j,n} u_{t_j}, \quad t \in [0, T_0]. \end{cases} \tag{5.6}$$

such that $\widehat{u}_n(t) \to u(t)$ *as* $n \to \infty$ *in* $\mathcal{B}_R$.

Proof We have

$$\begin{aligned} \mathbb{E}\|\widehat{u}_n(t) - u(t)\|_\alpha^2 &= \mathbb{E}\|P^n u_n(t) - P^n u(t) + P^n u(t) - u(t)\|_\alpha^2 \\ &\leqslant \mathbb{E}\|P^n(u_n(t) - u(t))\|_\alpha^2 + \mathbb{E}\|(P^n - I)u(t)\|_\alpha^2. \end{aligned}$$

By Theorem 4.2, we have

$$\lim_{n\to\infty} \sup_{t \in [-r, T]} \mathbb{E}\|u_n(t) - u(t)\|_\alpha^2 = 0.$$

Thus, this completes the proof of the theorem. □

Now for the convergence σ_j^n to σ_j, we have the following convergence result.

Theorem 5.3 *Let* $(H1)$–$(H6)$ *hold. Then*

(a) *if* $\widetilde{\phi}(0) \in \mathcal{L}_2^0(\Omega, D(A^\alpha)$ *for all* $t \in [-\tau, 0]$, *then*

$$\lim_{n\to\infty} \sup_{t\in[-\tau,T_0]} \left[\sum_{i=0}^{n} \lambda_i^{2\alpha} \mathbb{E}\|\sigma_i^n(t) - \sigma_i(t)\|^2\right] = 0. \tag{5.7}$$

(b) *if* $\widetilde{\phi}(0) \in \mathcal{L}_2^0(\Omega, D(A))$, *then*

$$\lim_{n\to\infty} \sup_{t\in[-\tau,T_0]} \left[\sum_{i=0}^{n} \lambda_i^{2\alpha} \mathbb{E}\|\sigma_i^n(t) - \sigma_i(t)\|^2\right] = 0. \tag{5.8}$$

Proof Since

$$\begin{aligned}\mathbb{E}\|A^\alpha(\widehat{u}_n(t) - u(t))\|^2 &= \sum_{i=0}^{\infty} \mathbb{E}\|A^\alpha(\sigma_i^n(t) - \sigma_i(t))u_i\|^2 \\ &= \sum_{i=0}^{\infty} \lambda_i^{2\alpha} \mathbb{E}\|(\sigma_i^n(t) - \sigma_i(t))u_i\|^2.\end{aligned}$$

Therefore

$$\mathbb{E}\|A^\alpha(\widehat{u}_n(t) - u(t))\|^2 \geqslant \sum_{i=0}^{n} \lambda_i^{2\alpha} \mathbb{E}\|(\sigma_i^n(t) - \sigma_i(t))u_i\|^2,$$

taking $\lim_{n\to\infty}$ both sides and using Theorem 5.2, we obtain the required result. □

6 Example

Consider the fractional differential equation in the separable Hilbert space $\mathcal{H} = \mathcal{L}^2(0, 1)$

$$\begin{cases} {}^c\mathbf{D}^\rho[V(t,x) + G(t,x,\partial_x V(t+\nu,x))] - \partial_x^2 V(t,x) = F_1(x, V(t,x)) + F_2(t,x,V(t+\nu,x)) \\ \qquad + \int_0^t a(t-s)K(s,x,\partial_x V(s,x))\partial\omega(s), \quad x \in (0,1),\ t \in (0,\frac{1}{2}) \cup (\frac{1}{2},1) \\ \Delta V(t,x)|_{t=1/2} = \frac{2V(1/2,x)^-}{2+V(1/2,x)^-}, \\ V(t,0) = V(t,1) = 0, \\ V(\nu,x) = \frac{1}{N^2}\cdot\frac{V(\nu,x)}{1+V(\nu,x)}, \quad -\tau \leqslant \nu \leqslant 0, \end{cases} \tag{6.1}$$

where

$$F_1(x, V(t,x)) = \int_0^x W(s,x)V(s,h(t)|V(t,s)|)ds,$$

and $F_2 : \mathbb{R}_+ \times [0,1] \times \mathbb{R}$ is measurable in x, locally Hölder continuous in t, locally Lipschitz continuous in V, uniformly in x, $N \in \mathbb{N}$. We also assume that $W \in C^1([0,1] \times [0,1], \mathbb{R})$

and $h : \mathbb{R}_+ \to \mathbb{R}_+$ is locally Hӧlder continuous in t with $h(0) = 0$. $K \in \mathcal{L}_Q(\mathcal{K}, \mathcal{H})$ and $a^2 \in \mathcal{L}^q(0, \infty)$ and ω is a standard Winer process.
Define operator A such that

$$Au = -u'' \quad \text{with} \quad u \in D(A) = \mathcal{H}_0^1(0,1) \cap \mathcal{H}^2(0,1). \tag{6.2}$$

Here $-A$ is infinitesimal generator of an analytic semigroup $S(t)$. Moreover, A has a discrete spectrum with the eigenvalues of the form $k^2\pi^2$ for $k \in \mathbb{N}$, whose corresponding(normalized) eigenfunctions are given by $e_k(x) = \sqrt{2}\sin k\pi x$. Therefore for $u \in D(A)$

$$u(x) = \sum_{k\in\mathbb{N}} < u(x), e_k(x) > e_k(x).$$

Now, for $\alpha = 1/2$, $D(A^{1/2})$ (denoted by $\mathcal{H}_{1/2}$) is a Banach space endowed with the norm

$$\|u\|_{1/2} = \|A^{1/2}u\| \quad \text{for } u \in D(A^{1/2}).$$

Also, define the space

$$C_t^{1/2} = C([-\tau, t], D(A^{1/2})), \quad t \in [0, T],$$

endowed with the sup norm

$$\|\varphi\|_{t,1/2} = \sup_{-\tau \leqslant \nu \leqslant t} \|\varphi(\nu)\|_\alpha, \quad \varphi \in C_t^{1/2}.$$

Then, we have

$$A^{1/2}u(x) = \sum_{k\in\mathbb{N}} k\pi < u(x), e_k(x) > e_k(x) \quad \text{with } u \in D(A^{1/2}).$$

The Eq. (6.1) can be reformulated as the following abstract stochastic fractional integro-differential equation with impulsive effects in $\mathcal{H} = \mathcal{L}^2(0,1)$

$$\begin{aligned} {}^c\mathbf{D}^\rho[u(t) + G(t, u_t)] + Au(t)dt =& F(t, u_t, u[b(u(t), t)])dt \\ &+ \int_0^t a(t-s)k(s, u_s)d\omega(s), \\ \Delta u(t)|_{t=1/2} =& I(u_t), \\ u(0) =& \widetilde{\phi}(0), \end{aligned} \tag{6.3}$$

where $u(t) = V(t, \cdot)$ i.e. $u(t)(x) = V(t, x)$, A is defined in (6.2), the function $G : [0, T] \times D(A^{1/2}) \to \mathcal{H}$ is defined as

$$G(t, u_t)(x) = G(t, x, \partial_x V(t + \nu, x)).$$

The function $F : [0, T] \times \mathcal{H}_{1/2} \times \mathcal{H}_{-1/2}$ is defined as

$$F(t, \widetilde{\phi}, \varphi)(x) = F_1(x, \varphi) + F_2(t, x, \widetilde{\phi}),$$

where

$$F_1(x, \varphi) = \int_0^x W(x, y)\varphi(y)dy, \quad F_2(t, x, \widetilde{\phi}) \leqslant Z(t, x)(1 + \|\widetilde{\phi}\|_{\mathcal{H}^2(0,1)}).$$

After a simple calculation, we get

$$\mathbb{E}\|F_1(x, \varphi_1) - F_1(x, \varphi_2)\|^2 \leqslant L_\varphi \mathbb{E}\|\varphi_1 - \varphi_2\|^2,$$

and

$$\mathbb{E}\|F_2(t,x,\widetilde{\phi}_1)-F_2(s,x,\widetilde{\phi}_2)\|^2 \leqslant L(|t-s|^{2\gamma_1}+\mathbb{E}\|\widetilde{\phi}_1-\widetilde{\phi}_2\|^2).$$

The function $b : H_0^1(0,1)\times\mathbb{R}^+\to\mathbb{R}^+$ defined by $b(u(t),t)=h(t)|u(t)|$ satisfies

$$|b(u,t)|=|h(t)|u(t)||\leqslant\|h\|_\infty\times\|u\|_\infty,\quad t\in[0,1]$$

As h is a Hölder continuous function, there exists a positive constant L_h and $\gamma\in(0,1]$ such that

$$|h(t)-h(s)|\leqslant L_h|t-s|^\gamma,\quad t,s\in[0,1].$$

For $u_1,u_2\in H_0^1(0,1)$, we have

$$\begin{aligned}|b(u_1,t)-b(u_2,s)|^2 &= |h(t)[|u_1|-|u_2|]+(h(t)-h(s))u_2|^2\\ &\leqslant \|h\|_\infty^2\mathbb{E}\|u_1-u_2\|^2_{H_0^1(0,1)}+L_h|t-s|^{2\gamma}\|u_2\|^2_\infty\\ &\leqslant \max\{\|h\|^2_\infty, L_h\|u_2\|^2_\infty\}\left(\mathbb{E}\|u_1-u_2\|^2_{H_0^1(0,1)}+|t-s|^{2\gamma}\right).\end{aligned}$$

For $u_1,u_2\in D(A^{1/2})$, we have

$$\mathbb{E}\|I(u_1)-I(u_2)\|^2_{1/2}\leqslant\frac{16\mathbb{E}\|u_1-u_2\|^2_{1/2}}{\|(2+u_1)(2+u_2)\|^2_{1/2}}.$$

It can be easily checked that the assumptions $(H1)$–$(H6)$ are satisfied. Therefore we may use the results established in the earlier sections to obtain approximate solutions and their convergence.

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References

1. Bazley, N.W.: Approximation of wave equations with reproducing nonlinearities. Nonlinear Anal. **3**, 539–546 (1979)
2. Bazley, N.W.: Global convergence of Faedo–Galerkin approximations to nonlinear wave equations. Nonlinear Anal. **4**, 503–507 (1980)
3. Bahuguna, D., Srivastava, S.K.: Approximation of solutions to evolution integrodifferential equations. J. Appl. Math. Stoch. Anal. **9**, 315–322 (1996)
4. Bahuguna, D., Shukla, R.: Approximations of solutions to nonlinear Sobolev type evolution equations. Electron. J. Differ. Equ. **31**, 1–16 (2003)
5. Benchohra, M., Henderson, J., Ntouyas, S.K.: Impulsive Differential Equations and Inclusions, Contemporary Mathematics and Its Applications, vol. 3. Hindawi Publishing Corporation, New York (2006)
6. Balasubramaniam, P., Ali, M.S., Kim, J.H.: Faedo–Galerkin approximate solutions for stochastic semilinear integrodifferential equations. Comput. Math. Appl. **58**, 48–57 (2009)
7. Balasubramaniam, P., Tamilalagan, P.: Approximate controllability of a class of fractional neutral stochastic integro-differential inclusions with infinite delay by using Mainardis function. Appl. Math. Comput. **256**, 232–246 (2015)
8. Cui, J., Yan, L.: Existence result for fractional neutral stochastic integro-differential equations with infinite delay. J. Phys. A **44**, 16 (2011)
9. Chaddha, A., Pandey, D.N.: Faedo–Galerkin approximation of solution for a nonlocal neutral fractional differential equation with deviating argument. Mediterr. J. Math. **13**, 3041–3067 (2016)
10. Chaddha, A., Pandey, D.N.: Existence and approximation of solution to neutral fractional differential equation with nonlocal conditions. Comput. Math. Appl. **69**, 893–908 (2015)

11. Chaudhary, R., Pandey, D.N.: Approximation of solutions to a delay equation with a random forcing term and non local conditions. J. Integral Equ. Appl. **28**, 481–507 (2016)
12. Chaudhary, R., Pandey, D.N.: Approximation of solutions to stochastic fractional integro-differential equation with deviated argument. Differ. Equ. Dyn. Syst. (2016). doi:10.1007/s12591-016-0329-3
13. Da Prato, G., Zabczyk, J.: Stochastic Equations in Infinite Dimensions, Encyclopedia of Mathematics and its Applications, vol. 44. Cambridge University Press, Cambridge (1992)
14. Dabas, J., Chauhan, A.: Existence and uniqueness of mild solutions for an impulsive neutral fractional integro-differential equation with infinite delay. Math. Comput. Model. **57**, 754–763 (2013)
15. El-Borai, M.M.: Some probability densities and fundamental solutions of fractional evolution equations. Chaos Solitons Fractals **14**, 433–440 (2002)
16. Feĉkan, M., Zhou, Y., Wang, J.-R.: On the concept and existence of solution for impulsive fractional differential equations. Commun. Nonlinear Sci. Numer. Simul. **17**, 3050–3060 (2012)
17. Feĉkan, M., Zhou, Y., Wang, J.-R.: Response to Comments on the concept of existence of solution for impulsive fractional differential equations. Commun. Nonlinear Sci. Numer. Simul. **19**, 401–403 (2014)
18. Gard, T.C.: Introduction to Stochastic Differential Equations, Monographs and Textbooks in Pure and Applied Mathematics, vol. 114. Dekker, New York (1988)
19. Heinz, E., Von Wahl, W.: Zn einem Satz von F.W. Browder uber nichtlineare Wellengleichungen. Math. Z. **141**, 33–45 (1974)
20. Hu, L., Ren, Y.: Existence results for impulsive neutral stochastic functional integro-differential equations with infinite delays. Acta Appl. Math. **111**, 303–317 (2010)
21. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
22. Kumar, P., Pandey, D.N., Bahuguna, D.: Existence of piecewise continuous mild solutions for impulsive functional differential equations with iterated deviating arguments. Electron. J. Differ. Equ. **241**, 1–15 (2013)
23. Kumar, P., Pandey, D.N., Bahuguna, D.: Approximation of solutions to a fractional differential equation with a deviating argument. Differ. Equ. Dyn. Syst. **22**, 333–352 (2014)
24. Lakshmikantham, V., Bainov, D., Simeonov, P.S.: Theory of Impulsive Differential Equations, Series in Modern Applied Mathematics. World Scientic, Teaneck (1989)
25. Lin, A., Ren, Y., Xia, N.: On neutral impulsive stochastic integro-differential equations with infinite delays via fractional operators. Math. Comput. Model. **51**, 413–424 (2010)
26. Murakami, H.: On non-linear ordinary and evolution equations. Funkcial. Ekvac. **9**, 151–162 (1966)
27. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations. A Wiley-Interscience Publication, Wiley, New York (1993)
28. Miletta, P.D.: Approximation of solutions to evolution equations. Math. Methods Appl. Sci. **17**, 753–763 (1994)
29. Mao, X.R.: Stochastic Differential Equations and Applications. Horwood, Chichester (1997)
30. Oksendal, B.: Stochastic Differential Equations, 5th edn. Springer, Berlin (2002)
31. Pazy, A.: Semi-Groups of Linear Operator and Applications of Partial Differential Equations. Springer, New York (1983)
32. Podlubny, I.: Fractional Differential Equations, Mathematics in Science and Engineering, vol. 198. Academic Press, San Diego (1999)
33. Park, J.Y., Jeong, J.U.: Existence results for impulsive neutral stochastic functional integro-differential inclusions with infinite delays. Adv. Differ. Equ. **2014**, 1–17 (2014)
34. Segal, I.: Non-linear semi-groups. Ann. Math. **2**(78), 339–364 (1963)
35. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives, Theory and Applications. Gordon and Breach, Yverdon (1993)
36. Tamilalagan, P., Balasubramaniam, P.: Approximate controllability of fractional stochastic differential equations driven by mixed fractional Brownian motion via resolvent operators. Int. J. Control (2016). doi:10.1080/00207179.2016.1219070
37. Wang, J.-R., Feĉkan, M., Zhou, Y.: On the new concept of solution and existence results for impulsive fractional evolution equations. Dyn. PDE **8**, 345–361 (2011)
38. Wang, G., Ahmad, B., Zhang, L., Nieto, J.J.: Comments on the concept of existence of solution for impulsive fractional differential equations. Commun. Nonlinear Sci. Numer. Simul. **19**, 401–403 (2014)
39. Yan, Z., Lu, F.: On approximate controllability of fractional stochastic neutral integro-differential inclusions with infiite delay. Appl. Anal. **2014**, 23 (2014)
40. Zhou, Y., Jiao, F.: Existence of mild solutions for fractional neutral evolution equations. Comput. Math. Appl. **59**, 1063–1077 (2010)