

# Nash multiplicities and resolution invariants

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**Abstract** The Nash multiplicity sequence was defined by Lejeune-Jalabert as a non-increasing sequence of integers attached to a germ of a curve inside a germ of a hypersurface. Hickel generalized this notion and described a sequence of blow ups which allows us to compute it and study its behavior. In this paper, we show how this sequence can be used to compute some invariants that appear in algorithmic resolution of singularities. Moreover, this indicates that these invariants from constructive resolution are intrinsic to the variety since they can be read in terms of its space of arcs. This result is a first step connecting explicitly arc spaces and algorithmic resolution of singularities.

**Keywords** Rees algebras · Resolution of singularities · Arc spaces

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## 1 Introduction

Consider a variety  $X$  of dimension  $d$  over a field  $k$ . By a *resolution of singularities* we mean a proper birational morphism

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$$X \xleftarrow{\phi} X'$$

such that  $X'$  is regular. In addition we require that  $\phi$  induces an isomorphism on the set of regular points of  $X$ , and that the exceptional divisor  $\phi^{-1}(\text{Sing}(X))$  has normal crossing support.

In [17], Hironaka proved that given a variety over a field of characteristic zero it is possible to find a resolution of singularities of  $X$  defined by a sequence of blow ups at smooth centers. Moreover, it is possible to construct such a sequence by means of some *invariants* attached to the points of  $X$  (see [3, 25, 26]). The study of those invariants becomes interesting as soon as they provide an algorithm for the construction of a resolution of singularities for any variety over a field of characteristic zero. Furthermore, they may also give insight into the resolution phenomenon, in order to solve the problem for more general fields. Through these invariants, one can define *resolution functions*, which stratify  $X$  in locally closed sets, so that there is a canonical (regular) center to blow up at each step of the resolution sequence. Then resolution is achieved via the construction of a finite sequence of blow ups.

One of the ingredients that one may take into account for this stratification is the *multiplicity function* (see [30]). The multiplicity is an upper semi-continuous function defined at each point  $\xi$  of a variety  $X$ . If  $X$  is defined over  $\mathbb{C}$  then the multiplicity at  $\xi$  is the smallest rank of the generic fiber for all possible local morphisms  $(X, \xi) \rightarrow (\mathbb{C}^d, 0)$ . If  $X$  is a reduced equidimensional scheme, then  $X$  is regular if and only if the multiplicity equals one at every point.

### 1.1 Constructive resolution of singularities

In short, a *constructive resolution of singularities* of  $X$  is given by an upper semi-continuous function

$$f : X \rightarrow (\Lambda, \geq),$$

where  $(\Lambda, \geq)$  is some well ordered set. The maximum value of  $f$  determines the first smooth center  $C \subset X$  to blow up:  $X \xrightarrow{\pi_1} X_1$ . Right after this blow up, a new upper semi-continuous function  $f_1 : X_1 \rightarrow (\Lambda, \geq)$  is defined, in such a way that  $f_1(\pi_1^{-1}(\xi)) = f(\xi)$  for any  $\xi \in X \setminus C$ , and  $f_1(\xi') < f(\xi)$  whenever  $\pi_1(\xi') = \xi \in C$ . If  $f$  is appropriately constructed so that it is constant if and only if  $X$  is smooth, then resolution is achieved after a finite number of steps. One way to construct such a function is to associate a string of invariants to each point.

Looking at the multiplicity function on  $X$  may be a good starting point when attempting to construct a resolution of singularities of  $X$ . But unfortunately, the strata defined by the multiplicity function may be non smooth. Thus, the use of other invariants becomes necessary in order to refine the stratification so that one can have a smooth stratum to choose as the center of the first blow up. The most important of these invariants is the so called Hironaka's *order function* (see [13] or Definition 2.5 in this paper). From it, many other invariants may be defined (see Sect. 2.8). If we choose the multiplicity function as the first coordinate of  $f$ ,  $C$  is contained in  $\text{Max mult}(X)$ , the closed subset of  $X$  where the multiplicity reaches its highest value. Now fix some point  $\xi \in \text{Max mult}(X)$ . Locally, in a neighbourhood of  $\xi$ , a finite local projection  $p$  to some smooth scheme of dimension  $d = \dim(X)$  can be defined, inducing a bijection between  $\text{Max mult}(X)$  and its image (see [5, 6]). There, Hironaka's order function can be defined at each point in the image of  $\text{Max mult}(X)$ . This function, which we will for the moment denote by  $\text{ord}_\xi^{(d)}(X)$ , does not depend on the projection (if it is general enough). Moreover, it can be shown that lowering the maximum multiplicity in a neighbourhood

of  $\xi$  is equivalent to solving a suitable problem in a  $d$ -dimensional smooth scheme. This gives the possibility of constructing a resolution of singularities of  $X$  by resolving such problems, which simplifies the process. We will refer to  $\text{ord}_\xi^{(d)}(X)$  as *Hironaka's order function in dimension  $d$*  (see Sect. 2.5 for full details). It can be shown that  $\text{ord}_\xi^{(d)}(X)$  is the next relevant coordinate of  $f$ , refining the stratum  $\text{Max mult}(X)$  (see Sect. 2.8), so we will consider

$$f(\xi) = (\text{mult}_\xi(X), \text{ord}_\xi^{(d)}(X), \dots).$$

Surprisingly,  $\text{ord}_\xi^{(d)}(X)$  can readily be read by looking at a sufficiently general arc in  $\mathcal{L}(X)$ , as our main result, Theorem 3.6, shows.

### 1.2 Arcs

There are many other approaches to the study of singularities. Jet and arc spaces of varieties often appear among them. Many properties of the jet schemes and the arc scheme of a variety are linked to its singularities. See for instance the works of Ein, Ishii, Mustařa and Yasuda where some singularity types are characterized through topological or geometrical properties of the associated arc schemes [10–12, 19, 22, 23].

It is in this context of arc spaces where the *Nash multiplicity sequence* appears. It was defined by Lejeune–Jalabert [20] as a non-increasing sequence of positive integers attached to a germ of a curve inside a germ of a hypersurface. Hickel generalized this notion to arbitrary codimension [16] and defined a sequence of blow ups (at points) that allows us to compute Nash multiplicity sequences and study their behaviour. Given a variety  $X$ , fix an arc through a point  $\xi \in \text{Max mult}(X)$  (not necessarily closed). By means of its graph  $\Gamma \subset X \times \mathbb{A}^1$ , the arc  $\varphi$  defines a sequence of blow ups at points:

$$\begin{array}{ccccccc} X_0 = X \times \mathbb{A}^1 & \xleftarrow{\pi_1} & X_1 & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_r} & X_r, \\ \xi_0 = (\xi, 0) & & \xi_1 & & \dots & & \xi_r \end{array}$$

where  $\xi_i$  is the intersection of the exceptional divisor of  $\pi_i$  and the strict transform of the graph  $\Gamma$  in  $X_i$  for  $i = 1, \dots, r$ . The Nash multiplicity sequence is then the sequence

$$m_0 \geq m_1 \geq \dots \geq m_r \geq 1,$$

in which  $m_i$  is the multiplicity of  $X_i$  at  $\xi_i$  for  $i = 0, \dots, r$  (see Sect. 3.2 for details).

### 1.3 Our results

In this work, we analyze a connection of arc spaces with the problem of resolution of singularities. We study the Nash multiplicity sequence for arcs in varieties, and find a relation between the structure of this sequence and some invariants of resolution. In particular, for an algebraic variety  $X$  of dimension  $d$ , we are in position to give a relation between the length  $\rho_{X,\varphi}$  of the first step of the sequence (before the Nash multiplicity decreases for the first time) and Hironaka's order function in dimension  $d$ . We introduce an invariant for  $X$  and  $\varphi$  at  $\xi$  which is sharper than  $\rho_{X,\varphi}$  and which we will denote by  $r_{X,\varphi}$ . More precisely, we will see that  $\rho_{X,\varphi} = \lceil r_{X,\varphi} \rceil$ . For this invariant, we prove the following result:

**Main Theorem (3.6)** Let  $X$  be a variety of dimension  $d$ . Let  $\xi$  be a point in  $\underline{\text{Max}} \text{mult}(X)$ . Then,

$$\text{ord}_\xi^{(d)}(X) = \min_\varphi \left\{ \frac{r_{X,\varphi}}{\text{ord}(\varphi)} \right\},$$

where  $\varphi$  runs through all arcs in  $X$  through  $\xi$ .

As we mentioned before, this minimum is achieved for any arc which is generic enough with respect to the tangent cone of  $X$  at  $\xi$ .

When we work with a hypersurface  $X$ , computing invariants and giving a local expression of the equation of  $X$  is much easier than when we deal with a variety of higher codimension. To avoid this difficulty, we rely on the results on *local presentations* attached, in this case, to the multiplicity (see [30]). They allow us to work locally with a set of hypersurfaces with weights.

*Rees algebras* happen to provide a useful tool for the study of these local presentations and their behaviour under blow ups. They keep track locally of the behaviour of resolution functions before and after blowing up at smooth centers. We will also see that our problem can be translated into a problem of resolution of Rees algebras.

Our work is organized as follows. In Sect. 2, we present some preliminary definitions and results on Rees algebras, as well as some examples motivating their use and their connection to algorithmic resolution. We also include some comments about the resolution invariants we want to focus on. Section 3 is devoted to arcs and the Nash multiplicity sequence. It is in Sect. 4 where we finally connect all the previous concepts and state our main result (Theorem 3.6). The proof of the main result is given in Sect. 5 where we first prove it in the simpler case of a hypersurface. Then we deduce the general case from this one, making use of what we know from [30] about local presentations attached to the multiplicity.

## 2 Rees algebras and their use in resolution of singularities

### 2.1 Rees algebras

**Definition 2.1** Let  $R$  be a Noetherian ring. A *Rees algebra*  $\mathcal{G}$  over  $R$  is a graded ring,<sup>1</sup> that is:

$$\mathcal{G} = \bigoplus_{l \in \mathbb{N}} I_l W^l \subset R[W]$$

for some ideals  $I_i \in R$ ,  $i \in \mathbb{N}$  such that  $I_0 = R$  and  $I_i I_j \subset I_{i+j}$ ,  $\forall i, j \in \mathbb{N}$ , which is also a finitely generated  $R$ -algebra. That is, there exist some  $f_1, \dots, f_r \in R$  and weights  $n_1, \dots, n_r \in \mathbb{N}$  such that

$$\mathcal{G} = R[f_1 W^{n_1}, \dots, f_r W^{n_r}]. \tag{2.1}$$

*Remark 2.2* Rees algebras can be defined over a Noetherian scheme  $V$  in the obvious manner, that is,  $\mathcal{G}$  will be locally at each  $\xi \in V$  as in (2.1), with  $\text{Spec}(R) \subset V$  an open affine subset.

**Definition 2.3** Let  $\mathcal{G}_1$  and  $\mathcal{G}_2$  be two Rees algebras. We denote by  $\mathcal{G}_1 \odot \mathcal{G}_2$  the smallest Rees algebra containing both of them. If  $\mathcal{G}_1 = R[f_1 W^{n_1}, \dots, f_r W^{n_r}]$  and  $\mathcal{G}_2 = R[g_1 W^{m_1}, \dots, g_l W^{m_l}]$ , then  $\mathcal{G}_1 \odot \mathcal{G}_2 = R[f_1 W^{n_1}, \dots, f_r W^{n_r}, g_1 W^{m_1}, \dots, g_l W^{m_l}]$ . If  $\mathcal{G}'_2 = R'[g_1 W^{m_1}, \dots, g_l W^{m_l}]$ , where  $R' \subset R$  is a subring, by abuse of notation we will

<sup>1</sup>  $W$  is just a variable in charge of the degree of the  $I_i$ .

sometimes denote by  $\mathcal{G}_1 \odot \mathcal{G}'_2$  the Rees algebra  $\mathcal{G}_1 \odot \mathcal{G}_2$ , where  $\mathcal{G}_2$  is the extension of  $\mathcal{G}'_2$  to a Rees algebra over  $R$ .

2.1.1 Notations and conventions

From now on we will assume  $k$  to be a field of characteristic zero, unless otherwise stated. We will also assume  $R$  to be a smooth  $k$ -algebra, or  $V$  to be a smooth scheme over  $k$ .

**Definition 2.4** Let  $\mathcal{G}$  be a Rees algebra over  $R$ . The *singular locus* of  $\mathcal{G}$ ,  $\text{Sing}(\mathcal{G})$ , is the closed set given by all the points  $\xi \in \text{Spec}(R)$  such that  $v_\xi(I_l) \geq l, \forall l \in \mathbb{N}$ .<sup>2</sup> Equivalently, if  $\mathcal{G} = R[f_1 W^{n_1}, \dots, f_r W^{n_r}]$ , then it can be shown [15, Proposition 1.4] that

$$\text{Sing}(\mathcal{G}) = \{ \xi \in \text{Spec}(R) : v_\xi(f_i) \geq n_i, \quad \forall i = 1, \dots, r \}.$$

Note that the singular locus of the Rees algebra over  $V$  generated by  $f_1 W^{n_1}, \dots, f_r W^{n_r}$  does not coincide with the usual definition of the singular locus of the subvariety of  $V$  defined by  $f_1, \dots, f_r$ .

We will sometimes refer to the singular locus of a Rees algebra as the *closed set attached to it*.

**Definition 2.5** We define the *order of an element*  $f W^n \in \mathcal{G}$  at  $\xi \in \text{Sing}(\mathcal{G})$  as

$$\text{ord}_\xi(f W^n) = \frac{v_\xi(f)}{n}.$$

We define the *order of the Rees algebra*  $\mathcal{G}$  at  $\xi \in \text{Sing}(\mathcal{G})$  as the infimum of the orders of the elements of  $\mathcal{G}$  at  $\xi$ , that is

$$\text{ord}_\xi(\mathcal{G}) = \inf_{f W^n \in \mathcal{G}} \{ \text{ord}_\xi(f W^n) \}.$$

Actually, one could define the order of an  $\mathcal{O}_V$ -Rees algebra  $\mathcal{G}$  at any point  $\xi \in V$ , but for our purposes, only the order at points in  $\text{Sing}(\mathcal{G})$  will be needed.

**Theorem 2.6** [15, Proposition 6.4.1] *Let  $\mathcal{G} = R[f_1 W^{n_1}, \dots, f_r W^{n_r}]$  be a Rees algebra and let  $\xi \in \text{Sing}(\mathcal{G})$ . Then*

$$\text{ord}_\xi(\mathcal{G}) = \min_{i=1 \dots r} \{ \text{ord}_\xi(f_i W^{n_i}) \}.$$

**Definition 2.7** Let  $\mathcal{G}$  be a Rees algebra over  $R$ . Let  $\mathcal{P} \subset R$  be a prime ideal. We say that  $\mathcal{P}$  is a *permissible center* for  $\mathcal{G}$  if  $R/\mathcal{P}$  is a regular ring and  $v_{\mathcal{P}}(\mathcal{G}) \geq 1$ . That is,  $\mathcal{P}$  is permissible for  $\mathcal{G}$  if it defines a smooth closed set in  $\text{Spec}(R)$  which is also contained in  $\text{Sing}(\mathcal{G})$ . If  $\mathcal{G}$  is a Rees algebra over  $V$ , a closed set  $Y \subset V$  is a permissible center for  $\mathcal{G}$  if it is a regular subvariety contained in  $\text{Sing}(\mathcal{G})$ .

**Definition 2.8** [28, Definition 6.1] Let  $\mathcal{G}$  be a Rees algebra on  $V$ . A  *$\mathcal{G}$ -permissible transformation*

$$V \xleftarrow{\pi} V_1,$$

<sup>2</sup> Here  $v_\xi(I)$  denotes the order of the ideal  $I$  in the regular local ring  $R_{\mathcal{M}_\xi}$ , where  $\mathcal{M}_\xi$  is the ideal defining the point  $\xi$ .

is the blow up of  $V$  at a permissible center  $Y \subset V$ . We denote then by  $\mathcal{G}_1$  the transform of  $\mathcal{G}$  by  $\pi$ , which is defined as

$$\mathcal{G}_1 := \bigoplus_{l \in \mathbb{N}} I_{l,1} W^l,$$

where

$$I_{l,1} = I_l \mathcal{O}_{V_1} \cdot I(E)^{-l} \tag{2.2}$$

for  $l \in \mathbb{N}$  and  $E$  the exceptional divisor of the blow up  $V \leftarrow V_1$ .

**Definition 2.9** Let  $\mathcal{G}$  be a Rees algebra over  $V$ . A *resolution* of  $\mathcal{G}$  is a finite sequence of blow ups,

$$\begin{aligned} V &= V_0 \xleftarrow{\pi_1} V_1 \xleftarrow{\pi_2} \dots \xleftarrow{\pi_l} V_l \\ \mathcal{G} &= \mathcal{G}_0 \xleftarrow{\quad} \mathcal{G}_1 \xleftarrow{\quad} \dots \xleftarrow{\quad} \mathcal{G}_l \end{aligned} \tag{2.3}$$

at permissible centers  $Y_i \subset \text{Sing}(\mathcal{G}_i)$ ,  $i = 0, \dots, l - 1$ , such that  $\text{Sing}(\mathcal{G}_l) = \emptyset$ , and the exceptional divisor of the composition  $V_0 \leftarrow V_l$  is a union of hypersurfaces with normal crossings. Recall that a set of hypersurfaces  $\{H_1, \dots, H_r\}$  in a smooth  $n$ -dimensional  $V$  has normal crossings at a point  $\xi \in V$  if there is a regular system of parameters  $x_1, \dots, x_n \in \mathcal{O}_{V,\xi}$  such that if  $\xi \in H_{i_1} \cap \dots \cap H_{i_s}$ , and  $\xi \notin H_l$  for  $l \in \{1, \dots, r\} \setminus \{i_1, \dots, i_s\}$ , then  $\mathcal{I}(H_{i_j})_\xi = \langle x_{i_j} \rangle$  for  $i_j \in \{i_1, \dots, i_s\}$ ; we say that  $H_1, \dots, H_r$  have normal crossings in  $V$  if they have normal crossings at each point of  $V$ .

In [17], Hironaka proves resolution of singularities of varieties over fields of characteristic zero by showing that the maximum value of the Hilbert Samuel function can be lowered after a sequence of blow ups at suitable regular centers. To this end, he proceeds as follows. Let  $X$  be an algebraic variety over a (perfect) field  $k$ , let  $\max \text{HS}(X)$  be the maximum value of the Hilbert Samuel function on  $X$ , let  $\underline{\text{Max}} \text{HS}(X)$  be the maximum stratum of this function, and let  $\xi \in \underline{\text{Max}} \text{HS}(X)$ . Then in some (étale) neighborhood of  $\xi$  there is an immersion of  $X$  in some smooth  $V$  and a Rees algebra  $\mathcal{G}$  strongly attached to  $\underline{\text{Max}} \text{HS}(X)$  (see Example 2.17 below; see also [18]). Then he shows that a resolution of  $\mathcal{G}$  induces a sequence of blow ups over  $X$  that ultimately leads to a lowering of  $\max \text{HS}(X)$ . To conclude, he proves that such resolution exists when the characteristic is zero:

**Theorem 2.10** [17] *Let  $k$  be a field of characteristic zero, and let  $R$  be a smooth  $k$ -algebra. Given a Rees algebra  $\mathcal{G}$  over  $R$ , there exists a resolution of  $\mathcal{G}$ .*

The previous result is existential. The following theorem says that, in fact, resolution of Rees algebras can be constructed; i.e., given a Rees algebra  $\mathcal{G}$  in a smooth  $V$  defined over a field of characteristic zero, there is a procedure that indicates how to actually construct a sequence of blow ups that leads to a resolution. See also [3,25].

**Theorem 2.11** [13, Theorem 3.1] *Let  $k$  be a field of characteristic zero, and let  $R$  be a smooth  $k$ -algebra. Given a Rees algebra  $\mathcal{G}$  over  $R$ , it is possible to construct a resolution of  $\mathcal{G}$ .*

For more details about transformations and resolution of Rees algebras, we refer to [8,13].

*Remark 2.12* To construct a resolution of  $\mathcal{G}$ , we use the so called *resolution invariants*. The most important resolution invariant is Hironaka’s order function,  $\text{ord}_\xi \mathcal{G}$ , at a point  $\xi \in \text{Sing}(\mathcal{G})$  [18]. All other invariants derive from it (see Sect. 2.8 and [8,9,14,18]).

*Remark 2.13* For some purposes, during the resolution, one may need to keep track of more information than that given by the Rees algebra itself. We refer to  $(V^{(n)}, \mathcal{G}^{(n)})$  as a *pair*, where  $V^{(n)}$  is an  $n$ -dimensional smooth scheme of finite type, and  $\mathcal{G}^{(n)}$  a Rees algebra over  $V^{(n)}$ . We understand by *basic object* a triple  $(V^{(n)}, \mathcal{G}^{(n)}, E)$ , where  $(V^{(n)}, \mathcal{G}^{(n)})$  is a pair and  $E$  is a set of smooth hypersurfaces in  $V^{(n)}$  (possibly empty) so that their union has normal crossings. For more details and the definition of transformations and resolution of pairs and basic objects, we refer to [13].

### 2.2 Motivation I

In general, Rees algebras represent a very interesting tool, since many problems in resolution of singularities can be codified by them. We mention here a few examples that may help getting an overall impression of their use.

*Example 2.14 Resolution of singularities of a hypersurface.* Consider a hypersurface  $X \subset V$ . Then  $I(X)$  is locally principal. Set  $\mathcal{G} = \mathcal{O}_V[I(X)W^b]$ , where  $b$  is the maximum multiplicity of  $X$  (see Example 2.25), which we will denote by  $\max \text{mult}(X)$ . A resolution of  $\mathcal{G}$  as (2.3) gives a simplification of the points of multiplicity  $b$  of  $X$ , that is, the induced sequence  $X \leftarrow X_I$  will be such that  $\max \text{mult}(X_I) < b$ . One can resolve the singularities of  $X$  by iterating this process until  $X_r$  is such that  $\max \text{mult}(X_r) = 1$ .

*Example 2.15 Resolution of  $\mathcal{G} = \mathcal{O}_V[I(X)W]$ .* Let  $V$  be a smooth scheme over a field of characteristic zero. Let now  $X \subset V$  be a closed reduced equidimensional subscheme, defined by  $I(X) \subset \mathcal{O}_V$ . Let  $\mathcal{G} = \mathcal{O}_V[I(X)W]$ . By Theorem 2.11, one can construct a resolution of Rees algebras for  $\mathcal{G}$ :

$$\begin{aligned} V &= V_0 \xleftarrow{\pi_1} V_1 \xleftarrow{\pi_2} \dots \xleftarrow{\pi_r} V_r \\ \mathcal{G} &= \mathcal{G}_0 \xleftarrow{\quad} \mathcal{G}_1 \xleftarrow{\quad} \dots \xleftarrow{\quad} \mathcal{G}_r \end{aligned} \tag{2.4}$$

such that  $\text{Sing}(\mathcal{G}_r) = \emptyset$ , and so that the exceptional locus of  $V \leftarrow V_r$  is a union of smooth hypersurfaces with normal crossings. Let us show now how a resolution of singularities of  $X$  can be obtained: For any  $i \in \{1, \dots, r\}$ , the transform  $I(X)^{(i)}$  of  $I(X)$  in  $\mathcal{O}_{V_i}$ , defined by  $I(X)^{(i)} := I_{1,i}$  as in (2.2), is supported in the exceptional locus (which has normal crossings) as well as in the strict transform of  $X$  by  $V \leftarrow V_i$ . The condition  $\text{Sing}(\mathcal{G}_r) = \emptyset$  implies that the maximum order of  $I(X)^{(r)}$  is less than one, so for some  $j \in \{1, \dots, r\}$ , the strict transform  $X_{j-1}$  of  $X$  in  $V_{j-1}$  is a connected component of the center of the transform  $\pi_j$ , and hence is permissible. In particular, this implies that  $X_{j-1}$  is regular and has normal crossings with the exceptional divisor. Therefore

$$\begin{aligned} V &= V_0 \xleftarrow{\pi_1} V_1 \xleftarrow{\pi_2} \dots \xleftarrow{\pi_j} V_j \\ \cup \cup \quad \quad \cup \quad \quad \quad \cup \\ X &= X_0 \xleftarrow{\quad} X_1 \xleftarrow{\quad} \dots \xleftarrow{\quad} X_j \end{aligned} \tag{2.5}$$

is a resolution of singularities of  $X$  (see [14, Proof of Theorem 1.5] for a precise proof of this result in the language of basic objects).

*Example 2.16 Log-resolution of ideals.* A *Log-resolution* of an ideal  $I$  on a smooth scheme  $V$  is a proper birational morphism of smooth schemes, say  $V' \rightarrow V$ , so that the total transform of  $I$ ,  $I\mathcal{O}_{V'}$ , is an invertible ideal in  $V'$  supported on smooth hypersurfaces having only normal crossings. A resolution of  $\mathcal{G} = R[IW]$  gives a Log-resolution of  $I$ . In [15],

Encinas and Villamayor proved, by using Rees algebras, that for two ideals with the same integral closure, one obtains the same algorithmic Log-resolution.

In this work, we use Rees algebras to give an answer to a problem of computing a sequence of multiplicities. As we will see, we translate our problem into a resolution of some specific Rees algebras (see Sect. 4).

### 2.3 Motivation II: local presentations

When one tries to study certain closed subsets of a variety  $X$ , one often needs to consider some equations  $\{f_1, \dots, f_r\} \subset R$  with weights  $\{n_1, \dots, n_r\} \subset \mathbb{Z}_{>0}$  that describe the closed set in question:

$$C = \cap_{i=1}^r \{\eta \in V : v_\eta(f_i) \geq n_i\},$$

in a way that the expression is stable under blow ups at suitably chosen centers. We call such a representation a *local presentation*. Example 2.14 is a particular case of this representation. Let us see another example:

*Example 2.17* Let  $X$  be a variety over a perfect field  $k$ . Let  $\text{HS}(X)$  be the Hilbert–Samuel function on  $X$ . This is an upper semicontinuous function<sup>3</sup> on  $X$ ,

$$\begin{aligned} \text{HS}(X) : X &\longrightarrow (\mathbb{N}^{\mathbb{N}}, \leq) \\ \xi &\longmapsto \text{HS}(X)(\xi), \end{aligned}$$

where  $\mathbb{N}^{\mathbb{N}}$  is ordered lexicographically. Let  $\max \text{HS}(X)$  and  $\underline{\text{Max}} \text{HS}(X)$  denote the maximum value of  $\text{HS}(X)$  in  $X$  and the closed subset of points where  $\text{HS}(X)$  reaches this value respectively. Pick  $\xi \in \underline{\text{Max}} \text{HS}(X)$ . Then (see [18]), it is possible to find, locally in an étale neighbourhood of  $\xi$ , an immersion of  $X$  in a smooth scheme  $V$  and equations  $f_1, \dots, f_r$  such that  $I(X) = \langle f_1, \dots, f_r \rangle$ ,

$$\underline{\text{Max}} \text{HS}(X) = \cap_{i=1}^r \underline{\text{Max}} \text{HS}(\{f_i = 0\}),$$

and such that this condition is preserved by blow ups with smooth centers in  $\underline{\text{Max}} \text{HS}(X)$  and by smooth morphisms, in terms of the strict transforms of  $X$  and of the  $f_i$ . Let us translate this it into the language of Rees algebras: let  $\mathcal{G} = \mathcal{O}_{V,\xi}[f_1 W^{\mu_1}, \dots, f_r W^{\mu_r}]$ , where  $\mu_i$  is the maximum order of  $f_i$  for  $i = 1, \dots, r$ . Then

$$\text{Sing}(\mathcal{G}) = \underline{\text{Max}} \text{HS}(X),$$

and for any sequence of  $\mathcal{G}$ -permissible transformations

$$\begin{aligned} V &= V_0 \xleftarrow{\pi_1} V_1 \xleftarrow{\pi_2} \dots \xleftarrow{\pi_l} V_l \\ \mathcal{G} &= \mathcal{G}_0 \longleftarrow \mathcal{G}_1 \longleftarrow \dots \longleftarrow \mathcal{G}_l, \end{aligned} \tag{2.6}$$

we have

$$\text{Sing}(\mathcal{G}_l) = \underline{\text{Max}} \text{HS}(X_l).$$

Resolving the Rees algebra  $\mathcal{G}$  is equivalent to making  $\max \text{HS}(X)$  decrease.

<sup>3</sup> Actually, the Hilbert–Samuel function has to be modified in order to be semicontinuous (see [2]). Here we refer to this modification of the Hilbert–Samuel function.



The previous example shows that Rees algebras appear as an appropriate language to represent such a set of equations and weights, and allow us to describe the transformations on the subset  $C$  we are interested in via well defined transformations of the associated Rees algebra (see (2.2)). It is very important to understand to which extent a given algebra can represent  $C$ . For this purpose we will consider the following transformations:

**Definition 2.18** A *local sequence* on a variety  $V$  is a sequence of morphisms

$$V = V_0 \xleftarrow{\phi_1} V_1 \xleftarrow{\phi_2} \dots \xleftarrow{\phi_l} V_l$$

where each  $\phi_i$  is either a blow up at a regular center or a smooth morphism, such as an open immersion or a projection from a product by some affine space (see for example (3.2)).

**Definition 2.19** Let  $\mathcal{G}$  be a Rees algebra over  $\mathcal{O}_V$ . A  *$\mathcal{G}$ -local sequence* over  $V$  is a local sequence over  $V$  as in Definition 2.18,

$$\begin{aligned} V &= V_0 \xleftarrow{\phi_1} V_1 \xleftarrow{\phi_2} \dots \xleftarrow{\phi_l} V_l \\ \mathcal{G} &= \mathcal{G}_0 \xleftarrow{\psi_1} \mathcal{G}_1 \xleftarrow{\psi_2} \dots \xleftarrow{\psi_l} \mathcal{G}_l, \end{aligned} \tag{2.7}$$

such that whenever  $\phi_i$  is a blow up, it is in particular a blow up at a permissible center  $Y_{i-1} \subset \text{Sing}(\mathcal{G}_{i-1}) \subset V_{i-1}$ , and then  $\mathcal{G}_i$  is the transform of  $\mathcal{G}_{i-1}$  by the rule in Definition 2.8; if  $\phi_i$  is a smooth morphism, then  $\mathcal{G}_i$  is the pullback of  $\mathcal{G}_{i-1}$  by  $\phi_i$  (see [7, Definition 3.2]).

**Definition 2.20** Let  $\mathcal{G}$  be a Rees algebra over  $V$ , and consider a  $\mathcal{G}$ -local sequence over  $V$  as in (2.7). This sequence determines a collection of closed sets, namely  $\{\text{Sing}(\mathcal{G}), \text{Sing}(\mathcal{G}_1), \dots, \text{Sing}(\mathcal{G}_l)\}$ . We will refer to this collection (or *branch*) of closed sets as the one defined by or attached to the sequence (2.7). If we consider all possible  $\mathcal{G}$ -local sequences over  $V$ , we obtain a *tree of closed sets* for  $\mathcal{G}$ , which we denote by  $\mathcal{F}_V(\mathcal{G})$  (see [7, Section 3]).

For the next examples, let us recall a few concepts and notations:

*Notation 2.21* Let  $F$  be an upper semicontinuous function defined on varieties, that is, for each variety, there is

$$F(X) = F_X : X \longrightarrow (\Lambda, \geq), \tag{2.8}$$

where  $(\Lambda, \geq)$  is a well ordered set. We will denote by  $\max F(X)$  the maximum value achieved by  $F_X$  in  $X$ . We will use  $\underline{\text{Max}} F(X)$  to denote the set of points of  $X$  in which  $F$  achieves this maximum value, that is:

$$\underline{\text{Max}} F(X) = \{\eta \in X : F_X(\eta) \geq \max F(X)\} = \{\eta \in X : F_X(\eta) = \max F(X)\}.$$

Note that  $\underline{\text{Max}} F(X)$  is a closed set.

**Definition 2.22** Let  $F$  be an upper semicontinuous function defined on varieties. An  *$F_X$ -local sequence* is a local sequence on  $X$  (Definition 2.18) such that, whenever  $\phi_i$  is a blow up, the center is contained in  $\underline{\text{Max}} F_{X_{i-1}}$ .

**Definition 2.23** (see [8, Definition 28.4]) An upper semicontinuous function  $F$  defined on varieties as (2.8) is said to be *representable via local embeddings* if, for each  $X$  and each  $\xi \in X$ , in an étale neighbourhood of  $\xi$ , we can find a closed immersion  $X \hookrightarrow V$  and a Rees algebra  $\mathcal{G}$  over  $\mathcal{O}_{V,\xi}$  such that

1. the Rees algebra  $\mathcal{G}$  satisfies:

$$\text{Sing}(\mathcal{G}) = \underline{\text{Max}} F_X; \tag{2.9}$$

2. any  $F_X$ -local sequence

$$X = X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_r \tag{2.10}$$

such that

$$m = \max F_X = \max F_{X_1} = \cdots = \max F_{X_{r-1}} \geq \max F_{X_r} \tag{2.11}$$

induces a  $\mathcal{G}$ -local sequence of Rees algebras over  $V$

$$\begin{aligned} V &= V_0 \leftarrow V_1 \leftarrow \cdots \leftarrow V_r \\ X &= X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_r \\ \mathcal{G} &= \mathcal{G}_0 \leftarrow \mathcal{G}_1 \leftarrow \cdots \leftarrow \mathcal{G}_r \end{aligned}$$

such that for  $i = 1, \dots, r$ ,

$$\text{Sing}(\mathcal{G}_i) = \{\eta \in X_i : F_{X_i}(\eta) = m\},$$

with  $\text{Sing}(\mathcal{G}_r) = \emptyset$  if and only if  $\max F_{X_r} < m$  and

3. any  $\mathcal{G}$ -local sequence over  $V$  induces an  $F_X$ -local sequence as (2.10) satisfying (2.11).

**Example 2.24 Hilbert–Samuel** The results of Hironaka [17, 18] show that it is possible to resolve the singularities of a variety (over a perfect field) if we know how to lower the maximum value of the Hilbert–Samuel function of the variety through a finite sequence of blow ups. Then, to construct a resolution of the singularities of a given variety  $X$ , one just needs to iterate the process a finite number of times.

On the other hand, the Hilbert–Samuel function is upper semicontinuous, and it is representable for any variety  $X$  via local embeddings (see [18] and Example 2.17). Thus, for each point  $\xi \in X$  we can find, in an étale neighbourhood of  $\xi$ , an immersion of  $X$  into a smooth scheme  $V$  and an  $\mathcal{O}_{V,\xi}$ -Rees algebra  $\mathcal{G}_X$  such that  $\text{Sing}(\mathcal{G}_X) = \underline{\text{Max}} \text{HS}(X)$  and this identity is preserved by  $\mathcal{G}$ -local sequences over  $V$  as long as the maximum value of the Hilbert–Samuel function of  $X$  does not decrease. From this, it will follow that finding a sequence of blow ups

$$X = X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_r$$

such that  $\max \text{HS}(X_0) = \cdots = \max \text{HS}(X_{r-1}) > \max \text{HS}(X_r)$  is equivalent to finding a resolution of the Rees algebra  $\mathcal{G}_X$ .

A similar statement holds for the multiplicity of a variety defined over a perfect field, see Example 2.38 and [30]:

**Example 2.25 Multiplicity** The multiplicity of an equidimensional variety  $X$  at a point  $\eta \in X$  is given by an upper semicontinuous function

$$\begin{aligned} \text{mult}(X) : X &\longrightarrow \mathbb{N} \\ \eta &\longmapsto \text{mult}(X)(\eta) = \text{mult}_\eta(X) = \text{mult}(\mathcal{O}_{X,\eta}) \end{aligned}$$

where  $\text{mult}(\mathcal{O}_{X,\eta})$  stands for the multiplicity of the local ring at the maximal ideal  $\mathcal{M}_\eta$ . Let  $m$  be the maximum multiplicity of  $X$ . The set

$$\underline{\text{Max}} \text{mult}(X) = \{\eta \in X : \text{mult}_\eta(X) \geq m\} = \{\eta \in X : \text{mult}_\eta(X) = m\}$$

is closed, and the multiplicity is representable via local embeddings for  $X$  (see [30, Proposition 5.7 and Theorem 7.1]).

Therefore, just as for the Hilbert–Samuel function in Example 2.24, we can attach a Rees algebra  $\mathcal{G}$  to  $\text{mult}(X)$  so that resolving  $\mathcal{G}$  is equivalent to decreasing the maximum value of  $\text{mult}(X)$ .

By Theorem 2.11, the resolution for such an algebra can be constructed whenever  $k$  is a field of characteristic zero. It is not known if this is true for fields of positive characteristic.

### 2.4 Equivalence of Rees algebras

Given an upper semicontinuous function  $F$  as in (2.8) which is representable via local embeddings, the choice of a Rees algebra satisfying the properties of Definition 2.23 is not unique. To begin with, for a given  $X$ , there are many possible choices for the immersion  $X \hookrightarrow V$ , but we will mention this problem later. On the other hand, once an immersion is fixed, we can attach a different Rees algebra to a neighbourhood of each point  $\xi \in X$ . This choice is not unique either. Therefore, given two possible choices of Rees algebras,  $\mathcal{G}$  and  $\mathcal{G}'$ , attached to a fixed point  $\xi \in \text{Max } F(X)$ , it would be desirable to compare the algorithmic resolution of  $\mathcal{G}$  to that of  $\mathcal{G}'$ , and vice versa. To deal with this problem, we need the notion of weak equivalence of Rees algebras.

**Definition 2.26** [7, Definition 3.5] We say that two  $\mathcal{O}_V$ -Rees algebras  $\mathcal{G}$  and  $\mathcal{H}$  are *weakly equivalent* if:

1.  $\text{Sing}(\mathcal{G}) = \text{Sing}(\mathcal{H})$ ,
2. Any  $\mathcal{G}$ -local sequence over  $V$

$$\mathcal{G} = \mathcal{G}_0 \longleftarrow \mathcal{G}_1 \longleftarrow \dots \longleftarrow \mathcal{G}_l$$

induces an  $\mathcal{H}$ -local sequence over  $V$

$$\mathcal{H} = \mathcal{H}_0 \longleftarrow \mathcal{H}_1 \longleftarrow \dots \longleftarrow \mathcal{H}_l$$

and vice versa, and moreover the equality in (1.) is preserved, that is

3.  $\text{Sing}(\mathcal{G}_j) = \text{Sing}(\mathcal{H}_j)$  for  $j = 0, \dots, l$ .

*Example 2.27* Let  $V$  be a smooth scheme over a field  $k$  of characteristic zero. Let  $X$  be a hypersurface in  $V$ . Denote now by  $b$  the maximum multiplicity of  $X$ . Then, locally at each point, there exists a Rees algebra  $\mathcal{G}$  representing  $\text{mult}(X)$  via local embeddings (see Examples 2.17, 2.38 and Definition 2.23). This algebra  $\mathcal{G}$  is unique up to weak equivalence.

The following definitions and results give a flavour of what this equivalence relation means:

**Definition 2.28** A Rees algebra over  $V$ ,  $\mathcal{G} = \bigoplus_{n \geq 0} I_n W^n$  is *integrally closed* if it is integrally closed as an  $\mathcal{O}_V$ -ring in  $\text{Quot}(\mathcal{O}_V)[W]$ . We denote by  $\overline{\mathcal{G}}$  the integral closure of  $\mathcal{G}$ .

**Definition 2.29** Two Rees algebras are *integrally equivalent* if their integral closure in  $\text{Quot}(\mathcal{O}_V)[W]$  coincides.

**Definition 2.30** A Rees algebra  $\mathcal{G} = \bigoplus_{n \geq 0} I_n W^n$  over  $V$  is *differentially closed* (or a *Diff-algebra*) if there is an affine open covering of  $V$ ,  $\{U_i\}$  such that for every  $D \in \text{Diff}^{(r)}(U_i)$  and  $h \in I_n(U_i)$ , we have  $D(h) \in I_{n-r}(U_i)$  whenever  $n \geq r$ , where  $\text{Diff}^{(r)}(U_i)$  is the locally free sheaf of  $k$ -linear differential operators of order  $r$  or less. In particular,  $I_{n+1} \subset I_n$  for  $n \geq 0$ . We denote by  $\text{Diff}(\mathcal{G})$  the smallest differential Rees algebra containing  $\mathcal{G}$  (its *differential closure*). (See [28, Theorem 3.4] for the construction.)

**Theorem 2.31** [7, Theorem 3.11] *Let  $\mathcal{G}_1$  and  $\mathcal{G}_2$  be two Rees algebras over  $V$ . Then  $\mathcal{G}_1$  and  $\mathcal{G}_2$  are weakly equivalent if and only if  $\overline{\text{Diff}}(\mathcal{G}_1) = \overline{\text{Diff}}(\mathcal{G}_2)$ .*

**Corollary 2.32** *Let  $\mathcal{G}_1$  and  $\mathcal{G}_2$  be two weakly equivalent Rees algebras over  $V$ . Then for all  $\eta \in \text{Sing}(\mathcal{G}_1) = \text{Sing}(\mathcal{G}_2)$ , we have  $\text{ord}_\eta \mathcal{G}_1 = \text{ord}_\eta \mathcal{G}_2$ .*

**Corollary 2.33** *Let  $\mathcal{G}_1$  and  $\mathcal{G}_2$  be two weakly equivalent Rees algebras. Then a constructive resolution of  $\mathcal{G}_1$  induces a constructive resolution of  $\mathcal{G}_2$  and vice versa (see [8, Remark 11.8]).*

*Remark 2.34* Let  $X$  be a variety, and fix an immersion  $X \hookrightarrow V$ . Any two local presentations of  $X$  attached to the multiplicity or to the Hilbert–Samuel function are weakly equivalent by definition, and therefore Corollary 2.32 applies: fixed an immersion for  $X$ , the order of a Rees algebra attached to a local presentation at any point of its singular locus does not depend on the local presentation, and neither does the resolution. The previous results give an answer to the problem of compatibility of Rees algebras over  $V$ .

### 2.5 Elimination algebras

In the following examples, one can observe that, in some cases, the relevant information regarding the simplification of the multiplicity of a variety  $X^{(d)} \hookrightarrow V^{(n)}$  can be reflected in a lower dimensional version of  $V^{(n)}$ . In order to generalize this idea, we have the concept of elimination, which we introduce next.

### 2.6 Example I: hypersurface case

*Example 2.35* Let  $S$  be a regular  $d$ -dimensional  $k$ -algebra of finite type, with  $d > 0$ . Let  $V^{(n)} = \text{Spec}(S[x])$ , where  $n = d + 1$ . There is an injective morphism

$$S \xrightarrow{\beta^*} S[x],$$

and an induced smooth projection

$$V^{(n)} \xrightarrow{\beta} V^{(d)} = \text{Spec}(S). \tag{2.12}$$

Let  $X$  be a hypersurface in  $V^{(n)}$ ,  $X = \text{Spec}(S[x]/f(x))$ , where  $f$  is a polynomial in  $x$  of degree  $b > 1$  with coefficients in  $S$ . Let  $\xi^{(n)}$  be a point in the closed set of multiplicity  $b$  of  $X$ . We can suppose that the maximal ideal  $\mathcal{M}_{\xi^{(n)}}$  of  $\xi^{(n)}$  in  $S[x]$  is given by  $\langle x, z_1, \dots, z_d \rangle$  for a regular system of parameters  $\{z_1, \dots, z_d\}$  in  $S$ . The image  $\xi^{(d)}$  of  $\xi^{(n)}$  by the projection (2.12) is defined by the maximal ideal  $\mathcal{M}_{\xi^{(d)}} = \langle z_1, \dots, z_d \rangle$ . Then, the Rees algebra  $\mathcal{G}_X^{(n)}$  over  $S[x]$

$$\mathcal{G}_X^{(n)} = \text{Diff}(S[x][fW^b]) \subset S[x][[W]]$$

represents the multiplicity function on  $X$  locally at  $\xi^{(n)}$ .

Let us suppose that, in addition,  $f$  has the form of Tschirnhausen:

$$f(x) = x^b + B_{b-2}x^{b-2} + \dots + B_i x^i + \dots + B_0 \in S[x], \tag{2.13}$$

where  $B_i \in S$  for  $i = 0, \dots, b - 2$  and<sup>4</sup>  $\text{ord}_\xi(B_i) \geq b - i$ . The following lemma shows that for  $X$  as in Example 2.35, the meaningful part of  $f \in S[x]$  (regarding the multiplicity) is given by the coefficients  $B_i$ , which are already in  $S$ .

<sup>4</sup> For simplicity, we will sometimes write  $\xi$  when we refer to the image of  $\xi^{(n)}$  by most of the maps we use in this article. In particular, we will often write  $\text{ord}_\xi$  meaning  $\text{ord}_{\xi^{(n)}}$  or  $\text{ord}_{\xi^{(d)}}$ .

**Lemma 2.36** *Let  $X$  be given by  $f$  as in (2.13). Then*

$$\mathcal{G}_X^{(n)} = S[x][xW] \odot \text{Diff}(S[x][B_{b-2}W^2, \dots, B_iW^{b-i}, \dots, B_0W^b]).$$

*Proof* In order to compute the differential closure of  $S[x][fW^b]$ , let us start by computing the  $(b - 1)$ -th derivative of  $fW^b$  with respect to  $x$ : one can see that  $xW \in \mathcal{G}_X^{(n)}$ . Therefore  $f_2W^b = fW^b - (xW)^b \in \mathcal{G}_X^{(n)}$  and, if we consider  $xW$  and  $f_2W^b$  among the generators of  $\mathcal{G}_X^{(n)}$ , there is no need to include  $fW^b$ . To continue, we compute the  $(b - 2)$ -th derivative of  $f_2W^b$  with respect to  $x$  obtaining, up to a nonzero constant,  $B_{b-2}W^2 \in \mathcal{G}_X^{(n)}$ . Just like in the previous step, it is possible to verify that  $f_3W^b = f_2W^b - (B_{b-2}W^2)(xW)^{b-2} \in \mathcal{G}_X^{(n)}$ , and that  $f_2W^b$  can be generated by  $xW$ ,  $B_{b-2}W^2$  and  $f_3W^b$ . By iterating this argument, one concludes that the set consisting of  $xW$  and  $B_iW^{b-i}$  for  $i = 0, \dots, b - 2$  is contained in  $\mathcal{G}_X^{(n)}$  and, in addition, the differential closure of the  $S[x]$ -Rees algebra generated by this set<sup>5</sup> corresponds exactly to  $\mathcal{G}_X^{(n)}$ . □

*Example 2.37* Instead of (2.13), suppose now that  $f$  is of the form

$$f(x) = x^b + D_{b-1}x^{b-1} + \dots + D_i x^i + \dots + D_0 \in S[x], \tag{2.14}$$

where  $D_i \in S$ ,  $D_{b-1} \neq 0$  and  $\text{ord}_\xi(D_i) \geq b - i$  for  $i = 0, \dots, b - 1$ . After a suitable change, namely  $\tilde{x} = x + \frac{D_{b-1}}{b}$ , we obtain

$$f(x) = \tilde{f}(\tilde{x}) = \tilde{x}^b + B_{b-2}\tilde{x}^{b-2} + \dots + B_0 \in S[\tilde{x}], \quad B_i \in S, \quad \text{ord}_\xi(B_i) \geq b - i.$$

**2.7 Example II: multiplicity of a variety**

*Example 2.38* (see [30, 7.1]) Let  $X$  be a variety of dimension  $d$  over  $k$  of maximum multiplicity  $b$ , and let  $\xi \in X$  be a point in  $\text{Max mult}(X)$ . We have, after possibly replacing  $X$  by an étale neighbourhood of  $\xi$ , a smooth  $k$ -algebra  $S$  of dimension  $d$  and a finite and transversal projection

$$\beta_X : X \longrightarrow \text{Spec}(S) = V^{(d)}, \tag{2.15}$$

that is, a finite projection of generic rank  $b$ . Note that  $\beta_X$  induces a homeomorphism between  $\text{Max mult}(X)$  and its image ([8, Appendix A], [30, 4.8]), and an injective finite morphism

$$S \longrightarrow B = S[\theta_1, \dots, \theta_{n-d}] \cong S[x_1, \dots, x_{n-d}]/I(X).$$

As a consequence, we have a local immersion of  $X$  in a smooth  $n$ -dimensional space

$$V^{(n)} = \text{Spec}(S[x_1, \dots, x_{n-d}])$$

in a neighbourhood of  $\xi$ , and it can be shown that there exist  $f_1, \dots, f_{n-d} \in I(X) \subset S[x_1, \dots, x_{n-d}]$  such that for some positive integers  $b_1, \dots, b_{n-d}$  the Rees algebra

$$\mathcal{G}_X^{(n)} = \text{Diff}(\mathcal{O}_{V^{(n)}, \xi}[f_1W^{b_1}, \dots, f_{n-d}W^{b_{n-d}}]) \tag{2.16}$$

represents  $\text{mult}(X) : X \longrightarrow \mathbb{N}$  locally at  $\xi$ . In addition, for  $i = 1, \dots, n - d$ ,

$$f_i \in S[x_i] \tag{2.17}$$

---

<sup>5</sup> Note that it is already differentially closed with respect to  $x$ .

and it is the minimal polynomial of  $\theta_i$  over  $S$  (see [30, 7.1] for more details). Note that  $S[x_1, \dots, x_{n-d}] \rightarrow B \cong S[x_1, \dots, x_{n-d}]/I(X)$  is a surjective map and that for any  $i = 1, \dots, n - d$  the following diagram commutes:

$$\begin{array}{ccccc}
 S[x_1, \dots, x_{n-d}] & \longrightarrow & S[x_1, \dots, x_{n-d}]/(f_1, \dots, f_{n-d}) & \longrightarrow & B \longrightarrow 0 & (2.18) \\
 \uparrow & & \uparrow & & & \\
 S[x_i] & \longrightarrow & S[x_i]/(f_i) & & & \\
 \uparrow & \nearrow & & & & \\
 S & & & & & 
 \end{array}$$

Due to (2.17), we can perform changes of variables for all of the  $x_i$  as in 2.37 in order to obtain an expression as in (2.13) for each of the  $f_i$ . We will therefore assume that, when we consider a local presentation attached to the multiplicity for  $X$  as (2.16), the  $f_i$  have the form of Tschirnhausen.

*Remark 2.39* In the particular case in which, locally at  $\xi$ ,  $B = S[\theta_1]$ , necessarily  $I(X) = (f_1)$ ,  $B \cong k[x_1]/(f_1)$ , and hence  $X$  is a hypersurface in  $V^{(n)}$ .

Given an  $n$ -dimensional smooth scheme of finite type  $V^{(n)}$ , and a Rees algebra  $\mathcal{G}^{(n)}$  over  $V^{(n)}$ , which we will consider as a pair from now on, it would be useful to find a new pair  $(V^{(n-e)}, \mathcal{G}^{(n-e)})$  of dimension  $n - e < n$ , so that a resolution of  $\mathcal{G}^{(n-e)}$  induces a resolution of  $\mathcal{G}^{(n)}$ , since the first one could be easier to find.

**Definition 2.40** Let  $\mathcal{G}^{(n)}$  be a differential Rees algebra over  $V^{(n)}$ , and let  $\xi \in \text{Sing}(\mathcal{G}^{(n)})$  be a closed point. For a suitable<sup>6</sup>  $e \geq 1$  and a smooth transversal<sup>7</sup> projection (also admissible<sup>8</sup>),

$$\beta : V^{(n)} \longrightarrow V^{(n-e)}$$

in a neighbourhood of  $\xi$ , we define an *elimination algebra*  $\mathcal{G}^{(n-e)}$  of  $\mathcal{G}^{(n)}$  as  $\mathcal{G}^{(n)} \cap \mathcal{O}_{V^{(n-e)}}$  up to integral closure.

For a complete description of the properties asked to the projections, and of elimination algebras, we refer to [5, 6], [8, 16 and Appendix A], [30] and [28, Theorem 4.11 and Theorem 4.13].

### 2.7.1 Properties

1. The projection  $\beta$  induces a homeomorphism between  $\text{Sing}(\mathcal{G}^{(n)})$  and  $\beta(\text{Sing}(\mathcal{G}^{(n)})) = \text{Sing}(\mathcal{G}^{(n-e)})$ .

<sup>6</sup> No larger than the invariant  $\tau$  at  $\xi$ , see [1] for more details.

<sup>7</sup> This condition just means that the intersection of  $\text{Ker}(d\beta)$  and the tangent space of  $\mathcal{G}^{(n)}$  at  $\xi$  is 0. This guarantees that  $\beta$  induces a homeomorphism between  $\text{Sing}(\mathcal{G}^{(n)})$  and  $\beta(\text{Sing}(\mathcal{G}^{(n)}))$ .

<sup>8</sup> For this, it suffices to have  $\mathcal{G}^{(n)}$  differentially closed with respect to  $\beta$ , that is, closed under the action of the sheaf of relative differential operators  $\text{Diff}_{V^{(n)}/V^{(n-e)}}$ .

2. Any  $\mathcal{G}^{(n)}$ -local sequence over  $V^{(n)}$  induces a  $\mathcal{G}^{(n-e)}$ -local sequence over  $V^{(n-e)}$  and a commutative diagram

$$\begin{array}{ccccccc}
 \mathcal{G}^{(n)} = \mathcal{G}_0^{(n)} & & \mathcal{G}_1^{(n)} & \dots & & \mathcal{G}_r^{(n)} & (2.19) \\
 \\
 V^{(n)} = V_0^{(n)} & \longleftarrow & V_1^{(n)} & \longleftarrow & \dots & \longleftarrow & V_r^{(n)} \\
 \beta \downarrow & & \beta_1 \downarrow & & & & \beta_r \downarrow \\
 V^{(n-e)} = V_0^{(n-e)} & \longleftarrow & V_1^{(n-e)} & \longleftarrow & \dots & \longleftarrow & V_r^{(n-e)} \\
 \\
 \mathcal{G}^{(n-e)} = \mathcal{G}_0^{(n-e)} & & \mathcal{G}_1^{(n-e)} & \dots & & \mathcal{G}_r^{(n-e)} & 
 \end{array}$$

where  $\mathcal{G}_i^{(n-e)}$  is an elimination algebra of  $\mathcal{G}_i^{(n)}$  for  $i = 0, \dots, r$ , and the  $\beta_i$  are smooth  $\mathcal{G}^{(n)}$ -admissible projections inducing therefore homeomorphisms between  $\text{Sing}(\mathcal{G}_i^{(n)})$  and  $\text{Sing}(\mathcal{G}_i^{(n-e)})$ .

3. Any  $\mathcal{G}^{(n-e)}$ -local sequence over  $V^{(n-e)}$  induces a  $\mathcal{G}^{(n)}$ -local sequence over  $V^{(n)}$  and a commutative diagram as above where  $\mathcal{G}_i^{(n-e)}$  is an elimination algebra of  $\mathcal{G}_i^{(n)}$  for  $i = 0, \dots, r$ , and with  $\beta_i$  smooth  $\mathcal{G}^{(n)}$ -admissible projections inducing homeomorphisms between  $\text{Sing}(\mathcal{G}_i^{(n)})$  and  $\text{Sing}(\mathcal{G}_i^{(n-e)})$ .
4. Properties 1-3 characterize the elimination algebra  $\mathcal{G}^{(n-e)}$  up to weak equivalence.
5. Any resolution of  $\mathcal{G}^{(n)}$  induces a resolution of  $\mathcal{G}^{(n-e)}$  and vice versa.
6. For any two elimination algebras  $\mathcal{G}^{(n-e)}$  and  $\check{\mathcal{G}}^{(n-e)}$  of  $\mathcal{G}^{(n)}$ , given by projections  $V^{(n)} \xrightarrow{\beta} V^{(n-e)}$  and  $V^{(n)} \xrightarrow{\check{\beta}} \check{V}^{(n-e)}$  respectively, we have the same order at the image of  $\xi$  (see [5, Theorem 10.1]). That is,

$$\text{ord}_\xi \mathcal{G}^{(n-e)} = \text{ord}_\xi \check{\mathcal{G}}^{(n-e)}.$$

Let us define

$$\text{ord}_\xi^{(n-e)}(\mathcal{G}^{(n)})$$

as the order  $\text{ord}_\xi \mathcal{G}^{(n-e)}$  (the order at the image of  $\xi$ ) for any elimination algebra  $\mathcal{G}^{(n-e)}$  of  $\mathcal{G}^{(n)}$  of dimension  $n - e$ . Hence  $\text{ord}_\xi^{(n-e)}(\mathcal{G}^{(n)})$  is an invariant for  $\mathcal{G}^{(n)}$  at  $\xi$ .

In particular, given  $X \subset V^{(n)}$  and a Rees algebra  $\mathcal{G}^{(n)}$  representing the multiplicity of  $X$ , as in Example 2.38, we wish to find a Rees algebra in dimension  $d = \dim(X)$  which is an elimination algebra of  $\mathcal{G}^{(n)}$ . The reason for this will be explained in Sect. 2.8. The following theorem guarantees that this is possible:

**Theorem 2.41** *Let  $X \subset V^{(n)}$  be a  $d$ -dimensional variety over a field of characteristic zero, and  $\mathcal{G}_X^{(n)}$  a Rees algebra over  $V^{(n)}$  representing the multiplicity of  $X$ . Then it is possible to find a smooth projection  $\beta : V^{(n)} \rightarrow V^{(d)}$  inducing an elimination algebra  $\mathcal{G}_X^{(d)}$  of  $\mathcal{G}_X^{(n)}$ . Moreover, the order  $\text{ord}_\xi^{(d)}(\mathcal{G}_X^{(n)}) := \text{ord}_{\beta\xi} \mathcal{G}_X^{(d)}$  does not depend on the choice of the embedding or of the algebra  $\mathcal{G}_X^{(n)}$ .*

*Proof* This fact follows from [8, Section 21, Theorem 28.8, Theorem 28.10 and Example 28.2]. □

*Example 2.42* Let us suppose that  $X$  is a hypersurface of dimension  $d$ , and consider the Rees algebra  $\mathcal{G}_X^{(n)}$  representing the multiplicity of  $X$ , as in Example 2.35. There is a Rees algebra  $\mathcal{G}_X^{(d)}$  over  $S$ , the elimination algebra of  $\mathcal{G}_X^{(n)}$ , given by

$$\mathcal{G}_X^{(d)} = \text{Diff}(S[x][fW^b]) \cap S[W] \tag{2.20}$$

describing the image by (2.12) of  $\text{Max mult}(X)$  (or equivalently, the set of points of maximum multiplicity of the image of  $X$  by (2.12)). For a description of this elimination algebra see Lemma 2.44 below.

*Example 2.43* Let us go back to Example 2.37. It is worth noting that  $\mathcal{G}_X^{(d)}$  is invariant under translations of the variable  $x$ , see [27], and hence the  $S[x]$ -Rees algebra generated by  $fW^b \in S[x][W]$  and the  $S[\tilde{x}]$ -Rees algebra generated by  $\tilde{f}W^b \in S[\tilde{x}][W]$  give equivalent elimination algebras  $\text{Diff}(S[x][fW^b]) \cap S[W]$  and  $\text{Diff}(S[\tilde{x}][\tilde{f}W^b]) \cap S[W]$  respectively (and now we are in the situation of Example 2.35).

**Lemma 2.44** *Let  $X$  be given by  $f$  as in Example 2.35. Then the elimination algebra of  $\mathcal{G}_X^{(n)}$  relative to (2.12) is (up to integral closure)*

$$\mathcal{G}_X^{(d)} = \text{Diff}(S[B_{b-2}W^2, \dots, B_iW^{b-i}, \dots, B_0W^b]). \tag{2.21}$$

*Proof* Considering the expression given by Lemma 2.36, (2.21) follows from the facts that  $B_i \in S$  for  $i = 0, \dots, b - 2$ , and that  $\mathcal{G}^{(n-e)} = \mathcal{G}^{(n)} \cap \mathcal{O}_{V^{(n-e)}}$ .  $\square$

*Remark 2.45* One can see  $\mathcal{G}_X^{(n)}$  as the smallest  $S[x]$ -Rees algebra containing  $xW$  and  $\mathcal{G}_X^{(d)}$ . By abuse of notation, we will simply write

$$\mathcal{G}_X^{(n)} = S[x][xW] \odot \mathcal{G}_X^{(d)},$$

meaning that we extend both algebras to Rees algebras over the same ring and apply  $\odot$  afterwards (see Definition 2.3).

**Lemma 2.46** *Let  $X$  be a hypersurface, given by  $f$  as in Example 2.35. Let  $\mathcal{G}_X^{(d)}$  be the elimination algebra of  $\mathcal{G}_X^{(n)}$  as in (2.20). Then for  $\xi \in \text{Sing}(\mathcal{G}_X^{(n)})$ ,*

$$\text{ord}_\xi(\mathcal{G}_X^{(d)}) = \min_{i=0, \dots, b-2} \left\{ \frac{\text{ord}_\xi(B_i)}{(b-i)} \right\}. \tag{2.22}$$

*Proof* By the expression of  $\mathcal{G}_X^{(d)}$  given in Lemma 2.44, it is clear that it is enough to prove that, for any  $i$ , the element  $B_iW^{b-i}$  has lower order than any of its derivatives in  $\xi$ . The element  $B_iW^{b-i}$  has order  $\frac{\text{ord}_\xi(B_i)}{b-i}$  in  $\xi$ , while the order of its  $j$ -th derivative (for  $j < b - i$ ) is greater than or equal to  $\frac{\text{ord}_\xi(B_i)-j}{b-i-j}$ , and for any pair of positive integers  $A \geq A'$ ,  $\frac{A}{A'} \leq \frac{A-k}{A'-k}$  for any  $k < A'$ . On the other hand, any element generated by the  $B_i$  and their derivatives has greater order (see [4, Proposition 3.11]).  $\square$

*Remark 2.47* Let  $X$  be a hypersurface given by  $f$  as in Example 2.37. Then the result in Lemma 2.46 can be applied after a variable change.

*Example 2.48* If  $X$  is as in Example 2.38, for any  $i \in \{1, \dots, n - d\}$ ,  $f_i \in S[x_i]$  is the equation of a hypersurface  $H_i$  in a scheme of dimension  $\bar{n} = d + 1$ ,  $\text{Spec}(S[x_i])$  and, by Remark 2.45:

$$\mathcal{G}_{H_i}^{(\bar{n})} = \text{Diff}(S[x_i][f_iW^{b_i}]) = S[x_i][x_iW] \odot \mathcal{G}_{H_i}^{(d)}.$$



By extending this algebra to  $\mathcal{O}_{V^{(n)},\xi}$ , we obtain

$$\mathcal{G}_{H_i}^{(n)} = \text{Diff}(\mathcal{O}_{V^{(n)},\xi}[f_i W^{b_i}]) = \mathcal{O}_{V^{(n)},\xi}[x_i W] \odot \mathcal{G}_{H_i}^{(d)}.$$

Hence, (2.16) can be written as

$$\mathcal{G}_X^{(n)} = \mathcal{G}_{H_1}^{(n)} \odot \dots \odot \mathcal{G}_{H_{n-d}}^{(n)} = \mathcal{O}_{V^{(n)},\xi}[x_1 W, \dots, x_{n-d} W] \odot \mathcal{G}_{H_1}^{(d)} \odot \dots \odot \mathcal{G}_{H_{n-d}}^{(d)}.$$

This gives an easy expression for the elimination algebra of  $\mathcal{G}_X^{(n)}$  relative to the projection

$$\text{Spec}(S[x_1, \dots, x_{n-d}]) = V^{(n)} \longrightarrow V^{(d)} = \text{Spec}(S),$$

namely

$$\mathcal{G}_X^{(d)} = \mathcal{G}_{H_1}^{(d)} \odot \dots \odot \mathcal{G}_{H_{n-d}}^{(d)}.$$

An explanation of this elimination can be found in [8, Remark 16.10]. The elimination algebra  $\mathcal{G}_X^{(d)}$  will be differentially closed (see [29, Proposition 5.1]). See Theorem 2.50 for the role of  $\mathcal{G}_X^{(d)}$  in algorithmic resolution.

### 2.8 Algorithmic resolution

A variety  $X$  of dimension  $d$  over a field of characteristic zero can be desingularized by a sequence of blow ups at smooth centers [17]. *Algorithmic resolutions* provide a way to construct such sequences, attending to suitable invariants associated to the points of  $X$  [3, 13,25,26].

#### 2.8.1 Resolution functions

For the construction of an algorithm of resolution [13], consider a well ordered set  $(\Lambda, \geq)$  and an upper semicontinuous function defined on varieties  $F(X) = F_X, F_X : X \longrightarrow (\Lambda, \geq)$  such that for any  $X, \text{Max } F_X \subset X$  is a closed and smooth subset, and  $F_X$  is constant on  $X$  if and only if  $X$  is smooth. Set  $\text{Max } F_X$  as the center of the first blow up  $X \xleftarrow{\pi_1} X_1$ . The function  $F_X$  must satisfy  $F_X(\xi) > F_{X_1}(\xi')$  whenever  $\xi = \pi_1(\xi') \in \text{Max } F_X$ . Given a variety  $X$ , the algorithm will give us a sequence of blow ups by iterating the process, that is,

$$X = X_0 \xleftarrow{\pi_1} X_1 \xleftarrow{\pi_2} \dots \xleftarrow{\pi_r} X_r,$$

with  $\pi_i$  being the blow up at  $\text{Max } F_{X_{i-1}}$  for  $i = 1, \dots, r$ .

#### 2.8.2 Invariants

When it comes to the construction of the resolution function, we use invariants of the varieties in order to assign a value (in fact, a set of values) to each point reflecting the complexity of the singularities. Examples 2.24 and 2.25 give upper semicontinuous functions which are often useful for this construction.

As a first coordinate of the resolution function  $F_X$ , we can consider the Hilbert–Samuel function or the multiplicity at each point. In particular, we will be interested in considering the multiplicity. We will compare the values of  $F_X$  at different points using the lexicographical order, and this first coordinate will allow us to focus already on the stratum of maximum value of the multiplicity in  $X$ .

For each  $\xi \in \underline{\text{Max}} \text{mult}(X)$ , we know that we can attach a local presentation and an algebra  $\mathcal{G}_X^{(n)}$  for the multiplicity. We have already said that the order of  $\mathcal{G}_X^{(n)}$  at  $\xi$  is the most important resolution invariant at  $\xi$ . Therefore, let us take it as the second coordinate of  $F_X$ .

If  $X$  is a  $d$ -dimensional variety, then it can be shown that there are suitable admissible projections to smooth  $(n - i)$ - dimensional schemes  $V^{(n-i)}$ , and elimination algebras  $\mathcal{G}^{(n-i)}$ ,  $i = 1, \dots, n - d$ . For the following coordinates, we will use the orders  $\text{ord}_\xi \mathcal{G}_X^{(n-i)}$  of the eliminations as in 2.7.1 (6), for  $i = 1, \dots, n - d$  (see 2.41):

$$F_X(\xi) = (\text{mult}_\xi(X), \text{ord}_\xi \mathcal{G}_X^{(n)}, \text{ord}_\xi^{(n-1)} \mathcal{G}_X^{(n)}, \dots, \text{ord}_\xi^{(d+1)} \mathcal{G}_X^{(n)}, \text{ord}_\xi^{(d)} \mathcal{G}_X^{(n)}, \dots). \tag{2.23}$$

These invariants behave well under weak equivalence of Rees algebras. More precisely:

*Remark 2.49* Two weakly equivalent Rees algebras  $\mathcal{G}$  and  $\mathcal{G}'$  share their resolution invariants and hence the constructive resolution of each of them induces the constructive resolution of the other one. This follows from the fact that all invariants that we consider for the construction or the resolution functions derive from Hironaka’s order function ([7, 10.3],[13, 4.11, 4.15]) together with Corollary 2.32. In particular, this is the case for Rees algebras coming from different local presentations once we have fixed an immersion (see 2.4).

Among the orders in (2.23), the next theorem will tell us that  $\text{ord}_\xi^{(d)} \mathcal{G}_X^{(n)}$  is the first interesting one, since all the previous are necessarily equal to 1, and therefore this will be the coordinate we will focus on for our results.

**Theorem 2.50** [8, 16.7] *Let  $X$  be a  $d$ -dimensional variety, and let  $(V^{(n)}, \mathcal{G}^{(n)})$  be an  $n$ -dimensional pair attached to  $X$  at a point  $\xi \in \underline{\text{Max}} \text{mult}(X)$ . Then for any  $e < n - d$  we have  $\text{ord}_\xi^{(n-e)} \mathcal{G}^{(n)} = 1$ .*

Thus,  $F_X$  can actually be constructed as

$$F_X(\xi) = (\text{mult}_\xi(X), \text{ord}_\xi^{(d)} \mathcal{G}_X^{(n)}, \dots). \tag{2.24}$$

It follows from 2.7.1 that  $\text{ord}_\xi^{(d)} \mathcal{G}_X^{(n)}$  does not depend on the choice of the elimination algebra. It neither depends on the immersion, by Theorem 2.41. Our main result (Theorem 3.6) will show that this invariant,  $\text{ord}_\xi^{(d)} \mathcal{G}_X^{(n)}$ , can be obtained from the arcs in  $X$  through  $\xi$ .

### 3 Arc spaces and Nash multiplicity sequences

#### 3.1 The space of arcs of $X$

Let  $X$  be an algebraic variety over a field  $k$  of characteristic zero. Let us suppose, for simplicity, that  $X$  is affine. Otherwise, since we will work locally, it would be enough to consider open affine subsets of  $X$ . Thus, say  $X = \text{Spec}(R)$  for some  $k$ -algebra of finite type  $R$ .

**Definition 3.1** The *space of arcs of  $X$* ,  $\mathcal{L}(X)$ , is a  $k$ -scheme whose  $K$ -valued points are the morphisms

$$\varphi : R \longrightarrow K[[t]] \tag{3.1}$$

for any extension  $K$  of  $k$ . We say that the prime  $\varphi^{-1}(\langle t \rangle) \subset R$  is the *center of the arc  $\varphi$*  in  $X$ . We denote by  $\mathcal{L}_\xi(X)$  the *space of arcs of  $X$  through a (not necessarily closed) point  $\xi \in X$* , i.e., those arcs in  $\mathcal{L}(X)$  with center  $\xi$ .

*Remark 3.2* It should be noticed that, for a given  $X$ ,  $\mathcal{L}(X)$  is not necessarily of finite type, and therefore it is not an algebraic variety over  $k$ .

There is a long bibliography where one can find the basics of arc spaces. For instance, we refer to [31] for more details on the construction of  $\mathcal{L}(X)$ .

**Definition 3.3** We define the *order of an arc*  $\varphi \in \mathcal{L}(X)$  through  $\xi \in X, \varphi : \mathcal{O}_{X,\xi} \longrightarrow K[[t]]$  as the largest positive integer  $n$  such that  $\varphi(\mathcal{M}_\xi) \subset (t^n)$ , where  $\mathcal{M}_\xi$  is the maximal ideal of the local ring  $\mathcal{O}_{X,\xi}$ , and denote it by  $\text{ord}(\varphi)$  if  $\xi$  is clear from the context.

### 3.2 Rees algebras and Nash multiplicity sequences

In [20], Lejeune-Jalabert introduced a sequence of positive integers attached to an arc in a germ of a hypersurface at a point, and she called it the *Nash multiplicity sequence*. This sequence is non increasing:

$$m_0 \geq m_1 \geq m_2 \geq \dots \geq m_k \geq 1$$

and stabilizes for some  $k \in \mathbb{N}$ .

Later, in [16], Hickel generalized this sequence for varieties of higher codimension. The way in which he constructs the sequence, involves a sequence of blow ups determined by the chosen arc. For this construction, Hickel works with arcs inside of a germ of a variety at a point (analytic context). We will work with arcs inside of a local neighbourhood of the variety at the point (local algebraic context). We will explain now this construction carefully, to show the computation of the Nash multiplicity sequence from this local algebraic point of view.

#### 3.2.1 Nash multiplicity sequence

Let  $X^{(d)}$  be an irreducible algebraic variety of dimension  $d$  over a perfect field  $k$ . Let  $\xi$  be a point contained in  $\text{Max mult}(X^{(d)})$ , the closed set of points of maximum multiplicity of  $X^{(d)}$ .<sup>9</sup> For simplicity, in this paper we will assume that  $\xi$  is a closed point. This will allow us to consider the blow up at  $\xi$ , since  $\xi$  is a smooth center in this case. In case one wants to consider non closed points, one needs just to localize  $X$  at  $\xi$  before performing the sequences that we will construct in this section.

Consider the product of  $X^{(d)}$  with an affine line. Then, we have a surjective morphism

$$X^{(d)} \xleftarrow{p} X_0^{(d+1)} = X^{(d)} \times \mathbb{A}_k^1, \tag{3.2}$$

given by the projection onto the first component. Let us write  $\xi_0 = (\xi, 0)$ , which is a point in  $X_0^{(d+1)}$ .

Consider the blow up of  $X_0^{(d+1)}$  at  $\xi_0$ , which we will denote by  $\pi_1$ . We will write  $X_1^{(d+1)}$  for the transform of  $X_0^{(d+1)}$  under  $\pi_1$ . After performing this blow up, we can choose a new point  $\xi_1 \in X_1^{(d+1)}$ , and call  $\pi_2$  the blow up of  $X_1^{(d+1)}$  at  $\xi_1$ .

Next, we will establish a criterion for the choice of each  $\xi_i \in X_i^{(d+1)}$  using an arc, so that we can perform a sequence of permissible blow ups at points in this way.

$$(X_0^{(d+1)}, \xi_0) \xleftarrow{\pi_1} (X_1^{(d+1)}, \xi_1) \xleftarrow{\pi_2} \dots \xleftarrow{\pi_r} (X_r^{(d+1)}, \xi_r). \tag{3.3}$$

<sup>9</sup> Note that we can always assume this situation for any  $\xi \in X$ , since one can always consider a neighbourhood of  $\xi$  where this is true.

Let  $\varphi \in \mathcal{L}(X^{(d)})$  be an arc in  $X^{(d)}$  through  $\xi$ . That is, we have a local homomorphism of local rings

$$\begin{aligned} \varphi : \mathcal{O}_{X^{(d)}, \xi} &\longrightarrow K[[t]] \\ \mathcal{M}_\xi &\longrightarrow \langle t \rangle, \end{aligned}$$

or, equivalently, a morphism  $\varphi^* : \text{Spec}(K[[t]]) \rightarrow X^{(d)}$ , mapping the closed point to  $\xi$ . This, together with the inclusion map  $i : k[t] \rightarrow K[[t]]$  gives an arc  $\Gamma_0$  in  $X_0^{(d+1)}$  through  $\xi_0$

$$\begin{aligned} \Gamma_0 : \mathcal{O}_{X_0^{(d+1)}, \xi_0} &\xrightarrow{\varphi \otimes i} K[[t]] \\ \mathcal{M}_{\xi_0} &\mapsto \langle t \rangle \end{aligned}$$

where  $\Gamma_0^*$  is the morphism given by the universal property of the fiber product:

$$\begin{array}{ccc} \text{Spec}(K[[t]]) & \xrightarrow{i^*} & \text{Spec}(K[t]) \\ \downarrow \varphi^* & \dashrightarrow \Gamma_0^* & \downarrow \\ (X^{(d)}, \xi) \times_k \text{Spec}(K[t]) = (X_0^{(d+1)}, \xi_0) & \longrightarrow & \text{Spec}(K[t]) \\ \downarrow & & \downarrow \\ (X^{(d)}, \xi) & \longrightarrow & \text{Spec}(k) \end{array} \tag{3.4}$$

Note that  $\Gamma_0$  is in fact the graph of  $\varphi$ .

Consider the blow up  $\pi_1$  of  $X_0^{(d+1)}$  at  $\xi_0$ . The initial Nash multiplicity of  $X$  at  $\xi$  is defined as

$$m = m_0 = \text{mult}_{\xi_0}(X_0^{(d+1)}) = \text{mult}_\xi(X^{(d)}),$$

where the last identity follows from the faithful flatness of (3.2).

After blowing up  $X_0^{(d+1)}$  at  $\xi_0$  (as in 3.3), the valuative criterion of properness ensures that we can lift  $\Gamma_0^*$  to a unique arc in  $X_1^{(d+1)}$ , which we will denote by  $\Gamma_1^*$ . Now  $\Gamma_1^*$  maps the closed point of  $\text{Spec}(K[[t]])$  to some closed point  $\xi_1 \in X_1^{(d+1)}$ . Furthermore,  $\xi_1 \in E_1 = \pi_1^{-1}(\xi_0)$  and  $\xi_1 \in \text{Im}(\Gamma_1^*)$ . This point  $\xi_1$  will be the center of the blow up  $\pi_2$ . We iterate this process: for  $i = 1, \dots, r$ , let  $\Gamma_i$  be the lifting of the arc  $\Gamma_{i-1} \in \mathcal{L}(X_{i-1}^{(d+1)})$  through  $\xi_{i-1}$  by the blow up  $\pi_i$  of  $X_{i-1}^{(d+1)}$  with center  $\xi_{i-1}$ . Then  $\Gamma_i$  is an arc in  $\mathcal{L}(X_i^{(d+1)})$  through a point  $\xi_i$  in the exceptional divisor  $E_i = \pi_i^{-1}(\xi_{i-1})$ . We will say that the sequence of transformations at points chosen in this way is the sequence directed by  $\varphi$  (or that the blow ups themselves are directed by  $\varphi$ ), meaning that  $\xi_0 = (\varphi(0), 0) = (\xi, 0)$  and  $\xi_i = \text{Im}(\Gamma_i^*) \cap E_i$  for  $i = 1, \dots, r$ :

$$\begin{array}{ccccccc} (X_0^{(d+1)}, \xi_0) & \xleftarrow{\pi_1} & (X_1^{(d+1)}, \xi_1) & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_r} & (X_r^{(d+1)}, \xi_r) \\ \uparrow \Gamma_0^* & & \uparrow \Gamma_1^* & & & & \uparrow \Gamma_r^* \\ (\text{Spec}(K[[t]]), 0) & \xleftarrow{id} & (\text{Spec}(K[[t]]), 0) & \xleftarrow{id} & \dots & \xleftarrow{id} & (\text{Spec}(K[[t]]), 0). \end{array} \tag{3.5}$$

For this sequence, the multiplicity of  $X_i^{(d+1)}$  at  $\xi_i$ , will be the  $i$ -th Nash multiplicity,  $m_i$ . The sequence  $m_0, m_1, \dots, m_r$  is non-increasing (see [16, Theorem 4.1] or [9]: the blow up at regular equimultiple centers does not increase the multiplicity) and eventually decreasing

whenever the generic point of the initial arc  $\varphi$  is not contained in  $\underline{\text{Max}} \text{mult}(X)$ . Indeed, if  $\varphi$  is contained in the stratum of  $X$  of multiplicity  $m'$  but not totally contained in any stratum of multiplicity greater than  $m'$ , then the sequence stabilizes at the value  $m'$ .<sup>10</sup> Thus, we can find some  $r$  so that for the diagram above the sequence of Nash multiplicities is such that  $m_0 = \dots = m_{r-1} > m_r$ . Our interest is in finding this  $r$ , namely the minimum number of blow ups at points directed by the arc  $\varphi$  as above which is necessary to perform in order to lower the Nash multiplicity of  $X$  at  $\xi$ . Since this can be done for any arc  $\varphi \in \mathcal{L}(X)$  through  $\xi$ , let us define:

**Definition 3.4** Let  $\varphi$  be an arc in  $X$  through  $\xi$ . We denote by  $\rho_{X,\varphi}$  the minimum number of blow ups directed by  $\varphi$  which are needed to lower the Nash multiplicity of  $X$  at  $\xi$ . That is,  $\rho_{X,\varphi}$  is such that  $m = m_0 = \dots = m_{\rho_{X,\varphi}-1} > m_{\rho_{X,\varphi}}$ . We will call  $\rho_{X,\varphi}$  the *persistance of  $\varphi$  in  $X$* . We denote by  $\rho_X(\xi)$  the infimum of the number of blow ups directed by some arc in  $X$  through  $\xi$  needed to lower the Nash multiplicity at  $\xi$ :

$$\rho_X : \underline{\text{Max}} \text{mult}(X) \longrightarrow \mathbb{N}$$

$$\xi \longmapsto \rho_X(\xi) = \inf_{\varphi \in \mathcal{L}_\xi(X)} \{ \rho_{X,\varphi} \}.$$

To keep the notation as simple as possible,  $\rho_{X,\varphi}$  does not contain a reference to the point  $\xi$ , even though it is clear that it is local. However, the point is determined by  $\varphi$ , and hence it is implicit, although not explicit in the notation. Similarly, we may refer to  $\rho_X(\xi)$  as  $\rho_X$  once the point is fixed.

Let us define normalized versions of  $\rho_{X,\varphi}$  and  $\rho_X$  in order to avoid the influence of the order of the arc in the number of blow ups needed to lower the Nash multiplicity.

**Definition 3.5** For a given arc  $\varphi$  in  $X$ , we will write

$$\bar{\rho}_{X,\varphi} = \frac{\rho_{X,\varphi}}{\text{ord}(\varphi)},$$

and similarly, we will denote

$$\bar{\rho}_X(\xi) = \inf_{\varphi \in \mathcal{L}_\xi(X)} \{ \bar{\rho}_{X,\varphi} \}.$$

Let us state our main theorem now, and develop afterwards the tools needed for the proof of this and some related results. Recall that  $\text{ord}_\xi \mathcal{G}_X^{(d)}$  is the first interesting coordinate of our resolution function (see Sect. 2.8). Theorem 5.20 at the end of Sect. 5 gives a relation between this invariant and the Nash multiplicity sequence.

In the following section, we will show that for  $X$ ,  $\xi \in X$  and  $\varphi \in \mathcal{L}_\xi(X)$ , we can attach a Rees algebra to the sequence of blow ups directed by  $\varphi$ . From this algebra, we will define a new quantity,  $r_{X,\varphi}$  (see Definition 4.7) which is a refinement of  $\rho_{X,\varphi}$ . In particular,  $\rho_{X,\varphi}$  is obtained by taking the integral part of  $r_{X,\varphi}$  (see Proposition 4.11). With this notation, the following result holds:

**Theorem 3.6** (Main Theorem) *Let  $X$  be a  $d$ -dimensional variety defined over a field of characteristic zero  $k$ . Let  $\xi$  be a point in  $\underline{\text{Max}} \text{mult}(X)$ . Then,*

$$\text{ord}_\xi \mathcal{G}_X^{(d)} = \min_{\varphi \in \mathcal{L}_\xi(X)} \left\{ \frac{r_{X,\varphi}}{\text{ord}(\varphi)} \right\}.$$

This result will be reformulated in Theorem 4.10 and the proof will be addressed in Sect. 5.

<sup>10</sup> Therefore, for our purpose, we need to choose arcs in a way such that they are not contained in the set of points of highest multiplicity of  $X$  (that is,  $\varphi^*( < 0 >) \not\subseteq \underline{\text{Max}} \text{mult}(X)$ ).

### 4 Rees algebras attached to Nash multiplicity sequences

In this section, the situation we consider for all constructions and results is always the same, specified in Sect. 4.1.

#### 4.1 Setting: notation and hypothesis

Let  $X$  be a  $d$ -dimensional variety over  $k$ . Locally in an étale neighbourhood  $\mathcal{U}_\eta$  of each point  $\eta \in X$ , we can find an immersion of  $\eta, \mathcal{U}_\eta \hookrightarrow V^{(n)}$ , and a Rees algebra  $\mathcal{G}_X^{(n)}$  over  $\mathcal{O}_{V^{(n)}, \xi}$  such that

$$\text{Sing}(\mathcal{G}_X^{(n)}) = \underline{\text{Max}} \text{ mult}(X), \tag{4.1}$$

and the equality is preserved by  $\mathcal{G}_X^{(n)}$ -local sequences over  $V^{(n)}$  as long as the maximum multiplicity does not decrease (see [30]). In other words, the multiplicity is represented by  $\mathcal{G}_X^{(n)}$  (see Definition 2.23). Let us recall that  $\mathcal{G}_X^{(n)}$  can be chosen to be differentially closed (see 2.16). For simplicity of the notation, we will also write  $X$  for this neighbourhood  $\mathcal{U}_\eta$  from now on.

Let us choose a point  $\xi \in \underline{\text{Max}} \text{ mult}(X)$ . If we go back to (3.2), after the product  $X^{(d)} \times \mathbb{A}_k^1$ , we also have an immersion, and thus a commutative diagram

$$\begin{CD} V^{(n)} @<p<< V_0^{(n+1)} = V^{(n)} \times \mathbb{A}_k^1 \\ @VVV @VVV \\ X^{(d)} @<p|_{X_0^{(d+1)}}<< X_0^{(d+1)} = X^{(d)} \times \mathbb{A}_k^1 \end{CD} \tag{4.2}$$

In particular,  $p$  is a local sequence on  $V^{(n)}$  and preserves (4.1), and thus the smallest  $\mathcal{O}_{V_0^{(n+1)}, \xi_0}$ -Rees algebra containing  $\mathcal{G}_X^{(n)}$  (the extended algebra) represents the function  $\text{mult}(X_0^{(d+1)})$ . We will refer to this algebra as the  $\mathcal{O}_{V_0^{(n+1)}, \xi_0}$ -Rees algebra  $\mathcal{G}_{X_0}^{(n+1)}$ .

Fix an arc  $\varphi \in \mathcal{L}(X)$  through  $\xi$  not contained in  $\underline{\text{Max}} \text{ mult}(X)$ . The sequence of blow ups at points directed by  $\varphi$  defined in (3.5) induces a sequence<sup>11</sup> of blow ups for  $V_0^{(n+1)}$ :

$$\begin{CD} (V_0^{(n+1)}, \xi_0) @<\pi_1<< (V_1^{(n+1)}, \xi_1) @<\pi_2<< \dots @<\pi_r<< (V_r^{(n+1)}, \xi_r) \\ @VVV @VVV @VVV @VVV \\ (X_0^{(d+1)}, \xi_0) @<\pi_1|_{X_1^{(d+1)}}<< (X_1^{(d+1)}, \xi_1) @<\pi_2|_{X_2^{(d+1)}}<< \dots @<\pi_r|_{X_r^{(d+1)}}<< (X_r^{(d+1)}, \xi_r) \\ @VV\Gamma_0^*V @VV\Gamma_1^*V @VVV @VV\Gamma_r^*V \\ (\text{Spec}(K[[t]]), 0) @<id<< (\text{Spec}(K[[t]]), 0) @<id<< \dots @<id<< (\text{Spec}(K[[t]]), 0) \end{CD} \tag{4.3}$$

Consider now the ring  $\mathcal{O}_{V^{(n)}, \xi} \otimes_k K[[t]]$ , and the localization at  $\xi_0 = (\xi, 0)$ :<sup>12</sup>

$$\delta : \mathcal{O}_{V^{(n)}, \xi} \longrightarrow (\mathcal{O}_{V^{(n)}, \xi} \otimes_k K[[t]])_{\xi_0}, \tag{4.4}$$

<sup>11</sup> For simplicity of the notation, we will often identify the points  $\xi_i$  in  $X_i^{(d+1)}$  with their images in  $V_i^{(n+1)}$ .

<sup>12</sup> We use the same notation for the image of  $\xi$  by  $p^*$  and by  $\bar{\delta}$ .

and let us denote  $\tilde{V}_0^{(n+1)} = \text{Spec}(\mathcal{O}_{V^{(n)}, \xi} \otimes_k K[[t]])_{\xi_0}$  and  $\tilde{X}_0^{(d+1)} = \text{Spec}(\mathcal{O}_{X^{(d)}, \xi} \otimes_k K[[t]])_{\xi_0}$ . Let us choose a regular system of parameters  $y_1, \dots, y_n \in \mathcal{O}_{V^{(n)}, \xi}$ , so that  $\{y_1, \dots, y_n, t\}$  is a regular system of parameters in  $(\mathcal{O}_{V^{(n)}, \xi} \otimes_k K[[t]])_{\xi_0}$ .

Note that if  $\beta_X : X \rightarrow \text{Spec}(S) = V^{(d)}$  is a finite morphism as in (2.15) then after the natural base extension,  $\tilde{X}_0^{(d+1)} \rightarrow \tilde{V}_0^{(d+1)}$  is also a finite morphism. We will need this fact in the proof of Proposition 4.4 below.

Now  $\Gamma_0$  can be described by the images of  $t$  and the classes  $\bar{y}_i$  of the  $y_i$  in  $\mathcal{O}_{X_0^{(d+1)}, \xi_0}$ , for  $i = 1, \dots, n$ :

$$\begin{aligned} \Gamma_0 : \mathcal{O}_{X_0^{(d+1)}, \xi_0} &\longrightarrow K[[t]] \\ \bar{y}_i &\longmapsto \varphi_{y_i} = \varphi(\bar{y}_i) \quad i = 1, \dots, n \\ t &\longmapsto t. \end{aligned}$$

Since both  $\varphi$  and  $\delta$  are continuous, there is a  $k$ -morphism  $\tilde{\Gamma}_0 : (\mathcal{O}_{V^{(n)}, \xi} \otimes_k K[[t]])_{\xi_0} \rightarrow K[[t]]$  which is completely determined by the images of the  $\bar{y}_i$  and  $t$ . The following commutative diagram provides an overview of the situation:

$$\begin{array}{ccccc} \mathcal{O}_{V_0^{(n+1)}, \xi_0} & \xleftarrow{p^*} & \mathcal{O}_{V^{(n)}, \xi} & \xrightarrow{\delta} & (\mathcal{O}_{V^{(n)}, \xi} \otimes_k K[[t]])_{\xi_0} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}_{X_0^{(d+1)}, \xi_0} & \xleftarrow{\quad} & \mathcal{O}_{X^{(d)}, \xi} & \xrightarrow{\quad} & (\mathcal{O}_{X^{(d)}, \xi} \otimes_k K[[t]])_{\xi_0} \\ & \searrow \Gamma_0 & \downarrow \varphi & & \downarrow \\ & \varphi_{y_i} = \varphi(\bar{y}_i), t & K[[t]] & \xleftarrow{\tilde{\Gamma}_0} & K[[t]] \end{array}$$

(4.5)

Note that  $\tilde{\Gamma}_0$  is an arc in  $\tilde{X}_0^{(d+1)}$  defining a curve  $C_0$  which is smooth, since it is given in  $\tilde{V}_0^{(n+1)}$  by the equations<sup>13</sup>  $y_i - \varphi_{y_i} = 0$  for  $i = 1, \dots, n$  where  $\varphi_{y_i} \in K[[t]]$  for  $i = 1, \dots, n$ . This curve is the closure of the image of  $\tilde{\Gamma}_0^* : \text{Spec}(K[[t]]) \rightarrow \tilde{V}_0^{(n+1)}$ , induced by  $\tilde{\Gamma}_0$ . We get an analogous diagram to that in (4.3):

$$\begin{array}{ccccccc} (\tilde{V}_0^{(n+1)}, \xi_0) & \xleftarrow{\tilde{\pi}_1} & (\tilde{V}_1^{(n+1)}, \xi_1) & \xleftarrow{\tilde{\pi}_2} & \dots & \xleftarrow{\tilde{\pi}_r} & (\tilde{V}_r^{(n+1)}, \xi_r) \\ \uparrow & & \uparrow & & & & \uparrow \\ (\tilde{X}_0^{(d+1)}, \xi_0) & \xleftarrow{\tilde{\pi}_1|_{\tilde{X}_1^{(d+1)}}} & (\tilde{X}_1^{(d+1)}, \xi_1) & \xleftarrow{\tilde{\pi}_2|_{\tilde{X}_2^{(d+1)}}} & \dots & \xleftarrow{\tilde{\pi}_r|_{\tilde{X}_r^{(d+1)}}} & (\tilde{X}_r^{(d+1)}, \xi_r) \\ \uparrow & & \uparrow & & & & \uparrow \\ (C_0, \xi_0) & \xleftarrow{\tilde{\pi}_1|_{C_1}} & (C_1, \xi_1) & \xleftarrow{\tilde{\pi}_2|_{C_2}} & \dots & \xleftarrow{\tilde{\pi}_r|_{C_r}} & (C_r, \xi_r) \end{array}$$

(4.6)

<sup>13</sup>  $C_0$  is a smooth curve in a local ring, and hence a complete intersection.

where we can see that the preimage  $\tilde{E}_i$  of  $\xi_{i-1}$  by  $\tilde{\pi}_i$  always intersects  $C_i$  at a single point. This point is  $\xi_i$ , the center of the blow up  $\tilde{\pi}_{i+1}$ .

### 4.2 Contact algebras

With the notation in Sect. 4.1, let us look now at the closed set  $C_0 \subset \tilde{V}_0^{(n+1)}$  defined by the arc  $\varphi$ . We can find an  $(\mathcal{O}_{V^{(n)},\xi} \otimes_k K[[t]])_{\xi_0}$ -Rees algebra  $\mathcal{G}_\varphi^{(n+1)}$  representing  $C_0$  in the sense of Definition 2.23. That is,  $\mathcal{G}_\varphi^{(n+1)}$  will satisfy  $\text{Sing}(\mathcal{G}_\varphi^{(n+1)}) = C_0$ , and for any local sequence as in (2.7),  $\text{Sing}(\mathcal{G}_{\varphi,i}^{(n+1)}) = C_i$ , where  $C_i$  is the strict transform of  $C_{i-1}$  by  $\phi_i$  if it is a blow up at a smooth center, or the pullback of  $C_{i-1}$  if  $\phi_i$  is a smooth morphism. It can be shown that

$$\mathcal{G}_\varphi^{(n+1)} = \mathcal{O}_{\tilde{V}_0^{(n+1)},\xi_0} [h_1 W, \dots, h_n W], \tag{4.7}$$

where  $h_i = (y_i - \varphi_{y_i})$  for  $i = 1, \dots, n$ . Consider now the closed set

$$Z_0 = C_0 \cap \left\{ \eta \in \tilde{X}_0^{(d+1)} : \text{mult}_\eta(\tilde{X}_0^{(d+1)}) = m \right\} \subset \tilde{V}_0^{(n+1)}. \tag{4.8}$$

For any local sequence

$$\tilde{V}_0^{(n+1)} \xleftarrow{\pi_1} \tilde{V}_1^{(n+1)} \xleftarrow{\pi_2} \dots \xleftarrow{\pi_r} \tilde{V}_r^{(n+1)} \tag{4.9}$$

we define  $Z_i$ , for  $i = 1, \dots, r$ , as the closed set

$$Z_i = C_i \cap \left\{ \eta \in \tilde{X}_i^{(d+1)} : \text{mult}_\eta(\tilde{X}_i^{(d+1)}) = m \right\}, \tag{4.10}$$

where  $C_i$  is the transform of  $C_{i-1}$  by  $\pi_i$  (that is, the pullback if  $\pi_{i-1}$  is a smooth morphism, and the strict transform if it is a blow up at a smooth center contained in  $Z_{i-1}$ ) and  $\tilde{X}_i^{(d+1)}$  is the transform of  $\tilde{X}_{i-1}^{(d+1)}$ .

**Definition 4.1** Let us suppose now that one can find an  $(\mathcal{O}_{V^{(n)},\xi} \otimes_k K[[t]])_{\xi_0}$ -Rees algebra  $\mathcal{H}$  whose singular locus is  $Z_0$ , and such that this is preserved by local sequences as in (4.9) (and in particular for sequences of blow ups of  $\tilde{X}_0^{(d+1)}$  directed by  $\varphi$ ). We will say that such an algebra, if it exists, is an *algebra of contact of  $\varphi$  with Max mult( $X$ )*.

*Remark 4.2* Lowering the Nash multiplicity of  $X$  at  $\xi$ ,  $m$ , is therefore equivalent to resolving this  $\mathcal{H}$ , and consequently  $\rho_{X,\varphi}$  as in Definition 3.4 is the number of induced transformations of this Rees algebra  $\mathcal{H}$  which are necessary to resolve it (see Definition 2.9).

*Remark 4.3* Note that, by the way in which it has been defined, the algebra of contact of  $\varphi$  with Max mult( $X$ ), if it exists, is unique up to weak equivalence.

Denote

$$\mathcal{G}_{X_0,\varphi}^{(n+1)} := \mathcal{G}_{\tilde{X}_0}^{(n+1)} \odot \mathcal{G}_\varphi^{(n+1)}, \tag{4.11}$$

where  $\mathcal{G}_{\tilde{X}_0}^{(n+1)}$  is the extension of  $\mathcal{G}_X^{(n)}$  to  $(\mathcal{O}_{V^{(n)},\xi} \otimes_k K[[t]])_{\xi_0}$  (see (4.2) and (4.4)) and  $\mathcal{G}_\varphi^{(n+1)}$  is as in (4.7).<sup>14</sup>

**Proposition 4.4** *Let  $X$  be a variety, let  $\xi$  be a point in Max mult( $X$ ), and let  $\varphi$  be an arc in  $X$  through  $\xi$  with the hypothesis and notation from Sect. 4.1. Then the Rees algebra  $\mathcal{G}_{X_0,\varphi}^{(n+1)}$*

<sup>14</sup> Note that  $\mathcal{G}_\varphi$  and  $\mathcal{G}_{\tilde{X}_0}^{(n+1)}$  are differentially closed by definition.



from (4.11) is an algebra of contact of  $\varphi$  with  $\underline{\text{Max}} \text{ mult}(X)$ . Moreover, the restriction  $\mathcal{G}_{X_0, \varphi}^{(1)}$  of the same Rees algebra to the curve  $C_0$  defined by  $\varphi$  is also an algebra of contact of  $\varphi$  with  $\underline{\text{Max}} \text{ mult}(X)$ . In particular, resolving  $\mathcal{G}_{X_0, \varphi}^{(n+1)}$  is equivalent to resolving  $\mathcal{G}_{X_0, \varphi}^{(1)}$ .

*Proof* By definition of  $\mathcal{G}_{X_0, \varphi}^{(n+1)}$ ,

$$\mathcal{F}_{\tilde{V}_0} \left( \mathcal{G}_{X_0, \varphi}^{(n+1)} \right) = \mathcal{F}_{\tilde{V}_0} \left( \mathcal{G}_{\tilde{X}_0}^{(n+1)} \right) \cap \mathcal{F}_{\tilde{V}_0} \left( \mathcal{G}_{\varphi}^{(n+1)} \right)$$

(see Definition 2.20). Then,  $\mathcal{G}_{X_0, \varphi}^{(n+1)}$  is an algebra of contact of  $\varphi$  with  $\underline{\text{Max}} \text{ mult}(X)$  as long as  $\mathcal{G}_{\tilde{X}_0}^{(n+1)}$  represents  $\underline{\text{Max}} \text{ mult}(\tilde{X}_0)$  and  $\mathcal{G}_{\varphi}^{(n+1)}$  represents  $C_0$  in the sense of Definition 2.23. The latter was already shown at the beginning of this section. For the first assertion, we may assume that locally we are in the situation of Example 2.38, and with the notation there, we have now that  $S \otimes_k K[[t]] \subset B \otimes_k K[[t]] = S[\theta_1, \dots, \theta_{n-d}] \otimes_k K[[t]]$  is a finite extension of rings satisfying the properties in [30, 4.5], and therefore the argument in [30, Proposition 5.7] is also valid for them:  $\xi \in \underline{\text{Max}} \text{ mult}(\tilde{X}_0)$  if and only if  $\text{ord}_{\xi} f_i \geq n_i$  for  $i = 1, \dots, n-d$ , so the  $f_i$  are also the minimal polynomials of the  $\theta_i$  over  $S \otimes K[[t]]$ .

On the other hand, by [7, Proposition 6.6]

$$\mathcal{F}_{C_0} \left( \mathcal{G}_{\tilde{X}_0}^{(n+1)} \Big|_{C_0} \right) = \mathcal{F}_{\tilde{V}_0} \left( \mathcal{G}_{\tilde{X}_0}^{(n+1)} \right) \cap \mathcal{F}_{\tilde{V}_0} \left( \mathcal{G}_{\varphi}^{(n+1)} \right),$$

since  $\mathcal{G}_{\tilde{X}_0}^{(n+1)}$  is differentially closed, and  $C_0$  is smooth. Hence, it is clear that the Rees algebra  $\mathcal{G}_{\tilde{X}_0}^{(n+1)} \Big|_{C_0}$  defines the same tree of closed sets as  $\mathcal{G}_{X_0, \varphi}^{(n+1)}$ . In addition, the restriction of  $\mathcal{G}_{X_0, \varphi}^{(n+1)}$  to  $C_0$  defines the very same tree, since

$$\begin{aligned} \mathcal{F}_{\tilde{V}_0} \left( \mathcal{G}_{X_0, \varphi}^{(1)} \right) &:= \mathcal{F}_{\tilde{V}_0} \left( \mathcal{G}_{\tilde{X}_0}^{(n+1)} \Big|_{C_0} \odot \mathcal{G}_{\varphi}^{(n+1)} \Big|_{C_0} \right) = \mathcal{F}_{\tilde{V}_0} \left( \mathcal{G}_{\tilde{X}_0}^{(n+1)} \Big|_{C_0} \right) \cap \mathcal{F}_{\tilde{V}_0} \left( \mathcal{G}_{\varphi}^{(n+1)} \Big|_{C_0} \right) \\ &= \mathcal{F}_{C_0} \left( \mathcal{G}_{\tilde{X}_0}^{(n+1)} \Big|_{C_0} \right), \end{aligned}$$

and the proposition is proved. □

The following definition will give us a tool to compute the algebra  $\mathcal{G}_{X_0, \varphi}^{(1)}$  that appears in the last Proposition. This will become quite useful in Sect. 5:

**Definition 4.5** With the notation in Sect. 4.1, let  $\mathcal{G}$  be a Rees algebra over  $V^{(n)}$  given as

$$\mathcal{G} = \mathcal{O}_{V^{(n)}, \xi} [g_1 W^{c_1}, \dots, g_s W^{c_s}]$$

locally at  $\xi$ . Then, for any arc  $\varphi \in \mathcal{L}_{\xi}(V^{(n)})$ , we define

$$\varphi(\mathcal{G}) = K[[t]][\varphi(g_1)W^{c_1}, \dots, \varphi(g_s)W^{c_s}].$$

*Remark 4.6* With the notation in Sect. 4.1, we may define the image by  $\tilde{\Gamma}_0$  of the Rees algebra  $\mathcal{G}_{X_0, \varphi}^{(n+1)}$  from (4.11). This algebra  $\tilde{\Gamma}_0(\mathcal{G}_{X_0, \varphi}^{(n+1)})$  happens to be the restriction of the algebra  $\mathcal{G}_{X_0, \varphi}^{(n+1)}$  to the curve  $C_0$  defined by  $\varphi$ , and the proof of Proposition 4.4 shows that if  $\mathcal{G}_{X_0}^{(n+1)} = \mathcal{O}_{V^{(n+1)}} [g_1 W^{c_1}, \dots, g_s W^{c_s}]$ , then

$$\tilde{\Gamma}_0(\mathcal{G}_{X_0, \varphi}^{(n+1)}) = K[[t]][\varphi(g_1)W^{c_1}, \dots, \varphi(g_s)W^{c_s}],$$

since  $\tilde{\Gamma}_0(h_i) = 0$  for  $i = 1, \dots, n$ .

Our goal now is to define an invariant for  $X$ ,  $\xi$  and  $\varphi$  using the algebra of contact of  $\varphi$  with  $\text{Max mult}(X)$ . However, Proposition 4.4 shows that it would also make sense to define it from the restriction  $\mathcal{G}_{X_0,\varphi}^{(1)}$  to  $C_0$ . In addition, from the way in which  $\mathcal{G}_{X_0,\varphi}^{(n+1)}$  is constructed, we know that it has elements of order 1 in weight 1, and hence has order 1 itself<sup>15</sup> at all points of its singular locus. On contrary, the order of  $\mathcal{G}_{X_0,\varphi}^{(1)}$  will be much more interesting, as we will see in Proposition 4.11.

**Definition 4.7** Let  $X$  be a variety, and let  $\varphi$  be an arc in  $X$  through  $\xi \in \text{Max mult}(X)$ . We define the *order of contact* of  $\varphi$  with  $\text{Max mult}(X)$  as the order<sup>16</sup> at  $\xi$  of the restriction  $\mathcal{G}_{X_0,\varphi}^{(1)}$  to  $C_0$  of the algebra of contact of  $\varphi$  with  $\text{Max mult}(X)$ , and we write it by

$$r_{X,\varphi} = \text{ord}_\xi(\mathcal{G}_{X_0,\varphi}^{(1)}) \in \mathbb{Q}.$$

We denote by  $r_X$  the infimum of the orders of contact of  $\text{Max mult}(X)$  with all arcs in  $X$  through  $\xi$ :

$$r_X = \inf_{\varphi \in \mathcal{L}_\xi(X)} \left\{ \text{ord}_\xi(\mathcal{G}_{X_0,\varphi}^{(1)}) \right\} \in \mathbb{R}.$$

*Remark 4.8* We have defined an invariant  $r_{X,\varphi}$  for the pair  $(X, \varphi)$  and another invariant  $r_X$  for  $X$ : by Hironaka’s trick (see [13, Section 7]), it can be shown that  $r_{X,\varphi}$  depends only on  $X$ ,  $\xi$  and  $\varphi$ , not on the choice of the algebra of contact (which is not unique). For the same reason  $r_X$  depends only on  $X$  and on the point  $\xi$  we are looking at.

**Definition 4.9** Normalizing  $r_{X,\varphi}$  and  $r_X$  by the order of the respective arcs (see Definition 3.3) we define new invariants. We denote

$$\bar{r}_{X,\varphi} = \frac{\text{ord}_\xi(\mathcal{G}_{X_0,\varphi}^{(1)})}{\text{ord}(\varphi)} \in \mathbb{Q},$$

and

$$\bar{r}_X = \inf_{\varphi \in \mathcal{L}_\xi(X)} \left\{ \frac{\text{ord}_\xi(\mathcal{G}_{X_0,\varphi}^{(1)})}{\text{ord}(\varphi)} \right\} \in \mathbb{R}.$$

We give now a more complete version of Theorem 3.6, which we will prove in Sect. 5:

**Theorem 4.10** *Let  $X$  be an algebraic variety of dimension  $d$  and  $\xi$  a point in  $\text{Max mult}(X)$ . Then*

$$\bar{r}_X = \text{ord}_\xi \mathcal{G}_X^{(d)} \in \mathbb{Q}.$$

*Moreover, the infimum  $\bar{r}_X$  is indeed a minimum.*

Equivalently, for every arc  $\varphi \in \mathcal{L}(X)$  through  $\xi$ ,

$$\bar{r}_{X,\varphi} \geq \text{ord}_\xi \mathcal{G}_X^{(d)},$$

and in addition, one can find an arc  $\varphi_0 \in \mathcal{L}(X)$  through  $\xi$  such that

$$\bar{r}_{X,\varphi_0} = \text{ord}_\xi \mathcal{G}_X^{(d)}.$$

We already mentioned at the end of Sect. 3.2 that  $r_{X,\varphi}$  is a refinement of  $\rho_{X,\varphi}$ . The following proposition shows that in fact  $\rho_{X,\varphi}$  may be obtained from  $r_{X,\varphi}$ .

<sup>15</sup> Note that  $\mathcal{G}_\varphi^{(n+1)}$  has order one (see (4.7)).

<sup>16</sup> As we have done already, we will write  $\xi$  for the image of  $\xi$  under most of the morphisms we use, as long as the identification between both points is clear.

**Proposition 4.11** *Let  $X$  be a variety, let  $\xi$  be a point in  $\underline{\text{Max}} \text{mult}(X)$  and let  $\varphi$  be an arc in  $X$  through  $\xi$ . Then*

$$\rho_{X,\varphi} = [r_{X,\varphi}]. \tag{4.12}$$

*That is, the persistence of  $\varphi$  in  $X$  (Definition 3.4) equals the integral part of the order of contact of  $\varphi$  with  $\underline{\text{Max}} \text{mult}(X)$ .*

*Proof* Since  $\mathcal{G}_{X_0,\varphi}^{(1)}$  is a Rees algebra over a smooth curve, it is defined over a regular local ring  $\mathcal{O}_{C_0,\xi}$  of dimension one. If the maximal ideal  $\mathcal{M}_\xi$  of  $\xi$  in  $\mathcal{O}_{C_0,\xi}$  is  $\mathcal{M}_\xi = \langle T \rangle$  for some regular parameter  $T$ , then  $\mathcal{G}_{X_0,\varphi}^{(1)}$  is necessarily generated by a finite set of elements of the form  $T^\alpha W^{l_\alpha}$ , where  $\alpha, l_\alpha$  are positive integers. Observe also that  $\mathcal{G}_{X_0,\varphi}^{(1)}$  is integrally equivalent to a Rees algebra generated by  $JW^l$  for some principal ideal  $J \subset \mathcal{O}_{C_0,\varphi}$  and some positive integer  $l$ , at least in a neighbourhood of  $\xi$  (see [7, Lemma 1.7]). Therefore, we can suppose that  $\mathcal{G}_{X_0,\varphi}^{(1)} = \mathcal{O}_{C_0,\xi}[T^\alpha W^l]$ . In this case, the order of  $\mathcal{G}_{X_0,\varphi}^{(1)}$  at  $\xi$  will be given by

$$\text{ord}_\xi(\mathcal{G}_{X_0,\varphi}^{(1)}) = \frac{\alpha}{l}.$$

By the transformation law (2.2), the first transform of  $\mathcal{G}_{X_0,\varphi}^{(1)}$  by blowing up at the closed point is

$$\mathcal{G}_{X_0,\varphi,1}^{(1)} = \mathcal{O}_{C_0,\xi}[T^{\alpha-l} W^l].$$

The order of the  $k$ -th transform will therefore be

$$\frac{\alpha - k \cdot l}{l},$$

and the number  $\rho_{X,\varphi}$  of blow ups needed to resolve  $\mathcal{G}_{X_0,\varphi}^{(1)}$  must satisfy:

$$0 \leq \alpha - \rho_{X,\varphi} \cdot l < l.$$

But this implies

$$0 \leq \frac{\alpha}{l} - \rho_{X,\varphi} < 1,$$

which means that  $\rho_{X,\varphi}$  is the integral part of  $\frac{\alpha}{l} = \text{ord}_\xi(\mathcal{G}_{X_0,\varphi}^{(1)})$ , which is precisely the order of contact of  $\varphi$  with  $\underline{\text{Max}} \text{mult}(X)$ . □

**Corollary 4.12** *For any variety  $X$ ,*

$$\begin{aligned} \rho_X &= [r_X], \\ [\bar{r}_X] &\leq \bar{\rho}_X \leq \bar{r}_X. \end{aligned}$$

The proof follows solely from the definitions of  $r_X, \bar{r}_X, \rho_X$  and  $\bar{\rho}_X$  together with Proposition 4.11, by means of algebraic manipulations of their integral parts.

In what follows, we will give the proof of Theorem 4.10 by focusing first on the hypersurface case and generalizing then to arbitrary codimension.

### 5 Proof of the main result

For the proof of Theorem 4.10, we assume first that  $X$  is a hypersurface in Theorems 5.10 and 5.12. Later on, we will see that we can deduce the proof of the general case from the hypersurface one in Theorems 5.17 and 5.19.

### 5.1 Rees algebras and orders for a hypersurface

For any variety  $X$  which is locally a hypersurface, we can always find a nice expression for  $X$  in an étale neighbourhood of each point. Using this expression, we will prove Theorem 4.10 for the hypersurface case by dividing it into two theorems: Theorem 5.10 states that  $\text{ord}_\xi \mathcal{G}_X^{(d)}$  from Sect. 2.8 is a lower bound of  $\bar{r}_{X,\varphi}$  for any arc  $\varphi \in \mathcal{L}_\xi(X)$ , and Theorem 5.12 shows that in fact we can find an arc giving the equality, so that  $\bar{r}_X$  is actually a minimum. For the proof of these two, we will define diagonal arcs, which will help us analyzing the orders of contact and the order  $\text{ord}_\xi \mathcal{G}_X^{(d)}$  (see 2.7.1 1-5, and Theorems 2.41, 2.50), and giving some conclusions and lemmas about them.

#### 5.1.1 Notation and hypothesis

Let  $X = X^{(d)}$  be a  $d$ -dimensional variety over  $k$  of maximum multiplicity  $b$ , and let  $\xi \in \underline{\text{Max}} \text{mult}(X)$ . Let us suppose that  $X$  at  $\xi$  is locally a hypersurface, given by  $\mathcal{O}_{X,\xi} \cong S[x]/(f)$  for a regular local  $k$ -algebra  $S$  and a variable  $x$ , as in Example 2.35. As we did in (2.13), we can suppose that  $f$  has an expression of the form

$$f(x) = x^b + B_{b-2}x^{b-2} + \dots + B_i x^i + \dots + B_0 \tag{5.1}$$

in some étale neighbourhood of  $\xi \in X$ , with  $B_0, \dots, B_{b-2} \in S$ , and where we write  $n = d + 1$  for the dimension of the ambient space  $V^{(n)} = \text{Spec}(S[x])$ . Consider  $\mathcal{G}_X^{(d)}$ , the elimination algebra of  $\mathcal{O}_{V^{(n)},\xi}[fW^b]$  in  $\mathcal{O}_{V^{(d)},\xi^{(d)}}$  induced by the projection  $\beta : V^{(n)} \rightarrow V^{(d)} = \text{Spec}(S)$  (see Theorem 2.41), as the diagram shows:

$$\begin{array}{ccccc}
 \mathcal{G}_{X_0}^{(n+1)} & \longleftarrow & \mathcal{G}_X^{(n)} & \longrightarrow & \mathcal{G}_{\tilde{X}_0}^{(n+1)} \\
 \mathcal{O}_{V_0^{(n+1)},\xi_0} & \xleftarrow{p^*} & \mathcal{O}_{V^{(n)},\xi} & \xrightarrow{\delta} & (\mathcal{O}_{V^{(n)},\xi} \otimes_k K[[t]])_{\xi_0} \\
 \uparrow & & \uparrow \beta^* & & \uparrow \\
 \mathcal{O}_{V_0^{(d+1)},\xi_0^{(d+1)}} & \longleftarrow & \mathcal{O}_{V^{(d)},\xi^{(d)}} & \longrightarrow & (\mathcal{O}_{V^{(d)},\xi_0^{(d)}} \otimes_k K[[t]])_{\xi_0^{(d+1)}} \\
 \mathcal{G}_{X_0}^{(d+1)} & \longleftarrow & \mathcal{G}_X^{(d)} & \longrightarrow & \mathcal{G}_{\tilde{X}_0}^{(d+1)}
 \end{array} \tag{5.2}$$

where  $\mathcal{G}_{X_0}^{(d+1)}$  is an elimination of  $\mathcal{G}_{X_0}^{(n+1)}$ . We have the following expression:

$$\mathcal{G}_X^{(n)} = \text{Diff}(\mathcal{O}_{V^{(n)},\xi}[fW^b]) = \mathcal{O}_{V^{(n)},\xi}[xW] \odot \mathcal{G}_X^{(d)} \tag{5.3}$$

(see Lemma 2.44 for  $\mathcal{G}_X^{(d)}$ ). Let  $\varphi$  be an arc in  $X$  through  $\xi$ , not contained in  $\text{Max mult}(X)$ . Suppose that  $\varphi$  is such that  $\varphi_x = u_0 t^{\alpha_0}$  and  $\varphi_{z_i} = u_i t^{\alpha_i}$  for a regular system of parameters  $\{z_1, \dots, z_d\} \in S$ , as in (4.5), where  $u_0, \dots, u_d$  are units in  $K[[t]]$  and  $\alpha_0, \dots, \alpha_d$  are positive integers. This gives the following expressions for the algebra of contact of  $\varphi$  with  $\text{Max mult}(X)$  (see Proposition 4.4):

$$\begin{aligned}
 \mathcal{G}_{X_0,\varphi}^{(n+1)} &= \text{Diff}(\mathcal{O}_{\tilde{V}_0^{(n+1)},\xi_0}[fW^b]) \odot \mathcal{O}_{\tilde{V}_0^{(n+1)},\xi_0}[(x - u_0 t^{\alpha_0})W, (z_i - u_i t^{\alpha_i})W; \quad i = 1, \dots, d] \\
 &= \mathcal{O}_{\tilde{V}_0^{(n+1)},\xi_0}[xW] \odot \mathcal{G}_X^{(d)} \odot \mathcal{O}_{\tilde{V}_0^{(n+1)},\xi_0}[(x - u_0 t^{\alpha_0})W, (z_i - u_i t^{\alpha_i})W; \quad i = 1, \dots, d].
 \end{aligned} \tag{5.4}$$

This expression will allow us to know the order of contact of  $\varphi$  with  $\underline{\text{Max}} \text{mult}(X)$  (see Definition 4.7), which is our real interest.

Let us recall that Properties 2.7.1 1-4 guarantee that  $\mathcal{G}_X^{(d)}$  represents  $\beta(\underline{\text{Max}} \text{mult}(X))$ . Note now that the corresponding projection of  $\varphi$  by  $\beta$  gives also an arc  $\varphi^{(d)}$  in  $V^{(d)}$  according to the following diagram

$$\begin{array}{ccc}
 \mathcal{O}_{V^{(n)}, \xi} & \xrightarrow{\varphi} & K[[t]] \\
 \beta^* \uparrow & \nearrow \varphi^{(d)} & \\
 \mathcal{O}_{V^{(d)}, \xi^{(d)}} & & 
 \end{array}
 \tag{5.5}$$

Consider then the elimination algebra  $\mathcal{G}_{X_0}^{(d+1)}$  above. We can construct an algebra of contact of  $\varphi^{(d)}$  with  $\beta(\underline{\text{Max}} \text{mult}(X))$  by an analogous construction to that in (4.11), using the fact that  $\mathcal{G}_X^{(d)}$  represents  $\beta(\underline{\text{Max}} \text{mult}(X))$ . Then we obtain the  $\mathcal{O}_{\tilde{V}^{(d+1)}, \xi^{(d+1)}}$ -Rees algebra

$$\mathcal{G}_{X_0, \varphi^{(d)}}^{(d+1)} = \mathcal{G}_X^{(d)} \odot \mathcal{G}_{\varphi^{(d)}}^{(d+1)}.
 \tag{5.6}$$

Also  $\mathcal{G}_{X_0, \varphi^{(d)}}^{(1)}$  will be the restriction of  $\mathcal{G}_{X_0, \varphi^{(d)}}^{(d+1)}$  to the image of  $C_0$  in  $\tilde{V}_0^{(d+1)}$  (which we will denote by  $C_0^{(d)}$ ). Note that  $\mathcal{G}_{X_0, \varphi^{(d)}}^{(1)} = \tilde{\Gamma}_0^{(d)}(\mathcal{G}_{X_0, \varphi^{(d)}}^{(d+1)})$ , where  $\tilde{\Gamma}_0^{(d)} : (\mathcal{O}_{V^{(d)}, \xi^{(d)}} \otimes_k K[[t]])_{\xi_0^{(d)}} \rightarrow K[[t]]$  is given by  $\varphi^{(d)} : \mathcal{O}_{V^{(d)}, \xi^{(d)}} \rightarrow K[[t]]$  as in (4.5). With this notation we can write,

$$\mathcal{G}_{X_0, \varphi}^{(n+1)} = \mathcal{O}_{\tilde{V}_0^{(n+1)}, \xi_0} [xW, t^{\alpha_0}W] \odot \mathcal{G}_X^{(d)} \odot \mathcal{G}_{\varphi^{(d)}}^{(d+1)} = \mathcal{O}_{\tilde{V}_0^{(n+1)}, \xi_0} [xW] \odot K[[t]][t^{\alpha_0}W] \odot \mathcal{G}_{X_0, \varphi^{(d)}}^{(d+1)}$$

by (5.4) and (5.6), and hence

$$\mathcal{G}_{X_0, \varphi}^{(1)} = K[[t]][t^{\alpha_0}W] \odot \mathcal{G}_{X_0, \varphi^{(d)}}^{(1)},$$

and

$$r_{X, \varphi} = \text{ord}_{\xi}(\mathcal{G}_{X_0, \varphi}^{(1)}) = \min\{\alpha_0, \text{ord}_{\xi}(\mathcal{G}_{X_0, \varphi^{(d)}}^{(1)})\}.
 \tag{5.7}$$

### 5.1.2 Auxiliary results

The following Lemma shows that, in fact,  $\alpha_0$  is not important for  $r_{X, \varphi}$ .

**Lemma 5.1** *Let  $X$  be as in Sect. 5.1.1. Let  $\xi \in \underline{\text{Max}} \text{mult}(X)$ . Then for any arc  $\varphi \in \mathcal{L}(X)$  through  $\xi$  as in Sect. 5.1.1:*

$$\text{ord}_{\xi}(\mathcal{G}_{X_0, \varphi}^{(1)}) = \text{ord}_{\xi}(\mathcal{G}_{X_0, \varphi^{(d)}}^{(1)}).$$

*Proof* Assume that  $X$  is given by  $f$  as in (5.1). Let us suppose that  $\varphi$  is given by  $(\varphi_x, \varphi_{z_1}, \dots, \varphi_{z_d}) = (u_0t^{\alpha_0}, u_1t^{\alpha_1}, \dots, u_dt^{\alpha_d})$ , with  $u_0, \dots, u_d$  units in  $K[[t]]$  and  $\alpha_0, \dots, \alpha_d$  positive integers, and recall that, since  $\varphi \in \mathcal{L}(X)$ ,

$$\varphi(f) = \varphi \left( x^b + \sum_{i=0}^{b-2} B_i x^i \right) = 0.
 \tag{5.8}$$

By (5.7), it suffices to prove that

$$\alpha_0 \geq \text{ord}_{\xi}(\mathcal{G}_{X_0, \varphi^{(d)}}^{(1)}).
 \tag{5.9}$$

On the other hand, from Lemma 2.44 and diagram (5.2) we know that

$$\mathcal{G}_{\tilde{X}_0}^{(d+1)} = \text{Diff}(\mathcal{O}_{\tilde{V}_0^{(d+1)}, \xi_0^{(d+1)}}[B_i W^{b-i} : i = 0, \dots, b - 2]).$$

Denote

$$\mathcal{H} = \mathcal{O}_{\tilde{V}_0^{(d+1)}, \xi_0^{(d+1)}}[B_i W^{b-i} : i = 0, \dots, b - 2] \subset \mathcal{G}_{\tilde{X}_0}^{(d+1)}.$$

The inclusion holds after restricting both algebras to  $C_0^{(d)}$ , and hence

$$\text{ord}_\xi(\varphi^{(d)}(\mathcal{H})) \geq \text{ord}_\xi(\varphi^{(d)}(\mathcal{G}_{\tilde{X}_0}^{(d+1)})) = \text{ord}_\xi(\mathcal{G}_{X_0, \varphi^{(d)}}^{(1)}).$$

We will show now that

$$\alpha_0 \geq \text{ord}_\xi(\varphi^{(d)}(\mathcal{H})), \tag{5.10}$$

which implies (5.9). On the contrary, let us suppose that

$$\alpha_0 < \text{ord}_\xi(\varphi^{(d)}(\mathcal{H})) = \min_{i=0, \dots, b-2} \left\{ \frac{\text{ord}_t \varphi^{(d)}(B_i)}{b-i} \right\}.$$

That is,

$$\alpha_0 < \left( \frac{\text{ord}_t(\varphi^{(d)}(B_i))}{b-i} \right), \quad \text{for } i = 0, \dots, b - 2,$$

or equivalently

$$(b-i)\alpha_0 < \text{ord}_t(\varphi^{(d)}(B_i)), \quad \text{for } i = 0, \dots, b - 2. \tag{5.11}$$

Now observe that this implies

$$\varphi(f - x^b) = \text{ord}_t \left( \sum_{i=0}^{b-2} \varphi^{(d)}(B_i) \right) u_0^i t^{i\alpha_0} \geq \min_{i=0, \dots, b-2} \{ \text{ord}_t(\varphi^{(d)}(B_i)) + i \cdot \alpha_0 \} > b \cdot \alpha_0.$$

But this contradicts (5.8), so necessarily (5.10) holds, concluding the proof of the Lemma.  $\square$

We know now that we can just focus on the projection of  $X$  over  $S$ , for the computation of the order of contact. We need to know now how the induced projection of arcs (5.5) behaves.

**Definition 5.2** We say that an arc  $\varphi \in \mathcal{L}(V^{(d)})$  through  $\xi^{(d)} \in V^{(d)}$  is a *diagonal arc* if there exists a regular system of parameters  $\{z_1, \dots, z_d\} \in \mathcal{O}_{V^{(d)}, \xi^{(d)}}$ , units  $u_1, \dots, u_d \in K[[t]]$  and  $\alpha \in \mathbb{N}$  such that  $\varphi(z_i) = u_i t^\alpha$  for  $i = 1, \dots, d$ .

*Remark 5.3* The following definition is equivalent to the previous one: We say that an arc  $\varphi \in \mathcal{L}(V^{(d)})$  through  $\xi^{(d)} \in V^{(d)}$  is a diagonal arc if there exists a regular system of parameters  $\{z_1, \dots, z_d\} \in \mathcal{O}_{V^{(d)}, \xi^{(d)}}$  inducing a diagram

$$\begin{array}{ccccc}
 0 & \longrightarrow & \text{Ker}(\Gamma_0) & \longrightarrow & \mathcal{O}_{V_0^{(d+1)}, \xi_0^{(d+1)}} & & \\
 & & & & \uparrow p^* & \searrow \Gamma_0 & \\
 0 & \longrightarrow & \text{Ker}(\varphi) & \longrightarrow & \mathcal{O}_{V^{(d)}, \xi^{(d)}} & \xrightarrow{\varphi} & K[[t]] \\
 & & & & \downarrow \delta & \nearrow \tilde{\Gamma}_0 & \\
 0 & \longrightarrow & \text{Ker}(\tilde{\Gamma}_0) & \longrightarrow & (\mathcal{O}_{V^{(n)}, \xi} \otimes_k K[[t]])_{\xi_0} & & 
 \end{array} \tag{5.12}$$

where the ideal  $\text{Ker}(\tilde{\Gamma}_0) \subset (\mathcal{O}_{V^{(n)}, \xi} \otimes_k K[[t]])_{\xi_0}$  is generated by elements of the form  $(u_j z_i - u_i z_j)$ , where  $u_l \in K[[t]]$  are units for  $l = 1, \dots, d$ .<sup>17</sup>

*Remark 5.4* Let  $\varphi$  and  $\varphi'$  be two arcs in  $\mathcal{L}(V^{(d)})$  through  $\xi \in V^{(d)}$  whose respective graphs are  $\Gamma_0$  and  $\Gamma'_0$ . If  $\varphi$  is diagonal and  $\text{Ker}(\Gamma_0) = \text{Ker}(\Gamma'_0)$ , then  $\varphi'$  is also diagonal. Moreover, since  $\varphi$  is given by  $\varphi(z_i) = u_i t^\alpha$  for some regular system of parameters  $\{z_1, \dots, z_d\}$ , where  $u_1, \dots, u_d$  are units in  $K[[t]]$  and  $\alpha$  is some positive integer, then  $\varphi'$  is given as  $\varphi'(z_i) = u_i g'(t)$  for some  $g'(t) \in K[[t]]$ .

**Lemma 5.5** *Let  $X$  be as in Sect. 5.1.1 and let  $\varphi^{(d)}$  be an arc in  $V^{(d)}$  through  $\xi^{(d)} \in V^{(d)}$ . Then*

$$\text{ord}_\xi(\mathcal{G}_{X_0, \varphi^{(d)}}^{(1)}) \geq \text{ord}_\xi(\mathcal{G}_X^{(d)}) \cdot \text{ord}(\varphi^{(d)}). \tag{5.13}$$

*Proof* Suppose, contrary to our claim, that  $\text{ord}_\xi(\mathcal{G}_{X_0, \varphi^{(d)}}^{(1)}) < \text{ord}_\xi(\mathcal{G}_X^{(d)}) \cdot \alpha$ , where  $\alpha = \text{ord}(\varphi^{(d)})$ . Let  $\varphi^{(d)}$  be given by  $\varphi^{(d)}(z_i) = u_i t^{\alpha_i}$  for some regular system of parameters  $\{z_1, \dots, z_d\}$  in  $\mathcal{O}_{V^{(d)}, \xi^{(d)}}$ , units  $u_1, \dots, u_d \in K[[t]]$  and positive integers  $\alpha_1, \dots, \alpha_d$ . Then for some  $qW^l \in \mathcal{G}_X^{(d)}$ ,

$$\frac{\text{ord}_t(\varphi^{(d)}(q))}{l} < \text{ord}_\xi(\mathcal{G}_X^{(d)}) \cdot \alpha. \tag{5.14}$$

But  $\text{ord}_t(\varphi^{(d)}(q)) \geq \alpha \cdot \text{ord}_\xi(q)$ , and hence

$$\frac{\text{ord}_t(\varphi^{(d)}(q))}{l} \geq \frac{\alpha \cdot \text{ord}_\xi(q)}{l} \geq \alpha \cdot \text{ord}_\xi(\mathcal{G}_X^{(d)}),$$

leading to a contradiction, and proving the Lemma. □

Note that in the Lemma  $\varphi^{(d)}$  is any arc in  $\mathcal{L}_\xi(V^{(d)})$ , not necessarily the projection of any arc  $\varphi \in \mathcal{L}_\xi(X)$ .

**Definition 5.6** Let  $\mathcal{G}^{(d)}$  be a Rees algebra over  $V^{(d)}$ . We say that an arc  $\varphi^{(d)} \in \mathcal{L}_\xi(V^{(d)})$  is *generic* for  $\mathcal{G}^{(d)}$  if

$$\text{ord}_\xi \left( (\mathcal{G}^{(d)} \odot \mathcal{G}_{\varphi^{(d)}}^{(d+1)}) \Big|_{C_0^{(d)}} \right) = \text{ord}(\varphi^{(d)}) \cdot \text{ord}_\xi(\mathcal{G}^{(d)}).$$

If  $\varphi^{(d)}$  is also diagonal, we say that it is *diagonal-generic*.

*Remark 5.7* In the situation of Lemma 5.5, an arc for which (5.13) is an equality is a generic arc for  $\mathcal{G}_X^{(d)}$ :  $\mathcal{G}_X^{(d)} \odot \mathcal{G}_{\varphi^{(d)}}^{(d+1)} \Big|_{C_0^{(d)}} = \mathcal{G}_{X_0, \varphi^{(d)}}^{(d+1)} \Big|_{C_0^{(d)}} = \mathcal{G}_{X_0, \varphi^{(d)}}^{(1)}$  shows it. Note that such an arc can always be found, by just considering a diagonal arc  $\varphi^{(d)}$  in  $V^{(d)}$  through  $\xi^{(d)} \in V^{(d)}$  given, in some regular system of parameters  $\{z_1, \dots, z_d\}$ , by  $(u_1 t^\alpha, \dots, u_d t^\alpha)$ , for some positive integer  $\alpha$  and units  $u_1, \dots, u_d \in k$  such that there exists some element  $qW^l \in \mathcal{G}_X^{(d)}$  with  $\frac{\text{ord}_t(q)}{l} = \text{ord}_\xi(\mathcal{G}_X^{(d)})$ , and for which<sup>18</sup>  $(\text{in}_\xi(q))(u_1, \dots, u_d) \neq 0$ . For this arc,

$$\text{ord}_t(\varphi^{(d)}(q)) = \alpha \cdot \text{ord}_\xi(q),$$

<sup>17</sup> Note that  $\text{Ker}(\tilde{\Gamma}_0) = \text{Ker}(\Gamma_0)(\mathcal{O}_{V^{(n)}, \xi} \otimes_k K[[t]])_{\xi_0}$ .

<sup>18</sup> If  $q \in R$  for a regular local ring  $R$  with maximal ideal  $\mathcal{M}$ , then we denote by  $\text{in}_\xi(q)$  the *initial part* of  $q$  at the closed point  $\xi$ , meaning the equivalence class of  $q$  in the quotient  $\mathcal{M}^n / \mathcal{M}^{n+1}$ , where  $n$  is such that  $q \in \mathcal{M}^n$  but  $q \notin \mathcal{M}^{n+1}$ . Therefore  $\text{in}_\xi(q) \in \text{Gr}_{R, \mathcal{M}} \cong k[z_1, \dots, z_d]$  is a homogeneous polynomial of degree  $n$ .

and hence

$$\text{ord}_\xi(\mathcal{G}_{X_0, \varphi^{(d)}}^{(1)}) \leq \frac{\text{ord}_l(\varphi^{(d)}(q))}{l} = \frac{\alpha \cdot \text{ord}_\xi(q)}{l} = \alpha \cdot \text{ord}_\xi(\mathcal{G}_X^{(d)}),$$

but Lemma 5.5 forces the last inequality to be an equality.

Even though in this section we are always under the assumption of  $X$  being locally a hypersurface, the following Lemma will be stated and proved for a variety of arbitrary codimension, since no extra work is needed and this generality will be necessary in the next section.

**Lemma 5.8** *Let  $X$  be a variety of dimension  $d$  over  $k$ . With the notation from Sect. 5.1.1, let  $\bar{\varphi}^{(d)}$  be a diagonal arc in  $V^{(d)}$  through  $\xi^{(d)} \in V^{(d)}$  which is diagonal-generic for  $\mathcal{G}_X^{(d)}$ . Then it is possible to find an arc  $\varphi \in \mathcal{L}(X)$  through  $\xi$  whose projection  $\varphi^{(d)}$  onto  $V^{(d)}$  via  $\beta_X$  is a diagonal arc which is also diagonal-generic for  $\mathcal{G}_X^{(d)}$ .*

*Proof* Consider a local presentation as in Example 2.38 for  $X$  at  $\xi$  attached to the multiplicity. Let us recall that not every arc in  $\{f_1 = \dots = f_{n-d} = 0\}$  is an arc in  $X$ , since

$$(f_1, \dots, f_{n-d}) \subset I(X) \implies X \subset \{f_1 = \dots = f_{n-d} = 0\}.$$

Assume that  $\bar{\varphi}^{(d)}(z_i) = u_i t^\alpha, i = 1, \dots, d$  for some units  $u_1, \dots, u_d \in K[[t]]$ . We need to choose an arc  $\varphi$  such that  $\varphi \in \mathcal{L}(V(f))$  for all  $f \in I(X)$ , or equivalently an arc such that  $\text{Ker}(\varphi) \supset I(X)$ . Consider the following diagram

$$\begin{array}{ccc} \mathcal{O}_{X, \xi} \cong \mathcal{O}_{V^{(d)}, \xi^{(d)}}[x_1, \dots, x_{n-d}] / I(X) & \longleftarrow & \mathcal{O}_{V^{(d)}, \xi^{(d)}}[x_1, \dots, x_{n-d}] \\ & \swarrow \beta_X^* & \nearrow \beta^* \\ & \mathcal{O}_{V^{(d)}, \xi^{(d)}} & \end{array}$$

where  $\beta_X^*$  (induced by  $\beta_X$  from (2.15)) is a finite morphism. Let  $\mathcal{P} = \text{Ker}(\bar{\varphi}^{(d)}) \subset \mathcal{O}_{V^{(d)}, \xi^{(d)}}$ . There is a prime ideal  $\mathcal{Q} \subset \mathcal{O}_{X, \xi}$  such that  $\mathcal{Q} \cap \mathcal{O}_{V^{(d)}, \xi^{(d)}} = \mathcal{P}$ . Note that  $\mathcal{Q}$  is lifted to a unique ideal  $\mathcal{Q}' \subset \mathcal{O}_{V^{(d)}, \xi^{(d)}}[x_1, \dots, x_{n-d}]$ , with the property that  $I(X) \subset \mathcal{Q}'$ . We have the following diagram

$$\begin{array}{ccc} \mathcal{Q} \subset \mathcal{O}_{X, \xi} & \longrightarrow & \mathcal{O}_{X, \xi} / \mathcal{Q} \\ \beta_X^* \uparrow & & \uparrow \\ \mathcal{P} \subset \mathcal{O}_{V^{(d)}, \xi^{(d)}} & \longrightarrow & \mathcal{O}_{V^{(d)}, \xi^{(d)}} / \mathcal{P} \end{array}$$

where the left vertical arrow is a finite morphism, forcing the right vertical one to be also finite. Then, the two rings in the right side of the diagram have the same dimension, and thus  $\mathcal{Q}$  defines a closed set of dimension 1 in  $X, C$ . There is an arc  $\varphi$  (different from the morphism 0) in  $C$  through  $\xi$ , and we know that, locally in a neighbourhood of  $\xi$ ,  $\mathcal{Q} = \text{Ker}(\varphi)$  and that  $\text{Ker}(\varphi) \cap \mathcal{O}_{V^{(d)}, \xi^{(d)}} = \text{Ker}(\varphi^{(d)}) = \text{Ker}(\bar{\varphi}^{(d)})$ , so the projection of  $\varphi$  onto  $V^{(d)}$ ,  $\varphi^{(d)}$ , is diagonal by Remark 5.4. To see that it is generic for  $\mathcal{G}_X^{(d)}$ , note that there exists some element  $qW^l \in \mathcal{G}_X^{(d)}$  with  $\frac{\text{ord}_\xi(q)}{l} = \text{ord}_\xi(\mathcal{G}_X^{(d)})$  for which  $(\text{in}_\xi(q))(u_1, \dots, u_d) \neq 0$ . By passing to



the completion of  $(\mathcal{O}_{V^{(n)},\xi} \otimes_k K[[t]])_{\xi_0}$  at its maximal ideal (see Remark 5.3) and using Remark 5.4, it can be checked that this implies that  $\varphi^{(d)}$  is also generic for  $\mathcal{G}_X^{(d)}$ .  $\square$

*Remark 5.9* The arc obtained in Lemma 5.8 is given (as in (4.5)) by

$$\varphi = (g_1(t), \dots, g_{n-d}(t), u_1 g'(t), \dots, u_d g'(t)) \tag{5.15}$$

for some  $g_1(t), \dots, g_{n-d}(t), g'(t) \in K[[t]]$  and  $u_1, \dots, u_d \in K[[t]]$ , because  $\text{Ker}(\varphi) \cap \mathcal{O}_{V^{(d)},\xi^{(d)}} = \text{Ker}(\tilde{\varphi}^{(d)}) = \text{Ker}(\varphi^{(d)})$  and  $\varphi^{(d)}$  is diagonal (see Remark 5.4).

### 5.1.3 Results for hypersurfaces

Now we return to the hypersurface case, and we have enough tools to prove the following theorem:

**Theorem 5.10** *Let  $X$  be a variety of dimension  $d$  which is locally a hypersurface at  $\xi \in \text{Max mult}(X)$ . For any  $\varphi \in \mathcal{L}(X)$  through  $\xi$ , with the notation from Sect. 5.1.1,*

$$\bar{r}_{X,\varphi} \geq \text{ord}_\xi(\mathcal{G}_X^{(d)}). \tag{5.16}$$

*Proof* We can assume that  $X$  is given locally by  $f$  is as in (5.1). Let us write  $\alpha = \text{ord}(\varphi) = \min\{\alpha_0, \dots, \alpha_d\}$ . From Lemma 5.5, for any diagonal arc  $\tilde{\varphi}$ , given as  $(\tilde{u}_0 t^\alpha, \dots, \tilde{u}_d t^\alpha)$

$$\alpha \cdot \text{ord}_\xi(\mathcal{G}_X^{(d)}) \leq \text{ord}_\xi(\mathcal{G}_{X_0, \tilde{\varphi}^{(d)}}^{(1)}).$$

It suffices to show that it is possible to choose units  $\tilde{u}_i \in K[[t]]$  for  $i = 0, \dots, d$  so that

$$\text{ord}_\xi(\mathcal{G}_{X_0, \tilde{\varphi}^{(d)}}^{(1)}) \leq \text{ord}_\xi(\mathcal{G}_{X_0, \varphi^{(d)}}^{(1)}). \tag{5.17}$$

This, together with Lemma 5.1, would imply that

$$\alpha \cdot \text{ord}_\xi(\mathcal{G}_X^{(d)}) \leq \text{ord}_\xi(\mathcal{G}_{X_0, \varphi^{(d)}}^{(1)}) = \text{ord}_\xi(\mathcal{G}_{X_0, \varphi}^{(1)}),$$

and complete the proof of the Theorem.

In order to prove (5.17), let us consider a finite set of generators of  $\mathcal{G}_X^{(d)}$ ,  $\{g_i W^{l_i}\}_{i=1, \dots, r}$ . Since this set is finite and  $k$  is infinite, it is possible to choose units  $\tilde{u}_1, \dots, \tilde{u}_d \in k$  in a way such that

$$\text{in}_\xi(g_i)(\tilde{u}_1, \dots, \tilde{u}_d) \neq 0, \quad \text{for } i = 1, \dots, r.$$

Let  $\lambda_i = \text{ord}_\xi(g_i)$  for  $i = 1, \dots, r$ . As  $\text{in}_\xi(g_i)$  is a homogeneous polynomial,

$$\text{in}_\xi(\tilde{\varphi}^{(d)}(g_i)) = t^{\alpha \cdot \lambda_i} \cdot \text{in}_\xi(g_i)(\tilde{u}_1, \dots, \tilde{u}_d)$$

and

$$\text{ord}_t(\tilde{\varphi}^{(d)}(g_i)) = \alpha \cdot \lambda_i.$$

On the other hand, observe that

$$\varphi^{(d)}(g_i) \in \langle t^{\alpha \cdot \lambda_i} \rangle,$$

so

$$\text{ord}_t(\varphi^{(d)}(g_i)) \geq \alpha \cdot \lambda_i = \text{ord}_t(\tilde{\varphi}^{(d)}(g_i)). \tag{5.18}$$

Since (5.18) holds for all  $i \in \{1, \dots, r\}$ , and for some  $k \in \{1, \dots, r\}$ ,

$$\frac{\text{ord}_t(\varphi^{(d)}(g_k))}{l_k} = \text{ord}_\xi(\mathcal{G}_{X_0, \varphi^{(d)}}^{(1)}),$$

it follows that

$$\text{ord}_\xi(\mathcal{G}_{X_0, \varphi^{(d)}}^{(1)}) = \frac{\text{ord}_t(\varphi^{(d)}(g_k))}{l_k} \geq \frac{\text{ord}_t(\tilde{\varphi}^{(d)}(g_k))}{l_k} \geq \text{ord}_\xi(\mathcal{G}_{X_0, \tilde{\varphi}^{(d)}}^{(1)})$$

concluding the proof of (5.17), and the proof of the Theorem. □

For the proof of the existence of an arc giving an equality in (5.16), we will use the following Lemma:

**Lemma 5.11** *Let  $X$  be as in Sect. 5.1.1, and let  $\varphi$  be an arc in  $X$  through  $\xi \in \text{Max mult}(X)$  with the notation used there where  $\varphi(x) = g_1(t)$  and  $\varphi(z_i) = u_i g^t(t)$ ,  $u_i$  a unit in  $K[[t]]$ , for  $i = 1, \dots, d$ . Assume that  $\varphi$  is such that the projection  $\varphi^{(d)}$  on  $V^{(d)}$  is a diagonal-generic arc for  $\mathcal{G}_X^{(d)}$ .<sup>19</sup> If  $\text{ord}(\varphi) = \text{ord}_t(g_1(t))$ , then*

$$\bar{r}_{X, \varphi} = \text{ord}_\xi(\mathcal{G}_X^{(d)}) = 1.$$

*Proof* Let us suppose that  $g^t(t) = t^L$  for some positive integer  $L$ , that is,  $\varphi_{z_i} = u_i t^L$  for  $i = 1, \dots, d$ . By Lemma 5.1,

$$\text{ord}_\xi(\mathcal{G}_{X_0, \varphi}^{(1)}) = \text{ord}_\xi(\mathcal{G}_{X_0, \varphi^{(d)}}^{(1)}),$$

and since  $\varphi^{(d)}$  is generic for  $\mathcal{G}_X^{(d)}$ , Remark 5.7 yields

$$\text{ord}_\xi(\mathcal{G}_{X_0, \varphi}^{(1)}) = L \cdot \text{ord}_\xi(\mathcal{G}_X^{(d)}). \tag{5.19}$$

It suffices to prove that

$$\text{ord}_t(g_1(t)) \geq L \cdot \text{ord}_\xi(\mathcal{G}_X^{(d)}), \tag{5.20}$$

since it implies

$$1 \leq \text{ord}_\xi(\mathcal{G}_X^{(d)}) \leq \bar{r}_{X, \varphi} = \frac{L \cdot \text{ord}_\xi(\mathcal{G}_X^{(d)})}{\text{ord}_t(g_1(t))} \leq 1, \tag{5.21}$$

where we have used Theorem 5.10 for the second inequality and (5.19) together with the definition of  $\bar{r}_{X, \varphi}$  for the equality. Hence  $\text{ord}_\xi(\mathcal{G}_X^{(d)}) = \bar{r}_{X, \varphi} = 1$ , concluding the proof of the Lemma. In order to prove (5.20), let us suppose that our claim is false, that is:

$$\text{ord}_t(g_1(t)) < L \cdot \text{ord}_\xi(\mathcal{G}_X^{(d)}). \tag{5.22}$$

Then, in particular,

$$\text{ord}_t(g_1(t)) < L \cdot \frac{\text{ord}_\xi(B_i)}{b-i} \leq \frac{\text{ord}_t(\varphi^{(d)}(B_i))}{b-i} \text{ for } i = 0, \dots, b-2 \tag{5.23}$$

where the first inequality follows from the same argument used in the proof of Lemma 5.1. Therefore

$$\text{ord}_t(\varphi^{(d)}(B_i)) > \text{ord}_t(g_1(t))(b-i)$$

---

<sup>19</sup> We know that such an arc exists by Remark 5.9.

and

$$\begin{aligned} \text{ord}_t(\varphi(f - x^b)) &= \text{ord}_t\left(\sum_{i=0}^{b-2} \varphi^{(d)}(B_i)g_1(t)^i\right) \geq \min_{i=0, \dots, b-2} \{\text{ord}_t(\varphi^{(d)}(B_i)) + i \cdot \text{ord}_t(g_1(t))\} \\ &> \min_{i=0, \dots, b-2} \{\text{ord}_t(g_1(t))(b - i) + i \cdot \text{ord}_t(g_1(t))\} = b \cdot \text{ord}_t(g_1(t)), \end{aligned}$$

where (5.23) is needed for the second inequality. But this contradicts  $\varphi(f) = 0$  and hence the fact that  $\varphi \in \mathcal{L}_\xi(X)$ , so necessarily (5.20) holds, concluding the proof.  $\square$

**Theorem 5.12** *Let  $X$  be a  $d$ -dimensional variety over a field  $k$  of characteristic zero which is locally a hypersurface in a neighbourhood of  $\xi \in \text{Max mult}(X)$ . Then there exists some  $\varphi \in \mathcal{L}(X)$  through  $\xi$ , with the notation from Sect. 5.1.1 such that*

$$\bar{r}_{X,\varphi} = \text{ord}_\xi(\mathcal{G}_X^{(d)}). \tag{5.24}$$

*Proof* We can assume again that  $X$  is locally given by  $f$  as in (5.1). Pick a diagonal-generic arc for  $\mathcal{G}_X^{(d)}$  (see Remark 5.7 for the existence). By Lemma 5.8 it can be lifted to an arc  $\varphi$  in  $X$  through  $\xi$  whose projection  $\varphi^{(d)}$  onto  $V^{(d)}$  is diagonal generic for  $\mathcal{G}_X^{(d)}$ . Remark 5.9 shows that  $\varphi$  is given (as in (4.5)) by

$$(g(t), u_1g'(t), \dots, u_dg'(t)) \tag{5.25}$$

for some  $g(t), g'(t) \in K[[t]]$  and  $u_1, \dots, u_d \in k$ . We only need to check that for such an arc (5.24) holds. Let  $N = \text{ord}_t(g'(t))$ . Note that, since  $\varphi^{(d)}$  is generic for  $\mathcal{G}_X^{(d)}$ ,  $\text{ord}_\xi(\mathcal{G}_{X_0,\varphi^{(d)}}^{(1)}) = N \cdot \text{ord}_\xi(\mathcal{G}_X^{(d)})$ . By Lemma 5.1,

$$\text{ord}_\xi(\mathcal{G}_{X_0,\varphi}^{(1)}) = N \cdot \text{ord}_\xi(\mathcal{G}_X^{(d)}). \tag{5.26}$$

Consider now two possible situations, depending on whether  $\text{ord}(\varphi) = \text{ord}_t(g(t))$  or not. If  $\text{ord}(\varphi) = \text{ord}_t(g(t))$ , then Lemma 5.11 implies

$$1 = \text{ord}_\xi(\mathcal{G}_X^{(d)}) = \bar{r}_{X,\varphi}.$$

Otherwise  $\text{ord}(\varphi) = N$ , and by definition of  $\bar{r}_{X,\varphi}$  and (5.26),  $\bar{r}_{X,\varphi} = \frac{N \cdot \text{ord}_\xi(\mathcal{G}_X^{(d)})}{N}$ , completing the proof.  $\square$

*Remark 5.13* Under the assumptions of Theorem 5.12, let  $\varphi$  be the arc (5.25) given by the proof. For this arc

$$\text{ord}(\varphi) = N. \tag{5.27}$$

To see this we observe that, since we have proved that  $\bar{r}_{X,\varphi} = \text{ord}_\xi(\mathcal{G}_X^{(d)})$ , it follows easily from (5.26) that:

$$\text{ord}_\xi(\mathcal{G}_X^{(d)}) = \bar{r}_{X,\varphi} = \frac{\text{ord}_\xi(\mathcal{G}_{X_0,\varphi}^{(1)})}{\text{ord}(\varphi)} = \frac{N \cdot \text{ord}_\xi(\mathcal{G}_X^{(d)})}{\text{ord}(\varphi)} \Rightarrow \frac{N}{\text{ord}(\varphi)} = 1.$$

### 5.2 Rees algebras and orders for the general case

As we have just done for the proof of Theorem 4.10 for hypersurfaces, we will use that we can find, in an étale neighbourhood of each point  $\xi$  of  $X$ , a local presentation (as in Example 2.38) given by a collection of hypersurfaces and integers. For each of these hypersurfaces

we will assume a nice expression in the line of Sect. 5.1.1. As a consequence, for any arc  $\varphi$  in  $X$  through  $\xi$  we will be able to give an expression of the algebra of contact of  $\varphi$  with  $\text{Max mult}(X)$  in terms of some algebras of contact of arcs with hypersurfaces. This will lead to an easy formula for  $r_{X,\varphi}$ . With these tools, we will prove in Theorem 5.17 that  $\text{ord}_\xi \mathcal{G}_X^{(d)}$  is again a lower bound for  $\bar{r}_{X,\varphi}$  for any arc  $\varphi$ , and that  $\bar{r}_X$  is also a minimum in this case in Theorem 5.19. They will come naturally from Theorems 5.10 and 5.12, respectively.

5.2.1 Notation and hypothesis for the general case

Let  $X$  be a variety of dimension  $d$ , and let  $\xi$  be a point in  $\text{Max mult}(X)$ . We already explained in Example 2.38 that, in an étale neighbourhood of  $\xi$ , we can find a local presentation for  $X$  attached to the multiplicity, meaning an immersion in  $V^{(n)}$ , elements  $f_i \in \mathcal{O}_{V^{(n)},\xi} = \mathcal{O}_{V^{(d),\xi^{(d)}}}[x_1, \dots, x_{n-d}]$  and positive integers  $b_i$  for  $i = 1, \dots, n - d$  as in (2.17), such that

$$\mathcal{G}_X^{(n)} = \text{Diff}(\mathcal{O}_{V^{(n)},\xi}[f_1 W^{b_1}, \dots, f_{n-d} W^{b_{n-d}}]) \tag{5.28}$$

represents the function  $\text{mult}(X)$ . Consider the differential closure of the  $\mathcal{O}_{V_0^{(n+1)},\xi_0^{(n+1)}}$ -Rees algebra generated by the  $f_i, \mathcal{G}_{X_0}^{(n+1)}$ . We already mentioned that  $f_i$  is the minimal polynomial of  $\theta_i$  over  $\mathcal{O}_{V^{(d),\xi^{(d)}}$ , where  $\mathcal{O}_{X,\xi} = \mathcal{O}_{V^{(d),\xi^{(d)}}}[\theta_1, \dots, \theta_{n-d}]$ , and we can assume (by 2.37) that each  $f_i$  is of the form:

$$f_i = x_i^{b_i} + B_{\{i\},b_i-2} x_i^{b_i-2} + \dots + B_{\{i\},0} \in \mathcal{O}_{V^{(d),\xi^{(d)}}}[x_i] \subset \mathcal{O}_{V^{(d),\xi^{(d)}}}[x_1, \dots, x_{n-d}],$$

where  $\{z_1, \dots, z_d, t\}$  is a regular system of parameters in  $\mathcal{O}_{V_0^{(d+1)},\xi_0}$  and  $\{x_1, \dots, x_{n-d}, z_1, \dots, z_d, t\}$  a regular system of parameters in  $(\mathcal{O}_{V_{i,0}^{(e),\xi}} \otimes_k K[[t]])_{\xi_0}$ ,  $B_{\{i\},b_i-j} \in \mathcal{O}_{V^{(d),\xi^{(d)}}$  and  $\text{ord}_\xi(B_{\{i\},b_i-j}) \geq j$  for  $j = 2, \dots, b_i, i = 1, \dots, n - d$ .

By Example 2.38, we know that

$$\begin{aligned} \mathcal{G}_X^{(n)} &= \text{Diff}(\mathcal{O}_{V^{(n)},\xi}[f_1 W^{b_1}, \dots, f_{n-d} W^{b_{n-d}}]) \\ &= \text{Diff}(\mathcal{O}_{V^{(n)},\xi}[f_1 W^{b_1}]) \odot \dots \odot \text{Diff}(\mathcal{O}_{V^{(n)},\xi}[f_{n-d} W^{b_{n-d}}]), \end{aligned}$$

where each  $\text{Diff}(\mathcal{O}_{V^{(n)},\xi}[f_i W^{b_i}])$  is the smallest differentially closed  $\mathcal{O}_{V^{(n)},\xi}$ -Rees algebra with the property of containing the algebra  $\text{Diff}(\mathcal{O}_{V^{(d),\xi^{(d)}}}[x_i][f_i W^i])$ , since  $f_i \in \mathcal{O}_{V^{(d),\xi^{(d)}}}[x_i]$ . Therefore we can write

$$\mathcal{G}_X^{(n)} = \text{Diff}(\mathcal{O}_{V^{(d),\xi^{(d)}}}[x_1][f_1 W^{b_1}]) \odot \dots \odot \text{Diff}(\mathcal{O}_{V^{(d),\xi^{(d)}}}[x_{n-d}][f_{n-d} W^{b_{n-d}}]). \tag{5.29}$$

Observe that, for each  $f_i, H_i = \{f_i = 0\}$  is a hypersurface in  $V_i^{(e)} = \text{Spec}(\mathcal{O}_{V^{(d),\xi^{(d)}}}[x_i])$ , where  $e = d + 1$ . Using the hypersurface case, the Rees algebra

$$\mathcal{G}_{H_i}^{(e)} = \text{Diff}(\mathcal{O}_{V^{(d),\xi^{(d)}}}[x_i][f_i W^{b_i}]) = \mathcal{O}_{V^{(d),\xi^{(d)}}}[x_i][f_i W] \odot \mathcal{G}_{H_i}^{(d)} \tag{5.30}$$

represents  $\text{mult}(H_i)$  (see Remark 2.45).

Remark 5.14 Using (5.29) we can rewrite  $\mathcal{G}_X^{(n)}$  in terms of the  $\mathcal{G}_{H_i}^{(e)}$  for  $i = 1, \dots, n - d$ :

$$\begin{aligned} \mathcal{G}_X^{(n)} &= \mathcal{G}_{H_1}^{(e)} \odot \dots \odot \mathcal{G}_{H_{n-d}}^{(e)} \\ &= \mathcal{O}_{V^{(d),\xi^{(d)}}}[x_1][x_1 W] \odot \mathcal{G}_{H_1}^{(d)} \odot \dots \odot \mathcal{O}_{V^{(d),\xi^{(d)}}}[x_{n-d}][x_{n-d} W] \odot \mathcal{G}_{H_{n-d}}^{(d)} \\ &= \mathcal{O}_{V^{(n),\xi}}[x_1 W, \dots, x_{n-d} W] \odot \mathcal{G}_{H_1}^{(d)} \odot \dots \odot \mathcal{G}_{H_{n-d}}^{(d)}. \end{aligned} \tag{5.31}$$

If one goes back to diagram (5.2), using the factorization

$$\begin{array}{ccc}
 \mathcal{O}_{V^{(n)}, \xi} & \xrightarrow{\varphi} & K[[t]] \\
 \uparrow \beta^* & \swarrow & \uparrow \varphi_i \\
 & \mathcal{O}_{V_i^{(e)}, \xi} = \mathcal{O}_{V^{(d)}, \xi^{(d)}}[x_i] & \\
 \mathcal{O}_{V^{(d)}, \xi^{(d)}} & \nearrow & 
 \end{array} \tag{5.32}$$

one can consider also the Rees algebras  $\mathcal{G}_{H_{i,0}}^{(e+1)}$  and  $\mathcal{G}_{H_{i,0}}^{(e+1)}$  induced by  $\mathcal{G}_{H_i}^{(e)}$  over  $\mathcal{O}_{V_{i,0}^{(e+1)}, \xi_0} = \mathcal{O}_{V_0^{(d+1)}, \xi_0^{(d+1)}}[x_i]$  and  $(\mathcal{O}_{V_{i,0}^{(e)}, \xi} \otimes_k K[[t]])_{\xi_0}$  respectively.

Consider now an arc  $\varphi \in \mathcal{L}(X)$  through  $\xi$ , and the  $\mathcal{O}_{\tilde{V}_0^{(n+1)}, \xi_0}$ -Rees algebra  $\mathcal{G}_{X_0, \varphi}^{(n+1)}$  of contact of  $\varphi$  with Max  $\text{mult}(X)$ . Let us suppose that  $\varphi$  is given by  $(\varphi_{x_1}, \dots, \varphi_{x_{n-d}}, \varphi_{z_1}, \dots, \varphi_{z_d})$  as in (4.5). At the same time, for  $i = 1, \dots, n-d$ , the projection of  $\varphi$  onto  $V_i^{(e)}$  by (5.32) is an arc  $\varphi_i$  given by  $(\varphi_{x_i}, \varphi_{z_1}, \dots, \varphi_{z_d})$  in  $\mathcal{L}(H_i)$ . Therefore we can define

$$\begin{aligned}
 \mathcal{G}_{H_{i,0}, \varphi_i}^{(e+1)} &= \text{Diff}((\mathcal{O}_{V^{(n)}, \xi} \otimes_k K[[t]])_{\xi_0} [f_i W^{b_i}, h_i W, h_{n-d+1} W, \dots, h_n W]) \\
 &= \mathcal{G}_{H_i}^{(e)} \odot \mathcal{G}_{\varphi_i}^{(e+1)},
 \end{aligned} \tag{5.33}$$

where  $h_i = x_i - \varphi_{x_i}$  for  $i = 1, \dots, n-d$  and  $h_{n-d+j} = z_j - \varphi_{z_j}$  for  $j = 1, \dots, d$ , and

$$\begin{aligned}
 \mathcal{G}_{\varphi}^{(n+1)} &= (\mathcal{O}_{V_{i,0}^{(e)}, \xi} \otimes_k K[[t]])_{\xi_0} [h_1 W, \dots, h_n W] \\
 &= (\mathcal{O}_{V_{i,0}^{(e)}, \xi} \otimes_k K[[t]])_{\xi_0} [h_1 W, h_{n-d+1} W, \dots, h_n W] \odot \dots \odot \\
 &\quad (\mathcal{O}_{V_{i,0}^{(e)}, \xi} \otimes_k K[[t]])_{\xi_0} [h_{n-d} W, h_{n-d+1} W, \dots, h_n W] \\
 &= \mathcal{G}_{\varphi_1}^{(e+1)} \odot \dots \odot \mathcal{G}_{\varphi_{n-d}}^{(e+1)}.
 \end{aligned} \tag{5.34}$$

Now we can use the result for hypersurfaces in Theorem 5.10 to assert that, for  $i = 1, \dots, n-d$ ,

$$\frac{\text{ord}_{\xi}(\mathcal{G}_{H_{i,0}, \varphi_i}^{(1)})}{\text{ord}(\varphi_i)} \geq \text{ord}_{\xi}(\mathcal{G}_{H_i}^{(d)}).$$

Note that

$$\text{ord}(\varphi) = \min_{i=1, \dots, n-d} \{\text{ord}(\varphi_i)\}. \tag{5.35}$$

The following remark will be important for the generalization of Theorem 5.10.

*Remark 5.15* The Rees algebra  $\mathcal{G}_{X_0, \varphi}^{(n+1)}$  can be written in terms of the  $\mathcal{G}_{H_{i,0}, \varphi_i}^{(e+1)}$ , by (4.11), (5.31), (5.34) and (5.33):

$$\begin{aligned}
 \mathcal{G}_{X_0, \varphi}^{(n+1)} &= \mathcal{G}_X^{(n+1)} \odot \mathcal{G}_{\varphi}^{(n+1)} = \mathcal{O}_{V_0^{(n+1)}, \xi_0^{(n+1)}}[x_1 W, \dots, x_{n-d} W] \odot \mathcal{G}_{H_1}^{(d)} \odot \dots \odot \mathcal{G}_{H_{n-d}}^{(d)} \odot \mathcal{G}_{\varphi}^{(n+1)} = \\
 &= \mathcal{G}_{H_1}^{(e)} \odot \dots \odot \mathcal{G}_{H_{n-d}}^{(e)} \odot \mathcal{G}_{\varphi_1}^{(e+1)} \odot \dots \odot \mathcal{G}_{\varphi_{n-d}}^{(e+1)} = \mathcal{G}_{H_1}^{(e)} \odot \mathcal{G}_{\varphi_1}^{(e+1)} \odot \dots \odot \mathcal{G}_{H_{n-d}}^{(e)} \odot \mathcal{G}_{\varphi_{n-d}}^{(e+1)} = \\
 &= \mathcal{G}_{H_{1,0}, \varphi_1}^{(e+1)} \odot \dots \odot \mathcal{G}_{H_{n-d,0}, \varphi_{n-d}}^{(e+1)}.
 \end{aligned} \tag{5.36}$$

By expressing the algebras  $\mathcal{G}_{X_0}^{(n+1)}$  and  $\mathcal{G}_{X_0,\varphi}^{(n+1)}$  in terms of Rees algebras attached to hypersurfaces as we have done in (5.31) and (5.36), it is easy to establish a relation among the order of all Rees algebras involved in both cases, as the following Lemma states:

**Lemma 5.16** *Let  $X$  be a  $d$ -dimensional variety.*

1. *Let  $\mathcal{G}_X^{(n)}$  and  $\mathcal{G}_{H_i}^{(e)}$  be as in (5.28) and (5.30). Let  $\mathcal{G}_X^{(d)}$  and  $\mathcal{G}_{H_i}^{(d)}$  be respectively the elimination Rees algebras associated to their projection over  $V^{(d)}$ . Then*

$$\mathcal{G}_X^{(d)} = \mathcal{G}_{H_1}^{(d)} \odot \dots \odot \mathcal{G}_{H_{n-d}}^{(d)}, \tag{5.37}$$

and thus

$$\text{ord}_\xi(\mathcal{G}_X^{(d)}) = \min_{i=1,\dots,n-d} \{\text{ord}_\xi(\mathcal{G}_{H_i}^{(d)})\}. \tag{5.38}$$

2. *Let  $\mathcal{G}_{X_0,\varphi}^{(n+1)}$  and  $\mathcal{G}_{H_{i,0},\varphi_i}^{(e+1)}$  be as in (5.36) and (5.33). Let  $\mathcal{G}_{X_0,\varphi}^{(1)}$  and  $\mathcal{G}_{H_{i,0},\varphi_i}^{(1)}$  be respectively their restrictions to the curves defined by the arcs  $\varphi, \varphi_1, \dots, \varphi_{n-d}$  (as in Proposition 4.4). Then*

$$\mathcal{G}_{X_0,\varphi}^{(1)} = \mathcal{G}_{H_{1,0},\varphi_1}^{(1)} \odot \dots \odot \mathcal{G}_{H_{n-d,0},\varphi_{n-d}}^{(1)}. \tag{5.39}$$

As a consequence

$$\text{ord}_\xi(\mathcal{G}_{X_0,\varphi}^{(1)}) = \min_{i=1,\dots,n-d} \{\text{ord}_\xi(\mathcal{G}_{H_{i,0},\varphi_i}^{(1)})\}. \tag{5.40}$$

*Proof* Part (1) follows from the elimination of  $\mathcal{G}_X^{(n)}$  associated to the projection  $V^{(n)} \rightarrow V^{(d)}$ , using the expression in (5.31). For (2), one must note, by looking at the expression in (5.36), that the restriction of  $\mathcal{G}_{X_0,\varphi}^{(n+1)}$  to the curve defined by  $\varphi$  equals the smallest algebra containing the restrictions of the  $\mathcal{G}_{H_{i,0},\varphi_i}^{(e+1)} = \mathcal{O}_{V_0^{(n+1)},\xi_0}[x_i W] \odot \mathcal{G}_{H_i}^{(d)} \odot \mathcal{G}_{\varphi_i}^{(e+1)}$  to the respective curves defined by the  $\varphi_i$ , since all the Rees algebras are differentially closed.  $\square$

### 5.2.2 Results for the general case

**Theorem 5.17** *Let  $X$  be a variety as in Sect. 5.2.1, let  $\xi \in \text{Max mult}(X)$  and let  $\varphi$  be an arc in  $X$  through  $\xi$  with the notation used there. Then*

$$\bar{r}_{X,\varphi} \geq \text{ord}_\xi(\mathcal{G}_X^{(d)}). \tag{5.41}$$

*Proof* From (5.40) we obtain

$$\bar{r}_{X,\varphi} = \frac{\text{ord}_\xi(\mathcal{G}_{X_0,\varphi}^{(1)})}{\text{ord}(\varphi)} = \frac{\min_{i=1,\dots,n-d} \{\text{ord}_\xi(\mathcal{G}_{H_{i,0},\varphi_i}^{(1)})\}}{\text{ord}(\varphi)}.$$

For every  $i \in \{1, \dots, n - d\}$ , Theorem 5.10 gives

$$\frac{\text{ord}_\xi(\mathcal{G}_{H_{i,0},\varphi_i}^{(1)})}{\text{ord}(\varphi_i)} \geq \text{ord}_\xi(\mathcal{G}_{H_i}^{(d)}),$$

and this together with (5.35) and (5.38) implies

$$\frac{\text{ord}_\xi(\mathcal{G}_{H_{i,0},\varphi_i}^{(1)})}{\text{ord}(\varphi)} \geq \frac{\text{ord}_\xi(\mathcal{G}_{H_{i,0},\varphi_i}^{(1)})}{\text{ord}(\varphi_i)} \geq \text{ord}_\xi(\mathcal{G}_{H_i}^{(d)}) \geq \text{ord}_\xi(\mathcal{G}_X^{(d)}), \quad \forall i = 1, \dots, n - d.$$

As a consequence, we get

$$\bar{r}_{X,\varphi} = \frac{\min_{i=1,\dots,n-d}\{\text{ord}_\xi(\mathcal{G}_{H_i,0,\varphi_i}^{(1)})\}}{\text{ord}(\varphi)} \geq \text{ord}_\xi(\mathcal{G}_X^{(d)}),$$

concluding the proof of the Theorem. □

*Remark 5.18* If  $k$  is a field of characteristic zero, it is always possible to find a diagonal arc  $\bar{\varphi}^{(d)}$  which is diagonal-generic for  $\mathcal{G}_{H_i}^{(d)}$  for  $i = 1, \dots, n - d$ . As we did in Remark 5.7, one needs only to consider for each  $i \in \{1, \dots, n - d\}$ , an element  $p_i W^{l_i} \in \mathcal{G}_{H_i}^{(d)}$  such that  $\text{ord}_\xi(\mathcal{G}_{H_i}^{(d)}) = \frac{\text{ord}_\xi(p_i)}{l_i}$  and find units  $u_1, \dots, u_d \in k$  such that  $\text{in}_\xi(p_i)(u_1, \dots, u_d) \neq 0$  for  $i = 1, \dots, n - d$ , which is possible because we are considering once more a finite set of elements  $\{p_1, \dots, p_{n-d}\}$  and an infinite field  $k$ . Now, the arc  $\bar{\varphi}^{(d)}$  given as  $(u_1 t^\alpha, \dots, u_d t^\alpha)$ , where  $\alpha$  is some positive integer, is diagonal-generic for  $\mathcal{G}_{H_i}^{(d)}$  for all  $i = 1, \dots, n - d$ . Note that, in particular,  $\bar{\varphi}^{(d)}$  is diagonal-generic for  $\mathcal{G}_X^{(d)}$  (this follows from Lemma 5.16). Note also that by Lemma 5.8,  $\bar{\varphi}^{(d)}$  can be lifted to an arc  $\varphi$  in  $X$  and the projection of  $\varphi$  onto  $V^{(d)}$  is diagonal-generic for every  $\mathcal{G}_{H_i}^{(d)}$ ,  $i = 1, \dots, n - d$ .

**Theorem 5.19** *Let  $X$  be a variety as in Sect. 5.2.1 and let  $\xi \in \text{Max mult}(X)$ . There exists an arc  $\varphi \in \mathcal{L}(X)$  through  $\xi$  such that*

$$\bar{r}_{X,\varphi} = \text{ord}_\xi(\mathcal{G}_X^{(d)}). \tag{5.42}$$

*Proof* By Remark 5.18, we can choose a diagonal arc which is diagonal-generic for  $\mathcal{G}_{H_1}^{(d)}, \dots, \mathcal{G}_{H_{n-d}}^{(d)}$  and  $\mathcal{G}_X^{(d)}$ . Let us denote it by  $\bar{\varphi}^{(d)}$ . We can lift  $\bar{\varphi}^{(d)}$  to an arc  $\varphi \in \mathcal{L}(X)$  through  $\xi$ . By Remark 5.9 we know that  $\varphi$  is given (as in (4.5)) by

$$(g_1(t), \dots, g_{n-d}(t), u_1 g'(t), \dots, u_d g'(t))$$

for some  $g_1(t), \dots, g_{n-d}(t), g'(t) \in K[[t]]$  and some  $u_1, \dots, u_d \in k$  by Lemma 5.8. By Remark 5.18,  $\varphi^{(d)}$  is also generic for  $\mathcal{G}_{H_i}^{(d)}$ ,  $i = 1, \dots, n - d$ . The proof will be complete by showing that any arc of this form satisfies (5.42).

Let us denote  $N = \text{ord}_t(g'(t))$ . As in (2.18),  $\beta$  factorizes via  $\mathcal{O}_{H_i,\xi}$  for  $i = 1, \dots, n - d$ :

$$\begin{array}{ccc}
 \mathcal{O}_{X^{(d)},\xi} \cong \mathcal{O}_{V^{(d)},\xi^{(d)}}[x_1, \dots, x_{n-d}]/I(X) & \longleftarrow & \mathcal{O}_{V^{(d)},\xi^{(d)}}[x_1, \dots, x_{n-d}] \\
 \uparrow & \swarrow \beta_X^* & \nearrow \beta^* \\
 \mathcal{O}_{H_i,\xi} \cong \mathcal{O}_{V^{(d)},\xi^{(d)}}[x_i]/(f_i) & & \\
 & \nwarrow & \\
 & \mathcal{O}_{V^{(d)},\xi^{(d)}} & 
 \end{array}
 \tag{5.43}$$

and hence the projection  $\varphi_i$  of  $\varphi$  onto  $V_i^{(d+1)}$  is, in particular, a lifting of  $\bar{\varphi}^{(d)}$  to  $H_i$ , and the projection of each  $\varphi_i$  to  $V^{(d)}$  is  $\varphi^{(d)}$ , which is diagonal-generic for  $\mathcal{G}_{H_i}^{(d)}$ . Thus, the result of Theorem 5.12 holds for each  $H_i$ , as well as Remark 5.13, implying

$$\text{ord}_\xi(\mathcal{G}_{H_i,0,\varphi_i}^{(1)}) = \text{ord}(\varphi_i) \cdot \text{ord}_\xi(\mathcal{G}_{H_i}^{(d)}) = N \cdot \text{ord}_\xi(\mathcal{G}_{H_i}^{(d)})$$

for  $i = 1, \dots, n - d$ . By (5.35) we also know that  $\text{ord}(\varphi) = N$ . From this, together with Lemma 5.16, it follows that

$$\begin{aligned} \bar{r}_{X,\varphi} &= \frac{\text{ord}_\xi(\mathcal{G}_{X_0,\varphi}^{(1)})}{\text{ord}(\varphi)} = \frac{\min_{i=1,\dots,n-d}\{\text{ord}_\xi(\mathcal{G}_{H_i,0,\varphi_i}^{(1)})\}}{N} \\ &= \frac{\min_{i=1,\dots,n-d}\{N \cdot \text{ord}_\xi(\mathcal{G}_{H_i}^{(d)})\}}{N} = \min_{i=1,\dots,n-d}\{\text{ord}_\xi(\mathcal{G}_{H_i}^{(d)})\} = \text{ord}_\xi(\mathcal{G}_X^{(d)}), \end{aligned}$$

which completes the proof. □

### 5.3 Consequences of the main result

When we first presented our results in Sect. 3.2, we gave there a version of Theorem 4.10, which relates the invariants  $\bar{r}_X$  and  $\text{ord}_\xi \mathcal{G}_X^{(d)}$  for any  $X$ . It is clear now that the statement there is a consequence of Theorems 5.17 and 5.19. In addition, we claimed to know a relation between  $\text{ord}_\xi \mathcal{G}_X^{(d)}$  and  $\rho_{X,\varphi}$  for any arc  $\varphi$  in  $X$  through  $\xi$ . The following theorem shows this relation, which is just a small step more than a consequence of Proposition 4.11 and Theorem 4.10.

**Theorem 5.20** *Let  $X$  be a variety of dimension  $d$ . Let  $\xi$  be a point in  $\text{Max mult}(X)$ . For any arc  $\varphi$  in  $X$  through  $\xi$ ,*

$$\rho_{X,\varphi} \geq [\text{ord}_\xi \mathcal{G}_X^{(d)}] \cdot \text{ord}(\varphi),$$

where  $\mathcal{G}_X^{(d)}$  is the elimination algebra described in Example 2.48. Moreover,

$$\inf_{\varphi \in \mathcal{L}_\xi(X)} \{[\bar{\rho}_{X,\varphi}]\} = [\text{ord}_\xi \mathcal{G}_X^{(d)}].$$

one can find an arc  $\varphi_0$  in  $X$  through  $\xi$  satisfying

$$\bar{\rho}_{X,\varphi_0} = \text{ord}_\xi \mathcal{G}_X^{(d)}.$$

*Proof* For the first formula we use Proposition 4.11 and Theorem 5.17

$$\rho_{X,\varphi} = [r_{X,\varphi}] = \left[ \frac{r_{X,\varphi}}{\text{ord}(\varphi)} \cdot \text{ord}(\varphi) \right] \geq \left[ \frac{r_{X,\varphi}}{\text{ord}(\varphi)} \right] \cdot \text{ord}(\varphi) \geq [\text{ord}_\xi \mathcal{G}_X^{(d)}] \cdot \text{ord}(\varphi).$$

As a consequence, of this result,

$$\frac{r_{X,\varphi}}{\text{ord}(\varphi)} \geq \frac{[r_{X,\varphi}]}{\text{ord}(\varphi)} = \frac{\rho_{X,\varphi}}{\text{ord}(\varphi)} \geq \frac{[\text{ord}_\xi \mathcal{G}_X^{(d)}] \cdot \text{ord}(\varphi)}{\text{ord}(\varphi)} = [\text{ord}_\xi \mathcal{G}_X^{(d)}].$$

That is,

$$\bar{r}_{X,\varphi} \geq \bar{\rho}_{X,\varphi} \geq [\text{ord}_\xi \mathcal{G}_X^{(d)}],$$

where we may take integral parts and then the minimums over all arcs in  $X$  through  $\xi$ , obtaining

$$\min_{\varphi \in \mathcal{L}_\xi(X)} \{[\bar{r}_{X,\varphi}]\} \geq \min_{\varphi \in \mathcal{L}_\xi(X)} \{[\bar{\rho}_{X,\varphi}]\} \geq [\text{ord}_\xi \mathcal{G}_X^{(d)}] = \min_{\varphi \in \mathcal{L}_\xi(X)} \{[\bar{r}_{X,\varphi}]\},$$

which implies the second formula of the Theorem.

Finally, for the third formula, let us go back to the proof of Theorem 5.19. It allows us to find an arc  $\varphi_1$  in  $X$  through  $\xi$  satisfying  $\bar{r}_{X,\varphi_1} = \text{ord}_\xi \mathcal{G}_X^{(d)}$ . This arc will be given



by  $(g_1(t), \dots, g_{n-d}(t), u_1g'(t), \dots, u_dg'(t))$  for some  $g_1(t), \dots, g_{n-d}(t), g'(t) \in K[[t]]$  and some  $u_1, \dots, u_d \in k$ , and the projection  $\varphi_1^{(d)}$  given by  $(u_1g'(t), \dots, u_dg'(t))$  will be diagonal generic for  $\mathcal{G}_X^{(d)}$ . Let us choose  $\varphi_0$  as the arc in  $X$  through  $\xi$  given by

$$(g_1(t^b), \dots, g_{n-d}(t^b), u_1g'(t^b), \dots, u_dg'(t^b)),$$

for which the projection  $\varphi_0^{(d)}$  is also diagonal generic for  $\mathcal{G}_X^{(d)}$ , so it is also valid for Theorem 5.19, having  $\bar{r}_{X,\varphi_0} = \text{ord}_\xi \mathcal{G}_X^{(d)}$ . In particular, this implies that  $\bar{r}_{X,\varphi_0} = \bar{r}_{X,\varphi_1}$ . Note also that  $\text{ord}(\varphi_0) = \text{ord}(\varphi_1) \cdot b$ .

We have found an arc such that

$$r_{X,\varphi_0} = \text{ord}_\xi \mathcal{G}_X^{(d)} \cdot \text{ord}(\varphi_0),$$

and for which

$$r_{X,\varphi_0} = [r_{X,\varphi_0}] = \rho_{X,\varphi_0},$$

since  $\mathcal{G}_X^{(d)} \in \frac{1}{b} \cdot \mathbb{Z}_{>0}$  and  $\text{ord}(\varphi_0) \in b \cdot \mathbb{Z}_{>0}$ , concluding the proof. □

The following Corollary gives a characterization of  $\text{ord}_\xi \mathcal{G}_X^{(d)}$  in terms of the  $\bar{\rho}_{X,\varphi}$ .

**Corollary 5.21** *Let  $X$  be a variety of dimension  $d$ . Let  $\xi$  be a point in  $\text{Max mult}(X)$ . Consider the subset  $\mathcal{C} \subset \mathcal{L}_\xi(X)$  of all arcs  $\varphi$  satisfying  $\bar{r}_{X,\varphi} = \text{ord}_\xi \mathcal{G}_X^{(d)}$ . Then:*

$$\text{ord}_\xi \mathcal{G}_X^{(d)} = \max_{\varphi \in \mathcal{C}} \{\bar{\rho}_{X,\varphi}\}.$$

*Proof* For any arc  $\varphi \in \mathcal{C}$ ,

$$\rho_{X,\varphi} = [r_{X,\varphi}] = [\text{ord}_\xi \mathcal{G}_X^{(d)} \cdot \text{ord}(\varphi)].$$

It follows that

$$\frac{\rho_{X,\varphi}}{\text{ord}(\varphi)} = \frac{[\text{ord}_\xi \mathcal{G}_X^{(d)} \cdot \text{ord}(\varphi)]}{\text{ord}(\varphi)} \leq \text{ord}_\xi \mathcal{G}_X^{(d)}.$$

The result is a consequence of this together with Theorem 5.20. □

The following relations hold for every arc  $\varphi \in \mathcal{L}(X)$  through  $\xi$ :

**Corollary 5.22** *For  $X$  as in Proposition 4.11, and for every arc  $\varphi \in \mathcal{L}(X)$  through  $\xi$ :*

1.  $\bar{r}_{X,\varphi} \geq \bar{\rho}_{X,\varphi}$
2.  $\bar{\rho}_{X,\varphi} \geq [\text{ord}_\xi \mathcal{G}_X^{(d)}]$
3. *Since  $\bar{r}_{X,\varphi} \geq \text{ord}_\xi \mathcal{G}_X^{(d)}$  and  $\bar{r}_{X,\varphi} \geq \bar{\rho}_{X,\varphi} \geq [\text{ord}_\xi \mathcal{G}_X^{(d)}]$ , two possible situations can happen for  $\bar{\rho}_{X,\varphi}$  and  $\text{ord}_\xi \mathcal{G}_X^{(d)}$ , namely:*
  - $\bar{r}_{X,\varphi} \geq \text{ord}_\xi \mathcal{G}_X^{(d)} \geq \bar{\rho}_{X,\varphi} \geq [\text{ord}_\xi \mathcal{G}_X^{(d)}]$
  - $\bar{r}_{X,\varphi} \geq \bar{\rho}_{X,\varphi} > \text{ord}_\xi \mathcal{G}_X^{(d)} \geq [\text{ord}_\xi \mathcal{G}_X^{(d)}]$

*Proof* 1. Follows from the definitions of  $\bar{r}_{X,\varphi}$  and  $\bar{\rho}_{X,\varphi}$  together with Proposition 4.11.

2. By Definition 3.5, Proposition 4.11, Theorem 5.17:

$$\bar{\rho}_{X,\varphi} = \frac{\rho_{X,\varphi}}{\text{ord}\varphi} = \frac{[r_{X,\varphi}]}{\text{ord}\varphi} \geq \frac{[\text{ord}_\xi \mathcal{G}_X^{(d)} \cdot \text{ord}\varphi]}{\text{ord}\varphi} \geq [\text{ord}_\xi \mathcal{G}_X^{(d)}].$$

3. This is just an observation which follows from (2) and (3). □

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## References

1. Benito, A.: The tau-invariant and elimination. *J. Algebra* **324**(8), 1903–1920 (2010)
2. Bennett, B.M.: On the characteristic functions of a local ring. *Ann. Math.* **91**(2), 25–87 (1970)
3. Bierstone, E., Milman, P.: Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant. *Invent. Math.* **128**(2), 207–302 (1997)
4. Blanco, R., Encinas, S.: Coefficient and elimination algebras in resolution of singularities. *Asian J. Math.* **15**(2), 251–271 (2011)
5. Bravo, A., Villamayor U, O.: Singularities in positive characteristic, stratification and simplification of the singular locus. *Adv. Math.* **224**(4), 1349–1418 (2010)
6. Bravo, A., Villamayor U, O.E.: Elimination algebras and inductive arguments in resolution of singularities. *Asian J. Math.* **15**(3), 321–355 (2011)
7. Bravo, A., García-Escamilla, M.L., Villamayor U, O.E.: On Rees algebras and invariants for singularities over perfect fields. *Indiana Univ. Math. J.* **61**(3), 1201–1251 (2012)
8. Bravo, A., Villamayor U, O.E.: On the behavior of the multiplicity on schemes: stratification and blow-ups. The resolution of singular algebraic varieties. *Clay Math. Proc.* **20**, 81–207 (2014)
9. Dade, E.C.: Multiplicity and monoidal transformations. Thesis (Ph.D.), Princeton University (1960)
10. Ein, L., Mustață, M., Yasuda, T.: Jet schemes, log discrepancies, and inversion of adjunction. *Invent. Math.* **153**, 519–535 (2003)
11. Ein, L., Mustață, M.: Inversion of adjunction for local complete intersection varieties. *Am. J. Math.* **126**, 1355–1365 (2004)
12. Ein, L., Mustață, M.: Jet schemes and singularities. *Proc. Symp. Pure Math.* **80**(2), 505–546 (2009)
13. Encinas, S., Villamayor, O.: A course on constructive desingularization and equivariance. *Resolution of singularities. Obergrugl: Progr. Math.*, vol. 181. Birkhäuser, Basel, pp 147–227 (1997)
14. Encinas, S., Villamayor, O.: A new proof of desingularization over fields of characteristic zero. Proceedings of the international conference on algebraic geometry and singularities (Spanish) (Sevilla, 2001). *Rev. Mat. Iberoamericana* **19**(2), 339–353 (2003)
15. Encinas, S., Villamayor, O.: Rees algebras and resolution of singularities. Proceedings of the XVth Latin American algebra colloquium (Spanish), (Colonia 2005). *Bibl. Rev. Mat. Iberoamericana, Rev. Mat. Iberoamericana*, Madrid, pp. 63–85 (2007)
16. Hickel, M.: Sur quelques aspects de la géométrie de l'espace des arcs tracés sur un espace analytique. *Annales de la faculté des sciences de Toulouse Mathématiques* (6) **14**(1), 1–50 (2005)
17. Hironaka, H.: Resolution of singularities of an algebraic variety over a field of characteristic zero I, II. *Ann. Math.* **79**(2), 109–326 (1964)
18. Hironaka, H.: Idealistic exponents of a singularity. *J.J. Sylvester Sympos.*, Baltimore, Md.: Algebraic Geometry. The Johns Hopkins Centennial Lectures, pp. 55–125. Johns Hopkins University Press, Baltimore, vol. 1977 (1976)
19. Ishii, S.: Geometric properties of jet schemes. *Commun. Algebra* **39**(5), 1872–1882 (2011)
20. Lejeune-Jalabert, M.: Courbes Tracées sur un Germe D'Hypersurface. *Am. J. Math.* **112**(4), 525–568 (1990)
21. Mustață, M.: Spaces of arcs in birational geometry. In: *Lecture notes (available at the author's personal web page)*
22. Mustață, M.: Jet schemes of locally complete intersection canonical singularities, with an appendix by David Eisenbud and Edward Frenkel. *Invent. Math.* **145**, 397–424 (2001)
23. Mustață, M.: Singularities of pairs via jet schemes. *J. Am. Math. Soc.* **15**, 599–615 (2002)
24. Nobile, A.: Equivalence and resolution of singularities. *J. Algebra* **420**, 161–185 (2014)
25. Villamayor U, O.E.: Constructiveness of Hironaka's resolution. *Ann. Sci. École. Norm. Sup* **22**(1), 1–32 (1989)
26. Villamayor U, O.E.: Patching local uniformizations. *Ann. Sci. École. Norm. Sup.* (4) **25**(6), 629–677 (1992)
27. Villamayor, O.E.: Tschirnhausen transformations revisited and the multiplicity of the embedded hypersurface. In: *Colloquium on Homology and Representation Theory (Spanish)* (Vaquerías: Boletín de la Academia nacional de Ciencias, vol. 65, pp. 233–243. Córdoba, Argentina (2000)

28. Villamayor U, O.: Hypersurface singularities in positive characteristic. *Adv. Math.* **213**(2), 687–733 (2007)
29. Villamayor U, O.: Rees algebras on smooth schemes: integral closure and higher differential operator. *Rev. Mat. Iberoam.* **24**(1), 213–242 (2008)
30. Villamayor U, O.E.: Equimultiplicity, algebraic elimination, and blowing-up. *Adv. Math.* **262**, 313–369 (2014)
31. Vojta, P.: Jets via Hasse–Schmidt derivations. In: *Diophantine Geometry. CRM Series*, 4th ed., pp. 335–361. Norm., Pisa (2007)