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Classifying local Artinian Gorenstein algebras

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Abstract The classification of local Artinian Gorenstein algebras is equivalent to the study of orbits of a certain non-reductive group action on a polynomial ring. We give an explicit formula for the orbits and their tangent spaces. We apply our technique to analyse when an algebra is isomorphic to its associated graded algebra. We classify algebras with Hilbert function $(1, 3, 3, 3, 1)$, obtaining finitely many isomorphism types, and those with Hilbert function $(1, 2, 2, 2, 1, 1, 1)$. We consider fields of arbitrary, large enough, characteristic.

Mathematics Subject Classification Primary 13H10; Secondary 13N10, 14C05

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1 Introduction

The problem of classifying Artinian Gorenstein algebras up to isomorphism is hard and long studied, a comprehensive reference is [29]. The results usually rest on additional assumptions: small length [5,7–9,37] or being a codimension two complete intersection [3,15,18]. See also [11,12,20,26] for other approaches.

Let k be a field. So far we do not impose any conditions such as k being algebraically closed or on its characteristic (see Sect. 3), however the reader may freely assume that $k = \mathbb{C}$.

Let $(A, \mathfrak{m}_A, k)$ be a local Artinian Gorenstein k-algebra. As the notation suggests, throughout the paper we assume that $k \to A/\mathfrak{m}_A$ is an isomorphism. This assumption is automatic if k is algebraically closed. It is standard and it is used to discard examples such as $k = \mathbb{R} \subset A = \mathbb{C}$.

An important numerical invariant of A is the Hilbert function $H_A(i) = \dim_k \mathfrak{m}_A^i/\mathfrak{m}_A^{i+1}$. The *socle degree* d is the maximal number such that $H_A(d) \neq 0$. If $H_A(d-i) = H_A(i)$ for all i, then the associated graded algebra $\operatorname{gr} A = \bigoplus_{i\geq 0} \mathfrak{m}_A^i/\mathfrak{m}_A^{i+1}$ is also Gorenstein, see [28, Proposition 1.7].

We may now formulate our motivating problems:

1. What is the classification up to isomorphism of local Artinian Gorenstein algebras with Hilbert function H_A?
2. If H_A is symmetric, what are the sufficient conditions for $A \simeq \operatorname{gr} A$?

The first of these problems naturally leads to the second: suppose we are classifying A whose Hilbert function H_A is symmetric. Since $\operatorname{gr} A$ is in this case Gorenstein, one natural way is to first determine possible $\operatorname{gr} A$ and then classify A having fixed $\operatorname{gr} A$. Note that it is convenient to classify graded Gorenstein algebras by using ideas from projective geometry: Waring, border and smoothable ranks, and secant varieties, see [22,29] for the overview of methods and [4,32] for the state of the art on secant varieties. See [36,40] and references therein for information about the possible Hilbert functions. See [28, Appendix] for a list of possible Hilbert functions for small values of $\dim_k A$.

Our main goal is to refine, simplify and make explicit the existing theory. Having done this, we are gratified with classification results in several cases:

1. We reprove and extend the result of Elias and Rossi:

Proposition 1.1 ([16,17], Corollary 3.14) *Let k be a field of characteristic other than 2 or 3. A local Artin Gorenstein k-algebra with Hilbert function $(1, n, n, 1)$ or Hilbert function $\left(1, n, \binom{n+1}{2}, n, 1\right)$ is isomorphic to its associated graded algebras.*

The original statement is proven in [16,17] for fields of characteristic zero. In Example 3.15 we show that the statement is not true for fields of characteristic two or three.

In Proposition 1.1 the condition on the Hilbert function can be rephrased by saying that the algebra is *compressed*, see Sect. 3.2. The algebras appearing in Proposition 1.1 are precisely compressed algebras of socle degree three and four. A analogous statement for compressed algebras of higher socle degree is false. However, we show that compressed algebras of all socle degrees are near to the graded algebras, see Corollary 3.13.

2. We prove that general algebras with Hilbert function $(1, 2, 3, 3, 2, 1)$ and general algebras with Hilbert function $(1, 2, 2, 2, \ldots, 2, 1)$ are isomorphic to their associated graded algebras (see Example 3.17 or Example 3.25 for a precise meaning of the word general).

3. An algebra is *canonically graded* if it isomorphic to its associated graded algebra. We investigate the set of (dual socle generators of) algebras which are canonically graded. In Proposition 3.19 we show that this set is irreducible, but in general neither open nor closed in the parameter space. This answers a question of Elias and Rossi [17, Remark 3.6].

We investigate density of the set canonically graded algebras and in Proposition 3.20 we prove the *only if* part of the following conjecture.

Conjecture 1.2 (Conjecture 3.21) *Assume that k is a field of characteristic not equal to* 2, 3, 5. *Say that an algebra A has* type (n, d) *if $H_A(1) = n$ and the socle degree of A is d. Then a general algebra of type (n, d) is canonically graded if and only if (n, d) belongs to the following list:*

(a) $d \leq 4$ *and n arbitrary,*
(b) $d = 5$ *and $n \leq 6$,*
(c) $d = 6$ *and $n = 2$.*
(d) *d arbitrary and $n = 1$.*

4. We classify algebras with Hilbert function $(1, 3, 3, 3, 1)$, obtaining eleven isomorphism types, see Example 3.27. We also discuss the case of $(1, 3, 4, 3, 1)$, show that there are infinitely many isomorphism types and classify algebras which are not canonically graded, see Example 3.28. Non-canonically graded algebras with Hilbert function $(1, 3, 4, 3, 1)$ are investigated independently in [34].
5. We classify algebras with Hilbert function $(1, 2, 2, 2, 1, 1, 1)$ obtaining $|k| + 1$ isomorphism types, see Example 3.29. This example was worked out by Elias and Valla [18].

We now explain our approach. In short, the classification problem may be reduced to representation theory, see [19]. Assume for a moment for simplicity of presentation that the characteristic of k is zero. Let $n = H_A(1)$. Then A may be seen as a quotient of a fixed power series ring $S = k[[\alpha_1, \ldots, \alpha_n]]$. The ring S acts on a polynomial ring $P = k[x_1, \ldots, x_n]$ by letting α_i act as the partial derivative with respect to x_i. It turns out that $A = S/\operatorname{Ann}(f)$ for an element $f \in P$, called the *dual socle generator* of A. Then the problem of classifying algebras reduces to classifying quotients of S, which in turn boils down to classifying $f \in P$. There is a certain group $\mathbb{G}$ acting on P such that $S/\operatorname{Ann}(f) \simeq S/\operatorname{Ann}(g)$ if and only if f and g lie in the same $\mathbb{G}$-orbit. We summarise this (for fields of arbitrary characteristic) in Proposition 2.14.

Our contribution to the above is in providing explicit formulas for all objects involved and in applying Lie-theoretic ideas. In particular, we give formulas for the $\mathbb{G}$-orbit and its tangent space, thus we are able to compute dimensions of orbits. We also find the unipotent radical of $\mathbb{G}$ and analyse its orbits, which are closed.

The explicit formulas make several known results, such as those by Elias and Rossi, corollaries of the presented theory. This is, in our opinion, the most important contribution of this paper. Accordingly, we have tried to keep the paper as elementary as possible.

The reader willing to get a quick idea of the contents of this paper should analyse Example 2.16, referring to previous results if necessary. Otherwise we recommend to read through the paper linearly. The paper is organized as follows. First, we recall and develop the abstract theory of inverse systems; in particular we give an explicit formula for the automorphism group action in Proposition 2.9. Then we explain the link between classifying algebras and elements of P in Proposition 2.14. We give some examples, in particular Example 2.16, and then discuss the basic Lie theory in Sect. 2.4. Finally, in Sect. 3 we present the applications mentioned above. Apart from the conjecture above, there are several natural questions that are in our opinion worth considering:

1. What are H such that there are finitely many isomorphism types of Artinian Gorenstein algebras with Hilbert function H? What is the classification in these cases?
2. An Artinian Gorenstein algebra is *rigid* if it cannot be deformed to a non-isomorphic algebra, see [39]. The $\mathbb{G}$-orbit of a rigid algebra should be large. For k algebraically closed, are there rigid k-algebras other than k?
3. Given the Hilbert function, are there non-trivial bounds on the dimensions of $\mathbb{G}$-orbits of algebras with this Hilbert function?
4. Can the above ideas of classification be generalised effectively to non-Gorenstein algebras?

2 Preliminaries and theoretical results

2.1 Power series ring S and its dual P

In this section we introduce our main objects of study: the power series ring S and its action on the dual divided power (or polynomial) ring P.

By $\mathbb{N}_0$ we denote the set of non-negative integers. Let k be an algebraically closed field of arbitrary characteristic. Let S be a power series ring over k of dimension $\dim S = n$ and let $\mathfrak{m}$ be its maximal ideal. By $\operatorname{ord}(\sigma)$ we denote the order of a non-zero $\sigma \in S$ i.e. the largest i such that $\sigma \in \mathfrak{m}^i$. Then $\operatorname{ord}(\sigma) = 0$ if and only if σ is invertible. Let $S^\vee = \operatorname{Hom}_k(S, k)$ be the space of functionals on S. We denote the pairing between S and $S^\vee$ by

$$\langle -, - \rangle : S \times S^\vee \to k.$$

Definition 2.1 The dual space $P \subset S^\vee$ is the linear subspace of functionals eventually equal to zero:

$$P = \left\{ f \in S^\vee \mid \forall_{D \gg 0} \left\langle \mathfrak{m}^D, f \right\rangle = 0 \right\}.$$

On P we have a structure of S-module via precomposition: for every $\sigma \in S$ and $f \in P$ the element $\sigma \lrcorner f \in P$ is defined via the equation

$$\langle \tau, \sigma \lrcorner f \rangle = \langle \tau\sigma, f \rangle \quad \text{for every } \tau \in S. \tag{1}$$

This action is called *contraction*.

Existence of contraction is a special case of the following construction, which is basic and foundational for our approach. Let $L : S \to S$ be a k-linear map. Assume that there exists an integer s such that $L(\mathfrak{m}^i) \subset \mathfrak{m}^{i+s}$ for all i. Then the dual map $L^\vee : S^\vee \to S^\vee$ restricts to $L^\vee : P \to P$. Explicitly, $L^\vee$ is given by the equation

$$\left\langle \tau, L^\vee(f) \right\rangle = \langle L(\tau), f \rangle \quad \text{for every } \tau \in S,\ f \in P. \tag{2}$$

To obtain contraction with respect to σ we use $L(\tau) = \sigma\tau$, the multiplication by σ. Later in this paper we will also consider maps L which are automorphisms or derivations.

To get a down to earth description of P, choose $\alpha_1, \dots, \alpha_n \in S$ so that $S = k[[\alpha_1, \dots, \alpha_n]]$. Write $\alpha^{\mathbf{a}}$ to denote $\alpha_1^{a_1} \dots \alpha_n^{a_n}$. For every $\mathbf{a} \in \mathbb{N}_0^n$ there is a unique element $\mathbf{x}^{[\mathbf{a}]} \in P$ dual to $\alpha^{\mathbf{a}}$, given by

$$\left\langle \alpha^{\mathbf{b}}, \mathbf{x}^{[\mathbf{a}]} \right\rangle = \begin{cases} 1 & \text{if } \mathbf{a} = \mathbf{b} \\ 0 & \text{otherwise.} \end{cases}$$

Additionally, we define x_i as the functional dual to α_i, so that $x_i = \mathbf{x}^{[(0,\dots 0,1,0,\dots,0)]}$ with one on i-th position.

Let us make a few remarks:

1. The functionals $\mathbf{x}^{[\mathbf{a}]}$ form a basis of P.
2. The contraction action is given by the formula
$$\alpha^{\mathbf{a}} \lrcorner \mathbf{x}^{[\mathbf{b}]} = \begin{cases} \mathbf{x}^{[\mathbf{b}-\mathbf{a}]} & \text{if } \mathbf{b} \geq \mathbf{a}, \text{ that is, } \forall_i\ b_i \geq a_i \\ 0 & \text{otherwise.} \end{cases}$$
Therefore our definition agrees with the standard, see [29, Definition 1.1, p. 4].

We say that $\mathbf{x}^{[\mathbf{a}]}$ has degree $\sum a_i$. We will freely speak about constant forms, linear forms, (divided) polynomials of bounded degree etc.

We endow P with a topology, which is the Zariski topology of an affine space. It will be used when speaking about general polynomials and closed orbits, but for most of the time it is not important.

Now we will give a ring structure on P. It will be used crucially in Proposition 2.9. For multi-indices $\mathbf{a}, \mathbf{b} \in \mathbb{N}_0^n$ we define $\mathbf{a}! = \prod(a_i!)$, $\sum \mathbf{a} = \sum a_i$ and $\binom{\mathbf{a}+\mathbf{b}}{\mathbf{a}} = \prod_i \binom{a_i+b_i}{a_i} = \binom{\mathbf{a}+\mathbf{b}}{\mathbf{b}}$.

Definition 2.2 We define multiplication on P by
$$\mathbf{x}^{[\mathbf{a}]} \cdot \mathbf{x}^{[\mathbf{b}]} := \binom{\mathbf{a}+\mathbf{b}}{\mathbf{a}} \mathbf{x}^{[\mathbf{a}+\mathbf{b}]}. \tag{3}$$
In this way P is a *divided power ring*.

The multiplicative structure on P can be defined in a coordinate-free manned using a natural comultiplication on S. We refer to [13, §A2.4] for details in much greater generality.

Example 2.3 Suppose that k is of characteristic 3. Then P is not isomorphic to a polynomial ring. Indeed, $x_1 \cdot x_1 \cdot x_1 = (2x_1^{[2]}) \cdot x_1 = 3x_1^{[3]} = 0$. Moreover $x_1^{[3]}$ is not in the subring generated by $x_1, \dots, x_n$.

Note that linear forms from S act on P as derivatives. Therefore we can interpret S as lying inside the ring of differential operators on P. We will need the following related fact.

Lemma 2.4 *Let $\sigma \in S$ and denote by $\sigma^{(i)}$ its i-th partial derivative. For every $f \in P$ we have*
$$\sigma \lrcorner (x_i \cdot f) - x_i \cdot (\sigma \lrcorner f) = \sigma^{(i)} \lrcorner f. \tag{4}$$

Proof Since the formula is linear in σ and f we may assume these are monomials. Let $\sigma = \alpha_i^r \tau$, where α_i does not appear in τ. Then $\sigma^{(i)} = r\alpha_i^{r-1}\tau$. Moreover $\tau \lrcorner (x_i \cdot f) = x_i \cdot (\tau \lrcorner f)$, thus we may replace f by $\tau \lrcorner f$ and reduce to the case $\tau = 1$, $\sigma = \alpha_i^r$.

Write $f = x_i^{[s]} g$ where g is a monomial in variables other than x_i. Then $x_i \cdot f = (s+1)x_i^{[s+1]} g$ according to (3). If $s+1 < r$ then both sides of (4) are zero. Otherwise
$$\sigma \lrcorner (x_i \cdot f) = (s+1)x_i^{[s+1-r]} g, \quad x_i \cdot (\sigma \lrcorner f) = x_i \cdot x_i^{[s-r]} g = (s-r+1)x_i^{[s-r+1]} g,$$
$$\sigma^{(i)} \lrcorner f = r x_i^{[s-(r-1)]} g,$$
so Eq. (4) is valid in this case also. $\square$

Remark 2.5 Lemma 2.4 applied to $\sigma = \alpha_i$ shows that $\alpha_i \lrcorner (x_i \cdot f) - x_i \cdot (\alpha_i \lrcorner f) = f$. This can be rephrased more abstractly by saying that α_i and x_i interpreted as linear operators on P generate a Weyl algebra.

Example 2.3 shows that P with its ring structure has certain properties distinguishing it from the polynomial ring, for example it contains nilpotent elements. Similar phenomena do not occur in degrees lower than the characteristic or in characteristic zero, as we show in Propositions 2.6 and 2.8 below.

Proposition 2.6 *Let $P_{\geq d}$ be the linear span of $\{\mathbf{x}^{[\mathbf{a}]} \mid \sum \mathbf{a} \geq d\}$. Then $P_{\geq d}$ is an ideal of P, for all d. Let k be a field of characteristic p. The ring $P/P_{\geq p}$ is isomorphic to the truncated polynomial ring. In fact*

$$\Omega : P/P_{\geq p} \to k[x_1, \dots, x_n]/(x_1, \dots, x_n)^p$$

defined by

$$\Omega\left(\mathbf{x}^{[\mathbf{a}]}\right) = \frac{x_1^{a_1} \dots x_n^{a_n}}{a_1! \dots a_n!}.$$

is an isomorphism.

Proof Since Ω maps a basis of P/I_p to a basis of $k[x_1, \dots, x_n]/(x_1, \dots, x_n)^p$, it is clearly well defined and bijective. The fact that Ω is a k-algebra homomorphism reduces to the equality $\binom{\mathbf{a}+\mathbf{b}}{\mathbf{a}} = \prod \frac{(a_i+b_i)!}{a_i! b_i!}$. □

Characteristic zero case In this paragraph we assume that k is of characteristic zero. This case is technically easier, but there are two competing conventions: contraction and partial differentiation. These agree up to an isomorphism. The main aim of this section is clarify this isomorphism and provide a dictionary between divided power rings used in the paper and polynomial rings in characteristic zero. Contraction was already defined above, now we define the action of S via partial differentiation.

Definition 2.7 Let $k[x_1, \dots, x_n]$ be a polynomial ring. There is a (unique) action of S on $k[x_1, \dots, x_n]$ such that the element α_i acts a $\frac{\partial}{\partial x_i}$. For $f \in k[x_1, \dots, x_n]$ and $\sigma \in S$ we denote this action as $\sigma \circ f$.

The following Proposition 2.8 shows that in characteristic zero the ring P is in fact polynomial and the isomorphism identifies the S-module structure on P with that from Definition 2.7 above.

Proposition 2.8 *Suppose that k is of characteristic zero. Let $k[x_1, \dots, x_n]$ be a polynomial ring with S-module structure as defined in Definition 2.7. Let $\Omega : P \to k[x_1, \dots, x_n]$ be defined via*

$$\Omega(\mathbf{x}^{[\mathbf{a}]}) = \frac{x_1^{a_1} \dots x_n^{a_n}}{a_1! \dots a_n!}.$$

Then Ω is an isomorphism of rings and an isomorphism of S-modules.

Proof The map Ω is an isomorphism of k-algebras by the same argument as in Proposition 2.6. We leave the check that Ω is a S-module homomorphism to the reader. □

Summarising, we get the following corresponding notions.

Arbitrary characteristic	Characteristic zero
Divided power series ring P	Polynomial ring $k[x_1, \dots, x_n]$
S-action by contraction (precomposition) denoted $\sigma \lrcorner f$	S action by derivations denoted $\sigma \circ f$
$\mathbf{x}^{[\mathbf{a}]}$	$\mathbf{x}^{\mathbf{a}}/\mathbf{a}!$
$x_i = \mathbf{x}^{[(0,\dots 0,1,0,\dots,0)]}$	x_i

2.2 Automorphisms and derivations of the power series ring

Let as before $S = k[[\alpha_1, \dots, \alpha_n]]$ be a power series ring with maximal ideal $\mathfrak{m}$. This ring has a huge automorphism group: for every choice of elements $\sigma_1, \dots, \sigma_n \in \mathfrak{m}$ whose images span $\mathfrak{m}/\mathfrak{m}^2$ there is a unique automorphism $\varphi : S \to S$ such that $\varphi(\alpha_i) = \sigma_i$. Note that φ preserves $\mathfrak{m}$ and its powers. Therefore the dual map $\varphi^\vee : S^\vee \to S^\vee$ restricts to $\varphi^\vee : P \to P$. The map $\varphi^\vee$ is defined (using the pairing of Definition 2.1) via the condition

$$\langle \varphi(\sigma), f\rangle = \left\langle \sigma, \varphi^\vee(f)\right\rangle \quad \text{for all } \sigma \in S,\ f \in P. \tag{5}$$

Now we will describe this action explicitly.

Proposition 2.9 *Let $\varphi : S \to S$ be an automorphism. Let $D_i = \varphi(\alpha_i) - \alpha_i$. For $\mathbf{a} \in \mathbb{N}_0^n$ denote $\mathbf{D}^{\mathbf{a}} = D_1^{a_1} \dots D_n^{a_n}$. Let $f \in P$. Then*

$$\varphi^\vee(f) = \sum_{\mathbf{a}\in\mathbb{N}_0^n} \mathbf{x}^{[\mathbf{a}]} \cdot (\mathbf{D}^{\mathbf{a}} \lrcorner f) = f + \sum_{i=1}^{n} x_i \cdot (D_i \lrcorner f) + \cdots .$$

Proof We need to show that $\left\langle \sigma, \varphi^\vee(f)\right\rangle = \langle \varphi(\sigma), f\rangle$ for all $\sigma \in S$. Since $f \in P$, it is enough to check this for all $\sigma \in k[\alpha_1, \dots, \alpha_n]$. By linearity, we may assume that $\sigma = \alpha^{\mathbf{a}}$.

For every $g \in P$ let $\varepsilon(g) = \langle 1, g\rangle \in k$. We have

$$\begin{aligned}\langle \varphi(\sigma), f\rangle &= \langle 1, \varphi(\sigma) \lrcorner f\rangle = \varepsilon\left(\varphi(\sigma) \lrcorner f\right) = \varepsilon\left(\sum_{\mathbf{b}\le\mathbf{a}} \binom{\mathbf{a}}{\mathbf{b}} (\alpha^{\mathbf{a}-\mathbf{b}}\mathbf{D}^{\mathbf{b}}) \lrcorner f\right)\\ &= \sum_{\mathbf{b}\le\mathbf{a}} \varepsilon\left(\binom{\mathbf{a}}{\mathbf{b}}\alpha^{\mathbf{a}-\mathbf{b}} \lrcorner (\mathbf{D}^{\mathbf{b}} \lrcorner f)\right).\end{aligned}$$

Consider a term of this sum. Observe that for every $g \in P$

$$\varepsilon\left(\binom{\mathbf{a}}{\mathbf{b}}\alpha^{\mathbf{a}-\mathbf{b}} \lrcorner g\right) = \varepsilon(\alpha^{\mathbf{a}} \lrcorner (\mathbf{x}^{[\mathbf{b}]} \cdot g)). \tag{6}$$

Indeed it is enough to check the above equality for $g = \mathbf{x}^{[\mathbf{c}]}$ and both sides are zero unless $\mathbf{c} = \mathbf{a} - \mathbf{b}$, thus it is enough to check the case $g = \mathbf{x}^{[\mathbf{a}-\mathbf{b}]}$, which is straightforward. Moreover note that if $\mathbf{b} \not\le \mathbf{a}$, then the right hand side is zero for all g, because ε is zero for all $\mathbf{x}^{[\mathbf{c}]}$ with non-zero $\mathbf{c}$. We can use (6) and remove the restriction $\mathbf{b} \le \mathbf{a}$, obtaining

$$\begin{aligned}\sum_{\mathbf{b}} \varepsilon(\alpha^{\mathbf{a}} \lrcorner (\mathbf{x}^{[\mathbf{b}]} \cdot (\mathbf{D}^{\mathbf{b}} \lrcorner f))) &= \varepsilon\left(\sum_{\mathbf{b}} \alpha^{\mathbf{a}} \lrcorner (\mathbf{x}^{[\mathbf{b}]} \cdot (\mathbf{D}^{\mathbf{b}} \lrcorner f))\right) = \left\langle \alpha^{\mathbf{a}}, \sum_{\mathbf{b}} \mathbf{x}^{[\mathbf{b}]} \cdot (\mathbf{D}^{\mathbf{b}} \lrcorner f)\right\rangle\\ &= \left\langle \alpha^{\mathbf{a}}, \varphi^\vee(f)\right\rangle = \left\langle \sigma, \varphi^\vee(f)\right\rangle.\end{aligned}$$

□

Consider now a derivation $D : S \to S$. Here and elsewhere a derivation is a k-linear map satisfying the Leibniz rule. The derivation D gives rise to a dual map $D^\vee : P \to P$. We wish to describe it explicitly.

Proposition 2.10 *Let $D : S \to S$ be a derivation and $D_i := D(\alpha_i)$. Let $f \in P$. Then*

$$D^\vee(f) = \sum_{i=1}^{n} x_i \cdot (D_i \lrcorner f).$$

Proof The proof is similar, though easier, to the proof of Proposition 2.9. □

Remark 2.11 Suppose $D : S \to S$ is a derivation such that $D(\mathfrak{m}) \subseteq \mathfrak{m}^2$. Then $\deg(D^\vee(f)) < \deg(f)$. We say that D lowers the degree.

Characteristic zero case. Let k be a field of characteristic zero. By $\mathbf{x}^{\mathbf{a}}$ we denote the monomial $x_1^{a_1} \dots x_n^{a_n}$ in the polynomial ring $k[x_1, \dots, x_n]$. Then, in the notation of Proposition 2.8, we have

$$\Omega(\mathbf{x}^{[\mathbf{a}]}) = \frac{1}{\mathbf{a}!}\mathbf{x}^{\mathbf{a}}.$$

Clearly, an automorphism of S gives rise to an linear map $k[x_1, \dots, x_n] \to k[x_1, \dots, x_n]$. We may restate Propositions 2.9 and 2.10 as

Corollary 2.12 *Let $\varphi : S \to S$ be an automorphism. Let $D_i = \varphi(\alpha_i) - \alpha_i$. For $\mathbf{a} \in \mathbb{N}_0^n$ denote $\mathbf{D}^{\mathbf{a}} = D_1^{a_1} \dots D_n^{a_n}$. Let $f \in k[x_1, \dots, x_n]$. Then*

$$\varphi^\vee(f) = \sum_{\mathbf{a} \in \mathbb{N}_0^n} \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!} \left(\mathbf{D}^{\mathbf{a}} \circ f\right) = f + \sum_{i=1}^{n} x_i (D_i \circ f) + \dots.$$

Let $D : S \to S$ be a derivation and $D_i := D(\alpha_i)$. Then

$$D^\vee(f) = \sum_{i=1}^{n} x_i (D_i \circ f).$$

Example 2.13 Let $n = 2$, so that $S = k[[\alpha_1, \alpha_2]]$ and consider a linear map $\varphi : S \to S$ given by $\varphi(\alpha_1) = \alpha_1$ and $\varphi(\alpha_2) = \alpha_1 + \alpha_2$. Dually, $\varphi^\vee(x_1) = x_1 + x_2$ and $\varphi^\vee(x_2) = x_2$. Since φ is linear, $\varphi^\vee$ is an automorphism of $k[x_1, x_2]$. Therefore $\varphi^\vee(x_1^3) = (x_1 + x_2)^3$. Let us check this equality using Proposition 2.12. We have $D_1 = \varphi(\alpha_1) - \alpha_1 = 0$ and $D_2 = \varphi(\alpha_2) - \alpha_2 = \alpha_1$. Therefore $\mathbf{D}^{(a,b)} = 0$ whenever $a > 0$ and $\mathbf{D}^{(0,b)} = \alpha_1^b$.

We have

$$\varphi^\vee\left(x_1^3\right) = \sum_{(a,b) \in \mathbb{N}^2} \frac{x_1^a x_2^b}{a!b!} (\mathbf{D}^{(a,b)} \circ x_1^3) = \sum_{b \in \mathbb{N}} \frac{x_2^b}{b!} (\alpha_1^b \circ x_1^3) = x_1^3 + \frac{x_2}{1} \cdot (3x_1^2) + \frac{x_2^2}{2} \cdot (6x_1) + \frac{x_2^3}{6} \cdot (6) = (x_1 + x_2)^3,$$

which indeed agrees with our previous computation.

When φ is not linear, $\varphi^\vee$ is not an endomorphism of $k[x_1, x_2]$ and computing it directly from definition becomes harder. For example, if $\varphi(\alpha_1) = \alpha_1$ and $\varphi(\alpha_2) = \alpha_2 + \alpha_1^2$, then

$$\varphi^\vee(x_1) = x_1, \quad \varphi^\vee(x_1^4) = x_1^4 + 12x_1^2 x_2 + 12x_2^2.$$

2.3 Local Artinian Gorenstein algebras

Let $(A, \mathfrak{m}_A, k)$ be a local Artinian Gorenstein algebra. Recall that the *socle degree* of A is defined as $d = \max\{i \mid \mathfrak{m}_A^i \neq 0\}$. The *embedding dimension* of A is $H_A(1) = \dim \mathfrak{m}_A/\mathfrak{m}_A^2$. Note that A may be presented as a quotient of a power series ring S over k if and only $\dim S \geq H_A(1)$. Therefore it is most convenient to analyse the case when $\dim S = H_A(1)$, which we will usually do.

Recall that by Macaulay's inverse systems, for every local Gorenstein quotient $A = S/I$ there is an $f \in P$ such that $I = \operatorname{Ann}(f)$, see [28,29], [13, Chapter 21] and [33, Chapter IV]. Such an f is called a dual socle generator of A. Given I, the choice of f is non-unique, but any two choices differ by a unit in S. Conversely, for $f \in P$ we denote

$$\operatorname{Apolar}(f) = S/\operatorname{Ann}(f),$$

which is called the *apolar algebra* of f and $\operatorname{Ann}(f)$ is called the *apolar ideal* of f. It is important that the apolar algebra of the top degree form of f is a quotient of $\operatorname{gr} A$, see [28, Chapter 1]. We may compare A and f as follows:

Algebra A	Module Sf
S-module A	$\simeq Sf$
$\dim_k A$	$= \dim_k Sf$
Socle degree of A	$= \deg(f)$
Certain quotient of $\operatorname{gr} A$	$\simeq \operatorname{Apolar}(f_{\deg(f)})$
If for all i we have $H_A(d-i) = H_A(i)$ then $\operatorname{gr} A$	$\simeq \operatorname{Apolar}(f_{\deg(f)})$

As mentioned in the introduction, our interest is in determining when two local Artinian Gorenstein quotients of the power series ring are isomorphic. The following Proposition 2.14 connects this problem with the results of the previous section. It is well-known, see e.g. [19]. Let S^* denote the group of invertible elements of S and let

$$\mathbb{G} := \operatorname{Aut}(S) \ltimes S^*$$

be the group generated by $\operatorname{Aut}(S)$ and S^* in the group of linear operators on S. As the notation suggests, the group $\mathbb{G}$ is a semidirect product of those groups: indeed $\varphi \circ \mu_s \circ \varphi^{-1} = \mu_{\varphi(s)}$, where φ is an automorphism, $s \in S$ is invertible and μ_s denotes multiplication by s. By Eq. (2) we have an action of $\mathbb{G}$ on P. Here S^* acts by contraction and $\operatorname{Aut}(S)$ acts as described in Proposition 2.9.

Proposition 2.14 *Let $A = S/I$ and $B = S/J$ be two local Artinian Gorenstein algebras. Choose $f, g \in P$ so that $I = \operatorname{Ann}(f)$ and $J = \operatorname{Ann}(g)$. The following conditions are equivalent:*

1. *A and B are isomorphic,*
2. *there exists an automorphism $\varphi : S \to S$ such that $\varphi(I) = J$,*
3. *there exists an automorphism $\varphi : S \to S$ such that $\varphi^\vee(f) = \sigma \lrcorner g$, for an invertible element $\sigma \in S$.*
4. *f and g lie in the same $\mathbb{G}$-orbit of P.*

Proof Taking an isomorphism $A \simeq B$, one obtains $\varphi' : S \to B = S/I$, which can be lifted to an automorphism of S by choosing lifts of linear forms. This proves 1. $\iff$ 2.

2. $\iff$ 3. Let φ be as in 2. Then $\operatorname{Ann}\left(\varphi^{\vee}(f)\right) = \varphi(\operatorname{Ann}(f)) = \varphi(I) = J$. Therefore the principal S-submodules of P generated by $\varphi^{\vee}(f)$ and g are equal, so that there is an invertible element $\sigma \in S$ such that $\varphi^{\vee}(f) = \sigma \lrcorner g$. The argument can be reversed.

Finally, 4. is just a reformulation of 3. □

The invertible element in Point 3 cannot be discarded, see Examples 2.16 and 3.27. The outcome of this theorem is that we are interested in the orbits of elements of P under the $\mathbb{G}$ action.

Example 2.15 (*quadrics*) Let $f \in P_2$ be a quadric of maximal rank. Then $\mathbb{G}\cdot f$ is the set of (divided) polynomials of degree two, whose quadric part is of maximal rank.

Example 2.16 (*compressed cubics*) Assume that k has characteristic not equal to two.

Let $f = f_3 + f_{\leq 2} \in P_{\leq 3}$ be an element of degree three such $H_{\operatorname{Apolar}(f)} = (1, n, n, 1)$, where $n = H_S(1)$. Such a polynomial f is called compressed, see [17,27] or Sect. 3.2 for a definition.

We claim that there is an element $\varphi \in \mathbb{G}$ such that $\varphi^{\vee}(f_3) = f$. This implies that $\operatorname{Apolar}(f) \simeq \operatorname{Apolar}(f_3) = \operatorname{gr} \operatorname{Apolar}(f)$. We say that the apolar algebra of f is *canonically graded*.

Let $A = \operatorname{Apolar}(f_3)$. Since $H_A(2) = n = H_S(1)$, every linear form in P may be obtained as $\delta \lrcorner f$ for some operator $\delta \in S$ of *order two*, see Remark 3.11 below. We pick operators $D_1, \dots, D_n$ so that $\sum x_i \cdot (D_i \lrcorner f_3) = f_2 + f_1$. Explicitly, D_i is such that $D_i \lrcorner f_3 = (\alpha_i \lrcorner f_2)/2 + \alpha_i \lrcorner f_1$. Here we use the assumption on the characteristic.

Let $\varphi : S \to S$ be an automorphism defined via $\varphi(\alpha_i) = \alpha_i + D_i$. Since $(D_i D_j) \lrcorner f = 0$ by degree reasons, the explicit formula in Proposition 2.9 takes the form

$$\varphi^{\vee}(f_3) = f_3 + \sum x_i \cdot (D_i \lrcorner f_3) = f_3 + f_2 + f_1.$$

The missing term f_0 is a constant, so that we may pick an order three operator $\sigma \in S$ with $\sigma \lrcorner \varphi^{\vee}(f_3) = f_0$. Then $(1+\sigma) \lrcorner (\varphi^{\vee}(f_3)) = f_3 + f_2 + f_1 + f_0 = f$, as claimed.

The isomorphism $\operatorname{Apolar}(f) \simeq \operatorname{Apolar}(f_3)$ was first proved by Elias and Rossi, see [16, Thm 3.3]. See Example 2.22 for a more conceptual proof for $k = \mathbb{C}$ and Sect. 3.2 for generalisations. In particular Corollary 3.14 extends this example to cubic and quartic forms.

Remark 2.17 (*Graded algebras*) In the setup of Proposition 2.14 one could specialize to homogeneous ideals I, J and homogeneous polynomials $f, g \in P$. Then Condition 1. is equivalent to the fact that f and g lie in the same $\operatorname{GL}(\mathfrak{m}/\mathfrak{m}^2)$-orbit. The proof of Proposition 2.14 easily restricts to this case, see [22].

Now we turn our attention to derivations. The notation and motivation come from Lie theory, see section on characteristic zero below. Let $\mathfrak{aut}$ denote the space of derivations of S preserving $\mathfrak{m}$, i.e. derivations such that $D(\mathfrak{m}) \subseteq \mathfrak{m}$. Let $S \subset \operatorname{Hom}_k(S, S)$ be given by sending $\sigma \in S$ to multiplication by σ. Let

$$\mathfrak{g} := \mathfrak{aut} + S,$$

where the sum is taken in the space of linear maps from S to S. Then $\mathfrak{g}$ acts on P as defined in Eq. (2). The space $\mathfrak{g}$ is naturally the tangent space to the group $\mathbb{G}$. Similarly, $\mathfrak{aut} \cdot f$ is naturally the tangent space of the orbit $\operatorname{Aut}(S) \cdot f$ and $\mathfrak{g} f$ is the tangent space to the orbit $\mathbb{G}\cdot f$ for every $f \in P$.

Sometimes it is more convenient to work with S than with P. For each subspace $W \subseteq P$ we may consider the orthogonal space $W^{\perp} \subseteq S$. Below we describe the linear space $(\mathfrak{g} f)^{\perp}$.

For $\sigma \in S$ by $\sigma^{(i)}$ we denote the i-th partial derivative of σ. We use the convention that $\deg(0) < 0$.

Proposition 2.18 (Tangent space description) *Let $f \in P$. Then*

$$\mathfrak{aut} \cdot f = \operatorname{span}\langle x_i \cdot (\delta \lrcorner f) \mid \delta \in \mathfrak{m},\ i = 1, \ldots, n\rangle, \qquad \mathfrak{g}f = Sf + \sum_{i=1}^{n} \mathfrak{m}(x_i \cdot f).$$

Moreover

$$(\mathfrak{g}f)^{\perp} = \{\sigma \in S \mid \sigma \lrcorner f = 0,\ \ \forall_i \ \ \deg(\sigma^{(i)} \lrcorner f) \le 0\}.$$

Suppose further that $f \in P$ is homogeneous of degree d. Then $(\mathfrak{g}f)^{\perp}$ is spanned by homogeneous operators and

$$(\mathfrak{g}f)^{\perp}_{\le d} = \{\sigma \in S \mid \sigma \lrcorner f = 0,\ \ \forall_i \ \ \sigma^{(i)} \lrcorner f = 0\}.$$

Proof Let $D \in \mathfrak{aut}$ and $D_i := D(\alpha_i)$. By Proposition 2.10 we have $D^{\vee}(f) = \sum_{i=1}^{n} x_i \cdot (D_i \lrcorner f)$. For any $\delta \in \mathfrak{m}$ we may choose D so that $D_i = \delta$ and all other D_j are zero. This proves the description of $\mathfrak{aut} \cdot f$. Now $\mathfrak{g}f = Sf + \operatorname{span}\langle x_i \cdot (\sigma \lrcorner f) \mid \sigma \in \mathfrak{m},\ i = 1, \ldots, n\rangle$. By Lemma 2.4 we have $x_i(\sigma \lrcorner f) \equiv \sigma \lrcorner (x_i f) \mod Sf$. Thus

$$\mathfrak{g}f = Sf + \operatorname{span}\langle \sigma \lrcorner (x_i \cdot f) \mid \sigma \in \mathfrak{m},\ i = 1, \ldots, n\rangle = Sf + \sum \mathfrak{m}(x_i f).$$

Now let $\sigma \in S$ be an operator such that $\langle \sigma, \mathfrak{g}f\rangle = 0$. This is equivalent to $\sigma \lrcorner (\mathfrak{g}f) = 0$, which simplifies to $\sigma \lrcorner f = 0$ and $(\sigma\mathfrak{m}) \lrcorner (x_i f) = 0$ for all i. We have $\sigma \lrcorner (x_i f) = x_i(\sigma \lrcorner f) + \sigma^{(i)} \lrcorner f = \sigma^{(i)} \lrcorner f$, thus we get equivalent conditions:

$$\sigma \lrcorner f = 0 \quad \text{and} \quad \mathfrak{m} \lrcorner (\sigma^{(i)} \lrcorner f) = 0,$$

and the claim follows. Finally, if f is homogeneous of degree d and $\sigma \in S$ is homogeneous of degree at most d then $\sigma^{(i)} \lrcorner f$ has no constant term and so $\deg(\sigma^{(i)} \lrcorner f) \le 0$ implies that $\sigma^{(i)} \lrcorner f = 0$. $\square$

Remark 2.19 Let $f \in P$ be homogeneous of degree d. Let $i \le d$ and $K_i := (\mathfrak{g}f)^{\perp}_i$. Proposition 2.18 gives a useful connection of K_i with the conormal sequence. Namely, let $I = \operatorname{Ann}(f)$ and $B = \operatorname{Apolar}(f) = S/I$. We have $(I^2)_i \subseteq K_i$ and the quotient space fits into the conormal sequence of $S \to B$, making that sequence exact:

$$0 \to (K/I^2)_i \to (I/I^2)_i \to (\Omega_{S/k} \otimes B)_i \to (\Omega_{B/k})_i \to 0. \tag{7}$$

This is expected from the point of view of deformation theory. Recall that by [24, Theorem 5.1] the deformations of B over $k[\varepsilon]/\varepsilon^2$ are in one-to-one correspondence with elements of a k-linear space $T^1(B/k, B)$. On the other hand, this space fits [24, Prop 3.10] into the sequence

$$0 \to \operatorname{Hom}_B(\Omega_{B/k}, B) \to \operatorname{Hom}_B(\Omega_{S/k} \otimes B, B) \to \operatorname{Hom}_B(I/I^2, B) \to T^1(B/k, B) \to 0.$$

Since B is Gorenstein, $\operatorname{Hom}_B(-, B)$ is exact and we have $T^1(B/k, B)_i \simeq \operatorname{Hom}(K/I^2, B)_i$ for all $i \ge 0$. The restriction $i \ge 0$ appears because $\operatorname{Hom}(K/I^2, B)$ is the tangent space to deformations of B inside S, whereas $T^1(B/k, B)$ parameterises all deformations.

Now we introduce a certain subgroup $\mathbb{G}^+$ of $\mathbb{G}$. It plays an important part in characteristic zero, because $\mathbb{G}^+$-orbits are closed in P. This group is also very useful in applications, because it preserves the top degree form, which allows induction on the degree type arguments.

Each automorphism of S induces a linear map on its cotangent space: we have a restriction $\mathrm{Aut}(S) \to \mathrm{GL}(\mathfrak{m}/\mathfrak{m}^2)$. Let us denote by $\mathrm{Aut}^+(S)$ the group of automorphisms which act as identity on the tangent space: $\mathrm{Aut}^+(S) = \left\{\varphi \in S \mid \forall_i\ \varphi(\alpha_i) - \alpha_i \in \mathfrak{m}^2\right\}$. We have the following sequence of groups:

$$1 \to \mathrm{Aut}^+(S) \to \mathrm{Aut}(S) \to \mathrm{GL}(\mathfrak{m}/\mathfrak{m}^2) \to 1. \tag{8}$$

We define

$$\mathbb{G}^+ = \mathrm{Aut}^+(S) \ltimes (1+\mathfrak{m}) \subseteq \mathbb{G}.$$

Note that we have the following exact sequence:

$$1 \to \mathbb{G}^+ \to \mathbb{G} \to \mathrm{GL}(\mathfrak{m}/\mathfrak{m}^2) \times k^* \to 1. \tag{9}$$

Correspondingly, let $\mathfrak{aut}$ denote the space of derivations preserving $\mathfrak{m}$, i.e. derivations such that $D(\mathfrak{m}) \subseteq \mathfrak{m}$. Let $\mathfrak{aut}^+$ denote the space of derivations such that $D(\mathfrak{m}) \subseteq \mathfrak{m}^2$. Denoting by $\mathfrak{gl}\left(\mathfrak{m}/\mathfrak{m}^2\right)$ the space of linear endomorphisms of $\mathfrak{m}/\mathfrak{m}^2$, we have we following sequence of linear spaces:

$$0 \to \mathfrak{aut}^+ \to \mathfrak{aut} \to \mathfrak{gl}\left(\mathfrak{m}/\mathfrak{m}^2\right) \to 0. \tag{10}$$

We define

$$\mathfrak{g}^+ = \mathfrak{aut}^+ + \mathfrak{m}.$$

Following the proof of Proposition 2.18 we get the following proposition.

Proposition 2.20 *Let $f \in P$. Then $\mathfrak{g}^+ f = \mathfrak{m} f + \sum \mathfrak{m}^2(x_i f)$ so that*

$$\left(\mathfrak{g}^+ f\right)^\perp = \{\sigma \in S \mid \deg(\sigma \lrcorner f) \le 0,\ \ \forall_i\ \deg(\sigma^{(i)} \lrcorner f) \le 1\}.$$

If f is homogeneous of degree d then $\mathfrak{g}^+ f$ is spanned by homogeneous polynomials and

$$\left(\mathfrak{g}^+ f\right)^\perp_{<d} = \{\sigma \in S \mid \sigma \lrcorner f = 0,\ \ \forall_i\ \sigma^{(i)} \lrcorner f = 0\} = (\mathfrak{g} f)^\perp_{<d}.$$

2.4 Characteristic zero

For simplicity we restrict to $k = \mathbb{C}$ in this section, to freely speak about Lie groups and algebras. As a general reference we suggest [21]. One technical problem is that the group $\mathrm{Aut}(S)$ is infinite-dimensional. In this section we implicitly replace S with $S/\mathfrak{m}^D$ and $\mathrm{Aut}(S)$ with the group $\mathrm{Aut}(S/\mathfrak{m}^D)$, where $D \gg 0$ is larger than the degree of any element of P considered. Then $\mathrm{Aut}(S)$ and $\mathbb{G}$ become Lie groups.

Note that the group $\mathbb{G}^+ \subseteq \mathrm{GL}(S)$ is *unipotent*: the map $\varphi - \mathrm{id}$ is nilpotent for every $\varphi \in \mathbb{G}^+$. Therefore, we get the following theorem.

Theorem 2.21 *For every $f \in P$ the orbit $\mathbb{G}^+ \cdot f$ is closed in P, in both Euclidean and Zariski topologies.*

Classically, the Lie algebra corresponding to $\mathrm{Aut}(S)$ is $\mathfrak{aut}$. Moreover by the exact sequences (8) and (10), the Lie algebra of $\mathrm{Aut}^+(S)$ is $\mathfrak{aut}^+$. Finally, the Lie algebra of $\mathbb{G}$ is $\mathfrak{aut} + S = \mathfrak{g}$ and the Lie algebra of $\mathbb{G}^+$ is $\mathfrak{aut}^+ + \mathfrak{m} = \mathfrak{g}^+$. In particular

$$\dim \mathbb{G} \cdot f = \dim(\mathfrak{aut} \cdot f + Sf) = \dim(Sf + \sum \mathfrak{m}(x_i f)).$$

We now give another proof of Elias–Rossi theorem on canonically graded algebras, in the following Example 2.22.

Example 2.22 (*compressed cubics, using Lie theoretic ideas*) Assume that $k = \mathbb{C}$.

Let $f = f_3 + f_{\leq 2} \in P_{\leq 3}$ be an element of degree three such $H_{\mathrm{Apolar}(f)} = (1, n, n, 1)$, where $n = H_S(1)$. Such a polynomial f is called compressed, see [17,27] and Sect. 3.2 for a definition. Then $H_{\mathrm{Apolar}(f)}$ is symmetric, thus $H_{\mathrm{Apolar}(f_3)} = (1, n, n, 1)$ as explained in Sect. 2.3.

We claim that there is an element φ of $\mathbb{G}$ such that $\varphi^\vee(f_3) = f$. This proves that $\mathrm{Apolar}(f) \simeq \mathrm{Apolar}(f_3) = \mathrm{gr}\ \mathrm{Apolar}(f)$. We say that the apolar algebra of f is *canonically graded*.

In fact, we claim that already $\mathbb{G}^+ \cdot f_3$ is the whole space:

$$\mathbb{G}^+ \cdot f_3 = f_3 + P_{\leq 2}.$$

From the explicit formula in Proposition 2.9 we see that $\mathbb{G}^+ \cdot f_3 \subseteq f_3 + P_{\leq 2}$. Then it is a Zariski closed subset. To prove equality it enough to prove that these spaces have the same dimension, in other words that $\mathfrak{g}^+ f_3 = P_{\leq 2}$.

Let $\sigma \in \left(\mathfrak{g}^+ f_3\right)^\perp_{\leq 2}$ be non-zero. By Proposition 2.20 we get that $\sigma \lrcorner f_3 = 0$ and $\sigma^{(i)} \lrcorner f_3 = 0$ for all i. Then there is a degree one operator annihilating f_3. But this contradicts the fact that $H_{\mathrm{Apolar}(f_3)}(1) = n = H_S(1)$. Therefore $\left(\mathfrak{g}^+ f_3\right)^\perp_{\leq 2} = 0$ and the claim follows.

3 Applications

In this part of the paper we give several new classification results using the above theory. The two most important ones are the t-compressed algebras in Sect. 3.2, and algebras having Hilbert function (1, 3, 3, 3, 1) in Example 3.27. We begin and end with some examples computable by hand, which are nevertheless useful, see in particular the reference in Example 3.1.

We make the following assumption

Assumption The field k is algebraically closed. It has characteristic zero or greater than the degree of all considered elements of P.

For example, when we analyse cubics, we assume that k has characteristic different from 2 or 3. When analysing degree d polynomials, as in Example 3.1, we assume that the characteristic is at least $d + 1$ or 0. The fact that k is algebraically closed is assumed because of various geometrical arguments involved, such as in Proposition 3.3, and because of assumptions required for the references: Example 3.27 refers to [6,32] which both work over an algebraically closed field. It seems plausible that the classification is feasible over non-algebraically closed fields, however then there will be more isomorphism types: already the classifications of homogeneous quadrics over $\mathbb{C}$ and $\mathbb{R}$ differ. In contrast, the assumption that k has large enough characteristic is hardly avoidable since key results are false without it, see Example 3.15 and compare with Example 2.16. This assumption is frequently used to guarantee that a non-constant element of S has a non-zero derivative, which is needed to effectively apply Proposition 2.18, see Proposition 3.12.

Under assumption, we may always think of P as a polynomial ring, see Proposition 2.6. Before we begin, we make the following observation. As shown by Eq. (9), the group $\mathbb{G}$ is build from $\mathbb{G}^+$, $\mathrm{GL}(\mathfrak{m}/\mathfrak{m}^2)$ and k^*. We will be mostly concerned with $\mathbb{G}^+$, in particular we will analyse its orbits. This often suffices to prove important results as in the cubic case above (Example 2.16) or below for t-compressed algebras. However sometimes the action of $\mathbb{G}^+$ is not sufficient and the difference between this group and $\mathbb{G}$ becomes the substance of the result. This is the case for algebras having Hilbert function (1, 3, 3, 3, 1). The group $\mathbb{G}^+$

is also convenient because it allows degree-by-degree analysis which is our main technical tool, see Proposition 3.3.

When classifying Artinian Gorenstein quotients of S with Hilbert function H we will usually assume that $\dim S = H(1)$. This does not reduce generality and simplifies the argument a bit.

3.1 Basic examples and tools

In this section we analyse the $\mathbb{G}$-orbits of "easy" homogeneous polynomials: those with small Waring and border Waring rank.

Example 3.1 (*Rank one*) The simplest example seems to be the case $f = \ell^d$ for some linear form ℓ. We may assume $\ell = x_1$ and apply Proposition 2.20 to see that

$$\left(\mathfrak{g}^+ x_1^d\right)^{\perp}_{<d} = (\alpha_2, \ldots, \alpha_n)^2, \quad \mathfrak{g}^+ x_1^d = \operatorname{span}\left\langle x_1^r m \mid m \in P_{\leq 1}, r+1 < d\right\rangle,$$

or, in an invariant form, $\mathfrak{g}^+ \ell^d = \operatorname{span}\left\langle \ell^r \cdot m \mid m \in P_{\leq 1},\ r+1 < d\right\rangle$. Using Proposition 2.9 one can even compute the orbit itself. For example, when $d = 4$, the orbit is equal to

$$\mathbb{G}^+ \cdot \ell^4 = \left\{\ell^4 + \ell^2 \cdot m_1 + \ell \cdot m_2 + m_3 + m_1^2 \mid m_i \in P_{\leq 1}\right\}.$$

This example together with Corollary 3.4 plays an important role in the paper [6], compare [6, Lemma 4.2, Example 4.4, Proposition 5.13].

Example 3.2 (Border rank two) Consider $f = x^{d-1}y$. Assume $S = k[[\alpha, \beta]]$ and $P = k_{dp}[x, y]$. As above, the apolar ideal of f is monomial, equal to (α^d, β^2). Using Proposition 2.20 we easily get that

$$(\mathfrak{g}^+ f)^{\perp}_{<d} = (\beta^3)_{<d},$$

so $\mathfrak{g}^+ f$ is spanned by monomials $x^a y^b$, where $b \leq 2$ and $a + b < d$. Note that in contrast with Example 3.1, the equality $\left(\mathfrak{g}^+ f\right)^{\perp}_{<d} = \operatorname{Ann}(f)^2_{<d}$ does *not* hold. This manifests the fact that $\operatorname{Apolar}(f)$ is a deformation of $\operatorname{Apolar}(x^d + y^d)$.

Before we analyse the rank two polynomials i.e. $x^d + y^d$, we need a few more observations.

Let for every $g \in P$ the symbol $\operatorname{tdf}(g)$ denote the top degree form of g, so that for example $\operatorname{tdf}(x_1^3 + x_2^2 x_3 + x_4^2) = x_1^3 + x_2^2 x_3$.

Proposition 3.3 *Let $f \in P$. Then the top degree form of every element of $\mathbb{G}^+ \cdot f$ is equal to the top degree form of f. Moreover,*

$$\left\{\operatorname{tdf}(g - f) \mid g \in \mathbb{G}^+ \cdot f\right\} = \left\{\operatorname{tdf}(h) \mid h \in \mathfrak{g}^+ f\right\}. \tag{11}$$

If f is homogeneous, then both sides of (11) *are equal to the set of homogeneous elements of $\mathfrak{g}^+ f$.*

Proof Consider the S-action on $P_{\leq d}$. This action descents to an $S/\mathfrak{m}^{d+1}$ action. Further in the proof we implicitly replace S by $S/\mathfrak{m}^{d+1}$, thus also replacing $\operatorname{Aut}(S)$ and $\mathbb{G}$ by appropriate truncations. Let $\varphi \in \mathbb{G}^+$. Since $(\mathrm{id} - \varphi)\left(\mathfrak{m}^i\right) \subseteq \mathfrak{m}^{i+1}$ for all i, we have $(\mathrm{id} - \varphi)^{d+1} = 0$. By our global assumption on the characteristic $D := \log(\varphi)$ is well-defined and $\varphi = \exp(D)$. We get an injective map $\exp : \mathfrak{g}^+ \to \mathbb{G}^+$ with left inverse log. Since exp is algebraic we see by dimension count that its image is open in $\mathbb{G}^+$. Since log is Zariski-continuous, we get that $\log(\mathbb{G}^+) \subseteq \mathfrak{g}^+$, then $\exp : \mathfrak{g}^+ \to \mathbb{G}^+$ is an isomorphism.

Therefore

$$\varphi^\vee(f) = f + \sum_{i=1}^{d} \frac{\left(D^\vee\right)^i(f)}{i!} = f + D^\vee(f) + \left(\sum_{i=1}^{d-1} \frac{\left(D^\vee\right)^i}{(i+1)!}\right) D^\vee(f).$$

By Remark 2.11 the derivation $D \in \mathfrak{g}^+$ lowers the degree, we see that $\mathrm{tdf}(\varphi^\vee f) = \mathrm{tdf}(f)$ and $\mathrm{tdf}(\varphi^\vee f - f) = \mathrm{tdf}(D^\vee(f))$. This proves (11). Finally, if f is homogeneous then $\mathfrak{g}^+ f$ is equal to the span $\langle \mathrm{tdf}(h) \mid h \in \mathfrak{g}^+ f\rangle$ by Proposition 2.20, and the last claim follows.

For an elementary proof, at least for the subgroup $\mathrm{Aut}^+(S)$, see [35, Proposition 1.2]. □

The following almost tautological Corollary 3.4 enables one to prove that a given apolar algebra is canonically graded inductively, by lowering the degree of the remainder.

Corollary 3.4 *Let F and f be polynomials. Suppose that the leading form of $F - f$ lies in $\mathfrak{g}^+ F$. Then there is an element $\varphi \in \mathbb{G}^+$ such that* $\deg(\varphi^\vee f - F) < \deg(f - F)$.

Proof Let G be the leading form of $f - F$ and e be its degree. By Proposition 3.3 we may find $\varphi \in \mathrm{Aut}^+(S)$ such that $\mathrm{tdf}(\varphi^\vee(F) - F) = -G$, so that $\varphi^\vee(F) \equiv F - G \mod P_{\le e-1}$. By the same proposition we have $\deg(\varphi^\vee(f - F) - (f - F)) < \deg(f - F) = e$, so that $\varphi^\vee(f - F) \equiv f - F \mod P_{\le e-1}$. Therefore $\varphi^\vee(f) - F = \varphi^\vee(F) + \varphi^\vee(f - F) - F \equiv f - G - F \equiv 0 \mod P_{\le e-1}$, as claimed. □

Example 3.5 (Rank two)

Let $P = k_{dp}[x, y]$, $S = k[[\alpha, \beta]]$ and $F = x^d + y^d$ for some $d \ge 2$. Then $H_{\mathrm{Apolar}(F)} = (1, 2, 2, 2, \ldots, 2, 1)$. We claim that the orbit $\mathbb{G}^+ \cdot F$ consists precisely of polynomials f having leading form F and such that $H_{\mathrm{Apolar}(f)} = H_{\mathrm{Apolar}(F)}$.

Let us first compute $\left(\mathfrak{g}^+ F\right)^\perp$. Since F is homogeneous, Proposition 2.20 shows that

$$\left(\mathfrak{g}^+ F\right)^\perp_{<d} = \{\sigma \in S_{<d} \mid \sigma^{(x)} \lrcorner F = \sigma^{(y)} \lrcorner F = \sigma \lrcorner F = 0\}.$$

Since $\mathrm{Ann}\,(F) = (\alpha\beta, \alpha^d - \beta^d)$, we see that

$$\left(\mathfrak{g}^+ F\right)^\perp_{<d} = (\alpha\beta)^2_{<d}.$$

Now we proceed to the description of $\mathbb{G}^+ \cdot F$. It is clear that every $f \in \mathbb{G}^+ \cdot F$ satisfies the conditions given. Conversely, suppose that $f \in P$ satisfies these conditions: the leading form of f is F and $H_{\mathrm{Apolar}(f)} = H_{\mathrm{Apolar}(F)}$. If $d = 2$, then the claim follows from Example 2.15, so we may assume $d \ge 3$. By applying α^{d-2} and β^{d-2} to f, we see that x^2 and y^2 are leading forms of partials of f. Since $H_{\mathrm{Apolar}(f)}(2) = 2$, these are the only leading forms of partials of degree two.

Let G be the leading form of $f - F$. Suppose that G contains a monomial $x^a y^b$, where $a, b \ge 2$. Then $\alpha^{a-1}\beta^{b-1} \lrcorner f = xy + l$, where l is linear, then we get a contradiction with the conclusion of the previous paragraph.

Since G contains no monomials of the form $x^a y^b$ with $a, b \ge 2$, we see that G is annihilated by $\left(\mathfrak{g}^+ F\right)^\perp$, so it lies in $\mathfrak{g}^+ F$. By Corollary 3.4 we may find $u \in \mathbb{G}^+$ such that $\deg(uf - F) < \deg(f - F)$. Thus, replacing f by uf we lower the degree of $f - F$. Repeating, we arrive to the point where $f - F = 0$, so that $f = F$.

The analysis made in Example 3.5 may be generalized to obtain the following Proposition 3.6.

Proposition 3.6 *Let $f \in P$ be a polynomial with leading form F. Let $I = \mathrm{Ann}(F)$. Fix an integer $t \geq 0$ and assume that*

1. $\dim_k \mathrm{Apolar}(f) = \dim_k \mathrm{Apolar}(F)$.
2. *we have* $\left(\mathfrak{g}^+ F\right)_i^{\perp} = I_i^2$ *for all i satisfying $t \leq i \leq d-1$.*

Then there is an element $F + g \in \mathbb{G}^+ \cdot f$ such that $\deg(g) < t$. *Equivalently,* $\mathrm{Apolar}(f) \simeq \mathrm{Apolar}(F+g)$ *for some polynomial g of degree less than t.*

Proof We apply induction with respect to $\deg(f - F)$. If $\deg(f - F) < t$ then we are done. Otherwise, it is enough to find $u \in \mathbb{G}^+$ such that $\deg(uf - F) < \deg(f - F)$.

Since $\mathrm{Apolar}(F)$ is a quotient of $\mathrm{gr}\ \mathrm{Apolar}(f)$, the first condition implies that

$$\mathrm{Apolar}(F) \simeq \mathrm{gr}\ \mathrm{Apolar}(f),$$

Thus for every element $i \in I = \mathrm{Ann}(F)$ we may find $\sigma \in S$ such that $(i + \sigma) \lrcorner f = 0$ and $\mathrm{ord}(\sigma) > \mathrm{ord}(i)$.

Let G be the leading form of $f - F$ and $t \leq r \leq d-1$ be its degree. We will now prove that $G \in \mathfrak{g}^+ F$. By assumption, it is enough to show that I_r^2 annihilates G. The ideal I is homogeneous so it is enough to show that for any elements $i, j \in I$ such that $\deg(ij) = r$ we have

$$(ij) \lrcorner G = 0.$$

Take $\sigma \in S$ such that $(j - \sigma) \lrcorner f = 0$. Then $\mathrm{ord}(i\sigma) > \mathrm{ord}(ij) = r = \deg(f - F)$, thus $(i\sigma) \lrcorner f = (i\sigma) \lrcorner F = 0$. Therefore,

$$(ij) \lrcorner G = (ij) \lrcorner (F + G) = (ij) \lrcorner f = i(j - \sigma) \lrcorner f = 0.$$

and the claims follow. □

Example 3.7 (Rank n) Let $f \in P = k_{dp}[x_1, \ldots, x_n]$ be a polynomial of degree $d \geq 4$ with leading form $F = x_1^d + \cdots + x_n^d$ and Hilbert function $H_{\mathrm{Apolar}(f)} = (1, n, n, n, \ldots, n, 1)$. Then the apolar algebra of f is isomorphic to an apolar algebra of $F + g$, where g is a polynomial of degree at most three.

Indeed, let $I = \mathrm{Ann}(F)$. Then the generators of $I_{<d}$ are $\alpha_i \alpha_j$ for $i \neq j$, thus the ideals $I_{<d}$ and $(I^2)_{<d}$ are monomial. Also the ideal $\left(\mathfrak{g}^+ F\right)^{\perp}$ is monomial by its description from Proposition 2.20. The only monomials of degree at least 4 which do not lie in I^2 are of the form $\alpha_i^{d-1}\alpha_j$ and these do not lie in $\left(\mathfrak{g}^+ F\right)^{\perp}$. Therefore $I_i^2 = \left(\mathfrak{g}^+ F\right)_i^{\perp}$ for all $4 \leq i \leq d-1$. The assumptions of Proposition 3.6 are satisfied with $t = 4$ and the claim follows.

It is worth noting that for every g of degree at most three the apolar algebra of $F + g$ has Hilbert function $(1, n, n, \ldots, n, 1)$, so the above considerations give a full classification of polynomials in P with this Hilbert function and leading form F.

Since $\left(\mathfrak{g}^+ F\right)_3^{\perp} = \mathrm{span}\left\langle \alpha_i \alpha_j \alpha_l \mid i < j < l \right\rangle$, we may also assume that $f = F + \sum_{i<j<l} \lambda_{ijl} x_i x_j x_l$ for some $\lambda_{ijl} \in k$, see also Example 3.27.

3.2 Compressed algebras and generalisations

In this section we use Proposition 2.18 to obtain a generalisation of wonderful results of Elias and Rossi on being canonically graded (see Corollary 3.14) and answer their question stated in [17, Remark 3.6].

Example 2.16 is concerned with a degree three polynomial f such that the Hilbert function of $\mathrm{Apolar}(f)$ is maximal i.e. equal to $(1, n, n, 1)$ for $n = H_S(1)$. Below we generalise the results obtained in this example to polynomials of arbitrary degree.

Recall that a local Artinian Gorenstein algebra A of socle degree d (see Sect. 2.3) is called *compressed* if

$$H_A(i) = \min\left(H_S(i), H_S(d-i)\right) = \min\left(\binom{i+n-1}{i}, \binom{d-i+n-1}{d-i}\right)$$
$$\text{for all } \; i = 0, 1, \ldots, d.$$

Here we introduce a slightly more general notation.

Definition 3.8 (t-*compressed*) Let $A = S/I$ be a local Artinian Gorenstein algebra of socle degree d. Let $t \geq 1$. Then A is called t-*compressed* if the following conditions are satisfied:

1. $H_A(i) = H_S(i) = \binom{i+n-1}{i}$ for all $0 \leq i \leq t$,
2. $H_A(d-1) = H_S(1)$.

Example 3.9 Let $n = 2$. Then $H_A = (1, 2, 2, 1, 1)$ is not t-compressed, for any t. The function $H_A = (1, 2, 3, 2, 2, 2, 1)$ is 2-compressed. For any sequence $*$ the function $(1, 2, *, 2, 1)$ is 1-compressed.

Note that it is always true that $H_A(d-1) \leq H_A(1) \leq H_S(1)$, thus both conditions above assert that the Hilbert function is maximal possible. Therefore they are open in $P_{\leq d}$.

Remark 3.10 The maximal value of t, for which t-compressed algebras exists, is $t = \lfloor d/2 \rfloor$. Every compressed algebra is t-compressed for $t = \lfloor d/2 \rfloor$ but not vice versa. If A is graded, then $H_A(1) = H_A(d-1)$, so the condition $H_A(d-1) = H_S(1)$ is satisfied automatically.

The following technical Remark 3.11 will be useful later. Up to some extent, it explains the importance of the second condition in the definition of t-compressed algebras.

Remark 3.11 Let $A = \mathrm{Apolar}(f)$ be a t-compressed algebra with maximal ideal $\mathfrak{m}_A$. We have $\dim P_{\leq 1} = H_A(d-1) + H_A(d) = \dim \mathfrak{m}_A^{d-1}/\mathfrak{m}_A^d + \dim \mathfrak{m}_A^d = \dim \mathfrak{m}_A^{d-1}$. Moreover $\mathfrak{m}_A^{d-1} \simeq \mathfrak{m}^{d-1} f$ as linear spaces and $\mathfrak{m}^{d-1} f \subseteq P_{\leq 1}$. Thus

$$\mathfrak{m}^{d-1} f = P_{\leq 1}.$$

The definition of t-compressed algebras explains itself in the following Proposition 3.12.

Proposition 3.12 *Let $f \in P$ be a polynomial of degree $d \geq 3$ and A be its apolar algebra. Suppose that A is t-compressed. Then the $\mathbb{G}^+$-orbit of f contains $f + P_{\leq t+1}$. In particular $f_{\geq t+2} \in \mathbb{G}^+ \cdot f$, so that* $\mathrm{Apolar}(f) \simeq \mathrm{Apolar}(f_{\geq t+2})$.

Proof First we show that $P_{\leq t+1} \subseteq \mathfrak{g}^+ f$, i.e. that no non-zero operator of order at most $t+1$ lies in $\left(\mathfrak{g}^+ f\right)^{\perp}$. Pick such an operator. By Proposition 2.20 it is not constant. Let σ' be any of its non-zero partial derivatives. Proposition 2.20 asserts that $\deg(\sigma' \lrcorner f) \leq 1$. Let $\ell := \sigma' \lrcorner f$. By Remark 3.11 every linear polynomial is contained in $\mathfrak{m}^{d-1} f$. Thus we may choose a $\delta \in \mathfrak{m}^{d-1}$ such that $\delta \lrcorner f = \ell$. Then $(\sigma' - \delta) \lrcorner f = 0$. Since $d \geq 3$, we have $d - 1 > \lfloor d/2 \rfloor \geq t$, so that $\sigma - \delta$ is an operator of order at most t annihilating f. This contradicts the fact that $H_A(i) = H_S(i)$ for all $i \leq t$. Therefore $P_{\leq t+1} \subseteq \mathfrak{g}^+ f$.

Second, pick a polynomial $g \in f + P_{\leq t+1}$. We prove that $g \in \mathbb{G}^+ \cdot f$ by induction on $\deg(g - f)$. The top degree form of $g - f$ lies in $\mathfrak{g}^+ f$. Using Corollary 3.4 we find $\varphi \in \mathbb{G}^+$ such that $\deg(\varphi^{\vee}(g) - f) < \deg(g - f)$. $\square$

For completeness, we state the following consequence of the previous result.

Corollary 3.13 *Let $f \in P$ be a polynomial of degree $d \geq 3$ and A be its apolar algebra. Suppose that A is compressed. Then $A \simeq \mathrm{Apolar}(f_{\geq \lfloor d/2 \rfloor + 2})$.*

Proof The algebra A is $\lfloor d/2 \rfloor$-compressed and the claim follows from Proposition 3.12. □

As a corollary we reobtain the result of Elias and Rossi, see [17, Thm 3.1].

Corollary 3.14 *Suppose that A is a compressed Artinian Gorenstein local k-algebra of socle degree $d \leq 4$. Then A is canonically graded i.e. isomorphic to its associated graded algebra* $\mathrm{gr}\, A$.

Proof The case $d \leq 2$ is easy and left to the reader. We assume $d \geq 3$, so that $3 \leq d \leq 4$.

Fix $n = H_A(1)$ and choose $f \in P = k_{dp}[x_1, \dots, x_n]$ such that $A \simeq \mathrm{Apolar}(f)$. This is possible by the existence of standard form, see [28, Thm 5.3AB] or [30]. Let f_d be the top degree part of f. Since $\lfloor d/2 \rfloor + 2 = d$, Corollary 3.13 implies that $f_d \in \mathbb{G}^+ \cdot f$. Therefore the apolar algebras of f and f_d are isomorphic. The algebra $\mathrm{Apolar}(f_d)$ is a quotient of $\mathrm{gr}\,\mathrm{Apolar}(f)$. Since $\dim_k \mathrm{gr}\,\mathrm{Apolar}(f) = \dim_k \mathrm{Apolar}(f) = \dim_k \mathrm{Apolar}(f_d)$ it follows that

$$\mathrm{Apolar}(f) \simeq \mathrm{Apolar}(f_d) \simeq \mathrm{gr}\,\mathrm{Apolar}(f),$$

which was to be proved. □

The above Corollary 3.14 holds under the assumptions that k is algebraically closed and of characteristic not equal to 2 or 3. The assumption that k is algebraically closed is unnecessary as proven for cubics in Example 2.16, the cases of quartics is similar. Surprisingly, the assumption on the characteristic is necessary.

Example 3.15 (compressed cubics in characteristic two) Let k be a field of characteristic two. Let $f_3 \in P_3$ be a cubic form such that $H_{\mathrm{Apolar}(f_3)} = (1, n, n, 1)$ and $\alpha_1^2 \lrcorner f_3 = 0$. Then there is a degree three polynomial f with leading form f_3, whose apolar algebra is compressed but not canonically graded.

Indeed, take $\sigma = \alpha_1^2$. Then all derivatives of σ are zero because the characteristic is two. By Proposition 2.18 the element σ lies in $(\mathfrak{g} f_3)^\perp$. Thus $\mathfrak{g} f_3$ does not contain $P_{\leq 2}$ and so $\mathbb{G} \cdot f_3$ does not contain $f_3 + P_{\leq 2}$. Taking any $f \in f_3 + P_{\leq 2}$ outside the orbit yields the desired polynomial. In fact, one can explicitly check that $f = f_3 + x_i^{[2]}$ is an example.

A similar example shows that over a field of characteristic three there are compressed quartics which are not canonically graded.

Compressed algebras of socle degree $d \geq 5$ in two variables. As noted in [17, Example 3.4], the claim of Corollary 3.14 is false for $d = 5$. Below we explain this from the point of view of our theory. First, we give an example of a compressed algebra of socle degree five, which is not canonically graded.

Example 3.16 ($H_A = (1, 2, 3, 3, 2, 1)$, special) Let $n = 2$, $P = k_{dp}[x_1, x_2]$ and $S = k[[\alpha_1, \alpha_2]]$. Take $F = x_1^3 x_2^2 \in P$ and $A = \mathrm{Apolar}(F) = S/(\alpha_1^4, \alpha_2^3)$. Since x_1^2, x_1x_2, x_2^2 are all partials of F we have $H_A(2) = 3$ and the algebra A is compressed. It is crucial to note that

$$\alpha_2^4 \in (\mathfrak{g} F)^\perp. \qquad (12)$$

Indeed, the only nontrivial derivative of this element is $4\alpha_2^3 \in \mathrm{Ann}\,(F)$, so (12) follows from Proposition 2.18. Therefore $\mathbb{G}\cdot F$ is strictly contained in $F + P_{\leq 4}$. Pick any element $f \in F + P_{\leq 4}$ not lying in $\mathbb{G}\cdot F$. The associated graded of $\mathrm{Apolar}(f)$ is equal to $\mathrm{Apolar}(F)$ but $\mathrm{Apolar}(f)$ is *not* isomorphic to $\mathrm{Apolar}(F)$, thus $\mathrm{Apolar}(f)$ is not canonically graded. In fact one may pick $f = F + x_2^4$, so that $\mathrm{Apolar}(f) = S/(\alpha_1^4, \alpha_2^3 - \alpha_1^3\alpha_2)$.

On the positive side, Corollary 3.13 implies that every compressed algebra of socle degree 5 is isomorphic to $\mathrm{Apolar}(f)$ for some $f = f_5 + f_4$. Thus we may always remove the cubic part of f. But even more is true: a *general* compressed algebra of socle degree five is canonically graded, as shown in Example 3.17 below.

Example 3.17 ($H_A = (1, 2, 3, 3, 2, 1)$, *general*) Let $f \in P = k_{dp}[x_1, x_2]$ be a *general* polynomial of degree five with respect to the natural affine space structure on $k_{dp}[x_1, x_2]_{\leq 5}$. Then $\mathrm{Apolar}(f)$ and $\mathrm{Apolar}(f_5)$ are compressed by [27]. The ideal $\mathrm{Ann}\,(f_5)$ is a complete intersection of a cubic and a quartic. Since we assumed that f is general, we may also assume that the cubic generator c of $\mathrm{Ann}\,(f_5)$ is not a power of a linear form.

We claim that $\mathrm{Apolar}(f)$ is canonically graded. Equivalently, we claim that $f \in \mathbb{G}\cdot f_5$. It is enough to show that

$$\mathbb{G}^+\cdot f_5 = f_5 + P_{\leq 4}. \tag{13}$$

We will show that $\left(\mathfrak{g}^+ f_5\right)^{\perp}_{\leq 4} = 0$. This space is spanned by homogeneous elements. Suppose that this space contains a non-zero homogeneous σ. Proposition 2.20 implies that σ is non-constant and that all partial derivatives of σ annihilate f_5. These derivatives have degree at most three, thus they are multiples of the cubic generator c of $\mathrm{Ann}\,(f_5)$. This implies that all partial derivatives of σ are proportional, so that σ is a power of a linear form. Then also c is a power of a linear form, which contradicts earlier assumption. We conclude that no non-zero $\sigma \in \left(\mathfrak{g}^+ f_5\right)^{\perp}_{\leq 4}$ exists. Now, the Eq. (13) follows from Corollary 3.4.

Example 3.18 (*general polynomials in two variables of large degree*) Let $F \in P = k_{dp}[x_1, x_2]$ be a homogeneous form of degree $d \geq 9$ and assume that no linear form annihilates F. The ideal $\mathrm{Ann}\,(F) = (q_1, q_2)$ is a complete intersection. Let $d_i := \deg q_i$ for $i = 1, 2$ then $d_1 + d_2 = d + 2$. Since $d_1, d_2 \geq 2$, we have $d_1, d_2 \leq d$.

We claim that the apolar algebra of a general (in the sense explained in Example 3.17) polynomial $f \in P$ with leading form F is not canonically graded. Indeed, we shall prove that $\mathbb{G}\cdot F \cap (F + P_{<d})$ is strictly contained in $F + P_{<d}$ which is the same as to show that $K := (\mathfrak{g}F)^{\perp}_{<d}$ is non-zero.

The sequence (7) from Remark 2.19 becomes

$$0 \to (K/I^2)_{d-1} \to (A[-d_1] \oplus A[-d_2])_{d-1} \to (A \oplus A)_{d-2}\,,$$

where the rightmost map has degree -1. Note that

$$\begin{aligned}\dim\,(A[-d_1] \oplus A[-d_2])_{d-1} &= (d - 1 - d_1 + 1) + (d - 1 - d_2 + 1)\\ &= 2d - (d_1 + d_2) = d - 2.\end{aligned}$$

By assumption $d - 2 > 6 = \dim\,(A \oplus A)_{d-2}$, so that $(K/I^2)_{d-1} \neq 0$ and our claim follows.

The set of canonically graded algebras is constructible but not necessarily open nor closed. Again consider $P_{\leq d}$ as an affine space with Zariski topology. Suppose that there exists a polynomial of degree d whose apolar algebra is not canonically graded. Elias and Rossi asked in [17, Remark 3.6] whether in this case also the apolar algebra of a *general* polynomial of this degree is not canonically graded. The following proposition answers this question negatively.

Proposition 3.19 *Fix the degree d and the number of variables $n = \dim S$. Consider the set $\mathcal{G} \subset P_{\leq d}$ of dual socle generators of canonically graded algebras of socle degree d. This set is irreducible and constructible, but in general it is neither open nor closed.*

Proof The set $\mathcal{G} = \mathbb{G} \cdot P_d$ is the image of $\mathbb{G} \times P_d$, thus irreducible and constructible. Examples 3.16 and 3.17 together show that for $n = 2$, $d = 5$ this set is not closed. Example 3.18 shows that for $n = 2$, $d \geq 9$ this set is not open. □

In view of Proposition 3.19 it is natural to ask for which degrees and numbers of variables the answer to the question above is positive; the apolar algebra of a *general* polynomial in $P_{\leq d}$ is a canonically graded algebra. Example 3.17 shows that this happens for $n = 2$, $d = 5$. However, the list of cases where the answer may be positive is short, as we shall see now.

Proposition 3.20 *Assume that k is a field of characteristic zero. Suppose that $n = \dim S$ and d are such that the apolar algebra of a general polynomial in $P_{\leq d}$ is canonically graded. Then (n, d) belongs to the following list:*

1. $d \leq 4$ *and n arbitrary,*
2. $d = 5$ *and* $n \leq 6$,
3. $d = 6$ *and* $n = 2$,
4. d *arbitrary and* $n = 1$.

Proof Fix n, d outside the above list and suppose that $\mathbb{G} \cdot P_d$ is dense in $P_{\leq d}$. Since we are over a field of characteristic zero, the tangent map $\mathbb{G} \times P_d \to P_{\leq d}$ at a general point (g, F) is surjective. Then by $\mathbb{G}$-action, this map is surjective also at $(1, F)$. Its image is $\mathfrak{g}F + P_d$. Thus

$$\mathfrak{g}F + P_d = P_{\leq d}. \tag{14}$$

Let us now analyse $\mathfrak{g}F$. By Proposition 2.18 the equations of $\mathfrak{g}F$ are given by $\sigma \in S$ satisfying $\sigma \lrcorner F = 0$ and

$$\sigma^{(i)} \lrcorner F = 0 \quad \text{for all} \quad i = 1, \dots, n. \tag{15}$$

Consider $\sigma \in S$ homogeneous of degree $d-1$. The space $\operatorname{Ann}(F)_{d-2}$ is of codimension $H_{\operatorname{Apolar}(F)}(d-2) = H_{\operatorname{Apolar}(F)}(2) \leq \binom{n+1}{2}$. Thus for fixed i the condition $\sigma^{(i)} \lrcorner F = 0$ amounts to at most $\binom{n+1}{2}$ linear conditions on the coefficients of σ. Summing over $i = 1, \dots, n$ we get $n \cdot \binom{n+1}{2}$ conditions. Now, if (n, d) is outside the list, then

$$n \cdot \binom{n+1}{2} < \binom{n-1+d-1}{d-1},$$

so that there exists a non-zero σ satisfying (15). Since $\sigma = (d-1)^{-1} \sum \alpha_i \sigma^{(i)}$ also $\sigma \lrcorner F = 0$. Thus $\sigma \in (\mathfrak{g}F)^{\perp}$ is a non-zero element. Therefore $\mathfrak{g}F$ does not contain the whole $P_{\leq d-1}$, which contradicts (14). □

We have some computational evidence that in all cases listed in Proposition 3.20 a general polynomial indeed gives a canonically graded algebra. Of course for $d \leq 4$ it is the result of Elias and Rossi reproved in Corollary 3.14. We put forward the following conjecture.

Conjecture 3.21 *Assume that k is a field of characteristic not equal to $2, 3, 5$. The pairs (n, d) such that the apolar algebra of a general polynomial in $P_{\leq d}$ is canonically graded are precisely the pairs listed below:*

1. $d \leq 4$ *and n arbitrary,*

2. *$d = 5$ and $n \leq 6$,*
3. *$d = 6$ and $n = 2$.*
4. *d arbitrary and $n = 1$.*

Here $n = \dim S$ is the number of variables.

We have some supporting evidence in all cases $d = 5, n \leq 6$ and $d = 6, n = 2$. Namely, for a pseudo-randomly chosen form F a machine computation of $(\mathfrak{g}F)^{\perp}$ via Proposition 2.18 reveals that

$$\mathfrak{g}F \supset P_{\leq d-1}.$$

Thus arguing by semicontinuity we get that the map $\mathbb{G} \times P_d \to P_{\leq d}$ is dominating and the conjecture follows. However we find the use of computer here far from satisfactory, since it dims the reasons for this result. Hopefully someone would invent a computer-free proof.

Improvements using symmetric decomposition. The condition $H_A(d-1) = n$ in definition of t-compressed algebras may be slightly weakened, which is sometimes useful. The price one pays is a more technical assumption. We present the result below. An example of use of Proposition 3.22 is given in Example 3.29. For information on the symmetric decomposition Δ_r of the Hilbert function, see [6,28] or [1, Section 2]. In short, the Hilbert function H of an apolar algebra of socle degree d admits a canonical decomposition $H = \sum_{i=0}^{d} \Delta_i$, where Δ_i is a vector of length $d - i$ which is symmetric: $\Delta_i(j) = \Delta_i(d - i - j)$ for all j.

Proposition 3.22 *Let $f \in P$ be a polynomial of degree $d \geq 3$ and A be its apolar algebra. Let $\Delta_\bullet$ be the symmetric decomposition of H_A. Suppose that*

1. *$H_A(r) = H_S(r) = \binom{r+n-1}{r}$ for all $0 \leq r \leq t$.*
2. *$\Delta_r(1) = 0$ for all $d - 1 - t \leq r$.*

Then the $\mathbb{G}^+$-orbit of f contains $f + P_{\leq t+1}$. In particular $f_{\geq t+2} \in \mathbb{G}^+ \cdot f$.

Proof In the notation of [1, Section 2] or [6] we see that

$$\Delta_a(1) = \dim C_{a,1}(1)/C_{a-1,1}(1), \quad \text{where} \quad C_{a,1}(1) = \{\text{linear polynomials in } \mathfrak{m}^{d-1-a} f\}.$$

Moreover, the assumption $H_A(1) = n$ guarantees that $C_{d-2,1}(1) = P_{\leq 1}$ is the full space of linear polynomials. The assumption $\Delta_r(1) = 0$ for all $d - 1 - t \leq r \leq d - 2$ shows that

$$P_{\leq 1} = C_{d-2,1}(1) = C_{d-3,1}(1) = \cdots = C_{d-2-t,1}(1) = \mathfrak{m}^{t+1} f. \tag{16}$$

Thus, for every $\ell \in P_{\leq 1}$ we have a δ of order greater than t such that $\delta \lrcorner f = \ell$.

Now we repeat the proof of Proposition 3.12 with one difference: instead of referring to Remark 3.11 we use Eq. (16) to obtain, for a given linear form ℓ, an element $\delta \in S$ of order greater than t and such that $\delta \lrcorner f = \ell$. □

3.3 Further examples

Below we present some more involved examples, which employ the tools developed in the previous section.

Example 3.23 (*Hilbert function* $(1, 2, 2, 1)$) By the result of Elias and Rossi (Proposition 3.14) every apolar algebra with Hilbert function $(1, 2, 2, 1)$ is canonically graded. Then it is isomorphic to $\mathrm{Apolar}(x^3 + y^3)$ or $\mathrm{Apolar}(x^2 y)$ and these algebras are not isomorphic.

The example was treated in [7, p. 12] and [16, Prop 3.6].

Example 3.24 (Hilbert function (1, 2, 2, 2, 1)) Let $f \in P_{\leq 4}$ be a polynomial such that $H_{\mathrm{Apolar}(f)} = (1, 2, 2, 2, 1)$. If $f_4 = x^4 + y^4$ then if fact $\mathrm{Apolar}(f) \simeq \mathrm{Apolar}(f_4)$ by Example 3.5, so we may assume $f_4 = x^3y$. Since f is 1-compressed we may also assume $f_{\leq 2} = 0$, whereas by Corollary 3.4 together with Example 3.2 we may assume $f_3 = cy^3$ for $c \in k$. Thus

$$\mathrm{Apolar}(f) \simeq \mathrm{Apolar}(x^3y + cy^3).$$

By multiplying variables by suitable constants, we may assume $c = 0$ or $c = 1$ and obtain three possibilities:

$$f = x^4 + y^4, \qquad f = x^3y, \qquad f = x^3y + y^3.$$

Note that the three types appearing above are pairwise non-isomorphic. Indeed, an isomorphism may only occur between $\mathrm{Apolar}(x^3y)$ and $\mathrm{Apolar}(x^3y+y^3)$. Suppose that such exists, so that there is $\varphi \in \mathbb{G}$ such that $\varphi^\vee(x^3y) = x^3y+y^3$. Write $\varphi = gu$, for g linear and $u \in \mathbb{G}^+$. Then g preserves x^3y and a direct check shows that g is diagonal. Then $u^\vee(x^3y) = x^3y+cy^3$ for some non-zero $c \in k$ and $y^3 = c^{-1}\,\mathrm{tdf}(u(x^3y) - x^3y) \in \mathfrak{g}^+(x^3y)$, by Proposition 3.3. This is a contradiction with Example 3.2.

This example was analysed, among others, in [5], where Casnati classifies all Artinian Gorenstein algebras of length at most 9. See especially [5, Theorem 4.4].

Example 3.25 (Hilbert function (1, 2, 2, ..., 2, 1), general) Consider the set of polynomials $f \in P = k_{dp}[x, y]_{\leq d}$ such that $H_{\mathrm{Apolar}(f)} = (1, 2, \ldots, 2, 1)$ with $d - 1$ twos occurring. This set is irreducible in Zariski topology, by [6, Proposition 4.8] which uses [25, Theorem 3.13]. The leading form of a general element of this set has, up to a coordinate change, the form $x^d + y^d$. Example 3.5 shows that the apolar algebra of such an element is canonically graded.

Remark 3.26 In general, there are $d - 1$ isomorphism types of almost-stretched algebras of socle degree d and with Hilbert function $(1, 2, 2, \ldots, 2, 1)$ as proved in [5, Theorem 4.4]; see also [18, Remark 5, p. 447]. The claim is recently generalised by Elias and Homs, see [15].

Example 3.27 (*Hilbert function* (1, 3, 3, 3, 1)) Consider now a polynomial $f \in P = k_{dp}[x, y, z]$ whose Hilbert function is (1, 3, 3, 3, 1). Let F denote the leading form of f. By [32] or [6, Prop 4.9] the form F is linearly equivalent to one of the following:

$$F_1 = x^4 + y^4 + z^4, \quad F_2 = x^3y + z^4, \quad F_3 = x^2(xy + z^2).$$

Since $\mathrm{Apolar}(f)$ is 1-compressed, we have $\mathrm{Apolar}(f) \simeq \mathrm{Apolar}(f_{\geq 3})$; we may assume that the quadratic part is zero. In fact by the explicit description of top degree form in Proposition 3.3 we see that

$$\mathbb{G}^+ \cdot f = f + \mathfrak{g}^+F + P_{\leq 2}.$$

Recall that $\mathbb{G}/\mathbb{G}^+$ is the product of the group of linear transformations and k^* acting by multiplication.

The case F_1. Example 3.7 shows that $\left(\mathfrak{g}^+F\right)^\perp_{\leq 3}$ is spanned by $\alpha\beta\gamma$. Therefore we may assume $f = F_1 + c \cdot xyz$ for some $c \in k$. By multiplying variables by suitable constants and then multiplying whole f by a constant, we may assume $c = 0$ or $c = 1$. As before, we get two non-isomorphic algebras. Summarising, we got two isomorphism types:

$$f_{1,0} = x^4 + y^4 + z^4, \quad f_{1,1} = x^4 + y^4 + z^4 + xyz.$$

Note that $f_{1,0}$ is canonically graded, whereas $f_{1,1}$ is a complete intersection.
The case F_2. We have $\operatorname{Ann}(F_2)_2 = (\alpha\gamma, \beta^2, \beta\gamma)$, so that $\left(\mathfrak{g}^+ F_2\right)^{\perp}_{\leq 3} = \operatorname{span}\left\langle \beta^3, \beta^2\gamma\right\rangle$. Thus we may assume $f = F_2 + c_1 y^3 + c_2 y^2 z$. As before, multiplying x, y and z by suitable constants we may assume $c_1, c_2 \in \{0, 1\}$. We get four isomorphism types:

$$f_{2,00} = x^3 y + z^4, \quad f_{2,10} = x^3 y + z^4 + y^3, \quad f_{2,01} = x^3 y + z^4 + y^2 z,$$
$$f_{2,11} = x^3 y + z^4 + y^3 + y^2 z.$$

To prove that the apolar algebras are pairwise non-isomorphic one shows that the only linear maps preserving F_2 are diagonal and argues as an in Example 3.24 or as described in the case of F_3 below.
The case F_3. We have $\operatorname{Ann}(F_3)_2 = (\beta^2,\ \beta\gamma,\ \alpha\beta - \gamma^2)$ and

$$\left(\mathfrak{g}^+ F_3\right)^{\perp}_{\leq 3} = \operatorname{span}\left\langle \beta^2\gamma,\ \beta^3,\ \alpha\beta^2 - 2\beta\gamma^2\right\rangle.$$

We may choose $\operatorname{span}\left\langle y^3, y^2 z, yz^2\right\rangle$ as the complement of $\mathfrak{g}^+ F_3$ in P_3. Therefore the apolar algebra of each f with top degree form F_3 is isomorphic to the apolar algebra of

$$f_{3,*} = x^3 y + x^2 z^2 + c_1 y^3 + c_2 y^2 z + c_3 yz^2$$

and two distinct such polynomials $f_{3,*1}$ and $f_{3,*2}$ lie in different $\mathbb{G}^+$-orbits. We identify the set of $\mathbb{G}^+$-orbits with $P_3/\mathfrak{g}^+ F_3 \simeq \operatorname{span}\left\langle y^3, y^2 z, yz^2\right\rangle$. We wish to determine isomorphism classes, that is, check which such $f_{3,*}$ lie in the same $\mathbb{G}$-orbit. A little care should be taken here, since $\mathbb{G}$-orbits will be bigger than expected.

Recall that $\mathbb{G}/\mathbb{G}^+ \simeq \mathrm{GL}(\mathfrak{m}/\mathfrak{m}^2) \times k^*$ preserves the degree. Therefore, it is enough to look at the operators stabilising F_3. These are $c \cdot g$, where $c \in k^*$ is a constant and $g \in \mathrm{GL}(\mathfrak{m}/\mathfrak{m}^2)$ stabilises $\operatorname{span}\langle F_3\rangle$, i.e. $g^{\vee}(\operatorname{span}\langle F_3\rangle) = \operatorname{span}\langle F_3\rangle$. Consider such a g. It is a linear automorphism of P and maps $\operatorname{Ann}(F)$ into itself. Since $\beta(\lambda_1\beta + \lambda_2\gamma)$ for $\lambda_i \in k$ are the only reducible quadrics in $\operatorname{Ann}(F_3)$ we see that g stabilises $\operatorname{span}\langle\beta, \gamma\rangle$, so that $g^{\vee}(x) = \lambda x$ for a non-zero λ. Now it is straightforward to check directly that the group of linear maps stabilising $\operatorname{span}\langle F_3\rangle$ is generated by the following elements

1. homotheties: for a fixed $\lambda \in k$ and for all linear $\ell \in P$ we have $g^{\vee}(\ell) = \lambda\ell$.
2. for every $a, b \in k$ with $b \neq 0$, the map $t_{a,b}$ given by

$$t_{a,b}(x) = x, \quad t_{a,b}(y) = -a^2 x + b^2 y - 2abz, \quad t_{a,b}(z) = ax + bz.$$

   which maps F_3 to $b^2 F_3$.

The action of $t_{a,b}$ on $P_3/\mathfrak{g}^+ F_3$ in the basis $(y^3, y^2 z, yz^2)$ is given by the matrix

$$\begin{pmatrix} b^6 & 0 & 0 \\ -6ab^5 & b^5 & 0 \\ \frac{27}{2}a^2 b^4 & -\frac{9}{2}ab^4 & b^4 \end{pmatrix}$$

Suppose that $f_{3,*} = x^3 y + x^2 z^2 + c_1 y^3 + c_2 y^2 z + c_3 yz^2$ has $c_1 \neq 0$. The above matrix shows that we may choose a and b and a homothety h so that

$$(h \circ t_{a,b})(f_{3,*}) = c(x^3 y + x^2 z^2 + y^3 + c_3 yz^2), \quad \text{where } c \neq 0,\ c_3 \in \{0, 1\}.$$

Suppose $c_1 = 0$. If $c_2 \neq 0$ then we may choose a, b and λ so that $(h \circ t_{a,b})(f_{3,*}) = x^3 y + x^2 z^2 + y^2 z$. Finally, if $c_1 = c_2 = 0$, then we may choose $a = 0$ and b, λ so that $c_3 = 0$ or $c_3 = 1$. We get at most five isomorphism types:

$$f_{3,100} = x^3y + x^2z^2 + y^3, \quad f_{3,101} = x^3y + x^2z^2 + y^3 + yz^2,$$
$$f_{3,010} = x^3y + x^2z^2 + y^2z, \quad f_{3,001} = x^3y + x^2z^2 + yz^2,$$
$$f_{3,000} = x^3y + x^2z^2.$$

By using the explicit description of the $\mathbb{G}$ action on $P_3/\mathfrak{g}^+F_3$ one checks that the apolar algebras of the above polynomials are pairwise non-isomorphic.

Conclusion There are 11 isomorphism types of algebras with Hilbert function $(1, 3, 3, 3, 1)$. We computed the tangent spaces to the corresponding orbits in characteristic zero, using a computer implementation of the description in Proposition 2.18. The dimensions of the orbits are as follows:

Orbit	Dimension	Orbit	Dimension
$\mathbb{G}\cdot(x^4 + y^4 + z^4 + xyz)$	29	$\mathbb{G}\cdot(x^3y + x^2z^2 + y^3 + yz^2)$	27
$\mathbb{G}\cdot(x^4 + y^4 + z^4)$	28	$\mathbb{G}\cdot(x^3y + x^2z^2 + y^3)$	26
$\mathbb{G}\cdot(x^3y + z^4 + y^3 + y^2z)$	28	$\mathbb{G}\cdot(x^3y + x^2z^2 + y^2z)$	26
$\mathbb{G}\cdot(x^3y + z^4 + y^3)$	27	$\mathbb{G}\cdot(x^3y + x^2z^2 + yz^2)$	25
$\mathbb{G}\cdot(x^3y + z^4 + y^2z)$	27	$\mathbb{G}\cdot(x^3y + x^2z^2)$	24
$\mathbb{G}\cdot(x^3y + z^4)$	26		

The closure of the orbit of $f_{1,1} = x^4+y^4+z^4+xyz$ is contained in $\mathrm{GL}_3(x^4+y^4+z^4)+P_{\leq 3}$, which is irreducible of dimension 29. Since the orbit itself has dimension 29 it follows that it is dense inside. Hence the orbit closure contains $\mathrm{GL}_3(x^4 + y^4 + z^4) + P_{\leq 3}$. Moreover, the set $\mathrm{GL}_3(x^4 + y^4 + z^4)$ is dense inside the set σ_3 of forms F whose apolar algebra has Hilbert function $(1, 3, 3, 3, 1)$. Thus the orbit of $f_{1,1}$ is dense inside the set of polynomials with Hilbert function $(1, 3, 3, 3, 1)$. Therefore, the latter set is irreducible and of dimension 29.

It would be interesting to see which specializations between different isomorphism types are possible. There are some obstructions. For example, the GL_3-orbit of $x^3y + x^2z^2$ has smaller dimension than the GL_3-orbit of $x^3y + z^4$. Thus $x^3y + x^2z^2 + y^3 + yz^2$ does not specialize to $x^3y + z^4$ even though its $\mathbb{G}$-orbit has higher dimension.

Example 3.28 (*Hilbert function* $(1, 3, 4, 3, 1)$)

Consider now a polynomial $f \in k_{dp}[x, y, z]$ whose apolar algebra has Hilbert function $H = (1, 3, 4, 3, 1)$. Let F be the leading form of f. Since H is symmetric, the apolar algebra of F also has Hilbert function H. In particular it is annihilated by 2-dimensional space of quadrics. Denote this space by Q. Suppose that the quadrics in Q do not share a common factor, then they are a complete intersection. By looking at the Betti numbers, we see that $\mathrm{Ann}\,(F)$ itself is a complete intersection of Q and a cubic. From this point of view, the classification of F using our ideas seems ineffective compared to classifying ideals directly. Therefore we will not attempt a full classification. Instead we show that there are infinitely many isomorphism types and discuss non-canonically graded algebras.

Let $V = V(Q) \subset \mathbb{P}^2$ be the zero set of Q inside the projective space with coordinates α, β, γ.

Infinite family of isomorphism types. Consider

$$F = F_\lambda = \lambda_1 x^4 + \lambda_2 y^4 + \lambda_3 z^4 + \lambda_4 (x + y + z)^4. \tag{17}$$

for some non-zero numbers λ_i. Then the Hilbert function of $\mathrm{Apolar}(f)$ is $(1, 3, 4, 3, 1)$ and $V(Q)$ is a set of four points, no three of them collinear. Suppose that F_λ and $F_{\lambda'}$ are in the same $\mathbb{G}$-orbit. By Remark 2.17 there is an element of $\mathrm{GL}(\mathfrak{m}/\mathfrak{m}^2)$ mapping F_λ to $F_{\lambda'}$. Such element stabilizes $V(Q)$, which is a set of four points, no three of them collinear. But the only elements of $\mathrm{GL}(\mathfrak{m}/\mathfrak{m}^2)$ stabilizing such set of four points are the scalar matrices. Therefore we conclude that the set of isomorphism classes of F_λ is the set of quadruples $\lambda_\bullet$ up to homothety. This set is in bijection with $(k^*)^4/k^* \simeq (k^*)^3$, thus infinite. It fact the set of F_λ with $\lambda_4 = 1$ is a threefold in the moduli space of finite algebras with a fixed basis, see [38] for construction of this space.

Non-canonically graded algebras. We now classify forms F such that all polynomials f with leading form F lie in $\mathbb{G}\cdot F$. As in Example 3.27 we see that

$$\mathbb{G}^+\cdot F = F + \mathfrak{g}^+F + P_{\leq 2}.$$

Thus we investigate $\left(\mathfrak{g}^+F\right)_3^\perp$ using Proposition 2.20. Let us suppose it is non-zero and pick a non-zero element $\sigma \in \left(\mathfrak{g}^+F\right)_3^\perp$. Then $\sigma^{(i)} \in \mathrm{Ann}\,(F)_2 = Q$ for all i. Since Q is two-dimensional we see that the derivatives of σ are linearly dependent. Thus up to coordinate change we may assume that $\sigma \in k[\alpha, \beta]$. If σ has one-dimensional space of derivatives, then $\sigma = \alpha^3$ up to coordinate change and $\frac{1}{3}\sigma^{(1)} = \alpha^2$ annihilates Q. If σ has two-dimensional space of derivatives Q, then Q intersects the space of pure squares in an non-zero element α^2, thus α^2 annihilates F in this case also. Conversely, if α^2 annihilates F, then α^3 lies in $\left(\mathfrak{g}^+F\right)_3^\perp$.

Summarizing, $F + P_{\leq 3} \subset \mathbb{G}\cdot F$ if and only if no square of a linear form annihilates F. For example, apolar algebras of all polynomials with leading form F_λ from (17) are canonically graded. On the other hand, for $F = xz^3 + z^2y^2 + y^4$ and $f_\lambda = F + ax^3$ the apolar algebra of f_λ is canonically graded if and only if $a = 0$.

Non-canonically graded algebras with Hilbert function $(1, 3, 4, 3, 1)$ are investigated independently in [34], where more generally Hilbert functions $(1, n, m, n, 1)$ are considered.

Any discussion of isomorphism types would be incomplete without tackling the example of Hilbert function $(1, 2, 2, 2, 1, 1, 1)$, which is the smallest example where infinitely many isomorphism types appear. It is also instructive as a non-homogeneous example. Strangely enough, the argument is similar to the previous examples. We will use some standard tools to deal with the dual socle generator, see [6, Chap 3].

Example 3.29 (Hilbert function $(1, 2, 2, 2, 1, 1, 1)$) Consider any $f \in P$ such that

$$H_{\mathrm{Apolar}(f)} = (1, 2, 2, 2, 1, 1, 1).$$

Then f is of degree 6. Using the standard form, see [28, Theorem 5.3] or [6, Section 3], we may assume, after a suitable change of f, that

$$f = x^6 + f_{\leq 4}.$$

If y^4 appears with non-zero coefficient in $f_{\leq 4}$, we conclude that $f \in \mathbb{G}^+\cdot(x^6 + y^4)$ arguing similarly as in Example 3.5. Otherwise, we may assume that β^2 is a leading form of an element of an annihilator of f, so that the only monomials in f_4 are x^4, x^3y and x^2y^2. By subtracting a suitable element of $\mathfrak{g}^+x^6$ and rescaling by homotheties, we may assume

$$f = x^6 + cx^2y^2 + f_{\leq 3},$$

for $c = 0$ or $c = 1$. If $c = 0$, then the Hilbert function of f is $(1, *, *, 1, 1, 1, 1)$, which is a contradiction; see [28, Lem 1.10] or [31, Lem 4.34] for details. Thus $c = 1$.

Consider elements of $\left(\mathfrak{g}^{+}(x^6 + x^2y^2)\right)^{\perp}$ with order at most three. Every non-zero partial derivative τ of such an element σ has order at most two and satisfies $\deg(\tau \lrcorner f) \leq 1$. This is only possible if $\tau_2 = \beta^2$, so that $\sigma_3 = \beta^3$ up to a non-zero constant. Therefore we may assume

$$f = x^6 + x^2y^2 + \lambda y^3 + f_{\leq 2} \quad \text{for some} \quad \lambda \in k.$$

The symmetric decomposition of $\mathrm{Apolar}(f)$ is $\Delta_0 = (1, 1, 1, 1, 1, 1, 1)$, $\Delta_1 = \mathbf{0}$, $\Delta_2 = (0, 1, 1, 1, 0)$, $\Delta_3 = \mathbf{0}$, $\Delta_4 = \mathbf{0}$. In particular $\Delta_4(1) = 0$, so that by Proposition 3.22 we may assume $f_{\leq 2} = 0$ and

$$f = f_\lambda = x^6 + x^2y^2 + \lambda y^3.$$

Similarly as in the final part of Example 3.24 we see that for every $\lambda \in k$ we get a distinct algebra $A_\lambda = \mathrm{Apolar}(f_\lambda)$ where $A_\lambda \simeq A_{\lambda'}$ if and only if $\lambda = \lambda'$. Thus in total we get $|k| + 1$ isomorphism types. Note that the $|k|$ types f_λ form a curve in the moduli space of finite algebras with a fixed basis, see [38] for information on this space. The point corresponding to $y^6 + x^4$ does not seem to be related to this curve.

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