Collect. Math. (2017) 68:69–85
DOI 10.1007/s13348-016-0174-2

Some extensions of Hilbert–Kunz multiplicity

Neil Epstein[1] · **Yongwei Yao**[2]

Received: 20 March 2016 / Accepted: 10 May 2016 / Published online: 24 May 2016

Abstract Let R be an excellent Noetherian ring of prime characteristic. Consider an arbitrary nested pair of ideals (or more generally, a nested pair of submodules of a fixed finite module). We do *not* assume that their quotient has finite length. In this paper, we develop various sufficient numerical criteria for when the tight closures of these ideals (or submodules) match. For some of the criteria we only prove sufficiency, while some are shown to be equivalent to the tight closures matching. We compare the various numerical measures (in some cases demonstrating that the different measures give truly different numerical results) and explore special cases where equivalence with matching tight closure can be shown. All of our measures derive ultimately from Hilbert–Kunz multiplicity.

Keywords Hilbert–Kunz multiplicity · Tight closure

Mathematics Subject Classification Primary 13A35; Secondary 13D40

1 Introduction

The classical notions of the Hilbert–Samuel function and multiplicity of a finite colength ideal have far-reaching implications in commutative algebra. They arose in (and have many strong links to) intersection theory, and the multiplicity may be used to characterize when a pair $J \subseteq I$ of (finite colength) ideals have the same integral closure.

The second author was partially supported by the National Science Foundation DMS-0700554.

✉ Neil Epstein
nepstei2@gmu.edu

Yongwei Yao
yyao@gsu.edu

1 Department of Mathematical Sciences, George Mason University, Fairfax, VA 22030, USA

2 Department of Math and Statistics, Georgia State University, 30 Pryor St., Atlanta, GA 30303, USA

The notion of Hilbert–Samuel multiplicity has been extended in various ways to arbitrary ideals. One method of extension, using the $\mathfrak{m}$-torsion functor $\Gamma_{\mathfrak{m}}$, is the j-multiplicity of Achilles and Manaersi [2,3], in the context of intersection theory, later shown by Flenner and Manaresi [12] to characterize when an arbitrary pair $J \subseteq I$ have the same integral closure. The more recent ϵ-multiplicity of Ulrich and Validashti [31], in an easier definition that nevertheless uses the same functor, also characterizes integral dependence of ideals under mild conditions.

In characteristic p algebra, Hilbert–Samuel multiplicity and function have natural analogues, namely the Hilbert–Kunz function and Hilbert–Kunz multiplicity of a finite colength ideal. Both multiplicities may be extended somewhat to modules and to relative situations. The two share many properties with each other. It is notable that Hilbert–Kunz multiplicity characterizes when a pair $J \subseteq I$ of finite colength ideals have the same *tight* closure, and that the Hilbert–Kunz function was used [5] to show that tight closure does not commute with localization, giving a negative answer to a very important question. It appears that Hilbert–Kunz multiplicity sometimes fails to be rational [4]. It should be noted that the Hilbert–Kunz function is also linked to intersection theory (e.g., [19]).

The question then arises: What can one do in the case of an *arbitrary* pair of nested ideals? In this article, we explore a variety of techniques to provide criteria for when a nested pair of arbitrary ideals shares the same tight closure, all of which extend Hilbert–Kunz multiplicity and most of which involve ideal torsion functors,

One possible approach would be: for each ideal, define a limit (or at least a finite limsup) based on the definition of j-multiplicity (or ϵ-multiplicity) but using bracket powers in place of ordinary powers of ideals. Such an approach would require that the numbers $\lambda_R(H^0_{\mathfrak{m}}(R/I^{[q]}))/q^d$ (where q varies over powers of p) be bounded above by a constant. Such a result has proved elusive even when R is essentially of finite type over a field and R/I has small dimension (cf. [1, Corollary 5.2] for a solution to the already difficult dimension 1 case). Thus, in this paper we limit ourselves here to *relative* measures for a nested pair of ideals or submodules, where vanishing will be the benchmark for expecting tight closures to coincide.

For a Noetherian local ring $(R, \mathfrak{m})$, the Hilbert–Kunz multiplicity of an $\mathfrak{m}$-primary ideal I is defined via:

$$e_{\mathrm{HK}}(I) := \lim_{q\to\infty} \frac{\lambda(R/I^{[q]})}{q^d},$$

where $d = \dim R$. Monsky [22] showed that this is always well-defined, finite, and ≥ 1 for any $\mathfrak{m}$-primary ideal.

As noted above, Hilbert–Kunz multiplicity characterizes when a nested pair of $\mathfrak{m}$-primary ideals has the same tight closure. In fact, more is true (due to Hochster and Huneke):

Theorem 1.1 [13, Theorem 8.17] *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d, and let $L \subseteq M \subseteq N$ be finitely generated R-modules such that $\lambda(M/L) < \infty$.*

(a) *If $M \subseteq L^*_N$, then $\lambda(M^{[q]}_N/L^{[q]}_N) \leq Cq^{d-1}$ for all q, for some constant C independent of q. Hence,*

$$\limsup_{q\to\infty} \frac{\lambda(M^{[q]}_N/L^{[q]}_N)}{q^d} = 0.$$

(b) *Suppose that R has a completely stable weak test element c, and that $\hat{R}$ is equidimensional and reduced.*[1] *If*

$$\liminf_{q \to \infty} \frac{\lambda(M_N^{[q]}/L_N^{[q]})}{q^d} = 0,$$

then $M \subseteq L_N^*$.

If one sets $N := R$ and $J \subseteq I$ are $\mathfrak{m}$-primary ideals, then this theorem gives the result on Hilbert–Kunz multiplicities as a special case.

In Sect. 2, we introduce the limit $u_N(L, M)$ for an arbitrary triple $L \subseteq M \subseteq N$ of finite R-modules where $(R, \mathfrak{m})$ is local (as well as the associated limsup $u_N^+(L, M)$ and liminf $u_N^-(L, M)$), based partly on the ideas of Theorem 1.1. In Theorem 2.4, we show that vanishing of the liminf version gives a one-way implication for $M \subseteq L_N^*$.

In Sect. 3, we exhibit several situations where a strong converse holds, including finite projective dimension, finite F-representation type, and finite generation of a module over a certain non-commutative ring. For an additional such example, see the preprint [11].

In Sect. 4, we give several variants of the notion from Sect. 2, and we show in Theorem 4.4 that in cases where tight closure commutes with localization for $L \subseteq N$, a certain numerical vanishing condition is *equivalent* to $M \subseteq L_N^*$. We then show in Proposition 4.5 that two of these notions of relative Hilbert–Kunz multiplicity are, in general, quite distinct.

In Sect. 5, we provide a numerical criterion, based on a tight closure variant of the Nakayama lemma previously proved by the first named author, which determines exactly when a pair of nested submodules have the same tight closure. However, the measurement involved in this criterion transforms the lower module (or ideal) in such a profound way that it is hard to see how it could be used to analyze individual submodules, unlike the previous measurements and criteria.

Finally in Sect. 6, we introduce a notion that looks more closely related to j-multiplicity than any of our other definitions. In Theorem 6.2, we show that it gives another criterion that determines exactly when two nested ideals $J \subseteq I$ with $\lambda(I/J) < \infty$ have the same tight closure. Theorem 6.3 is a global version of this for rings which are F-regular on the punctured spectrum.

To conclude this introductory section, we recall some standard definitions and fix some notational conventions:

All rings are Noetherian and have prime characteristic $p > 0$. For a nonnegative integer e, denote $q := p^e$. All R-modules are considered as left modules unless noted otherwise. For an R-module M, let ${}^eM$ be the $(R$-$R)$-bimodule which equals M as an abelian group, and whose bimodule structure is given by $r \cdot z \cdot s := r^{p^e} sz$ for $z \in {}^eM$ and $r, s \in R$. For an R-module M, $F^e(M)$ denotes the *right* R-module structure on $M \otimes_R {}^eR$, but considered as a *left* R-module. Hence, ${}^e(F^e(M)) \cong M \otimes_R {}^eR$ as left R-modules.

Let $L \subseteq M$ be R-modules. For $z \in M$, z_M^q denotes the image $z \otimes {}^e1$ of z under the map $M \to F^e(M) = M \otimes_R {}^eR$. Similarly, $L_M^{[q]}$ is the image of the map $F^e(L) \to F^e(M)$ which is induced by the inclusion $L \hookrightarrow M$. The *tight closure* of L in M, denoted L_M^*, is the submodule of M consisting of all elements $z \in M$ such that there exists an element $c \in R^\circ$ (i.e., not in any minimal prime of R), possibly dependent on z, such that for all $q \gg 0$, we have $cz_M^q \in L_M^{[q]}$. If $M = R$, we omit the subscript. A ring R is *weakly F-regular* if all ideals are tightly closed (as R-submodules of R.) R is *F-regular* if $R_\mathfrak{p}$ is weakly F-regular for all

[1] We remark here that by the methods used in proving our Theorem 2.4, the assumption that $\hat{R}$ is reduced is unnecessary.

$\mathfrak{p} \in \operatorname{Spec} R$. An element $c \in R^\circ$ is a (q_0-)*weak test element* if there is some power q_0 of p such that for all finitely generated R-modules M, all submodules $L \subseteq M$, and all $x \in M$, $x \in L^*_M$ if and only if $cx^q = L^{[q]}_M$ for all $e \geq e_0$, where $q = p^e$ and $q_0 = p^{e_0}$. If we can take $q_0 = 1$, we say c is a *test element*. For a more thorough review of these concepts, we recommend the seminal paper [13] and the monograph [16].

Remark 1.2 Throughout this paper, many results are stated in general for nested triples of modules $L \subseteq M \subseteq N$. However, the reader should note that in all such cases, the results are new even when $N = R$ and L, M are ideals (which is probably the case of greatest interest).

Remark 1.3 This article has been available as a preprint, in essentially the current form, for several years. In that time, several authors have made use of some of the ideas in this article to positive effect, most notably in Brenner's construction of a non-rational Hilbert–Kunz multiplicity. See [4,7,8,32].

Another recent development is a preprint of Polstra [24]. Polstra's techniques should establish that the limits in Sects. 4 and 5 exist in some generality.

2 Relative multiplicity

Throughout, let R be a Noetherian ring of prime characteristic $p > 0$.

When $(R, \mathfrak{m})$ is local and M an R-module, we let

$$\Gamma_{\mathfrak{m}}(M) := \{z \in M \mid \mathfrak{m}^n z = 0 \text{ for some } n \in \mathbb{N}\}.$$

Recall that $\Gamma_{\mathfrak{m}}$ is a left-exact functor, and that a finitely generated R-module M has finite length if and only if $M = \Gamma_{\mathfrak{m}}(M)$. We start with the following definition:

Definition 2.1 Let $L \subseteq M \subseteq N$ be R-modules, where $(R, \mathfrak{m})$ is local of dimension d. Then the *relative multiplicity of* L *against* M (*in* N) is

$$u^+_N(L, M) := \limsup_{q \to \infty} \frac{\lambda(\Gamma_{\mathfrak{m}}(M^{[q]}_N / L^{[q]}_N))}{q^d}.$$

(resp.

$$u^-_N(L, M) := \liminf_{q \to \infty} \frac{\lambda(\Gamma_{\mathfrak{m}}(M^{[q]}_N / L^{[q]}_N))}{q^d}.$$

If these are equal (i.e., the limit is well-defined), then the common number is written $u_N(L, M)$.

When N is understood (e.g., when $N = R$), we omit it from the notation.

Hence, if $\lambda(M/L) < \infty$ and N/L is finite, then Theorem 1.1 shows that under the conditions of that theorem, $M \subseteq L^*_N$ if and only if $u^+_N(L, M) = 0$ if and only if $u^-_N(L, M) = 0$. (The assumption in Theorem 1.1 that $L \subseteq M \subseteq N$ are finitely generated is unnecessary.)

At this point, the reader may wonder the following: if $J \subseteq I$ are ideals with the same tight closure, and the ring is reasonable enough, can it happen that their quotient has infinite length? In fact it can, as the following example shows, in which we "add a variable":

Example 2.2 Let $(A, \mathfrak{n})$ be *any* Noetherian local ring that is not weakly F-regular, and let $\mathfrak{b} \subsetneq \mathfrak{a}$ be a pair of nested distinct ideals that have the same tight closure. Let $z \in \mathfrak{a} \setminus \mathfrak{b}$. Let X

be an indeterminate over A, let $Q := A[X]$, and let $R := Q_{\mathfrak{n}Q+XQ}$. Let $\mathfrak{m} := \mathfrak{n}R + XR$ be the unique maximal ideal of R. Let $J := \mathfrak{b}R$ and $I := \mathfrak{a}R$. Then $J^* = I^*$. We have:

$$\frac{I}{J} \supseteq \frac{J + Rz}{J} \cong \frac{Rz}{J \cap Rz} \cong \frac{R}{J :_R z}.$$

Note that for all integers $n \geq 0$, we have $X^n z \notin J$. Hence $X \notin \sqrt{(J :_R z)}$, which shows that $(J :_R z)$ is a non-$\mathfrak{m}$-primary ideal. By the containments displayed above, then, we have

$$\lambda_R(I/J) \geq \lambda_R(R/(J :_R z)) = \infty.$$

In the following theorem, we extend one of the implications of Theorem 1.1 to the infinite length case. First, recall the following definition from [15], here generalized to the module case (see [9, p. 4850] for further explanation):

Definition 2.3 Let L be a submodule of N and $z \in N$. Then $\mathfrak{q} \in \operatorname{Spec} R$ is a *stable prime associated to* $L \subseteq N$ *and* z if $z \notin (L_\mathfrak{q})^*_{N_\mathfrak{q}}$, but for all primes $\mathfrak{p} \subsetneq \mathfrak{q}$, $z \in (L_\mathfrak{p})^*_{N_\mathfrak{p}}$. The set of all such primes is denoted $T_L^N(z)$, and we set

$$T_L^N := \bigcup_{z \in N} T_L^N(z).$$

Theorem 2.4 *Let R be a Noetherian ring, and let $L \subseteq M \subseteq N$ be R-modules such that N/L is finitely generated. Suppose that R contains a completely stable weak test element c, and that $\widehat{R_\mathfrak{p}}$ is equidimensional for all $\mathfrak{p} \in T_L^N$. If $M \nsubseteq L_N^*$, then for any $x \in M \backslash L_N^*$, we have $u^-_{N_\mathfrak{p}}(L_\mathfrak{p}, M_\mathfrak{p}) > 0$ for all $\mathfrak{p} \in T_L^N(x)$. (Hence if $u^-_{N_\mathfrak{p}}(L_\mathfrak{p}, M_\mathfrak{p}) = 0$ for all $\mathfrak{p} \in T_L^N$, then $M \subseteq L_N^*$.)*

Proof Let x be as in the theorem. We may assume that all modules are finite and that $M = L + Rx$. By [15, Prop 3.3(g)], we have $T_L^N(x) \neq \emptyset$. Take any $\mathfrak{p} \in T_L^N(x)$. Let $Q := \{q = p^e \mid \mathfrak{p} \text{ is minimal over } (L_N^{[q]} :_R cx_N^q)\}$. By [15, Prop 3.1], $\mathbb{N} \backslash \{e \mid p^e \in Q\}$ is finite. Moreover, $R_\mathfrak{p}/((L_N^{[q]})_\mathfrak{p} :_{R_\mathfrak{p}} cx^q_{N_\mathfrak{p}})$ has finite length over $R_\mathfrak{p}$ for all $q \in Q$.

Now we can localize at $\mathfrak{p}$ and complete the ring. Replacing R by $\widehat{R_\mathfrak{p}}$, the new ring R is complete and equidimensional. For simplicity, we denote the maximal ideal of R by $\mathfrak{m}$.

We have an exact sequence

$$0 \to \frac{R}{L_N^{[q]} :_R cx_N^q} \xrightarrow{\cdot c} \frac{R}{L_N^{[q]} :_R x_N^q}$$

because $L_N^{[q]} :_R cx_N^q = (L_N^{[q]} :_R x_N^q) :_R c$ for all $q \in Q$ (hence for all $q \gg 0$). Applying the left-exact functor $\Gamma_\mathfrak{m}(-)$ to it, and using the fact that $\lambda(R/(L_N^{[q]} :_R cx_N^q)) < \infty$, we get another exact sequence

$$0 \to \frac{R}{L_N^{[q]} :_R cx_N^q} \xrightarrow{\cdot c} \Gamma_\mathfrak{m}\left(\frac{R}{L_N^{[q]} :_R x_N^q}\right).$$

Hence, $\lambda(R/(L_N^{[q]} :_R cx_N^q)) \leq \lambda(\Gamma_\mathfrak{m}(R/(L_N^{[q]} :_R x_N^q)))$ for all $q \gg 0$. Therefore, to show the claim of the theorem, we need only show that $\lambda_R(R/(L_N^{[q]} :_R cx_N^q))$ is bounded below by a constant multiple of q^d.

Since tight closure can be checked modulo minimal primes, there is some minimal prime $\mathfrak{q}$ of R such that the image of x is not in $(L + \mathfrak{q}N/\mathfrak{q}N)^*_{N/\mathfrak{q}N}$ as $(R/\mathfrak{q})$-modules. Since

$\dim(R/\mathfrak{q}) = \dim R = d$ (by the assumption of equidimensionality), and since the length of the desired quotient can only decrease when the colon is computed modulo $\mathfrak{q}$, we may replace R by $R/\mathfrak{q}$ and assume that R is a complete local *domain*. After doing this, we note that by the Cohen structure theorem, R is module-finite and torsion-free over a complete regular local ring $(A, \mathfrak{n})$, and by replacing c by a multiple we may assume that $c \in A^\circ$.

Next, we see that

$$\frac{R}{L_N^{[q]} :_R cx_N^q} \cong \frac{L_N^{[q]} + Rcx_N^q}{L_N^{[q]}} \supseteq \frac{L_N^{[q]} + Acx_N^q}{L_N^{[q]}} \cong \frac{A}{L_N^{[q]} :_A cx_N^q}.$$

In the above, the Frobenius computations are being done over R. The first isomorphism is as R-modules, hence also as A-modules, and the second isomorphism is as A-modules. However, the A-module length of an R-module is the same as its R-module length, so it makes sense (and is true) to say that $\lambda(A/(L_N^{[q]} :_A cx_N^q)) \leq \lambda(R/(L_N^{[q]} :_R cx_N^q))$ for all $q \gg 0$.

Since $x \notin L_N^*$, then by the last paragraph of the proof of [13, Theorem 8.17], there is some power q_1 of p such that

$$L_N^{[q]} :_A x_N^q \subseteq \mathfrak{n}^{[q/q_1]}$$

for all $q \gg 0$. Thus,

$$L_N^{[q]} :_A cx_N^q = (L_N^{[q]} :_A x_N^q) :_A c \subseteq \mathfrak{n}^{[q/q_1]} :_A c,$$

which implies that $\lambda(A/(\mathfrak{n}^{[q/q_1]} :_A c)) \leq \lambda(A/(L_N^{[q]} :_A cx_N^q))$.

Next, we have the following short exact sequence of A-modules:

$$0 \to \frac{A}{\mathfrak{n}^{[q/q_1]} :_A c} \xrightarrow{\cdot c} \frac{A}{\mathfrak{n}^{[q/q_1]}} \to \frac{\bar{A}}{\mathfrak{n}^{[q/q_1]}\bar{A}} \to 0,$$

where $\bar{A} := A/cA$. Combining the length equality we get from this sequence with the inequalities we have thus far, we have for all $q \in Q$:

$$\lambda\left(\frac{R}{L_N^{[q]} :_R cx_N^q}\right) \geq \lambda\left(\frac{A}{L_N^{[q]} :_A cx_N^q}\right) \geq \lambda\left(\frac{A}{\mathfrak{n}^{[q/q_1]} :_A c}\right)$$
$$= \lambda\left(\frac{A}{\mathfrak{n}^{[q/q_1]}}\right) - \lambda\left(\frac{\bar{A}}{\mathfrak{n}^{[q/q_1]}\bar{A}}\right).$$

Dividing the difference in the last line by q^d and taking the limit as q approaches infinity, we get $1/q_1^d$ which, as required, is positive. □

Remark By [15, Proposition 3.3(a)] (in light of [9, footnote 6]), if tight closure commutes with localization for the inclusion $L \subseteq N$, then $T_L^N = \mathrm{Ass}_R(N/L_N^*)$, which is a finite set. So in this situation, only finitely many primes need to be checked in order to use the test in Theorem 2.4.

3 Relative multiplicity: special cases

In this section, we give conditions under which a converse to Theorem 2.4 holds, so that we have a criterion that determines exactly when two submodules share the same tight closure.

For a first example, we note that if R is weakly F-regular at all non-maximal primes, then $L \subseteq M \subseteq L_N^*$ implies that $M_\mathfrak{m}/L_\mathfrak{m}$ has finite length for all maximal ideals $\mathfrak{m}$, so that Theorem 1.1 then yields the converse to Theorem 2.4 in this case.

3.1 Relative multiplicity and finite projective dimension

Let J be an ideal of finite projective dimension with no embedded primes (e.g., a parameter ideal in a Cohen–Macaulay local ring), and I an ideal such that $J \subseteq I \subseteq J^*$. Then $u_{R_\mathfrak{p}}(J_\mathfrak{p}, I_\mathfrak{p}) = 0$ for all prime ideals $\mathfrak{p}$.

To see this, let $G.$ be a finite projective resolution of R/J. Then for any $q = p^e$, we have that $F^e(G.)$ is a projective resolution of $R/J^{[q]}$, by [23]. Then since projective resolutions commute with localization and by the conditions for exactness of a complex [6], J and $J^{[q]}$ have the same associated primes (namely, the minimal primes of J). Thus we have

$$\operatorname{Ass}(I^{[q]}/J^{[q]}) \subseteq \operatorname{Ass}(R/J^{[q]}) = \operatorname{Ass}(R/J) = \operatorname{Min}(R/J).$$

Hence for any prime $\mathfrak{p} \notin \operatorname{Min}(R/J)$, we have $H^0_\mathfrak{p}(I^{[q]}/J^{[q]}) = 0$ for all q, so that $u_{R_\mathfrak{p}}(J_\mathfrak{p}, I_\mathfrak{p}) = 0$. On the other hand, for any $\mathfrak{p} \in \operatorname{Min}(R/J)$, we have that $J_\mathfrak{p}$ and $I_\mathfrak{p}$ are primary to the maximal ideal of $R_\mathfrak{p}$, so that for these primes,

$$u_{R_\mathfrak{p}}(J_\mathfrak{p}, I_\mathfrak{p}) = e_{\mathrm{HK}}(J_\mathfrak{p}) - e_{\mathrm{HK}}(I_\mathfrak{p}) = 0,$$

with the last equality holding because $J_\mathfrak{p} \subseteq I_\mathfrak{p} \subseteq (J^*)_\mathfrak{p} \subseteq (J_\mathfrak{p})^*$.

The same argument shows something slightly more general. Namely,

Proposition 3.1 *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and let $L \subseteq M \subseteq N$ be R-modules such that N/L is a finite module of finite projective dimension with no embedded primes. Then $u_{N_\mathfrak{p}}(L_\mathfrak{p}, M_\mathfrak{p})$ is well-defined and finite for all $\mathfrak{p} \in \operatorname{Spec} R$. Moreover, $M \subseteq L_N^*$ if and only if $u_{N_\mathfrak{p}}(L_\mathfrak{p}, M_\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \operatorname{Spec} R$.*

Proof As usual, we may assume all the modules are finitely generated. Then:

$$0 \leq u_{N_\mathfrak{p}}(L_\mathfrak{p}, M_\mathfrak{p}) = \begin{cases} 0, & \mathfrak{p} \notin \operatorname{Min}(N/L) \\ \lim\limits_{q\to\infty} \dfrac{\lambda_{R_\mathfrak{p}}(M_\mathfrak{p}^{[q]}/L_\mathfrak{p}^{[q]})}{q^{\operatorname{ht}\mathfrak{p}}}, & \mathfrak{p} \in \operatorname{Min}(N/L), \end{cases}$$

and hence if $M \subseteq L_N^*$, Theorem 1.1 applied to $R_\mathfrak{p}$ shows that $u_{N_\mathfrak{p}}(L_\mathfrak{p}, M_\mathfrak{p}) = 0$ for all $\mathfrak{p}$. □

3.2 Relative multiplicity and finite F-representation type

The concept of finite F-representation type is due to Smith and van der Bergh [30].

Definition 3.2 Let R be an F-finite Noetherian local ring. It has *finite F-representation type* (abbreviated *FFRT*) if there is a finite set of finitely generated R-modules $M_1, \dots, M_s$ and integers $c_{i,e}$ for all $1 \leq i \leq s$ and all positive integers e such that ${}^eR \cong \bigoplus_{i=1}^s M_i^{\oplus c_{i,e}}$ as R-modules, for all e.

Smith and van der Bergh observed that the following classes of local F-finite rings have finite F-representation type:

- regular rings
- rings of finite Cohen–Macaulay type

- any direct summand of a ring of finite F-representation type

On the other hand, they showed that the cubical cone $k[\![X, Y, Z]\!]/(X^3 + Y^3 + Z^3)$ does *not* have FFRT.[2]

The second named author took the study of such rings further. From this point on, we fix the modules $M_1, \dots, M_s$ in the definition, and we assume that they are indecomposable, nonzero, and of distinct isomorphism classes. For simplicity, we assume in the following that R is complete, so that it satisfies the Krull–Schmidt condition and the numbers $c_{i,e}$ are uniquely determined. Let $a := [k : k^p]$. Then

Theorem 3.3 [33] *Under the above circumstances,*

(1) *The limit*

$$\ell_i := \lim_{e\to\infty} \frac{c_{i,e}}{(ap^d)^e}$$

is well-defined and finite for $1 \le i \le s$.
(*Without loss of generality, we assume from this point on that* $\ell_i > 0$ *for* $1 \le i \le r$ *and* $\ell_i = 0$ *for* $r < i \le s$. *The modules* $M_1, \dots, M_r$ *are called the* F-contributors.)

(2) $r \ge 1$. *(That is, there is at least one F-contributor.)*

(3) *Let* $U := \bigoplus_{i=1}^r M_i$. *For any finitely generated modules* $L \subseteq N$, *we have*

$$L^*_N \subseteq \ker(N \to \operatorname{Hom}_R(U, (N/L) \otimes_R U)),$$

where the map is defined by $n \mapsto (u \mapsto \bar{n} \otimes u)$. *If* $N = R$, *this just means that* $L^* \subseteq \operatorname{ann}_R(U/LU)$.

(4) *If* R *has a completely stable test element and* $\hat{R}$ *is reduced*[3] *and equidimensional, then the displayed containment above becomes an equality* (*so that when* $N = R$, $L^* = \operatorname{ann}_R(U/LU)$).

Here we compute $u_N(L, M)$ when R is complete and has FFRT:

For any finitely generated R-module Z, and any e, we have ${}^e(F^e(Z)) \cong Z \otimes_R {}^eR$. Using this and the fact that ${}^e(-)$ is an exact functor, if R is complete we have

$$\begin{aligned} {}^e(M_N^{[q]}/L_N^{[q]}) &= {}^e\ker(F^e(N/L) \to F^e(N/M)) \\ &\cong \ker({}^e(F^e(N/L)) \to {}^e(F^e(N/M))) \\ &\cong \ker((N/L) \otimes {}^eR \to (N/M) \otimes {}^eR) \\ &= \bigoplus_{i=1}^s \ker((N/L) \otimes_R M_i \to (N/M) \otimes_R M_i)^{\oplus c_{i,e}}. \end{aligned}$$

Since $\Gamma_\mathfrak{m}$ is left-exact and commutes with ${}^e(-)$, we have

$${}^e\Gamma_\mathfrak{m}(M_N^{[q]}/L_N^{[q]}) \cong \bigoplus_{i=1}^s \Gamma_\mathfrak{m}(\ker((N/L) \otimes_R M_i \to (N/M) \otimes_R M_i))^{\oplus c_{i,e}}.$$

Set $K_i := \Gamma_\mathfrak{m}(\ker((N/L)\otimes_R M_i \to (N/M)\otimes_R M_i))$ for $1 \le i \le s$. Since $\lambda({}^eZ) = a^e\lambda(Z)$ for any finite length R-module Z, we have

---

[2] Note that rings with FFRT need not be F-regular (or even F-rational). Shibuta [29] proved that if R is any 1-dimensional complete local domain of prime characteristic whose residue field is either finite or algebraically closed, then R has finite F-representation type.

[3] Here too, the reducedness assumption appears to be unnecessary, in light of methods used in the proof of Theorem 2.4.

$$u_N(L, M) = \lim_{q\to\infty} \frac{\lambda(\Gamma_{\mathfrak{m}}(M_N^{[q]}/L_N^{[q]}))}{p^{de}} = \lim_{q\to\infty} \frac{\lambda({}^e\Gamma_{\mathfrak{m}}(M_N^{[q]}/L_N^{[q]})}{(ap^d)^e}$$
$$= \sum_{i=1}^{s} \lambda(K_i) \lim_{e\to\infty} \frac{c_{i,e}}{(ap^d)^e} = \sum_{i=1}^{r} \lambda(K_i)\ell_i.$$

(The sum only goes to r, since the limits equal 0 for $r < i \le s$.)

Now we can state a converse to Theorem 2.4 for rings with FFRT.

Theorem 3.4 *Let R be an F-finite ring with FFRT, and let $L \subseteq M \subseteq N$ be R-modules such that N/L is finitely generated. If $M \subseteq L_N^*$, then $u_{N_{\mathfrak{p}}}(L_{\mathfrak{p}}, M_{\mathfrak{p}}) = 0$ for all $\mathfrak{p} \in \operatorname{Spec} R$.*

Proof Since tight closure persists in localizations [14, Theorem 6.24], and since the FFRT property localizes, we may assume that $(R, \mathfrak{m})$ is local and just show that $u_N(L, M) = 0$.

Let M_i, s, r, ℓ_i, and U be as in Theorem 3.3, and let K_i be as in the discussion above. Since $M \subseteq L_N^*$, part (3) of that theorem implies that $M \subseteq \ker(N \to N/L \to \operatorname{Hom}_R(U, (N/L)\otimes_R U))$. Translating, this means that the natural map

$$(N/L) \otimes_R U \to (N/M) \otimes_R U$$

is an isomorphism, whence the natural maps $(N/L) \otimes_R M_i \to (N/M) \otimes_R M_i$ are isomorphisms, and hence $K_i = \Gamma_{\mathfrak{m}}(\ker((N/L) \otimes_R M_i \to (N/M) \otimes_R M_i)) = \Gamma_{\mathfrak{m}}(0) = 0$, for $1 \le i \le r$. Thus we have

$$u_N(L, M) = \sum_{i=1}^{r} \lambda(K_i)\ell_i = 0.$$

□

3.3 Relative multiplicity and finite generation of $R[x; f]$-modules

One of the standard constructions in noncommutative ring theory is the *skew polynomial ring* (cf. [20, Example 1.7], or almost any other introductory text on noncommutative rings). Given a ring R, an indeterminate x, and a ring endomorphism $f : R \to R$, the skew polynomial ring $S := R[x; f]$ is an $\mathbb{N}$-graded R-algebra, which looks like $\bigoplus_{n\ge0} Rx^n$ as a graded R-module, with multiplication given by $(rx^n)(sx^m) := rf^n(s)x^{m+n}$ for $r, s \in R$. When R is a commutative Noetherian ring of prime characteristic p, this ring and its modules were first studied by Yuji Yoshino [34], and were studied much further, to great effect, by Lyubeznik [21] (who called some of the S-modules ‘F-modules’), and by Rodney Sharp and his collaborators (e.g., [18,26–28]). In particular, Sharp studied various right and left $R[x; f]$-module structures on top local cohomology modules of finite R-modules, obtaining striking results on parameter test elements.

Let $\mathcal{M} := \bigoplus_{e\ge0}(M_N^{[p^e]}/L_N^{[p^e]})X^e$, and $\mathcal{H} := \Gamma_{\mathfrak{m}}(\mathcal{M}) = \bigoplus_{e\ge0} \Gamma_{\mathfrak{m}}(M_N^{[p^e]}/L_N^{[p^e]})X^e$, where $L \subseteq M \subseteq N$ are fixed R-modules such that M/L is finite.

Then $\mathcal{M}$ is a graded left $R[x; f]$-module, with $R[x; f]$-action given by $x \cdot \overline{m}X^e := \overline{m^p}X^{e+1}$. Moreover, it is finitely generated in degree 0 by the R-generators of M/L, and $\mathcal{H}$ is a graded left $R[x; f]$-submodule of $\mathcal{M}$. Since $R[x; f]$ is almost never left- (or right-) Noetherian [34], we cannot assume that an arbitrary left submodule of a finite left $R[x; f]$-module is finitely generated. However:

Theorem 3.5 *Suppose $(R, \mathfrak{m})$ is local and $\mathcal{H}$ is finitely generated as a left $R[x; f]$-module. If $M \subseteq L_N^*$, then $u_N(L, M) = 0$.*

Proof We may immediately assume that $L = 0$.

The hypotheses imply that there is some power q_0 of p such that for all $q \geq q_0$, we have $\Gamma_{\mathfrak{m}}(M_N^{[q]}) = H^{[q/q_0]}_{F^{e_0}(N)}$, where $H := \Gamma_{\mathfrak{m}}(M_N^{[q_0]})$. Then we have

$$\lambda(\Gamma_{\mathfrak{m}}(M_N^{[q]})) = \lambda(H^{[q/q_0]}_{F^{e_0}(N)}) \leq C(q/q_0)^{d-1} = (C/q_0^{d-1})q^{d-1},$$

where C is a constant which is independent of q. The last inequality is by Theorem 1.1, since $H \subseteq M_N^{[q_0]} \subseteq 0^*_{F^{e_0}(N)}$. So $u_N(0, M) = 0$. □

Example 3.6 In [11], we provide a specific example of a ring R, ideals I and J with $J \subseteq I = J^*$, $\lambda(I/J) = \infty$, and a prime ideal $\mathfrak{p}$ of R such that $\mathfrak{p} \subsetneq \mathfrak{m}$, $\mathfrak{m}$ a maximal ideal, and $\mathfrak{p}, \mathfrak{m} \in \operatorname{Ass}(I^{[q]}/J^{[q]})$ for all $q = p^e$. In that example, we have $u_{R_\mathfrak{m}}(J_\mathfrak{m}, I_\mathfrak{m}) = u_{R_\mathfrak{p}}(J_\mathfrak{p}, I_\mathfrak{p}) = 0$, so the expected converse is verified there as well. Moreover, the example does not fit into any of the other situations outlined in this Section.

4 Variants of relative multiplicity

The proof of Theorem 3.5 suggests a slight variant on Definition 2.1. Namely:

Definition 4.1 For modules $L \subseteq M \subseteq N$ over a local ring $(R, \mathfrak{m})$ of dimension d, we set

$$v_N^+(L, M) := \limsup_{q\to\infty} \frac{\lambda([\Gamma_\mathfrak{m}(M/L)]^{[q]}_{N/L})}{q^d}$$

and

$$v_N^-(L, M) := \liminf_{q\to\infty} \frac{\lambda([\Gamma_\mathfrak{m}(M/L)]^{[q]}_{N/L})}{q^d}.$$

If these are equal (i.e., the limit is well-defined),[4] then the common number is written $v_N(L, M)$.

We note that $u_N^-(L, M)$ (resp. $u_N^+(L, M)$) is an upper bound for $v_N^-(L, M)$ (resp. $v_N^+(L, M)$). Also, this new concept has the advantage of being *bounded above*:

Lemma 4.2 *If M/L is finitely generated over a local ring $(R, \mathfrak{m})$, then $v_N^+(L, M) < \infty$.*

Proof Let H be the submodule of M such that $H/L = \Gamma_\mathfrak{m}(M/L)$. Then since H/L has finite length and $H_N^{[q]}/L_N^{[q]}$ is the image of the map $F^e(H/L) \to F^e(N/L)$, we have

$$\limsup_{q\to\infty} \frac{\lambda(\Gamma_\mathfrak{m}(M/L)^{[q]}_{N/L})}{q^d} \leq \lim_{q\to\infty} \frac{\lambda(F^e(H/L))}{q^d}$$

But since H/L has finite length, this last limit is well-defined and finite by Seibert [25, Proposition 2]. □

In preparation for the next theorem, we define two more slight variants of relative multiplicity:

[4] Recent work of Polstra [24] indicates that this limit exists in some generality, as well as the other limits defined in this section and the next.

Definition 4.3 For modules $L \subseteq M \subseteq N$ over a local ring $(R, \mathfrak{m})$ of dimension d, we set

$$w_N^+(L, M) := \sup\{u_N^+(T, M) \mid L \subseteq T \subseteq M \text{ and } \lambda(M/T) < \infty\}$$

and

$$w_N^-(L, M) := \sup\{u_N^-(T, M) \mid L \subseteq T \subseteq M \text{ and } \lambda(M/T) < \infty\}.$$

Finally, we set

$$x_N^-(L, M) := v_N^-(L_N^* \cap M, M).$$

Theorem 4.4 *Let R be a Noetherian ring of prime characteristic $p > 0$, and let $L \subseteq M \subseteq N$ be R-modules such that N/L is finitely generated. Consider the following conditions:*

(a) $M \subseteq L_N^*$.
(b) $w_{N_\mathfrak{p}}^+(L_\mathfrak{p}, M_\mathfrak{p}) = 0$ *for all* $\mathfrak{p} \in \operatorname{Spec} R$.
(c) $w_{N_\mathfrak{p}}^-(L_\mathfrak{p}, M_\mathfrak{p}) = 0$ *for all* $\mathfrak{p} \in \operatorname{Spec} R$.
(d) $x_{N_\mathfrak{p}}^-(L_\mathfrak{p}, M_\mathfrak{p}) = 0$ *for all* $\mathfrak{p}$ *such that* $\lambda(M_\mathfrak{p}/((L_\mathfrak{p})_{N_\mathfrak{p}}^* \cap M_\mathfrak{p})) < \infty$.

Then (a) $\implies$ (b) $\implies$ (c) $\implies$ (d). *If, moreover, R contains a completely stable weak test element, $\widehat{R_\mathfrak{p}}$ is reduced and equidimensional for all* $\mathfrak{p} \in \operatorname{Spec} R$, *and tight closure commutes with localization for the pair $L \subseteq N$, then* (d) $\implies$ (a).

Proof To see that (a) $\implies$ (b), note that for any module T between L and M, we have $M_\mathfrak{p} \subseteq (T_N^*)_\mathfrak{p} \subseteq (T_\mathfrak{p})_{N_\mathfrak{p}}^*$, so that when $\lambda(M_\mathfrak{p}/T_\mathfrak{p}) < \infty$ over $R_\mathfrak{p}$, Theorem 1.1 shows that $u_{N_\mathfrak{p}}^+(T_\mathfrak{p}, M_\mathfrak{p}) = 0$. Since this holds for all such T, we have $w_{N_\mathfrak{p}}^+(L_\mathfrak{p}, M_\mathfrak{p}) = 0$. We have (b) $\implies$ (c) $\implies$ (d) by the definitions.

We need only show that the additional conditions require that (d) $\implies$ (a). We will prove the contrapositive. That is, suppose that $M \nsubseteq L_N^*$. Then let $T := L_N^* \cap M$. Since $T \subsetneq M$, there is some minimal prime $\mathfrak{p}$ of M/T, which means that $0 < \lambda(M_\mathfrak{p}/T_\mathfrak{p}) < \infty$ as $R_\mathfrak{p}$-modules. Moreover, $T_\mathfrak{p} = (L_N^*)_\mathfrak{p} \cap M_\mathfrak{p} = (L_\mathfrak{p})_{N_\mathfrak{p}}^* \cap M_\mathfrak{p}$. Since $M_\mathfrak{p} \nsubseteq (L_N^*)_\mathfrak{p} = (L_\mathfrak{p})_{N_\mathfrak{p}}^* \supseteq (T_\mathfrak{p})_{N_\mathfrak{p}}^*$, it follows that $M_\mathfrak{p} \nsubseteq (T_\mathfrak{p})_{N_\mathfrak{p}}^*$. Then

$$x_{N_\mathfrak{p}}^-(L_\mathfrak{p}, M_\mathfrak{p}) = u_{N_\mathfrak{p}}^-(T_\mathfrak{p}, M_\mathfrak{p}) = \liminf_{q \to \infty} \frac{\lambda((M_\mathfrak{p})_{N_\mathfrak{p}}^{[q]}/(T_\mathfrak{p})_{N_\mathfrak{p}}^{[q]})}{q^{\operatorname{ht} \mathfrak{p}}} > 0,$$

where the last inequality is by Theorem 1.1. □

We think of this theorem as an avatar of the fact [13, Proposition 6.1] that every tightly closed ideal is an intersection of finite colength tightly closed ideals.

It is natural to ask whether Definitions 2.1 and 4.1 are equivalent. In fact they are not:

Proposition 4.5 *Let R be a Noetherian ring of prime characteristic $p > 0$. Consider the following conditions:*

(a) *R is regular.*
(b) $u_{N_\mathfrak{p}}(L_\mathfrak{p}, M_\mathfrak{p}) = v_{N_\mathfrak{p}}(L_\mathfrak{p}, M_\mathfrak{p})$ *for all submodule inclusions* $L \subseteq M \subseteq N$ *and all* $\mathfrak{p} \in \operatorname{Spec} R$.
(c) $u_{N_\mathfrak{p}}(0, N_\mathfrak{p}) = v_{N_\mathfrak{p}}(0, N_\mathfrak{p})$ *for all finite R-modules N and all* $\mathfrak{p} \in \operatorname{Spec} R$.

Then (a) $\implies$ (b) $\implies$ (c), *and if R is reduced and has finite F-representation type then* (c) $\implies$ (a).

Proof (a) $\implies$ (b) because when R is regular, the functors F^e and $H^0_{\mathfrak{m}}$ commute with each other. Obviously (b) $\implies$ (c).

So suppose (c) holds, and assume R is reduced and has FFRT. We want to show $R_{\mathfrak{m}}$ is regular for all maximal ideals $\mathfrak{m}$, so we may replace R by $R_{\mathfrak{m}}$ for a maximal ideal $\mathfrak{m}$, and let $d = \dim R = \operatorname{ht} \mathfrak{m}$. We adopt the notation and terminology from Theorem 3.3. For a fixed finite R-module N and prime ideal $\mathfrak{p}$, we have

$$u_{N_{\mathfrak{p}}}(0, N_{\mathfrak{p}}) = \sum_{i=1}^{r} \lambda_{R_{\mathfrak{p}}}(H^0_{\mathfrak{p}R_{\mathfrak{p}}}(N_{\mathfrak{p}} \otimes (M_i)_{\mathfrak{p}})) \cdot \ell_i,$$

whereas

$$v_{N_{\mathfrak{p}}}(0, N_{\mathfrak{p}}) = \sum_{i=1}^{r} \lambda_{R_{\mathfrak{p}}}(\operatorname{im}(H^0_{\mathfrak{p}R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \otimes (M_i)_{\mathfrak{p}} \longrightarrow N_{\mathfrak{p}} \otimes (M_i)_{\mathfrak{p}})) \cdot \ell_i.$$

The fact that these are equal amounts to saying that $\lambda_{R_{\mathfrak{p}}}(H^0_{\mathfrak{p}R_{\mathfrak{p}}}(N_{\mathfrak{p}} \otimes (M_i)_{\mathfrak{p}})) = \lambda_{R_{\mathfrak{p}}}(\operatorname{im}(H^0_{\mathfrak{p}R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \otimes (M_i)_{\mathfrak{p}} \to N_{\mathfrak{p}} \otimes (M_i)_{\mathfrak{p}}))$ for each $1 \le i \le r$. In particular, if $\mathfrak{p} \in \operatorname{Ass}(N \otimes M_i)$, then $\lambda_{R_{\mathfrak{p}}}(H^0_{\mathfrak{p}R_{\mathfrak{p}}}(N_{\mathfrak{p}} \otimes (M_i)_{\mathfrak{p}})) \neq 0$, whence $\operatorname{im}(H^0_{\mathfrak{p}R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \otimes (M_i)_{\mathfrak{p}} \to N_{\mathfrak{p}} \otimes (M_i)_{\mathfrak{p}}) \neq 0$, which implies that $H^0_{\mathfrak{p}R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \neq 0$, so that $\mathfrak{p} \in \operatorname{Ass} N$.

That is, for all finitely generated R-modules N and all F-contributors M_i, we have $\operatorname{Ass}(N \otimes M_i) \subseteq \operatorname{Ass} N$. But the authors have shown in [10] that any R-module V such that $\operatorname{Ass}(N \otimes V) \subseteq \operatorname{Ass} N$ for all finite N must be flat, provided that R is reduced. Thus, each M_i is flat, hence (since they are finitely generated) free. Thus, we may arrange it (by re-grouping the summands) so that R *is the only F-contributor!* That is, $r = 1$ and $M_1 = R$.

Hence,

$$1 \le e_{\mathrm{HK}}(\mathfrak{m}) = \lim_{q \to \infty} \frac{\lambda(R/\mathfrak{m}^{[q]})}{q^d} = \lim_{e \to \infty} \frac{\lambda((R/\mathfrak{m}) \otimes_R {}^e R)}{(ap^d)^e}$$
$$= \sum_{i=1}^{r} \left(\lim_{e \to \infty} \frac{c_{i,e}}{(ap^d)^e} \right) \cdot \lambda(R/\mathfrak{m} \otimes_R M_i) = \ell_1 \cdot \lambda(R/\mathfrak{m}) = \ell_1.$$

However, note that ℓ_1 is the *F-signature* of the ring R (cf. Huneke and Leuschke [17]), so that by [17, Theorem 11 and Proposition 14], $\ell_1 \le 1$. Thus, $\ell_1 = 1$, and then [17, Corollary 16] shows that R is regular. □

Remark In [10], the authors have in fact shown that if R is reduced and V is an R-module, then V is flat if and only if $\operatorname{Ass}(Q \otimes V) \subseteq \operatorname{Ass} Q$ for all *prime ideals* Q. Given this, the proof of Proposition 4.5 yields a stronger result. Namely, if R is a reduced Noetherian ring of prime characteristic and finite F-representation type, then R is regular if and only if $u_{Q_{\mathfrak{p}}}(0, Q_{\mathfrak{p}}) = v_{Q_{\mathfrak{p}}}(0, Q_{\mathfrak{p}})$ for all $\mathfrak{p}, Q \in \operatorname{Spec} R$.

5 A numerical criterion based on a Nakayama-type lemma

If all we wanted to do was to get a numerical criterion determining exactly when two modules have the same tight closure, it already exists, in view of the following result of the first named author, for which we provide a new, slightly simplified proof here in order to make the paper more self-contained:

Proposition 5.1 (Nakayama lemma for tight closure) [9, Corollary 3.2] *Let $(R, \mathfrak{m})$ be a Noetherian local ring of prime characteristic $p > 0$ which possesses a weak test element (e.g. this holds whenever R is excellent* [14, Theorem 6.1a]). *Let $L \subseteq M \subseteq N$ be R-modules such that N/L is finitely generated and $L \subseteq M \subseteq (L + \mathfrak{m}M)^*_N$. Then $M \subseteq L^*_N$.*

Proof First, let q_0 be some power of p such that there exists a q_0-weak test element c. We show by induction on r that $M \subseteq (L + m^r M)^*_N$ for all $r \geq 1$.

The case $r = 1$ holds by assumption. So assume inductively that $r > 1$ and $M \subseteq (L + \mathfrak{m}^{r-1} M)^*_N$. From now on all bracket powers (except on $\mathfrak{m}$) are taken as submodules of N. Since M is finitely generated, it follows that for all $q \geq q_0$, we have $cM^{[q]} \subseteq (L+\mathfrak{m}^{r-1} M)^{[q]}$, so that $c^2 M^{[q]} \subseteq L^{[q]}+(\mathfrak{m}^{[q]})^{r-1} cM^{[q]} \subseteq L^{[q]}+(\mathfrak{m}^{[q]})^{r-1}(L+\mathfrak{m}M)^{[q]} \subseteq L^{[q]} + (\mathfrak{m}^{[q]})^r M^{[q]} = (L + \mathfrak{m}^r M)^{[q]}$. Since this holds for all $q \geq q_0$, it follows that $M \subseteq (L + \mathfrak{m}^r M)^*_N$, completing the induction.

Now fix any power $q \geq q_0$ of p. Since $M \subseteq (L + \mathfrak{m}^r M)^*_N$ for all $r \geq 1$, we have that $cM^{[q]} \subseteq L^{[q]} + (\mathfrak{m}^{[q]})^r M^{[q]}$ for all r (for this particular q), so that $cM^{[q]} \subseteq \bigcap_{r \geq 1} L^{[q]} + (\mathfrak{m}^{[q]})^r M^{[q]}$. Now going mod $L^{[q]}$ and taking bracket powers in N/L, we have that $c(M/L)^{[q]} \subseteq \bigcap_{r \geq 1} (\mathfrak{m}^{[q]})^r (M/L)^{[q]} = 0$, by the Krull intersection theorem. Now 'unfix' q, so that we have $cM^{[q]}_N \subseteq L^{[q]}_N$ for all $q \geq q_0$, whence $M \subseteq L^*_N$. □

Now define

$$y^-_N(L, M) := \liminf_{q \to \infty} \frac{\lambda(M^{[q]}_N / (L + \mathfrak{m}M)^{[q]}_N)}{q^d}.$$

and let $y^+_N(L, M)$ be the corresponding lim sup.

Proposition 5.2 *Let $(R, \mathfrak{m})$ be a Noetherian local ring of prime characteristic $p > 0$ with a weak test element. Let $L \subseteq M \subseteq N$ be R-modules such that N/L is finitely generated. The following are equivalent:*

(1) $M \subseteq L^*_N$.
(2) $y^+_N(L, M) = 0$.
(3) $y^-_N(L, M) = 0$.

Proof By Proposition 5.1, $M \subseteq L^*_N \iff M \subseteq (L + \mathfrak{m}M)^*_N$. By Theorem 1.1 and since $\lambda(M/(L + \mathfrak{m}M)) \leq \lambda(M/\mathfrak{m}M) = \mu(M) < \infty$, $M \subseteq (L + \mathfrak{m}M)^*_N \iff y^+_N(L, M) = 0 \iff y^-_N(L, M) = 0$. □

Here is a global version:

Proposition 5.3 *Let R be a Noetherian ring of prime characteristic $p > 0$ which is reduced, locally equidimensional, and essentially of finite type over an excellent local ring. Let $L \subseteq M \subseteq N$ be R-modules such that N/L is finitely generated. The following are equivalent:*

(a) $M \subseteq L^*_N$.
(b) $M \subseteq (L + \mathfrak{m}M)^*_N$ *for all maximal ideals* $\mathfrak{m}$.
(c) $M_\mathfrak{m} \subseteq (L_\mathfrak{m} + \mathfrak{m}M_\mathfrak{m})^*_{N_\mathfrak{m}}$ *for all maximal ideals* $\mathfrak{m}$.
(d) $y^+_{N_\mathfrak{m}}(L_\mathfrak{m}, M_\mathfrak{m}) = 0$ *for all maximal ideals* $\mathfrak{m}$.
(e) $y^-_{N_\mathfrak{m}}(L_\mathfrak{m}, M_\mathfrak{m}) = 0$ *for all maximal ideals* $\mathfrak{m}$.
(f) $M_\mathfrak{m} \subseteq (L_\mathfrak{m})^*_{N_\mathfrak{m}}$ *for all maximal ideals* $\mathfrak{m}$.

Proof (a) $\implies$ (b) $\implies$ (c): Clear.

(c) $\iff$ (d) $\iff$ (e): Since R is excellent, each $\widehat{R_{\mathfrak{m}}}$ is still equidimensional and reduced. Then the equivalence follows from Theorem 1.1 applied to each $R_{\mathfrak{m}}$.

(c) $\implies$ (f), by Proposition 5.1, since R has a locally stable weak test element by [14, Theorem 6.1].

(f) $\implies$ (a): Let $c \in R^\circ$ be a locally stable q_0-weak test element. Fix $q \geq q_0$. Then for all maximal ideals $\mathfrak{m}$, we have

$$\left(cM_N^{[q]}\right)_{\mathfrak{m}} = \frac{c}{1} \cdot (M_{\mathfrak{m}})_{N_{\mathfrak{m}}}^{[q]} \subseteq (L_{\mathfrak{m}})_{N_{\mathfrak{m}}}^{[q]} = \left(L_N^{[q]}\right)_{\mathfrak{m}}.$$

Since containment is a local property, it follows that $cM_N^{[q]} \subseteq L_N^{[q]}$. Since this holds for all $q \geq q_0$, it follows that $M \subseteq L_N^*$. □

6 Another numerical characterization of tight closure when $\lambda(I/J) < \infty$

Inspired by j-multiplicity, we make the following definitions:

Definition 6.1 For an ideal K of R and an integer $e \geq -1$, (using the convention $K^{[p^{-1}]} := R$) we set

$$l_e(K) := \lambda(\Gamma_{\mathfrak{m}}(K^{[p^e]}/K^{[p^{e+1}]}))$$

and

$$f_e(K) := \sum_{n=-1}^{e-1} l_n(K).$$

For a pair of ideals $J \subseteq I$, set

$$f^-(J, I) := \liminf_{q \to \infty} \frac{f_e(J) - f_e(I)}{q^d},$$

and

$$f^+(J, I) := \limsup_{q \to \infty} \frac{f_e(J) - f_e(I)}{q^d}.$$

If these two quantities are equal, we denote the common number by $f(J, I)$.

We have the following:

Theorem 6.2 *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring of dimension d and prime characteristic* $p > 0$*, and let* J, I *be ideals such that* $J \subseteq I$ *and* $\lambda(I/J) < \infty$*. Consider the following conditions:*

(a) $I^* = J^*$.
(b) $f(J, I) = 0$.
(c) $f^-(J, I) \leq 0$.

Then (a) $\implies$ (b) $\implies$ (c)*. If moreover R has a completely stable weak test element and* $\hat{R}$ *is equidimensional and reduced, or if* $\dim R = 0$*, then* (c) $\implies$ (a) *as well.*

Proof We dispense first with the case where $\dim R = 0$. In this case, for any proper ideal K we have $K^{[q]} = 0$ for $q \gg 0$, and hence $f_n(K) = \lambda(R) > 0$ for $n \gg 0$, whereas $f_n(R) = 0$ for all n. Also, $I^* = J^*$ if and only if both ideals are proper or both are the unit ideal. If both ideals are proper, then $f_n(I) = \lambda(R) = f_n(J)$ for $n \gg 0$, whence (b) holds. If both ideals are improper, then $I = J = R$, so that (b) holds. So we see that (a) $\implies$ (b). Conversely, suppose that (c) holds. If $I = R$ then $f_n(I) = 0$, whence $f_n(J) = 0$ for infinitely many values of n, which forces $J = R$. On the other hand, if I is proper, then J is proper since $J \subseteq I$. In either case, (a) holds, so (c) $\implies$ (a).

From now on, we assume that $d = \dim R > 0$. First note that we have the following short exact sequences for all $q = p^e$, $e \geq -1$:

$$0 \to I^{[pq]}/J^{[pq]} \to I^{[q]}/J^{[pq]} \to I^{[q]}/I^{[pq]} \to 0 \tag{1}$$

and

$$0 \to J^{[q]}/J^{[pq]} \to I^{[q]}/J^{[pq]} \to I^{[q]}/J^{[q]} \to 0. \tag{2}$$

Applying $\Gamma_{\mathfrak{m}}$ to sequence (1), and using the fact that $I^{[pq]}/J^{[pq]}$ has finite length, we get the short exact sequence:

$$0 \to I^{[pq]}/J^{[pq]} \to \Gamma_{\mathfrak{m}}(I^{[q]}/J^{[pq]}) \to \Gamma_{\mathfrak{m}}(I^{[q]}/I^{[pq]}) \to 0. \tag{3}$$

Hence,

$$\lambda(\Gamma_{\mathfrak{m}}(I^{[q]}/J^{[pq]})) = \lambda(I^{[pq]}/J^{[pq]}) + l_e(I). \tag{4}$$

Now, applying $\Gamma_{\mathfrak{m}}$ to the sequence (2) and using the fact that $I^{[q]}/J^{[q]}$ has finite length, we get the following exact sequence:

$$0 \to \Gamma_{\mathfrak{m}}(J^{[q]}/J^{[pq]}) \to \Gamma_{\mathfrak{m}}(I^{[q]}/J^{[pq]}) \to I^{[q]}/J^{[q]}, \tag{5}$$

which leads to the inequalities:

$$l_e(J) \leq \lambda(\Gamma_{\mathfrak{m}}(I^{[q]}/J^{[pq]})) \leq l_e(J) + \lambda(I^{[q]}/J^{[q]}). \tag{6}$$

Combining Eq. 4 with Inequalities 6, we get:

$$l_e(J) \leq \lambda(I^{[pq]}/J^{[pq]}) + l_e(I) \leq l_e(J) + \lambda(I^{[q]}/J^{[q]}),$$

which are equivalent to the following:

$$\lambda(I^{[pq]}/J^{[pq]}) - \lambda(I^{[q]}/J^{[q]}) \leq l_e(J) - l_e(I) \leq \lambda(I^{[pq]}/J^{[pq]}). \tag{7}$$

Taking the sum of Inequalities 7 from $e = -1$ to $n - 1$, we get:

$$\lambda(I^{[p^n]}/J^{[p^n]}) \leq f_n(J) - f_n(I) \leq \sum_{j=0}^{n} \lambda(I^{[p^j]}/J^{[p^j]}). \tag{8}$$

Following these preliminaries, we proceed to the implications in the proof. It is obvious that (b) implies (c). So suppose that (a) is true. Then by Theorem 1.1, there is a constant C such that $\lambda(I^{[q]}/J^{[q]}) \leq Cq^{d-1}$ for all $q = p^e$, $e \geq -1$. Thus,

$$\sum_{j=0}^{n} \lambda(I^{[p^j]}/J^{[p^j]}) \leq C \cdot \sum_{j=0}^{n} p^{j(d-1)} = \frac{C(p^{(n+1)(d-1)} - 1)}{p^{d-1} - 1} < C' p^{n(d-1)},$$

where $C' := Cp^{d-1}/(p^{d-1} - 1)$. Combining with Inequalities 8 and dividing by p^{nd}, we get

$$0 \leq \frac{\lambda(I^{[p^n]}/J^{[p^n]})}{p^{nd}} \leq \frac{f_n(J) - f_n(I)}{p^{nd}} < \frac{C' p^{nd-n}}{p^{nd}} = \frac{C'}{p^n}.$$

Since both the leftmost and rightmost terms clearly have a limit of 0 as $n \to \infty$, statement (b) follows.

Conversely, suppose R satisfies the additional specified conditions and that (c) holds. Using (c) and Inequalities 8, we have:

$$0 \leq \liminf_{n\to\infty} \frac{\lambda(I^{[p^n]}/J^{[p^n]})}{p^{nd}} \leq \liminf_{n\to\infty} \frac{f_n(J) - f_n(I)}{p^{nd}} \leq 0$$

Thus, $\liminf_{n\to\infty} \frac{\lambda(I^{[p^n]}/J^{[p^n]})}{p^{nd}} = 0$, so by Theorem 1.1, $I^* = J^*$. □

We also get a global version:

Theorem 6.3 *Let R be a Noetherian ring of prime characteristic $p > 0$ which is F-regular on the punctured spectrum, and let $J \subseteq I$ be ideals. Consider the following conditions:*

(a) $(I_\mathfrak{m})^* = (J_\mathfrak{m})^*$ *for all maximal ideals* $\mathfrak{m}$.
(b) $f(J_\mathfrak{p}, I_\mathfrak{p}) = 0$ *for all* $\mathfrak{p} \in \operatorname{Spec} R$.
(c) $f^-(J_\mathfrak{p}, I_\mathfrak{p}) \leq 0$ *for all* $\mathfrak{p} \in \operatorname{Spec} R$.

Then (a) $\implies$ (b) $\implies$ (c). *If moreover R has a completely stable weak test element and $\widehat{R_\mathfrak{p}}$ is equidimensional and reduced for all* $\mathfrak{p} \in \operatorname{Spec} R$, *or if* $\dim R = 0$, *then* (c) $\implies$ (a) *as well.*

Proof First we show that (a) $\implies$ (b): First, suppose $\mathfrak{p}$ is non-maximal, and choose a maximal ideal $\mathfrak{m}$ such that $\mathfrak{p} \subseteq \mathfrak{m}$. Then $J_\mathfrak{p} \subseteq I_\mathfrak{p} = (I_\mathfrak{m})_\mathfrak{p} \subseteq ((J_\mathfrak{m})^*)_\mathfrak{p} \subseteq ((J_\mathfrak{m})_\mathfrak{p})^* = J_\mathfrak{p}$. That is, $J_\mathfrak{p} = I_\mathfrak{p}$ for all non-maximal ideals $\mathfrak{p}$, so it follows that I/J has finite length and that (b) holds for non-maximal $\mathfrak{p}$. However, for any maximal ideal $\mathfrak{m}$, since $I_\mathfrak{m}/J_\mathfrak{m}$ has finite length, the implication follows for maximal ideals by Theorem 6.2.

Next, we show that (c) $\implies$ (a) under the stated conditions. We first show that $I_\mathfrak{p} = J_\mathfrak{p}$ for all non-maximal ideals $\mathfrak{p}$. By Noetherian induction, we may assume that $I_\mathfrak{q} = J_\mathfrak{q}$ for all prime ideals $\mathfrak{q} \subsetneq \mathfrak{p}$. Thus, $\lambda(I_\mathfrak{p}/J_\mathfrak{p}) < \infty$, so that Theorem 6.2 applied to the ideals $J_\mathfrak{p} \subseteq I_\mathfrak{p}$ shows that $(J_\mathfrak{p})^* = (I_\mathfrak{p})^*$. But $R_\mathfrak{p}$ is F-regular, so $J_\mathfrak{p} = I_\mathfrak{p}$.

We have now that $\lambda(I/J) < \infty$, so that for any maximal ideal $\mathfrak{m}$, Theorem 6.2 shows that $(J_\mathfrak{m})^* = (I_\mathfrak{m})^*$. □

References

1. Aberbach, I.M.: The existence of the F-signature for rings with large $\mathbb{Q}$-Gorenstein locus. J. Algebra **319**(7), 2994–3005 (2008)
2. Achilles, R., Manaresi, M.: Multiplicity for ideals of maximal analytic spread and intersection theory. J. Math. Kyoto Univ. **33**(4), 1029–1046 (1993)
3. Achilles, R., Manaresi, M.: Multiplicities of a bigraded ring and intersection theory. Math. Ann. **309**(4), 573–591 (1997)
4. Brenner, H.: Irrational Hilbert-Kunz multiplicities. arXiv:1305.5873 (2013)
5. Brenner, H., Monsky, P.: Tight closure does not commute with localization. Ann. of Math. (2) **171**(1), 571–588 (2010)

6. Buchsbaum, D.A., Eisenbud, D.: What makes a complex exact? J. Algebra **25**, 259–268 (1973)
7. Dao, H., Smirnov, I.: On generalized Hilbert–Kunz function and multiplicity. arXiv:1305.1833 (2013)
8. Dao, H., Watanabe, K.: Some computations of generalized Hilbert–Kunz function and multiplicity. arXiv:1503.00894 (2015)
9. Epstein, N.: Phantom depth and stable phantom exactness. Trans. Am. Math. Soc. **359**(10), 4829–4864 (2007)
10. Epstein, N., Yao, Y.: Criteria for flatness and injectivity. Math. Z. **271**(3–4), 1193–1210 (2012)
11. Epstein, N., Yao, Y.: A computation concerning relative Hilbert–Kunz multiplicities. arXiv:1605.01807 (2016)
12. Flenner, H., Manaresi, M.: A numerical characterization of reduction ideals. Math. Z. **238**(1), 205–214 (2001)
13. Hochster, M., Huneke, C.: Tight closure, invariant theory, and the Briançon-Skoda theorem. J. Am. Math. Soc. **3**(1), 31–116 (1990)
14. Hochster, M., Huneke, C.: F-regularity, test elements, and smooth base change. Trans. Am. Math. Soc. **346**(1), 1–62 (1994)
15. Hochster, M., Huneke, C.: Localization and test exponents for tight closure. Michigan Math. J. **48**, 305–329 (2000)
16. Huneke, C.: Tight Closure and its Applications. In: CBMS Reg. Conf. Ser. in Math., vol. **88**, Amer. Math. Soc., Providence (1996)
17. Huneke, C., Leuschke, G.J.: Two theorems about maximal Cohen–Macaulay modules. Math. Ann. **324**(2), 391–404 (2002)
18. Katzman, M., Sharp, R.Y.: Uniform behaviour of the Frobenius closures of ideals generated by regular sequences. J. Algebra **295**(1), 231–246 (2006)
19. Kurano, K.: The singular Riemann–Roch theorem and Hilbert–Kunz functions. J. Algebra **304**(1), 487–499 (2006)
20. Lam, T.Y.: A First Course in Noncommutative Rings. Graduate Texts in Mathematics. Springer, New York (1991)
21. Lyubeznik, G.: F-modules: applications to local cohomology and D-modules in characteristic $p > 0$. J. Reine Angew. Math. **491**, 65–130 (1997)
22. Monsky, P.: The Hilbert–Kunz function. Math. Ann. **263**(1), 43–49 (1983)
23. Peskine, C., Szpiro, L.: Dimension projective finie et cohomologie locale. applications à la démonstration de conjectures de M. Auslander, H. Bass et A. Grothendieck, Inst. Hautes Études Sci. Publ. Math. **42**, 47–119 (1973)
24. Polstra, T.: Uniform bounds in F-finite rings and lower semi-continuity of the F-signature. arXiv:1506.01073, (2015)
25. Seibert, G.: Complexes with homology of finite length and Frobenius functors. J. Algebra **125**(2), 278–287 (1989)
26. Sharp, R.Y.: Graded annihilators of modules over the Frobenius skew polynomial ring, and tight closure. Trans. Am. Math. Soc **359**(9), 4237–4258 (2007) (electronic)
27. Sharp, R.Y.: On the Hartshorne-Speiser-Lyubeznik theorem about Artinian modules with a Frobenius action. Proc. Am. Math. Soc. **135**(3), 665–670 (2007) (electronic)
28. Sharp, R.Y., Nossem, N.: Ideals in a perfect closure, linear growth of primary decompositions, and tight closure. Trans. Am. Math. Soc. **356**(9), 3687–3720 (2004)
29. Shibuta, T.: One-dimensional rings of finite F-representation type. J. Algebra **332**(1), 434–441 (2011)
30. Smith, K.E., van der Bergh, M.: Simplicity of rings of differential operators in prime characteristic. Proc. Lond. Math. Soc. **75**(1), 32–62 (1997)
31. Ulrich, B., Validashti, J.: Numerical criteria for integral dependence. Math. Proc. Camb. Philos. Soc. **151**(1), 95–102 (2011)
32. Vraciu, A.: An observation on generalized Hilbert-Kunz functions. arXiv:1510.00668, (2015)
33. Yao, Y.: Modules with finite F-representation type. J. Lond. Math. Soc. **2**(72), 53–72 (2005)
34. Yoshino, Y.: Skew-polynomial rings of Frobenius type and the theory of tight closure. Comm. Algebra **22**(7), 2473–2502 (1994)