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On the complete spacelike hypersurfaces immersed with two distinct principal curvatures in a locally symmetric Lorentz space

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Abstract Our purpose in this paper is to study the geometry of complete linear Weingarten spacelike hypersurfaces immersed with two distinct principal curvatures in a locally symmetric Lorentz space, which is supposed to obey standard curvature constrains. In this setting, we apply some appropriated generalized maximum principles to a suitable Cheng-Yau modified operator in order to guarantee that such a spacelike hypersurface must be isometric to an isoparametric hypersurface of the ambient space.

Keywords Locally symmetric Lorentz spaces · Einstein spacetimes · Complete linear Weingarten spacelike hypersurfaces · Isoparametric hypersurfaces

Mathematics Subject Classification Primary 53C42; Secondary 53A10 · 53C20 · 53C50

1 Introduction

Let L_1^{n+1} be an $(n+1)$-dimensional Lorentz space, that is, a semi-Riemannian manifold of index 1. When the Lorentz space L_1^{n+1} is simply connected and has constant sectional

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curvature, it is called a Lorentz space form. The Lorentz-Minkowski space $\mathbb{L}^{n+1}$, the de Sitter space $\mathbb{S}_1^{n+1}$ and the anti-de Sitter space $\mathbb{H}_1^{n+1}$ are the standard Lorentz space forms of constant sectional curvature 0, 1 and -1, respectively. We also recall that a hypersurface M^n immersed in a Lorentz space L_1^{n+1} is said to be spacelike if the metric on M^n induced from that of the ambient space L_1^{n+1} is positive definite.

The last few decades have seen a steadily growing interest in the study of the geometry of spacelike hypersurfaces immersed in a Lorentz space. Apart from physical motivations, from the mathematical point of view this is mostly due to the fact that such hypersurfaces exhibit nice Bernstein-type properties, and one can truly say that the first remarkable results in this branch were the rigidity theorems of Calabi [4] and Cheng and Yau [11], who showed (the former for $n \leq 4$, and the latter for general n) that the only maximal complete, non-compact, spacelike hypersurfaces of the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ are the spacelike hyperplanes. However, in the case that the mean curvature is a positive constant, Treibergs [27] astonishingly showed that there are many entire solutions of the corresponding constant mean curvature equation in $\mathbb{L}^{n+1}$, which he was able to classify by their projective boundary values at infinity.

Later, Nishikawa obtained extended Calabi and Cheng-Yau results showing that a complete maximal spacelike hypersurface immersed in a locally symmetric obeying certain curvature constraints must be totally geodesic. We recall that a Lorentz space is said locally symmetric when all the covariant derivative components of its curvature tensor vanish identically (see Theorem B of [23]).

As for the case of the de Sitter space $\mathbb{S}_1^{n+1}$, Goddard [17] conjectured that every complete spacelike hypersurface with constant mean curvature H in $\mathbb{S}_1^{n+1}$ should be totally umbilical. Although the conjecture turned out to be false in its original statement, it motivated a great deal of work of several authors trying to find a positive answer to the conjecture under appropriate additional hypotheses. For instance, in [2], Akutagawa showed that Goddard's conjecture is true when $0 \leq H^2 \leq 1$ in the case $n = 2$, and when $0 \leq H^2 < 4(n-1)/n^2$ in the case $n \geq 3$. Afterwards, Montiel [22] solved Goddard's problem in the compact case proving that the only closed spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ with constant mean curvature are the totally umbilical round spheres. Furthermore, he exhibited examples of complete spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ with constant H satisfying $H^2 \geq \frac{4(n-1)}{n^2}$ and being non totally umbilical, the so-called hyperbolic cylinders, which are isometric to the Riemannian product $\mathbb{H}^1(1-\coth^2 r) \times \mathbb{S}^{n-1}(1-\tanh^2 r)$ of a hyperbolic line and an $(n-1)$-dimensional Euclidean sphere.

When the ambient space is the anti-de Sitter space $\mathbb{H}_1^{n+1}$, Cao and Wei [6] showed that, if $n \geq 3$, then every n-dimensional complete maximal spacelike hypersurface in $\mathbb{H}_1^{n+1}$ with exactly two principal curvatures everywhere is isometric to some hyperbolic cylinder under an additional condition on these curvatures. Later, Perdomo [25] studied the 2-dimensional case and constructed new examples of complete maximal surfaces in $\mathbb{H}_1^3$. More recently, Chaves et al. [9] studied complete maximal spacelike hypersurfaces in $\mathbb{H}_1^{n+1}$ with either constant scalar curvature or constant non-zero Gauss-Kronecker curvature. In this context, they characterized the hyperbolic cylinders of $\mathbb{H}_1^{n+1}$ as the only such hypersurfaces with $(n-1)$ principal curvatures with the same sign everywhere.

Proceeding in this branch, an interesting question is to characterize complete linear Weingarten spacelike hypersurfaces (that is, complete spacelike hypersurfaces whose mean and scalar curvatures are linearly related) immersed in a certain Lorentz space. Many authors have approached problems in this subject. For instance, when the ambient space is a Lorentz space form, we refer to the readers the works [8,10,16,18,19,21].

Here, our aim is to study the geometry of complete linear Weingarten spacelike hypersurfaces with two distinct principal curvatures in a wide class of Lorentz spaces, the locally symmetric Lorentz spaces. In this setting, under appropriated constrains on the values of the mean curvature and on the norm of the traceless part of the second fundamental form, we extend the techniques developed in the recent papers [14,15,18] in order to characterize such spacelike hypersurfaces as being isometric to isoparametric hypersurfaces of the ambient space (see our several characterization results along Sects. 4 and 5). Our approach is based on the use of a Simons type formula jointly with the application of some generalized maximum principles to a suitable Cheng-Yau modified operator (for more details, see Sects. 2 and 3).

2 A Simons type formula in Lorentz spaces

Let M^n be a spacelike hypersurface immersed in a Lorentz space L_1^{n+1}, which means that the metric on M^n induced from L_1^{n+1} is positive defined. In this context, let us choose a local field of semi-Riemannian orthonormal frame $\{e_A\}_{A=1}^{n+1}$ in L_1^{n+1}, with dual coframe $\{\omega_A\}_{A=1}^{n+1}$, such that, at each point of M^n, $e_1, \dots, e_n$ are tangent to M^n and e_{n+1} is normal to M^n. We will use the following convention for indices:

$$1 \leq A, B, C, \dots \leq n+1, \ \ 1 \leq i, j, k, \dots \leq n.$$

We denote by $\{\omega_{AB}\}$ the connection forms of L_1^{n+1}. Thus, the structure equations of L_1^{n+1} are given by:

$$d\omega_A = -\sum_B \varepsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \quad \varepsilon_i = 1, \varepsilon_{n+1} = -1,$$

$$d\omega_{AB} = -\sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \varepsilon_C \varepsilon_D \overline{R}_{ABCD} \omega_C \wedge \omega_D.$$

Here, $\overline{R}_{ABCD}$, $\overline{R}_{CD}$ and $\overline{R}$ denote respectively the Riemannian curvature tensor, the Ricci tensor and the scalar curvature of the Lorentz space L_1^{n+1}. In this setting, we have

$$\overline{R}_{CD} = \sum_B \varepsilon_B \overline{R}_{BCDB} \quad \overline{R} = \sum_A \varepsilon_A \overline{R}_{AA}.$$

Moreover, the components $\overline{R}_{ABCD;E}$ of the covariant derivative of the Riemannian curvature tensor L_1^{n+1} are defined by

$$\sum_E \varepsilon_E \overline{R}_{ABCD;E} \omega_E = d\overline{R}_{ABCD} - \sum_E \varepsilon_E (\overline{R}_{EBCD} \omega_{EA} + \overline{R}_{AECD} \omega_{EB} + \overline{R}_{ABED} \omega_{EC} + \overline{R}_{ABCE} \omega_{ED}).$$

Now, we restrict all the tensors to the spacelike hypersurface M^n in L_1^{n+1}. First of all, $\omega_{n+1} = 0$ on M^n, so $\sum_i \omega_{(n+1)i} \wedge \omega_i = d\omega_{n+1} = 0$. Consequently, by Cartan's Lemma [7], there are on M^n smooth functions h_{ij} such that

$$\omega_{(n+1)i} = \sum_j h_{ij} \omega_j \quad \text{and} \quad h_{ij} = h_{ji}. \tag{2.1}$$

From (2.1), we have that the second fundamental form of M^n is given by $B = \sum_{i,j} h_{ij}\omega_i\omega_j e_{n+1}$, and its square length from second fundamental form is $S = \sum_{i,j} h_{ij}^2$. Furthermore, the mean curvature H of M^n is defined by $H = \frac{1}{n}\sum_i h_{ii}$.

The connection forms $\{\omega_{ij}\}$ of M^n are characterized by structure equations of M^n:

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$
$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l,$$

where R_{ijkl} are the components of curvature tensor of M^n.

From structure equations, we obtain Gauss equation

$$R_{ijkl} = \overline{R}_{ijkl} - (h_{ik}h_{jl} - h_{il}h_{jk}). \tag{2.2}$$

The components R_{ij} of the *Ricci* tensor and the scalar curvature R of M^n are given, respectively, by

$$R_{ij} = \sum_k \overline{R}_{kijk} - nHh_{ij} + \sum_k h_{ik}h_{kj}$$

and

$$n(n-1)R = \sum_{i,j} \overline{R}_{ijji} - n^2H^2 + S. \tag{2.3}$$

The first covariant derivatives h_{ijk} of h_{ij} satisfy

$$\sum_k h_{ijk}\omega_k = dh_{ij} - \sum_k h_{ik}\omega_{kj} - \sum_k h_{jk}\omega_{ki}. \tag{2.4}$$

Then, by exterior differentiation of (2.1), we obtain the Codazzi equation

$$h_{ijk} - h_{ikj} = \overline{R}_{(n+1)ijk}. \tag{2.5}$$

The second covariant derivative h_{ijkl} of h_{ij} are given by

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} - \sum_l h_{ljk}\omega_{li} - \sum_l h_{ilk}\omega_{lj} - \sum_l h_{ijl}\omega_{lk}.$$

By exterior differentiation of (2.4), we can get the following Ricci formula

$$h_{ijkl} - h_{ijlk} = -\sum_m h_{im}R_{mjkl} - \sum_m h_{jm}R_{mikl}. \tag{2.6}$$

Restricting the covariant derivative $\overline{R}_{ABCD;E}$ of $\overline{R}_{ABCD}$ on M^n, then $\overline{R}_{(n+1)ijk;l}$ is given by

$$\begin{aligned}\overline{R}_{(n+1)ijk;l} = \overline{R}_{(n+1)ijkl} + \overline{R}_{(n+1)i(n+1)k}h_{jl} \\ +\overline{R}_{(n+1)ij(n+1)}h_{kl} + \sum_m \overline{R}_{mijk}h_{ml},\end{aligned} \tag{2.7}$$

where $\overline{R}_{(n+1)ijkl}$ denotes the covariant derivative of $\overline{R}_{(n+1)ijk}$ as a tensor on M^n so that

$$\sum_l \overline{R}_{(n+1)ijkl}\omega_l = d\overline{R}_{(n+1)ijk} - \sum_l \overline{R}_{(n+1)ljk}\omega_{li} - \sum_l \overline{R}_{(n+1)ilk}\omega_{lj} - \sum_l \overline{R}_{(n+1)ijl}\omega_{lk}.$$

The Laplacian Δh_{ij} of h_{ij} is defined by $\Delta h_{ij} = \sum_k h_{ijkk}$. From (2.5)–(2.7), after a straightforward computation we obtain

$$\begin{aligned}\Delta h_{ij} = (nH)_{ij} - nH\sum_l h_{il}h_{lj} + Sh_{ij} + \sum_k \left(\overline{R}_{(n+1)ijk;k} + \overline{R}_{(n+1)kik;j}\right) \\ - \sum_k \left(h_{kk}\overline{R}_{(n+1)ij(n+1)} + h_{ij}\overline{R}_{(n+1)k(n+1)k}\right) \\ - \sum_{k,l}\left(2h_{kl}\overline{R}_{lijk} + h_{jl}\overline{R}_{lkik} + h_{il}\overline{R}_{lkjk}\right).\end{aligned} \tag{2.8}$$

Since $\Delta S = 2\left(\sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} h_{ij}\Delta h_{ij}\right)$, from (2.8) we get the following Simons type formula

$$\begin{aligned}\frac{1}{2}\Delta S = S^2 + \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} h_{ij}(nH)_{ij} + \sum_{i,j,k} h_{ij}\left(\overline{R}_{(n+1)ijk;k} + \overline{R}_{(n+1)kik;j}\right) \\ - \left(nH\sum_{i,j} h_{ij}\overline{R}_{(n+1)ij(n+1)} + S\sum_k \overline{R}_{(n+1)k(n+1)k}\right) \\ -2\sum_{i,j,k,l}\left(h_{kl}h_{ij}\overline{R}_{lijk} + h_{il}h_{ij}\overline{R}_{lkjk}\right) - nH\sum_{i,j,l} h_{il}h_{lj}h_{ij}.\end{aligned} \tag{2.9}$$

Now, we consider $\Psi = \sum_{i,j}\psi_{ij}\omega_i \otimes \omega_j$ a symmetric tensor on M^n defined by

$$\psi_{ij} = nH\delta_{ij} - h_{ij}.$$

Following Cheng-Yau [12], we introduce an operator $\square$ associated to Ψ acting on any smooth function f by

$$\square f = \sum_{i,j}\psi_{ij}f_{ij} = \sum_{i,j}\left(nH\delta_{ij} - h_{ij}\right)f_{ij}. \tag{2.10}$$

Taking $f = nH$ in (2.10) and taking a (local) orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n such that $h_{ij} = \lambda_i\delta_{ij}$, from equation (2.3) we obtain the following

$$\begin{aligned}\square(nH) &= \frac{1}{2}\Delta(nH)^2 - \sum_i (nH)_i^2 - \sum_i \lambda_i(nH)_{ii} \\ &= \frac{1}{2}\Delta S - n^2|\nabla H|^2 - \sum_i \lambda_i(nH)_{ii} + \frac{1}{2}\Delta\left(\sum_{i,j}\overline{R}_{ijji} - n(n-1)R\right).\end{aligned} \tag{2.11}$$

3 Locally symmetric Lorentz spaces

Following the ideas of Nishikawa [23] and Choi et al. [13,26], along this work we will assume that there exist constants c_1 and c_2 such that the sectional curvature K of the ambient space L_1^{n+1} satisfies the following two constraints

$$K(u, v) = -\frac{c_1}{n}, \tag{3.1}$$

for any spacelike vectors u and timelike v, and

$$K(u, v) \geq c_2, \tag{3.2}$$

for any spacelike vectors u and v.

We observe that the Lorentz space forms $L_1^{n+1}(c)$ of constant sectional curvature c satisfy curvature conditions (3.1) and (3.2) for $-\frac{c_1}{n} = c_2 = c$. On the other hand, Choi et al. [13] exhibited examples of Lorentz spaces which are not Lorentz space forms satisfying (3.1) and (3.2).

As mentioned before, a Lorentz space L_1^{n+1} is said locally symmetric when all the covariant derivative components $\overline{R}_{ABCD;E}$ of its curvature tensor vanish identically. In this setting, denoting by $\overline{R}_{AB}$ the components of the Ricci tensor of L_1^{n+1} satisfying curvature condition (3.1), the scalar curvature $\overline{R}$ of L_1^{n+1} is given by

$$\overline{R} = \sum_A \varepsilon_A \overline{R}_{AA} = \sum_{i,j} \overline{R}_{ijji} - 2\sum_i \overline{R}_{(n+1)ii(n+1)} = \sum_{i,j} \overline{R}_{ijji} + 2c_1.$$

Consequently, since the scalar curvature $\overline{R}$ of a locally symmetric Lorentz space is constant, we have that $\sum_{i,j} \overline{R}_{ijji}$ is a constant naturally attached to a locally symmetric Lorentz space satisfying curvature condition (3.1).

In what follows, we will quote some key lemmas in order to prove the results of the next section. The first one corresponds to Lemma 3.2 of [15].

Lemma 1 *Let L_1^{n+1} be a locally symmetric Lorentz space which satisfies curvature condition (3.1) and let M^n be a linear Weingarten spacelike hypersurface immersed in L_1^{n+1}, such that $R = aH + b$ for some $a, b \in \mathbb{R}$. Suppose that $b \neq \frac{1}{n(n-1)} \sum_{i,j} \overline{R}_{ijji}$ and that the following inequality is satisfied*

$$(n-1)^2 a^2 + 4\sum_{i,j} \overline{R}_{ijji} - 4n(n-1)\, b \geq 0. \tag{3.3}$$

Then,

$$\sum_{i,j,k} h_{ijk}^2 \geq n^2 |\nabla H|^2. \tag{3.4}$$

Moreover, if the inequality (3.3) is strict and the equality occurs in (3.4), then H is constant on M^n.

Now, we will consider a Cheng-Yau modified operator given by

$$L = \Box + \frac{n-1}{2} a\Delta. \tag{3.5}$$

The next result gives ellipticity criteria for the operator L is elliptic. For its proof, see Lemma 3.3 of [14].

Lemma 2 *Let L_1^{n+1} be a locally symmetric Lorentz space and let M^n be a linear Weingarten spacelike hypersurface immersed in L_1^{n+1}, such that $R = aH + b$ for some $a, b \in \mathbb{R}$ with $b < \frac{1}{n(n-1)} \sum_{i,j} \overline{R}_{ijji}$. Then, H has strict sign and L is elliptic.*

To close this section, we will reason as in the proof of Proposition 2.3 of [8] to get the following

Lemma 3 *Let L_1^{n+1} be a locally symmetric Lorentz space which satisfies curvature condition (3.2) and let M^n be complete linear Weingarten spacelike hypersurface immersed in L_1^{n+1}, such that $R = aH + b$ for some $a, b \in \mathbb{R}$ satisfying inequality (3.3) and with $a \geq 0$. If H is bounded on M^n, then there exists a sequence of points $\{q_k\}_{k\in\mathbb{N}} \subset M^n$ such that*

$$\lim_{k\to+\infty} nH(q_k) = \sup_M nH, \quad \lim_{k\to+\infty} |\nabla nH(q_k)| = 0 \quad \textit{and} \quad \limsup_{k\to+\infty} L(nH(q_k)) \leq 0.$$

Proof From (3.5), taking a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n such that $h_{ij} = \lambda_i \delta_{ij}$, we obtain

$$L(nH) = \sum_i \left(nH + \frac{n-1}{2}a - \lambda_i\right)(nH)_{ii}. \tag{3.6}$$

On the other hand, we observe that if H vanishes identically on M^n, then the result is valid. So, let us suppose that H is not identically zero. This way, we can choose the oriented of M^n so that $\sup_M H > 0$.

Thus, for all $i = 1, \ldots, n$ from (2.3) and (3.3) we have

$$\begin{aligned}
\lambda_i^2 \leq \sum_i \lambda_i^2 = S &= n^2H^2 + n(n-1)aH + n(n-1)b - \sum_{i,j} \overline{R}_{ijji} \\
&= \left(nH + \frac{n-1}{2}a\right)^2 - \frac{1}{4}\left((n-1)^2a^2 + 4\sum_{i,j} \overline{R}_{ijji} - 4n(n-1)b\right) \\
&\leq \left(nH + \frac{n-1}{2}a\right)^2.
\end{aligned}$$

Consequently, we get

$$-\lambda_i\lambda_j \leq \left(nH + \frac{n-1}{2}a\right)^2 \text{ and } |\lambda_i| \leq \left|nH + \frac{n-1}{2}a\right|. \tag{3.7}$$

From Gauss equation (2.2), taking into account (3.2) and (3.7), for $i \neq j$ we obtain

$$R_{ijji} = \bar{R}_{ijji} + \lambda_i\lambda_j \geq c_2 - \left(nH + \frac{n-1}{2}a\right)^2. \tag{3.8}$$

Since we are supposing that H is bounded on M^n, it follows from (3.8) that the sectional curvature of M^n is bounded below. Thus, we can apply the generalized maximum principle of Omori [24] to the function nH in order to obtain a sequence of points $\{q_k\}_{k\in\mathbb{N}} \subset M^n$ satisfying $\lim_{k\to+\infty} nH(q_k) = \sup_M nH$, $\lim_{k\to+\infty} |\nabla nH(q_k)| = 0$ and

$$\lim_{k\to+\infty} \sup \sum_i (nH)_{ii}(q_k) \leq 0. \tag{3.9}$$

Since $\sup_M H > 0$, passing subsequence if necessary, we can consider that such a sequence $\{q_k\}_{k\in\mathbb{N}}$ satisfies $H(q_k) \geq 0$.

Hence, since $a \geq 0$, from (3.7) we obtain

$$0 \leq nH(q_k) + \frac{n-1}{2}a - |\lambda_i(q_k)| \leq nH(q_k) + \frac{n-1}{2}a - \lambda_i(q_k)$$
$$\leq nH(q_k) + \frac{n-1}{2}a + |\lambda_i(q_k)| \leq 2nH(q_k) + (n-1)a.$$

This previous estimate shows that the function $nH(q_k) + \frac{n-1}{2}a - \lambda_i(q_k)$ is nonnegative and bounded on M^n, for all $k \in \mathbb{N}$. Therefore, taking into account inequality (3.9), we obtain

$$\lim_{k\to+\infty} \sup L(nH(q_k)) \leq \sum_i \lim_{k\to+\infty} \sup \left[\left(nH + \frac{n-1}{2}a - \lambda_i\right)(q_k)(nH)_{ii}(q_k)\right] \leq 0. \quad \square$$

4 Spacelike hypersurfaces with two distinct principal curvatures

In this section, proceeding with the context of the previous one, we will establish our characterization results concerning complete linear Weingarten hypersurfaces immersed in a locally symmetric Lorentz space. For this, given $\phi_{ij} = h_{ij} - H\delta_{ij}$, we will consider the following symmetric tensor

$$\Phi = \sum_{i,j} \phi_{ij}\omega_i \otimes \omega_j.$$

So, let $|\Phi|^2 = \sum_{i,j} \phi_{ij}^2$ be the square of the length of Φ. It is not difficult to see that Φ is traceless and that holds the following relation

$$|\Phi|^2 = S - nH^2. \tag{4.1}$$

Consequently, assuming that $R = aH + b$ for some $a, b \in \mathbb{R}$, from (2.3) and (4.1) we get

$$|\Phi|^2 = n(n-1)\left(H^2 + aH + b - \frac{1}{n(n-1)}\sum_{i,j} \overline{R}_{ijji}\right). \tag{4.2}$$

In order to prove our characterization results, it will be essential the following lower boundedness for the operator L acting on the mean curvature function of a linear Weingarten spacelike hypersurface.

Proposition 1 *Let L_1^{n+1} be a Lorentz locally symmetric space which satisfies curvature conditions (3.1) and (3.2). Let M^n be a linear Weingarten spacelike hypersurface immersed in L_1^{n+1} having two distinct principal curvatures with multiplicity p and $n-p$, where $1 \leq p \leq \frac{n}{2}$, and such that $R = aH + b$ for some $a, b \in \mathbb{R}$. Suppose that $b \neq \frac{1}{n(n-1)}\sum_{i,j}\overline{R}_{ijji}$ and that inequality (3.3) is satisfied. Then,*

$$L(nH) \geq |\Phi|^2 P_{H,p,c}(|\Phi|), \tag{4.3}$$

where

$$P_{H,p,c}(x) = x^2 - \frac{n(n-2p)}{\sqrt{pn(n-p)}}|H|x - n(H^2 - c), \tag{4.4}$$

with $c = \frac{c_1}{n} + 2c_2$.

Proof Let us choose a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n such that $h_{ij} = \lambda_i \delta_{ij}$ and $\phi_{ij} = \mu_i \delta_{ij}$. Since M^n has distinct principle curvatures with multiplicity p and $n-p$, then there exist μ and ν such that

$$\begin{cases} \mu_1 = \cdots = \mu_p = \mu, \\ \mu_{p+1} = \cdots = \mu_n = \nu, \\ \lambda_1 = \cdots = \lambda_p = \mu + H, \\ \lambda_{p+1} = \cdots = \lambda_n = \nu + H. \end{cases}$$

Thus, we obtain

$$0 = \sum_{i=1}^n \mu_i = p\mu + (n-p)\nu, \quad |\Phi|^2 = \sum_{i=1}^n \mu_i^2 = p\mu^2 + (n-p)\nu^2,$$

and

$$\mathrm{tr}(\Phi^3) = \sum_{i=1}^n \mu_i^3 = p\mu^3 + (n-p)\nu^3.$$

From expressions above,

$$\mu = -\frac{n-p}{p}\nu \quad \text{and} \quad \nu = \pm\sqrt{\frac{p}{n(n-p)}}|\Phi|,$$

and, hence,

$$\mathrm{tr}(\Phi^3) = \sum_{i=1}^n \mu_i^3 = \left((n-p) - \frac{(n-p)^3}{p^2}\right)\nu^3$$
$$= n(n-p)(2p-n)\frac{\nu^3}{p^2} = \pm\frac{(2p-n)}{\sqrt{pn(n-p)}}|\Phi|^3.$$

Consequently,

$$\left|\sum_{i=1}^n \mu_i^3\right| = \frac{(n-2p)}{\sqrt{pn(n-p)}}|\Phi|^3. \tag{4.5}$$

Taking into account that $h_{ij} = \lambda_i \delta_{ij}$ and since $R = aH + b$, from (2.9), (2.11) and (3.5) we obtain

$$\begin{aligned} L(nH) = {} & S^2 - nH\sum_i \lambda_i^3 + \sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 \\ & -2\sum_{i,k}\left(\lambda_i\lambda_k \overline{R}_{kiik} + \lambda_i^2 \overline{R}_{ikik}\right) + \sum_{i,k}\lambda_i\left(\overline{R}_{(n+1)iik;k} + \overline{R}_{(n+1)kik;i}\right) \\ & -\left(nH\sum_i \lambda_i \overline{R}_{(n+1)ii(n+1)} + S\sum_k \overline{R}_{(n+1)k(n+1)k}\right). \end{aligned} \tag{4.6}$$

Since L_1^{n+1} locally symmetric, we have

$$\sum_{i,k}\lambda_i\left(\overline{R}_{(n+1)iik;k} + \overline{R}_{(n+1)kik;i}\right) = 0.$$

On the other hand, since we are assuming that it holds relation (3.3), we can apply Lemma 1 to guarantee that

$$\sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 \geq 0.$$

Thus, from (4.6) we have

$$L(nH) \geq S^2 - nH\sum_i \lambda_i^3 - 2\sum_{i,k}\left(\lambda_i\lambda_k\overline{R}_{kiik} + \lambda_i^2\overline{R}_{ikik}\right) - \left(nH\sum_i \lambda_i\overline{R}_{(n+1)ii(n+1)} + S\sum_k \overline{R}_{(n+1)k(n+1)k}\right). \tag{4.7}$$

Now, we note that

$$\sum_i \mu_i^3 = \sum_i (\lambda_i - H)^3 = \sum_i \lambda_i^3 - 3H|\Phi|^2 - nH^3. \tag{4.8}$$

Hence, from (4.1), (4.5) and (4.8) we have

$$\begin{aligned} S^2 - nH\sum_i \lambda_i^3 &= \left(|\Phi|^2 + nH^2\right)^2 - nH\sum_i \mu_i^3 - 3nH^2|\Phi|^2 - n^2H^4 \\ &= |\Phi|^4 - nH^2|\Phi|^2 - nH\sum_i \mu_i^3 \\ &\geq |\Phi|^4 - nH^2|\Phi|^2 - n|H|\left|\sum_i \mu_i^3\right| \\ &\geq |\Phi|^2\left(|\Phi|^2 - \frac{n(n-2p)}{\sqrt{pn(n-p)}}|H||\Phi| - nH^2\right). \end{aligned} \tag{4.9}$$

On the other hand, using curvature conditions (3.1) and (3.2), after straightforward computations we get

$$-\left(\sum_i nH\lambda_i\bar{R}_{(n+1)ii(n+1)} + S\sum_k \bar{R}_{(n+1)k(n+1)k}\right) = c_1(S - nH^2) \tag{4.10}$$

and

$$\begin{aligned} -2\sum_{i,k}\left(\lambda_i\lambda_k\bar{R}_{kiik} + \lambda_i^2\bar{R}_{ikik}\right) &\geq c_2\sum_{i,k}(\lambda_i - \lambda_k)^2 \\ &= 2nc_2(S - nH^2). \end{aligned} \tag{4.11}$$

Therefore, inserting (4.9)–(4.11) in (4.7), we conclude that

$$\begin{aligned} L(nH) &\geq |\Phi|^2\left(|\Phi|^2 - \frac{n(n-2p)}{\sqrt{pn(n-p)}}|H||\Phi| - nH^2\right) + c_1|\Phi|^2 + 2nc_2|\Phi|^2 \\ &= |\Phi|^2\left(|\Phi|^2 - \frac{n(n-2p)}{\sqrt{pn(n-p)}}|H||\Phi| - nH^2\right) + n|\Phi|^2\left(\frac{c_1}{n} + 2c_2\right) \\ &= |\Phi|^2\left(|\Phi|^2 - \frac{n(n-2p)}{\sqrt{pn(n-p)}}|H||\Phi| - n(H^2 - c)\right), \end{aligned}$$

where $c = \dfrac{c_1}{n} + 2c_2$. □

Before to present our main results, we also need to make a brief analysis of the behavior of the polynomial $P_{H,p,c}$ defined in (4.4), in terms of the sign of its parameter c.

(a) *Case* $c > 0$.
In this case, if $n^2H^2 - 4p(n-p)c < 0$, then $H^2 < \dfrac{4p(n-p)c}{n^2}$ and, hence, $P_{H,p,c}(x) > 0$ for all $x \in \mathbb{R}$.
If $H^2 = \dfrac{4p(n-p)c}{n^2}$, then we can write $|H| = \dfrac{2\sqrt{p(n-p)c}}{n}$ and the polynomial $P_{H,p,c}$ has just a real root, namely

$$x^* = \frac{\sqrt{n}}{2\sqrt{p(n-p)}}(n-2p)|H| = \frac{(n-2p)\sqrt{c}}{\sqrt{n}}.$$

Thus, in this case,

$$P_{H,p,c}(x) = \left(x - \frac{(n-2p)\sqrt{c}}{\sqrt{n}}\right)^2 \geq 0,$$

for all $x \in \mathbb{R}$.
If $H^2 > \dfrac{4p(n-p)c}{n^2}$, then $P_{H,p,c}$ has two distinct real roots, which are given by

$$x^*_{\pm} = \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left((n-2p)|H| \pm \sqrt{n^2H^2 - 4p(n-p)c}\right). \tag{4.12}$$

Observe that x^*_+ is always positive, while x^*_- is positive if, and only if,

$$\frac{4p(n-p)c}{n^2} \leq H^2 < c.$$

(b) *Case* $c \leq 0$.
In this case, $P_{H,p,c}$ has two distinct real roots which coincide with (4.12). Observe that x^*_+ is always positive, while x^*_- is always negative. Consequently, $P_{H,p,c}(x) \geq 0$ if, and only if, $x \geq x^*_+$, where

$$x^*_+ = \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left((n-2p)|H| + \sqrt{n^2H^2 - 4p(n-p)c}\right).$$

At this point, we are in a position to present our first characterization results concerning linear Weingarten spacelike hypersurfaces immersed in a locally symmetric Lorentz space, having two distinct principal curvatures.

Theorem 1 *Let L_1^{n+1} be a locally symmetric Lorentz space which satisfies curvature conditions* (3.1) *and* (3.2)*. Let M^n be a complete linear Weingarten spacelike hypersurface immersed in L_1^{n+1}, such that $R = aH + b$ with $b < \frac{1}{n(n-1)}\sum_{i,j}\overline{R}_{ijji}$ and having two distinct principal curvatures with multiplicity p and $n-p$, where $1 \leq p < \frac{n}{2}$. Suppose that $H^2 \leq \dfrac{4p(n-p)c}{n^2}$, where $c = \dfrac{c_1}{n} + 2c_2 > 0$. If H attains a maximum on M^n, then M^n is an isoparametric hypersurface of L_1^{n+1}, with $|H| = \dfrac{2\sqrt{p(n-p)c}}{n}$ and $|\Phi| = \dfrac{(n-2p)\sqrt{c}}{\sqrt{n}}$.*

Proof We observe that, from our restriction on the parameter b, Lemma 2 guarantees that H is nonzero and that the operator L is elliptic.

On the other hand, from Proposition 1, we have

$$L(nH) \geq |\Phi|^2 P_{H,p,c}(|\Phi|),$$

where

$$P_{H,p,c}(|\Phi|) = |\Phi|^2 - \frac{n(n-2p)}{\sqrt{pn(n-p)}}|H||\Phi| - n(H^2 - c).$$

Since we are assuming $c > 0$ and $H^2 \leq \frac{4p(n-p)c}{n^2}$, we have that $P_{H,p,c}(|\Phi|) \geq 0$. Thus, since we are supposing that H attains its maximum on M^n, Hopf's strong maximum principle guarantees that H is constant on M^n. Then, from (4.7) we get

$$0 = L(nH) \geq \sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 + |\Phi|^2 P_{H,p,c}(|\Phi|) \geq 0. \tag{4.13}$$

Consequently, from Lemma 1

$$\sum_{i,j,k} h_{ijk}^2 = n^2|\nabla H|^2 = 0.$$

Hence, M^n is an isoparametric hypersurface of L_1^{n+1}.

Moreover, from (4.13) also we obtain that $|\Phi|^2 P_{H,p,c}(|\Phi|) = 0$. Since we are supposing that M^n has two distinct principal curvatures, we conclude that $P_{H,p,c}(|\Phi|) = 0$ and, hence, we must that $|H| = \frac{2\sqrt{p(n-p)c}}{n}$ and $|\Phi| = \frac{(n-2p)\sqrt{c}}{\sqrt{n}}$. □

Proceeding, we get the following nonexistence result

Proposition 2 *There are not exist complete linear Weingarten spacelike hypersurfaces immersed in a locally symmetric Lorentz space L_1^{2m+1}, satisfying curvature conditions (3.1) and (3.2), such that $R = aH + b$ with $(2m-1)^2a^2 + 4\sum_{i,j} \overline{R}_{ijji} - 8m(2m-1)b \geq 0$, $a \geq 0$, $b \neq \frac{1}{n(n-1)} \sum_{i,j} \overline{R}_{ijji}$, $H^2 \leq c$, where $c = \frac{c_1}{n} + 2c_2 > 0$, and having two distinct principal curvatures with the same multiplicity.*

Proof Suppose by contradiction that there exists such a complete linear Weingarten spacelike hypersurface M^{2m}. From Lemma 3 applied to the function $2mH$, we obtain a sequence of points $\{q_k\}_{k\in\mathbb{N}} \subset M^{2m}$ such that

$$\lim_{k\to+\infty} (2mH(q_k)) = \sup_M 2mH, \quad \text{and} \quad \limsup_{k\to+\infty} L(2mH)(q_k) \leq 0. \tag{4.14}$$

Thus, since we are assuming that $a \geq 0$, from equations (4.2), (4.3) and (4.14) we obtain

$$0 \geq \limsup_{k\to+\infty} L(2mH)(q_k) \geq \sup_M |\Phi|^2 P_{\sup H,m,c}\left(\sup_M |\Phi|\right).$$

Hence, since M^{2m} is supposed to have two distinct principal curvatures, we conclude that

$$P_{\sup H,m,c}\left(\sup_M |\Phi|\right) \leq 0.$$

Consequently, taking into account our restriction on H, from (4.4) we get

$$0 \leq \sup_M |\Phi|^2 \leq 2m(\sup_M H^2 - c) \leq 0.$$

Therefore, we must have $|\Phi|^2 = 0$ on M^{2m} and, hence, we reach at a contradiction. □

Returning to the characterization of linear Weingarten spacelike hypersurfaces, we get

Theorem 2 *Let L_1^{n+1} be a locally symmetric Lorentz space which satisfies curvature conditions (3.1) and (3.2). Let M^n be a complete linear Weingarten spacelike hypersurface immersed in L_1^{n+1}, such that $R = aH + b$ with $b < \frac{1}{n(n-1)} \sum_{i,j} \overline{R}_{ijji}$, and having two distinct principal curvatures with multiplicity p and $n-p$, where $1 \le p < \dfrac{n}{2}$. Suppose that $\dfrac{4p(n-p)c}{n^2} \le H^2 < c$, where $c = \frac{c_1}{n} + 2c_2 > 0$, and*

$$|\Phi| \le \frac{\sqrt{n}}{2\sqrt{p(n-p)}} \left((n-2p)|H| - \sqrt{n^2H^2 - 4p(n-p)c} \right). \tag{4.15}$$

If H attains a maximum on M^n, then M^n is an isoparametric hypersurface of L_1^{n+1}, with equality occurring in (4.15).

Proof From our restrictions on H and $|\Phi|$, we obtain that $P_{H,p,c}(|\Phi|) \ge 0$ with $P_{H,p,c}(|\Phi|) = 0$ if, and only if, equality occurs in (4.15). Now, proceeding as in the proof of Theorem 1, we conclude that M^n is an isoparametric hypersurface of L_1^{n+1}, with equality in (4.15). □

In a similar way of the proof of Theorem 2, we obtain

Corollary 1 *Let L_1^{n+1} be a locally symmetric Lorentz space which satisfies curvature conditions (3.1) and (3.2). Let M^n be a complete linear Weingarten spacelike hypersurface immersed in L_1^{n+1}, such that $R = aH + b$ with $b < \frac{1}{n(n-1)} \sum_{i,j} \overline{R}_{ijji}$, and having two distinct principal curvatures with multiplicity p and $n-p$. Suppose that either $1 \le p \le \dfrac{n}{2}$ and $H^2 > \dfrac{4p(n-p)c}{n^2}$, or $1 \le p < \dfrac{n}{2}$ and $H^2 \ge \dfrac{4p(n-p)c}{n^2}$, where $c = \dfrac{c_1}{n} + 2c_2 > 0$, and that*

$$|\Phi| \ge \frac{\sqrt{n}}{2\sqrt{p(n-p)}} \left((n-2p)|H| + \sqrt{n^2H^2 - 4p(n-p)c} \right). \tag{4.16}$$

If H attains a maximum on M^n, then M^n is an isoparametric hypersurface of L_1^{n+1}, with equality occurring in (4.16).

Proceeding, we also get the following result

Theorem 3 *Let L_1^{n+1} be a locally symmetric Lorentz space which satisfies curvature conditions (3.1) and (3.2). Let M^n be a complete linear Weingarten spacelike hypersurface immersed in L_1^{n+1}, such that $R = aH + b$ for some $a, b \in \mathbb{R}$ satisfying inequality (3.3) with $a \ge 0$, $b \ne \frac{1}{n(n-1)} \sum_{i,j} \overline{R}_{ijji}$ and having two distinct principal curvatures with multiplicity p and $n-p$. Suppose that either $1 \le p \le \dfrac{n}{2}$ and $H^2 > \dfrac{4p(n-p)c}{n^2}$, or $1 \le p < \dfrac{n}{2}$ and $H^2 \ge \dfrac{4p(n-p)c}{n^2}$, where $c = \dfrac{c_1}{n} + 2c_2 > 0$, and that*

$$|\Phi| \ge \frac{\sqrt{n}}{2\sqrt{p(n-p)}} \left((n-2p) \sup_M |H| + \sqrt{n^2 \sup_M H^2 - 4p(n-p)c} \right). \tag{4.17}$$

If H is bounded on M^n, then M^n is an isoparametric hypersurface of L_1^{n+1}, with equality occurring in (4.17).

Proof From Lemma 3 applied to the function H, we obtain a sequence of points $\{q_k\}_{k\in\mathbb{N}} \subset M^n$ such that

$$\lim_{k\to+\infty} H(q_k) = \sup_M H, \quad \text{and} \quad \limsup_{k\to+\infty} L(H)(q_k) \le 0. \tag{4.18}$$

In viewing of $\lim_{k\to+\infty} H(q_k) = \sup_M H$ and $a \ge 0$, Eq. (4.2) implies that

$$\lim_{k\to+\infty} |\Phi|(q_k) = \sup_M |\Phi|. \tag{4.19}$$

Thus, taking into account (4.18) and (4.19), from Proposition 1 we have

$$0 \ge \limsup_{k\to+\infty} L(nH)(q_k) \ge \sup_M |\Phi|^2 P_{\sup H,p,c}\left(\sup_M |\Phi|\right). \tag{4.20}$$

Hence, since M^n is supposed to have two distinct principal curvatures, from (4.20) we conclude that

$$P_{\sup H,p,c}\left(\sup_M |\Phi|\right) \le 0. \tag{4.21}$$

On the other hand, from ours restrictions on H and on $|\Phi|$, we have that $P_{H,p,c}(|\Phi|) \ge 0$, with $P_{H,p,c}(|\Phi|) = 0$ if, and only if,

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left((n-2p)|H| + \sqrt{n^2H^2 - 4p(n-p)c}\right).$$

Consequently, from (4.21) we get that

$$\sup_M |\Phi| = \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left((n-2p)\sup_M |H| + \sqrt{n^2 \sup_M H^2 - 4p(n-p)c}\right)$$

and, taking into account once more our restriction on $|\Phi|$, we have that $|\Phi|$ is constant on M^n. Thus, since M^n is a linear Weingarten hypersurface, from (4.2) we have that H is also constant on M^n. At this point, the proof proceed as in that one of Theorem 1. □

Finally, when the parameter c in nonpositive, we can reason as before to get the following results

Corollary 2 *Let L_1^{n+1} be a locally symmetric Lorentz space which satisfies curvature conditions (3.1) and (3.2). Let M^n be a complete linear Weingarten spacelike hypersurface immersed in L_1^{n+1}, such that $R = aH + b$ with $b < \frac{1}{n(n-1)}\sum_{i,j}\overline{R}_{ijji}$ and having two distinct principal curvatures with multiplicity p and $n-p$, where $1 \le p \le \frac{n}{2}$. Suppose that*

$$|\Phi| \ge \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left((n-2p)|H| + \sqrt{n^2H^2 - 4p(n-p)c}\right), \tag{4.22}$$

where $c = \frac{c_1}{n} + 2c_2 \le 0$. If H attains a maximum on M^n, then M^n is an isoparametric hypersurface, with equality occurring in (4.22).

Corollary 3 *Let L_1^{n+1} be a locally symmetric Lorentz space which satisfies curvature conditions (3.1) and (3.2). Let M^n be a complete linear Weingarten spacelike hypersurface immersed in L_1^{n+1}, such that $R = aH + b$ for some $a, b \in \mathbb{R}$ satisfying inequality (3.3) with $a \geq 0$, $b \neq \frac{1}{n(n-1)} \sum_{i,j} \overline{R}_{ijji}$ and having two distinct principal curvatures with multiplicity p and $n-p$, where $1 \leq p \leq \dfrac{n}{2}$. Suppose that*

$$|\Phi| \geq \frac{\sqrt{n}}{2\sqrt{p(n-p)}} \left((n-2p) \sup_M |H| + \sqrt{n^2 \sup_M H^2 - 4p(n-p)c} \right), \tag{4.23}$$

where $c = \dfrac{c_1}{n} + 2c_2 \leq 0$. If H is bounded on M^n, then M^n is an isoparametric hypersurface, with equality occurring in (4.23).

Remark 1 When the ambient space is a Lorentz space form $L_1^{n+1}(c)$ of constant sectional curvature c, according to the examples of Section 4 of [1], we have that the isoparametric hypersurfaces of $L_1^{n+1}(c)$ with two distinct principal curvatures are such that the norm of their traceless operator Φ verifies the equality in the hypothesis (4.15), (4.16) and (4.22). Hence, we conclude that the assumed restrictions on $|\Phi|$ along our previous results are, in fact, mild hypothesis.

5 Locally symmetric Einstein spacetimes

From (2.10) we have that

$$\square f = \mathrm{tr}(P_1 \circ \nabla^2 f),$$

where, denoting by I the identity in the algebra of smooth vector fields on M^n, $P_1 = nHI - B$ and $\nabla^2 f$ stands for the self-adjoint linear operator metrically equivalent to the hessian of f. Let us choose a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n. By using the standard notation $\langle\, , \rangle$ for the (induced) metric of M^n, we get

$$\square f = \sum_i \langle P_1(\nabla_{e_i} \nabla f), e_i \rangle. \tag{5.1}$$

From (5.1), we have

$$\begin{aligned} \mathrm{div}(P_1 \nabla f) &= \sum_i \langle \nabla_{e_i}(P_1 \nabla f), e_i \rangle \\ &= \sum_i \langle (\nabla_{e_i} P_1) \nabla f, e_i \rangle + \sum_i \langle P_1(\nabla_{e_i} \nabla f), e_i \rangle \\ &= \sum_i \langle \nabla f, (\nabla_{e_i} P_1) e_i \rangle + \square f \\ &= \langle \mathrm{div} P_1, \nabla f \rangle + \square f, \end{aligned} \tag{5.2}$$

where

$$\mathrm{div} P_1 = \mathrm{tr}(\nabla P_1) = \sum_i (\nabla_{e_i} P_1) e_i.$$

Hence, from Lemma 3.1 of [3] we have

$$\langle \mathrm{div} P_1, \nabla f \rangle = -\sum_i \langle \overline{R}(N, e_i) e_i, \nabla f \rangle = -\overline{\mathrm{Ric}}(N, \nabla f), \tag{5.3}$$

where $\overline{R}$ and $\overline{\mathrm{Ric}}$ are the curvature and Ricci tensors of L_1^{n+1}, respectively, and N denotes the Gauss map of M^n. Consequently, if we assume that L_1^{n+1} is an Einstein spacetime, from (5.3) we get

$$\langle \mathrm{div} P_1, \nabla f\rangle = 0.$$

Thus, in this case, from (5.2) we conclude that

$$\Box f = \mathrm{div}(P_1(\nabla f)).$$

Therefore, returning to the operator L and taking $f = nH$, we get

$$L(nH) = \mathrm{div}(P(\nabla H)), \tag{5.4}$$

where

$$P = nP_1 + \frac{n(n-1)}{2} aI. \tag{5.5}$$

Motivated by the previous digression, we will treat the case when the ambient space L_1^{n+1} is a locally symmetric Einstein spacetime. In order to establish our next results, we will also need the following result obtained by Caminha [5], which extends a result of Yau [28] on a version of Stokes theorem for an n-dimensional, complete noncompact Riemannian manifold (cf. Proposition 2.1 of [5]; see also the Theorem due to Karp in [20]). In what follows, let $\mathcal{L}^1(M)$ denote the space of Lebesgue integrable functions on M^n.

Lemma 4 *Let X be a smooth vector field on an n-dimensional complete oriented Riemannian manifold M^n, such that* $\mathrm{div} X$ *does not change sign on M^n. If $|X| \in \mathcal{L}^1(M)$, then* $\mathrm{div} X = 0$ *on M^n.*

Now, we are in position to present our next results.

Theorem 4 *Let L_1^{n+1} be a locally symmetric Einstein spacetime which satisfies curvature conditions (3.1) and (3.2). Let M^n be a complete linear Weingarten spacelike hypersurface immersed in L_1^{n+1}, such that $R = aH + b$ with $(n-1)^2a^2 + 4\sum_{i,j}\overline{R}_{ijji} - 4n(n-1)b > 0$, $b \neq \frac{1}{n(n-1)}\sum_{i,j}\overline{R}_{ijji}$ and having two distinct principal curvatures with multiplicity p and $n-p$, where $1 \leq p < \frac{n}{2}$. If $|\nabla H| \in \mathcal{L}^1(M)$ and $H^2 \leq \frac{4p(n-p)c}{n^2}$, where $c = \frac{c_1}{n} + 2c_2 > 0$, then M^n is an isoparametric hypersurface of L_1^{n+1}, with $|H| = \frac{2\sqrt{p(n-p)c}}{n}$ and $|\Phi| = \frac{(n-2p)\sqrt{c}}{\sqrt{n}}$.*

Proof Since $R = aH + b$ and H is bounded, from (2.3) it follows that the second fundamental form B of M^n is bounded. Consequently, the operator P defined in (5.5) is also bounded and, since we are assuming that $|\nabla H| \in \mathcal{L}^1(M)$, we obtain that

$$|P(\nabla H)| \leq |P||\nabla H| \in \mathcal{L}^1(M).$$

Thus, Lemma 4 guarantees that $\mathrm{div}(P(\nabla H)) = 0$ on M^n and, hence, from (5.4) we get $L(nH) = 0$ on M^n.

Since $H^2 \leq \frac{4p(n-p)c}{n^2}$, we have that $P_{H,p,c}(|\Phi|) \geq 0$ and from (4.13) we obtain

$$\sum_{i,j,k} h_{ijk}^2 = n^2|\nabla H|^2.$$

Consequently, taking into account that $(n-1)^2a^2 + 4\sum_{i,j}\overline{R}_{ijji} - 4n(n-1)b > 0$, from Lemma 1 we have H is constant on M^n and, hence, M^n must be an isoparametric hypersurface of L_1^{n+1}, with $|H| = \dfrac{2\sqrt{p(n-p)c}}{n}$ and $|\Phi| = \dfrac{(n-2p)\sqrt{c}}{\sqrt{n}}$. □

We can reason as in the proof of the previous theorem in order to get the following results

Corollary 4 *Let L_1^{n+1} be a locally symmetric Einstein spacetime which satisfies curvature conditions (3.1) and (3.2). Let M^n be a complete linear Weingarten spacelike hypersurface immersed in L_1^{n+1}, such that $R = aH + b$ with $(n-1)^2a^2 + 4\sum_{i,j}\overline{R}_{ijji} - 4n(n-1)b > 0$, $b \neq \frac{1}{n(n-1)}\sum_{i,j}\overline{R}_{ijji}$ and having two distinct principal curvatures with multiplicity p and $n-p$, where $1 \leq p < \frac{n}{2}$. Suppose that $\dfrac{4p(n-p)c}{n^2} \leq H^2 < c$, where $c = \dfrac{c_1}{n} + 2c_2 > 0$, and*

$$|\Phi| \leq \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left((n-2p)|H| - \sqrt{n^2H^2 - 4p(n-p)c}\right). \tag{5.6}$$

If $|\nabla H| \in \mathcal{L}^1(M)$, then M^n is an isoparametric hypersurface of L_1^{n+1}, with equality occurring in (5.6).

Corollary 5 *Let L_1^{n+1} be a locally symmetric Einstein spacetime which satisfies curvature conditions (3.1) and (3.2). Let M^n be a complete linear Weingarten spacelike hypersurface immersed in L_1^{n+1}, such that $R = aH + b$ with $(n-1)^2a^2 + 4\sum_{i,j}\overline{R}_{ijji} - 4n(n-1)b > 0$, $b \neq \frac{1}{n(n-1)}\sum_{i,j}\overline{R}_{ijji}$ and having two distinct principal curvatures with multiplicity p and $n-p$. Suppose that either $1 \leq p \leq \dfrac{n}{2}$ and $H^2 > \dfrac{4p(n-p)c}{n^2}$, or $1 \leq p < \dfrac{n}{2}$ and $H^2 \geq \dfrac{4p(n-p)c}{n^2}$, where $c = \dfrac{c_1}{n} + 2c_2 > 0$, and that*

$$|\Phi| \geq \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left((n-2p)|H| + \sqrt{n^2H^2 - 4p(n-p)c}\right). \tag{5.7}$$

If $|\nabla H| \in \mathcal{L}^1(M)$, then M^n is an isoparametric hypersurface of L_1^{n+1}, with equality occurring in (5.7).

As in Sect. 4, we also contemplate the case $c \leq 0$ in the context of Einstein spacetimes.

Corollary 6 *Let L_1^{n+1} be a locally symmetric Einstein spacetime which satisfies curvature conditions (3.1) and (3.2). Let M^n be a complete linear Weingarten spacelike hypersurface immersed in L_1^{n+1}, such that $R = aH + b$ with $(n-1)^2a^2 + 4\sum_{i,j}\overline{R}_{ijji} - 4n(n-1)b > 0$, $b \neq \frac{1}{n(n-1)}\sum_{i,j}\overline{R}_{ijji}$ and having two distinct principal curvatures with multiplicity p and $n-p$, where $1 \leq p \leq \dfrac{n}{2}$. If $|\nabla H| \in \mathcal{L}^1(M)$ and*

$$|\Phi| \geq \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left((n-2p)|H| + \sqrt{n^2H^2 - 4p(n-p)c}\right), \tag{5.8}$$

where $c = \dfrac{c_1}{n} + 2c_2 \leq 0$, then M^n is an isoparametric hypersurface of L_1^{n+1}, with equality occurring in (5.8).

To closed our paper, we establish the following nonexistence result.

Proposition 3 *There are not exist closed linear Weingarten spacelike hypersurfaces immersed in a locally symmetric Einstein spacetime* L_1^{n+1}*, satisfying curvature conditions* (3.1) *and* (3.2)*, such that* $R = aH + b$ *with* $(n-1)^2a^2 + 4\sum_{i,j}\overline{R}_{ijji} - 4n(n-1)b \geq 0$*,* $b \neq \frac{1}{n(n-1)}\sum_{i,j}\overline{R}_{ijji}$*, having two distinct principal curvatures with multiplicity* p *and* $n-p$*, and* $H^2 < \dfrac{4p(n-p)c}{n^2}$*, where* $c = \dfrac{c_1}{n} + 2c_2 > 0$*.*

Proof Suppose by contradiction that there exists such a closed linear Weingarten spacelike hypersurface M^n immersed a locally symmetric Einstein spacetime L_1^{n+1}, satisfying curvature conditions (3.1) and (3.2). Thus, assuming that $(n-1)^2a^2+4\sum_{i,j}\overline{R}_{ijji}-4n(n-1)b \geq 0$, from Proposition 1 and (5.4) we obtain

$$0 = \int_M L(nH)dM \geq \int_M |\Phi|^2 P_{H,p,c}(|\Phi|)dM. \tag{5.9}$$

On the other hand, since $H^2 < \frac{4p(n-p)c}{n^2}$, we have that $P_{H,p,c}(|\Phi|) > 0$ on M^n. Thus, from (5.9) we obtain that $|\Phi|^2 = 0$ on M^n, that is, M^n is a totally umbilical hypersurface of L_1^{n+1}. Therefore, taking into account that M^n is supposed to have two distinct principal curvatures, we reach at a contradiction. □

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