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Practical exponential stability in mean square of stochastic partial differential equations

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Abstract The main aim of this paper is to establish some criteria for the mean square and almost sure practical exponential stability of a nonlinear monotone stochastic partial differential equations.

Keywords Stochastic partial differential equations · Almost sure uniform stability · Mean square practical stability

Mathematics Subject Classification Primary 93E03 · Secondary 60H10

1 Introduction

We are mainly interested in the stability of a class of nonlinear stochastic partial differential equations of monotone type. The question of the asymptotic stability of the second moment of X_t (which is the solution of Eq. (2.1) below) has received considerable attention in the literature. Willems [7,18] have established sufficient conditions which guarantee asymptotic stability when the spaces are finite dimensional. Willems [19] and Wonham [20] and have considered a related problem, the stabilization problem, again in finite dimension. Recently Ichikawa [14] have extended these results to infinite dimensions. In fact, a coercivity condition, extending the one considered by Caraballo and Real [8] and Chow [11], is introduced

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and will play the role of a stability criterion. To be precise, under the coercivity condition from Caraballo and Real [8], almost sure exponential stability of solutions is obtained, while in Chow [11] pathwise asymptotic stability is proved. However, as we will explain later, coercivity criteria from Caraballo and Real [8] are too restrictive to be applied to a number of interesting and, in our opinion, important examples, especially in the non-autonomous case. In this work, we shall improve their results to cover the general non-autonomous stochastic differential equations in Hilbert spaces.

The organization of the paper is as follows. In Sect. 2, we introduce the basic notations and assumptions. In Sect. 3, we prove some sufficient conditions ensuring almost sure practical exponential stability in mean square of solutions of a class of nonlinear stochastic partial differential equation, and study an example to illustrate these results.

2 Preliminaries

Let V be a Banach space and H, K real, separable Hilbert spaces such that

$$V \hookrightarrow H \equiv H^{'} \hookrightarrow V',$$

where the injections are continuous and dense.

We denote by $\|\ .\ \|$, $|\ .\ |$ and $\|\ .\ \|_*$ the norms in V, H and $V^{'}$ respectively, by $(.,.)$ the inner product in H, and by $\langle \cdots \rangle$ the duality product between V and $V^{'}$, and β is a constant such that

$$|x| \leq \beta ||x||, \quad \forall x \in V.$$

Let W_t be a Wiener process defined on some complete probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ and taking its values in the separable Hilbert space K, with increment covariance operator $\mathcal{Q}$, and let $(\mathcal{F}_t)_{t\geq 0}$ be the usual family of subt-σ-algebras of $\mathcal{F}$ such that, for each $t \geq 0$, $\mathcal{F}_t$ is generated by $\{W_s, 0 \leq s \leq t\}$.

Consider the following nonlinear stochastic diffusion equation:

$$X_t = X_0 + \int_0^t A(s, X_s)ds + \int_0^t B(s, X_s)dW_s, \tag{2.1}$$

where $A(t,.) : V \to V^{'}$ is a family of nonlinear operators defined a.e.t. satisfying there exists $t \in \mathbb{R}_+$ such that $A(t, 0) \neq 0$, and where $B(t,.) : V \to \mathcal{L}(K, H)$, the family of all bounded linear operators from K into H, satisfies

(b.1) There exists $t \in \mathbb{R}_+$ such that $B(t, 0) \neq 0$,

(b.2) There exist continuous non-negative functions $k(t)$, $\psi(t)$ and positive constants θ and ξ such that

$$\theta := \int_0^{+\infty} k^2(t)dt, \quad \xi := \int_0^{+\infty} \psi^2(t)dt,$$

and

$$||B(t,x)||_2 \leq k(t)||x|| + \psi(t), \quad \text{for all } x \in V, \quad \text{a.e.}t.,$$

where $||.||_2$ denotes the Hilbert-Shmidt norm of nuclear operators, i.e.,

$$||B(t,x)||_2^2 = tr(B(t,x)\mathcal{Q}B(t,x)^*).$$

(b.3) The map $t \in (0, T) \mapsto B(t, x) \in \mathcal{L}(K, H)$ is Lebesgue-measurable $\forall x \in V, \forall T > 0$.

Definition 2.1 Let $\{\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\}$ be the stochastic filter associated to the K-valued Wiener process W_t with covariance operator $\mathcal{Q}$. Suppose that $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, i.e, X_0 is an H-valued $\mathcal{F}_0$-measurable random variable such that $\mathbb{E}|X_0|^2 < \infty$. A stochastic process X_t is said to be a strong solution on Ω to the SDE (2.1) for $t \in [0, T]$ if the following conditions are satisfied (see [12]):

(a) X_t is a V-valued $\mathcal{F}_t$-measurable random variable;
(b) $X_t \in I^p(0, T; V) \cap L^2(\Omega; C(0, T; H))$, $p < 1$, $T > 0$, where $I^p(0, T; V)$ denotes the space of all V-valued processes $(X_t)_{t\in[0,T]}$ (we will write X_t for short) measurable (from $[0, T] \times \Omega$ into V), satisfying that X_t is $\mathcal{F}_t$-measurable (hence X_t is $\mathcal{F}_t$-adapted) for almost all $t \in [0, T]$, and

$$\mathbb{E}\int_0^T ||X_t||^p dt < \infty.$$

Here $C(0, T; H)$ denotes the space of all continuous functions from $[0, T]$ into H.
(c) $\mathbb{E}\int_0^T ||A(t, X_t)||_*^2 dt < \infty$.
(d) Equation (2.1) is satisfied for every $t \in [0, T]$ with probability one.

If T is replaced by ∞, X_t is called a global strong solution of (2.1).

As we are mainly interested in the stability analysis, we always assume that for each $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, there exists a global strong solution to (2.1). This happens, for instance, if the following assumptions hold true (see, for example, Pardoux [17]).

(a.1) Coercivity: there exist $\alpha > 0$, $p > 1$ and $\lambda, \gamma \in \mathbb{R}^*$ such that

$$2 < A(t, x), x > +||B(t, x)||_2^2 \leq -\alpha||x||^p + \lambda|x|^2 + \gamma \quad \text{for all } x \in V, \quad \text{a.e.} t.$$

(a.2) Boundedness: there exists $\beta > 0$, $c > 0$ such that

$$||A(t, x)||_* \leq c||x||^{p-1} + \beta \quad \text{for all } x \in V, \quad \text{a.e.} t.$$

(a.3) Monotonicity:

$$||B(t, x) - B(t, y)||^2 \leq \lambda|x - y|^2 - \big(2 < A(t, x) - A(t, y), x - y > \big)$$
$$\text{for all } x, y \in V, \quad \text{a.e.} t.$$

(a.4) Hemicontinuity: The map $\theta \in \mathbb{R} \mapsto < A(t, x + \theta y), z > \in \mathbb{R}$ is continuous for every $x, y, z \in V$, a.e. t.
(a.5) Measurability: for every $x \in V$, the map $t \in (0, T) \mapsto A(t, x) \in V'$ is Lebesgue measurable, a.e. t., $\forall T > 0$.

Now we establish a version of the Itô formula (see Pardoux [17]) which will be needed later in this paper. Let $C^{(1,2)}([0, \infty) \times H, \mathbb{R}^+)$ denote the space of all $\mathbb{R}^+$-valued functions Ψ defined on $[0, \infty) \times H$ with the following properties:

(1) $\Psi(t, x)$ is differentiable in $t \in [0, \infty)$ and twice Frechet differentiable in x with $\Psi_t(t, .)$, $\Psi_x(t, .)$ and $\Psi_{xx}(t, .)$ locally bounded on H,
(2) $\Psi(t, .)$, $\Psi_t(t, .)$ and $\Psi_x(t, .)$ are continuous on H,
(3) for all trace class operators R, tr $(\Psi_{xx}(t, .)R)$ is continuous from H into $\mathbb{R}$,

(4) if $v \in V$ then $\Psi_x(t,v) \in V$, and $u \to < \Psi_x(t,u), v^* >$ is continuous for each $v^* \in V'$,
(5) $\|\Psi_x(t,v)\| \le C_0(t)(1+\|v\|)$, $C_0(t) > 0$, for all $v \in V$.

Theorem 2.1 (Itô's formula). *Let $\Psi \in C^{(1,2)}([0,\infty) \times H, \mathbb{R}^+)$. If the stochastic process $X(t)$ is a weak solution to (2.1), then it holds that*

$$\Psi(t, X(t)) = \Psi(0, X(0)) + \int_0^t L\Psi(s, X(s))ds,$$
$$+ \int_0^t (\Psi_x(s, X(s)), B(s, X(s))dW(s)),$$

where

$$\begin{aligned} L\Psi(s, X(s)) = {} & \Psi_t(s, X(s)), \\ & + < A(s, X(s)), \Psi_x(s, X(s)) >, \\ & + \frac{1}{2} tr(\Psi_{xx}(s, X(s))B(s, X(s))QB(s, X(s))^*). \end{aligned}$$

Remark 2.2 Notice that any strong solution in the sense of Definition 2.1 is a weak solution in the weak or variational sense in Theorem 2.1 (see e.g. [8,9,17]).

We state now the definitions of the almost surely convergence of solutions to a small closed ball $B_r \subset H$ centered at zero with radius r (see [1–6], [10]), and we will consider initial values in the space $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$.

Definition 2.2 The ball B_r is said to be almost surely globally practically uniformly exponentially stable if:

for any initial value $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, such that its corresponding strong solution $X(t) := X(t, X_0)$ to (2.1) satisfies $0 < |X(t)| - r$, for all $t \ge 0$, it holds that

$$\limsup_{t\to\infty} \frac{1}{t} \ln(|X(t, X_0)| - r) < 0, \text{ a.s.} \tag{2.2}$$

System (2.1) is said to be almost surely globally practically uniformly exponentially stable if there exists $r > 0$ such that B_r is almost surely globally practically uniformly exponentially stable.

Definition 2.3 The ball B_r is said to be almost surely globally practically uniformly exponentially stable in mean square if:

For any initial value $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, such that its corresponding strong solution $X(t) := X(t, X_0)$ to (2.1) satisfies $0 < \mathbb{E}\big(|X(t, X_0)|^2\big) - r$, for all $t \ge 0$, it holds that

$$\limsup_{t\to\infty} \frac{1}{t} \ln\left(\mathbb{E}(|X(t, X_0)|^2) - r\right) < 0, \text{ a.s.} \tag{2.3}$$

System (2.1) is said to be almost surely globally practically uniformly exponentially stable in mean square if there exists $r > 0$ such that B_r is almost surely globally practically uniformly exponentially stable in the mean square.

Definition 2.4 The system (2.1) is said to be almost surely globally practically uniformly exponentially convergent to zero in mean square if there exists a function $r(\cdot)$ such that:

For any initial value $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, such that its corresponding strong solution $X(t) := X(t, X_0)$ to (2.1) satisfies $0 < \mathbb{E}\big(|X(t, X_0)|^2\big) - r(t)$, for all $t \ge 0$, it holds that

$$\limsup_{t\to\infty} \frac{1}{t} \ln \left(\mathbb{E}(|X(t, X_0)|^2) - r(t)\right) < 0, \text{ a.s.} \tag{2.4}$$

with $\lim_{t\to+\infty} r(t) = 0$.

Definition 2.5 The ball B_r is said to be uniformly stable in probability if the strong solution $X(t) := X(t, X_0)$ to (2.1) satisfies:

For each $\epsilon \in]0, 1[$ and $k > r$, there exists $\delta = \delta(\epsilon, k) > r$ such that

$$\mathbb{P}\big(|X(t, X_0)| < k, \forall t \geq 0\big) \geq 1 - \epsilon \quad \text{for all} \quad |X_0| < \delta. \tag{2.5}$$

Remark 2.3 Noting that if $r \to 0$ we have the classical definition of the stability in probability. We write in the Definition 2.5 that $\delta = \delta(\epsilon, k) > r$ because if we take $\delta = \delta(\epsilon, k) < r$ and letting $r \to 0$ we get $|X_0| < 0$ which contradicts with the classical definition of the stability in probability when 0 is an equilibrium point.

3 Practical exponential stability in mean square

Now we shall impose the following coercivity condition (CC):

There exist constants $\alpha > 0$, $\mu > 0$, $\lambda \in \mathbb{R}$, and a nonnegative continuous function $\gamma(t)$, $t \in \mathbb{R}_+$, such that

$$2 < A(t, v), v > +||B(t, v)||_2^2 \leq -\alpha||v||^p + \lambda|v|^2 + \gamma(t)e^{-\mu t}, \quad v \in V, \tag{3.1}$$

where $p > 1$ and, for arbitrary $\delta > 0$, $\gamma(t)$ satisfies $\gamma(t) = o(e^{\delta t})$, as $t \to \infty$, i.e., $\lim_{t\to\infty} \frac{\gamma(t)}{e^{\delta t}} = 0$ and $\int_0^{+\infty} \gamma(t)e^{-\delta t} dt \leq K$ with $K > 0$.

Remark 3.1 Observe that, owing to the continuity and subexponential growth of the term $\gamma(t)e^{-\mu t}$, there exists a positive constant $\widetilde{\gamma}$ such that $\gamma(t)e^{-\mu t} \leq \widetilde{\gamma}$ for all $t \in \mathbb{R}_+$.

As a consequence, (3.1) implies $(a.1)$ (by replacing γ by $\widetilde{\gamma}$), i.e., this assumption is compatible with the existence of the strong solutions to (2.1).

Theorem 3.2 *Assuming conditions (CC) and $(b.3)$, there exists a constant $\tau > 0$ such that if X_t is a global strong solution to Eq. (2.1) corresponding to an initial value $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, satisfying that $\mathbb{E}|X_t|^2 > r(t) := Ke^{-\tau t}$, for all $t \geq 0$, then*

$$\mathbb{E}|X_t|^2 \leq \mathbb{E}|X_0|^2 e^{-\tau t} + r(t), \quad \forall t \geq 0, \tag{3.2}$$

if either one of the following hypotheses holds

(i) $\lambda < 0$, $(\forall p > 1)$;
(ii) $\lambda\beta^2 - \alpha < 0$, $(p = 2)$.

Then, system (2.1) is almost surely globally practically uniformly exponentially convergent to zero in mean square.

Proof Firstly, let us denote $\nu = \frac{(\alpha-\lambda\beta^2)}{\beta^2}$ for case (ii) and $\nu = \dfrac{-\lambda}{\beta^2}$ for case (i), which are positive by assumption (ii) and (i) respectively, and the rest of the proof is the same for both cases. Then, if $\mu - \nu \leq 0$, we can choose $\delta > 0$ small enough such that $\mu - \delta > 0$ and define $\tau := \mu - \delta$. If, on the other hand, $\mu - \nu > 0$, then we can choose $\delta > 0$ small enough such that $\mu - \nu - \delta > 0$ and, in this case, we define $\tau := \nu$. Now, let us suppose that $\mathbb{E}|X_t|^2 > r(t)$, for all $t \geq 0$. Then, Itô's formula implies

$$
\begin{aligned}
e^{(\mu-\delta)t}|X_t|^2 - |X_0|^2 = (\mu-\delta)\int_0^t e^{(\mu-\delta)s}|X_s|^2 ds + 2\int_0^t e^{(\mu-\delta)s}\langle A(s,X_s), X_s\rangle ds,\\
+ 2\int_0^t e^{(\mu-\delta)s}\langle X_s, B(s,X_s)dW_s\rangle\\
+ \int_0^t e^{(\mu-\delta)s} tr(B(s,X_s)QB(s,X_s)^*)ds. \qquad (3.3)
\end{aligned}
$$

Now, since $\int_0^t e^{(\mu-\delta)s}\langle X_s, B(s,X_s)dW_s\rangle$, $t \in \mathbb{R}_+$, is a continuous martingale, it follows that

$$
\mathbb{E}\left(\int_0^t e^{(\mu-\delta)s}\langle X_s, B(s,X_s)dW_s\rangle\right) = 0, \quad t \in \mathbb{R}_+.
$$

Therefore, condition (3.1) and the continuous injection $V \hookrightarrow H$ yield

$$
e^{(\mu-\delta)t}\mathbb{E}|X_t|^2 \le \mathbb{E}|X_0|^2 + (\mu-\delta-\nu)\int_0^t e^{(\mu-\delta)s}\mathbb{E}|X_s|^2 ds + \int_0^t \gamma(s)e^{-\delta s}ds. \qquad (3.4)
$$

If $\mu - \nu \le 0$, it follows immediately

$$
e^{(\mu-\delta)t}\mathbb{E}|X_t|^2 \le \mathbb{E}|X_0|^2 + \int_0^t \gamma(s)e^{-\delta s}ds \le \mathbb{E}|X_0|^2 + K,
$$

thus

$$
\mathbb{E}|X_t|^2 \le \mathbb{E}|X_0|^2 e^{-(\mu-\delta)t} + Ke^{-(\mu-\delta)t} \le \mathbb{E}|X_0|^2 e^{-\tau t} + r(t).
$$

On the other hand, if $\mu-\nu > 0$, as we have chosen $\delta > 0$ small enough such that $\mu-\nu-\delta > 0$, then, from (3.4) and Gronwall's lemma one can obtain

$$
e^{(\mu-\delta)t}\mathbb{E}|X_t|^2 \le \left(\mathbb{E}|X_0|^2 + \int_0^t \gamma(s)e^{-\delta s}ds\right)e^{(\mu-\delta-\nu)t} \le \left(\mathbb{E}|X_0|^2 + K\right)e^{(\mu-\delta-\nu)t},
$$

finally

$$
\mathbb{E}|X_t|^2 \le \mathbb{E}|X_0|^2 e^{-\nu t} + Ke^{-\nu t} \le \mathbb{E}|X_0|^2 e^{-\tau t} + r(t),
$$

as required. □

Remark 3.3 Notice that we can have a second version of Theorem 3.2 under the same hypotheses as it is straightforward to prove that

$$
\mathbb{E}|X_t|^2 \le \mathbb{E}|X_0|^2 e^{-\tau t} + K, \quad \forall t \ge 0.
$$

Then, system (2.1) is almost surely globally practically uniformly exponentially in mean square.

Theorem 3.4 *In addition to hypotheses in Theorem 3.2, assume that (b.2) also holds and* $\int_0^{+\infty}\gamma(s)e^{-\mu s}ds \le \eta < +\infty$ *and* $\sup_{u\in[s,t)} k^2(u) \le \varphi < +\infty$ *for* $0 \le s \le t$, $\mu > 0$, $\eta > 0$ *and* φ *is a positive constant independent of* t *and* s. *Then, there exist positive constants* M, ϵ *and a subset* $N_0 \subset \Omega$ *with* $\mathbb{P}(N_0) = 0$ *such that, for each* $\omega \in \Omega \setminus N_0$, *there exists a positive random number* $T(\omega)$ *such that*

$$
|X_t|^2 \le Me^{-\epsilon t} + \eta, \quad \forall t \ge T(\omega). \qquad (3.5)
$$

Then, the ball $B_{\sqrt{\eta}} \subset H$ *is uniformly stable in probability.*

Proof We only prove case (ii). Case (i) can be proved similarly. We shall split our proof into several steps, as follows.

Step 1: We will find three constants $C = C(\delta, X_0) > 0$, $\zeta > 0$ and $\tau > 0$, independent of $t \in \mathbb{R}_+$, such that

$$\int_s^t \mathbb{E}||B(u, X_u)||_2^2 du \leq Ce^{-\tau s} + \zeta, \quad 0 \leq s \leq t. \tag{3.6}$$

Applying Itô's formula to (2.1) as in Theorem 3.2, we get that for any $\delta > 0$ with $\mu - \delta > 0$

$$e^{(\mu-\delta)t}\mathbb{E}|X_t|^2 \leq \mathbb{E}|X_0|^2 + (\mu - \delta - \nu)\int_0^t e^{(\mu-\delta)s}\mathbb{E}|X_s|^2 ds + \int_0^t \gamma(s)e^{-\delta s} ds, \tag{3.7}$$

and

$$\begin{aligned} e^{(\mu-\delta)t}\mathbb{E}|X_t|^2 \leq \mathbb{E}|X_0|^2 + (\mu - \delta + \lambda)\int_0^t e^{(\mu-\delta)s}\mathbb{E}|X_s|^2 ds \\ + \int_0^t \gamma(s)e^{-\delta s} ds - \alpha \int_0^t e^{(\mu-\delta)s}\mathbb{E}||X_s||^2 ds, \end{aligned} \tag{3.8}$$

where $\nu = \frac{(\alpha - \lambda\beta^2)}{\beta^2}$.

Now, if $\mu - \nu \leq 0$, it follows from (3.7) that

$$\int_0^t e^{(\mu-\delta)s}\mathbb{E}|X_s|^2 ds \leq \frac{\mathbb{E}|X_0|^2 + \int_0^t \gamma(s)e^{-\delta s} ds}{\nu + \delta - \mu}, \tag{3.9}$$

which, together with (3.8), immediately implies

$$\begin{aligned} \int_0^t e^{(\mu-\delta)s}\mathbb{E}||X_s||^2 ds &\leq \frac{1}{\alpha}\Big[\mathbb{E}|X_0|^2 + \int_0^t \gamma(s)e^{-\delta s} ds\Big], \\ &\quad + \frac{\mu - \delta + \lambda}{\alpha}\int_0^t e^{(\mu-\delta)s}\mathbb{E}|X_s|^2 ds, \\ &\leq \frac{1}{\alpha}\Big[\frac{\mu - \delta + \lambda}{\nu + \delta - \mu} + 1\Big]\Big[\mathbb{E}|X_0|^2 + \int_0^t \gamma(s)e^{-\delta s} ds\Big], \\ &\leq \frac{1}{\alpha}\Big[\frac{\mu - \delta + \lambda}{\nu + \delta - \mu} + 1\Big]\big[\mathbb{E}|X_0|^2 + K\big]. \end{aligned} \tag{3.10}$$

Consequently, for $0 \leq s \leq t$,

$$\begin{aligned} \int_s^t \mathbb{E}||X_u||^2 du &\leq \int_s^t e^{(\mu-\delta)(u-s)}\mathbb{E}||X_u||^2 du, \\ &\leq e^{-(\mu-\delta)s}\int_0^t e^{(\mu-\delta)u}\mathbb{E}||X_u||^2 du, \end{aligned}$$

thus,

$$\int_s^t \mathbb{E}||X_u||^2 du \leq \frac{1}{\alpha}\Big[\frac{\mu - \delta + \lambda}{\nu + \delta - \mu} + 1\Big]\big[\mathbb{E}|X_0|^2 + K\big]e^{-(\mu-\delta)s}, \tag{3.11}$$

which, together with (b.2) immediately yields that

$$\begin{aligned}\int_s^t \mathbb{E}||B(u,X_u)||_2^2 du &\le 2\int_s^t k^2(u)\mathbb{E}||X_u||^2 du + 2\int_s^t \psi(u)^2 du\\ &\le 2\sup_{u\in[s,t)} k^2(u)\int_s^t \mathbb{E}||X_u||^2 du + 2\int_0^{+\infty}\psi(u)^2 du\\ &\le 2\varphi\int_s^t \mathbb{E}||X_u||^2 du + 2\xi\end{aligned}$$

therefore,

$$\int_s^t \mathbb{E}||B(u,X_u)||_2^2 du \le Ce^{-(\mu-\delta)s} + \zeta, \tag{3.12}$$

where k_1 is a positive constant, $C = C(\delta, X_0) = \dfrac{2\varphi}{\alpha}\Big[\dfrac{\mu-\delta+\lambda}{\nu+\delta-\mu}+1\Big]\Big[\mathbb{E}|X_0|^2+K\Big]$ and $\zeta = 2\xi$.

On the other hand, if $\mu-\nu>0$, it is always possible to choose a suitable $\delta>0$ such that $\nu-\delta>0$. Then, by applying Itô's lemma to the strong solution X_t, it is easy to deduce

$$\begin{aligned}e^{(\nu-\delta)t}\mathbb{E}|X_t|^2 &\le \mathbb{E}|X_0|^2 + (\nu-\delta+\lambda)\int_0^t e^{(\nu-\delta)s}\mathbb{E}|X_s|^2 ds,\\ &\quad + \int_0^t \gamma(s)e^{-(\mu-\nu+\delta)s}ds - \alpha\int_0^t e^{(\nu-\delta)s}\mathbb{E}||X_s||^2 ds,\\ &\le \mathbb{E}|X_0|^2 + (\nu-\delta+\lambda)\int_0^t e^{(\nu-\delta)s}\mathbb{E}|X_s|^2 ds,\\ &\quad + \int_0^t \gamma(s)e^{-\delta s}ds - \alpha\int_0^t e^{(\nu-\delta)s}\mathbb{E}||X_s||^2 ds. \end{aligned}\tag{3.13}$$

Noticing that, in this case, the parameter τ in theorem 3.2 turns out to be ν, (3.13) yields

$$\alpha\int_0^t e^{(\nu-\delta)s}\mathbb{E}||X_s||^2 ds \le \mathbb{E}|X_0|^2 + (\nu-\delta+\lambda)\int_0^t e^{-\delta s}ds + K,$$

and we can argue in a similar manner as we did previously. Hence our claim is proved.

Step 2: We claim that there exists a positive constant $M>0$ such that

$$\mathbb{E}\left(\sup_{0\le t<\infty}|X_t|^2\right) \le M.$$

Indeed, Itô's formula implies

$$\begin{aligned}|X_t|^2 - |X_0|^2 &= 2\int_0^t < A(s,X_s), X_s > ds + \int_0^t tr\big(B(s,X_s)QB(s,X_s)^*\big)ds,\\ &\quad +2\int_0^t < X_s, B(s,X_s)dW_s > . \end{aligned}\tag{3.14}$$

On the other hand, from Burkholder–Davis–Gundy's inequality, we get for any $T \in \mathbb{R}_+$

$$
\begin{aligned}
2\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t \langle X_s, B(s,X_s)dW_s\rangle\Big|\Big], \\
&\leq K_1\mathbb{E}\left[\left(\int_0^T |X_s|^2||B(s,X_s)||_2^2 ds\right)^{\frac{1}{2}}\right], \\
&\leq K_1\mathbb{E}\left\{\sup_{0\leq s\leq T}|X_s|\left[\int_0^T ||B(s,X_s)||_2^2 ds\right]^{\frac{1}{2}}\right\}, \\
&\leq \frac{1}{2}\mathbb{E}\left[\sup_{0\leq s\leq T}|X_s|^2\right] + K_2\int_0^T ||B(s,X_s)||_2^2 ds, \qquad (3.15)
\end{aligned}
$$

where K_1; K_2 are two positive constants. Therefore, in addition to condition (CC), (3.14) and (3.15) imply

$$
\begin{aligned}
\mathbb{E}\left(\sup_{0\leq s\leq T}|X_s|^2\right) \leq \mathbb{E}|X_0|^2 + \nu\int_0^T \mathbb{E}|X_s|^2 ds + \int_0^T \gamma(s)e^{-\mu s}ds, \\
+\frac{1}{2}\mathbb{E}\left[\sup_{0\leq s\leq T}|X_s|^2\right] + K_2\int_0^T \mathbb{E}||B(s,X_s)||_2^2 ds. \qquad (3.16)
\end{aligned}
$$

Thus, our claim can be easily obtained owing to (3.2), (3.6) and condition (CC).

Step 3: Now, we can finish our proof. We only sketch it because it is similar to that in Caraballo [9] and Haussmann [13].

Firstly, the coercivity condition (CC) and (3.14) imply

$$
\begin{aligned}
|X_T|^2 &\leq |X_N|^2 + \nu\int_N^T |X_s|^2 ds + \int_N^T \gamma(s)e^{-\mu s}ds, \\
&\quad + 2\left[\sup_{t\in[N,T]}\left|\int_N^t < X_s, B(s,X_s)dW_s >\right|\right], \\
&\leq |X_N|^2 + \nu\int_N^T |X_s|^2 ds + \int_0^{+\infty} \gamma(s)e^{-\mu s}ds, \\
&\quad + 2\left[\sup_{t\in[N,T]}\left|\int_N^t < X_s, B(s,X_s)dW_s >\right|\right], \\
&\leq |X_N|^2 + \nu\int_N^T |X_s|^2 ds + \eta, \\
&\quad + 2\left[\sup_{t\in[N,T]}\left|\int_N^t < X_s, B(s,X_s)dW_s >\right|\right],
\end{aligned}
$$

Consequently, we obtain

$$
|X_T|^2 - \eta \leq |X_N|^2 + \nu\int_N^T |X_s|^2 ds + 2\left[\sup_{t\in[N,T]}\left|\int_N^t < X_s, B(s,X_s)dW_s >\right|\right], \quad (3.17)
$$

for $T \geq N$, where N is a natural number.

In particular, taking $N \in \mathbb{N}$ large enough, we can easily obtain

$$\mathbb{P}\Big\{\sup_{t\in[N,N+1]} |X_t|^2 - \eta \geq \epsilon_N^2\Big\} \leq \mathbb{P}\Big\{\Big[\sup_{t\in[N,N+1]}\Big|\int_N^t < X_s, B(s,X_s)dW_s >\Big|\Big] \geq \frac{\epsilon_N^2}{6}\Big\}, \tag{3.18}$$

$$+\,\mathbb{P}\left\{|X_N|^2 \geq \frac{\epsilon_N^2}{3}\right\}, +\mathbb{P}\left\{\nu\int_N^{N+1}|X_s|^2 ds \geq \frac{\epsilon_N^2}{3}\right\},$$

where $\epsilon_N^2 = Ce^{-\frac{\tau(N+1)}{4}}$.

Now, we can estimate the terms on the right-hand side of (3.18) using Kolmogorov's inequality and (3.2) for the last two terms, and Burkholder–Davis–Gundy's lemma, Hölder inequality and an argument similar to that used in Steps 1 and 2 for the first one. Consequently, there exists a positive constant $K_3 > 0$ such that

$$\mathbb{P}\left\{\sup_{t\in[N,N+1]} |X_t|^2 - \eta \geq \epsilon_N^2\right\} \leq K_3 e^{-\frac{\tau N}{4}}.$$

Finally, a Borel-Cantelli's lemma-type there exist a subset $N_0 \subset \Omega$ with $\mathbb{P}(N_0) = 0$ such that, for each $\omega \in \Omega \setminus N_0$, there exists a positive random number $T(\omega)$ such that

$$|X_t|^2 \leq \eta + Ce^{-\frac{\tau(N+1)}{4}}, \quad \forall t \geq T(\omega).$$

Noting that $Ce^{-\frac{\tau(N+1)}{4}} \leq Ce^{-\frac{\tau t}{4}}$. Then we have

$$|X_t|^2 \leq \eta + Ce^{-\frac{\tau t}{4}}, \quad \forall t \geq T(\omega).$$

as desired. □

Next, we give an example to illustrate our results.

Example 3.5 We consider the following semi-linear stochastic partial differential equation, which models the heat production by an exothermic reaction taking place inside a rod of length π whose ends are maintained at 0° and whose sides are insulated (see Haussmann [13] for a similar situation in the linear case):

$$\begin{cases} dY_t(x) = \Big[\frac{\partial^2 Y_t(x)}{\partial x^2} + r_0 Y_t(x)\Big]dt + \alpha(t, Y_t(x))dW(t), & t > 0, x \in (0,\pi), \\ Y_0(x) = y_0(x), \quad Y_t(0) = Y_t(\pi) = 0, & t \geq 0. \end{cases} \tag{3.19}$$

Here W_t is a real standard Wiener process (so, $K = \mathbb{R}$ and $\mathcal{Q} = 1$), $r_0 \in \mathbb{R}$, and $\alpha(\cdot,\cdot) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a continuous function such that $\alpha(t,0) \neq 0$, for some $t \in \mathbb{R}$, and $|\alpha(t,u)| \leq e^{-t}|u| + te^{-at}$ with $a > 0$. We can set this problem in our formulation by taking $H = L^2[0,\pi]$, $V = W_0^{1,2}([0,\pi])$ (a Sobolev space with elements satisfying the boundary conditions above), $K = \mathbb{R}$, $A(t,u) = (d^2/dx^2)u(x) + r_0 u(x)$, and $B(t,u) = \alpha(t,u)$.

Clearly, operator B satisfies (b.2) and (b.3). On the other hand, it is easy to deduce for arbitrary $u \in V$ that

$$\begin{aligned} 2 < A(t,u), u >> +||B(t,u)||_2^2 &\leq -2||u||^2 + 2r_0|u|^2 + 2e^{-2t}|u|^2 + 2t^2e^{-2at}, \\ &\leq -2||u||^2 + (2r_0+2)|u|^2 + 2t^2e^{-2at}. \end{aligned}$$

The norm in V is given by

$$||u||^2 = \int_0^\pi (u'(x))^2 dx.$$

Therefore, it follows that hypothesis (b) in Theorems 3.2 and 3.4 is fulfilled provided $(2 + 2r_0)\beta^2 <<< 2$ (observe that we can set $\beta = \frac{\pi}{\sqrt{2}}$ in this case). We can take $\alpha = 2, \gamma(t) = 2t^2$, $\mu = 2a$, $\lambda = 2r_0 + 2$ and $r(t) = \dfrac{4}{\delta^3}e^{-\nu t}$ where $\nu = \dfrac{2}{\beta^2} - (2 + 2r_0)$.

Consequently, we easily deduce that the strong solution of the equation is almost surely practically exponentially stable in mean square.

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