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Sobolev embeddings for Riesz potentials of functions in non-doubling Morrey spaces of variable exponents

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Abstract The target is potential theory in connection with Morrey spaces on general metric measure spaces. The present paper is oritented to investigating Sobolev's inequality, Trudinger exponential integrability and continuity for Riesz potentials of functions in non-doubling Morrey spaces of variable exponents. A counterexample shows that our results are reasonable. In addition to the example above, what is new about this paper is that everything can be developed once the underlying measure does not charge any point.

Keywords Riesz potentials · Maximal functions · Sobolev's inequality · Trudinger's inequality · Continuity · Morrey spaces of variable exponents · Non-doubling measure · Generalized smoothness

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1 Introduction

We shall show that the Adams theorem about the boundedness of fractional integral operators and the related theorems can be extended even to general metric measure spaces by a slight modification of Morrey norms. We present an example showing that the modification is essential.

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Morrey spaces date back to the work of Morrey [25] in 1938. His observation has become a useful tool for partial differential equations and with this tool we can study the existence and regularity of solutions of partial differential equations. Nowadays, his technique turned out to be a wide theory of function spaces called Morrey spaces. The (original) space $M^p_u(\mathbf{R}^d)$ with $1 \le u \le p < \infty$ is a normed space whose norm is given by

$$||f||_{M^p_u} \equiv \sup_{x\in\mathbf{R}^d,\ r>0} r^{\frac{d}{p}-\frac{d}{u}} \left(\int_{B(x,r)} |f(y)|^u \, dy \right)^{\frac{1}{u}} \quad \text{for } f \in L^u_{\rm loc}(\mathbf{R}^d).$$

In the present paper, we are oriented to Sobolev's inequality for Riesz potentials of functions in Morrey spaces of variable exponents in the non-doubling setting, which will extend the results in [3,15,21,22,24,35]. We also establish Trudinger exponential integrability for Riesz potentials of functions in Morrey spaces of variable exponents in the non-doubling setting, as extensions of our earlier papers [21,22,34]. Further, we discuss continuity of Riesz potentials of variable order, which extends [7,12,21,23].

Let X be a separable metric space equipped with a non-negative Radon measure μ. Assume that X is a bounded set and we denote by d_X its diameter. By $B(x,r)$ we denote the open ball centered at x of radius $r > 0$. We write $d(x, y)$ for the distance of the points x and y in X. We assume that

$$\mu(\{x\}) = 0 \tag{1.1}$$

for $x \in X$ and that $0 < \mu(B(x,r)) < \infty$ for $x \in X$ and $r > 0$ for simplicity. In the present paper, we do not postulate on μ the "so-called" doubling condition. Recall that a Radon measure μ is said to be doubling, if there exists a constant $C > 0$ such that $\mu(B(x, 2r)) \le C\mu(B(x,r))$ for all $x \in \mathrm{supp}(\mu)(= X)$ and $r > 0$. Otherwise μ is said to be non-doubling. In connection with the $5r$-covering lemma, the doubling condition had been a key condition in harmonic analysis. However, Nazarov, Treil and Volberg [29,30] showed that the doubling condition is not necessary by using the modified maximal operator. In the present paper, we shall show that this is the case for Riesz potentials.

Let $p \ge 1$ and $\kappa > 0$. Define the Morrey norm $\|f\|_{L^{p,\kappa,\nu}(\mu)}$ by

$$||f||_{L^{p,\kappa,\nu}(\mu)} \equiv \sup \left\{ \left(\frac{r^\nu}{\mu(B(x,\kappa r))} \int_{B(x,r)} |f(y)|^p \, d\mu(y) \right)^{1/p} : x \in X,\ r \in (0, d_X),\ \mu(B(x,r)) > 0 \right\}$$

for μ-measurable functions f. The Morrey space $L^{p,\kappa,\nu}(\mu)$ is the set of all μ-measurable functions f for which the norm $||f||_{L^{p,\kappa,\nu}(\mu)}$ is finite.

The parameter κ affects the definition of the Morrey space $L^{p,\kappa,\nu}(\mu)$, as shall be illustrated by the following proposition. We state one of the main results in this paper.

Theorem 1.1 *There does exist a separable metric space (X, d, μ) such that the function spaces $L^{p,4,\nu}(\mu)$ and $L^{p,2,\nu}(\mu)$ do not coincide as sets.*

About the modified Morrey norm, we have the following remarks. The proof is simple and we omit it.

Remark 1.2 Let f be a μ-measurable function.

1. From the definition of the norm we learn $||f||_{L^{p,\kappa_2,\nu}(\mu)} \le ||f||_{L^{p,\kappa_1,\nu}(\mu)}$ for all $\kappa_2 > \kappa_1 > 0$ and $p \ge 1$.

2. If $p_2 \geq p_1 \geq 1, \kappa \geq 1$ and $\nu_1/p_1 = \nu_2/p_2 > 0$, then, $||f||_{L^{p_1,\kappa,\nu_1}(\mu)} \leq ||f||_{L^{p_2,\kappa,\nu_2}(\mu)}$ by the Hölder inequality.
3. If μ is a doubling measure, then $||f||_{L^{p,\kappa,\nu}(\mu)}$ and $||f||_{L^{p,1,\nu}(\mu)}$ are equivalent for all $p \geq 1, \kappa > 0$ and $\nu > 0$.

Our result can be readily translated into the Morrey space $M^p_q(\Omega)$, where $M^p_q(\Omega)$ is the set of all functions $f \in L^q_{\text{loc}}(\Omega)$ for which the norm

$$||f||_{M^p_q(\Omega)} \equiv \sup_{x\in\Omega,\, r>0} r^{d/p-d/q} \left(\int\limits_{B(x,r)\cap\Omega} |f(y)|^q \, dy \right)^{1/q} < \infty,$$

where Ω is an open bounded set in $\mathbb{R}^d$. In the present paper, we also show that a modification enables us to obtain boundedness results even in the variable Lebesgue setting. We consider variable exponents $p(\cdot)$ and $q(\cdot)$ on X such that
(P1) $1 < p_- \equiv \inf_{x\in X} p(x) \leq \sup_{x\in X} p(x) \equiv p_+ < \infty$;
(P2) $|p(x) - p(y)| \leq C/\log(e + d(x, y)^{-1})$ whenever $x \in X$ and $y \in X$;
(Q1) $-\infty < q_- \equiv \inf_{x\in X} q(x) \leq \sup_{x\in X} q(x) \equiv q_+ < \infty$;
(Q2) $|q(x) - q(y)| \leq C/\log(e + (\log(e + d(x, y)^{-1})))$ whenever $x \in X$ and $y \in X$.

In general, if $p(\cdot)$ satisfies (P2) (resp. $q(\cdot)$ satisfies (Q2)), then $p(\cdot)$ (resp. $q(\cdot)$) is said to satisfy the log-Hölder (resp. loglog-Hölder) condition.

Let G be a bounded open set in X. For a bounded μ-measurable function $\alpha : X \to (0, \infty)$ and $\tau > 0$, we define the Riesz potential of (variable) order α for a non-negative μ-measurable function f on G by

$$U_{\alpha(\cdot),\tau} f(x) \equiv \int\limits_G \frac{d(x, y)^{\alpha(x)} f(y)}{\mu(B(x, \tau d(x, y)))} d\mu(y).$$

The assumption (1.1) will be necessary for the definition of $U_{\alpha(\cdot),\tau} f$ in order that the integral is not infinite. Here and in what follows we tacitly assume that $f = 0$ outside G. Observe that this naturally extends the Riesz potential operator

$$U_\alpha f(x) \equiv \int\limits_{\mathbf{R}^d} \frac{f(y)}{|x - y|^{d-\alpha}} \, dy$$

when (X, d) is the d-dimensional Euclidean space and $\mu = dx$.

We also assume

$$\alpha_- \equiv \inf_{x\in X} \alpha(x) > 0 \tag{1.2}$$

for $\alpha(\cdot)$ appearing in the definition of the operator $U_{\alpha(\cdot),\tau}$.

Now we are going to formulate our results in full generality. First of all, we set

$$\Phi(x, r) = \Phi_{p(\cdot),q(\cdot)}(x, r) \equiv r^{p(x)}(\log(C_0 + r))^{q(x)} \quad (x \in X,\ r > 0); \tag{1.3}$$

here the constant $C_0 > e$ is chosen so that the following condition (Φ) holds:

$$(\Phi) \quad \Phi_{p(\cdot),q(\cdot)}(x, \cdot) \quad \text{is convex on } [0, \infty) \text{ for every } x \in X$$

(see [17, Theorem 5.1]). Note from (Φ) that the function $t^{-1}\Phi(x, t)$ is nondecreasing on $(0, \infty)$ for fixed $x \in X$.

Let $\kappa > 1$ be a fixed parameter and let G be a bounded subset of X. Let us denote by d_G the diameter of G. For bounded μ-measurable functions $\nu : X \to (0, \infty)$ and $\beta : X \to (-\infty, \infty)$, we introduce the family $L^{\Phi,\nu,\beta;\kappa}(G)$ of all μ-measurable functions f on G such that for some $\lambda \in (0, \infty)$

$$\sup_{x\in G, 0<r<d_G} \frac{r^{\nu(x)}(\log(e+1/r))^{\beta(x)}}{\mu(B(x,\kappa r))} \int_{G\cap B(x,r)} \Phi(y, |f(y)|/\lambda)\, d\mu(y) \le 1. \tag{1.4}$$

Denote by $||f||_{L^{\Phi,\nu,\beta;\kappa}(G)}$ the smallest value of λ satisfying (1.4).

The space $L^{\Phi,\nu,\beta;\kappa}(G)$ is a kind of generalized Morrey spaces. Generalized Morrey spaces with non-doubling measures on $\mathbf{R}^d$ are taken up in [11,33]. However, it appears in a natural context. Nowadays, generalized Morrey spaces are not generalization for its own sake. Note that generalized Morrey spaces occur naturally when we consider the limiting case as Proposition 1.3 below shows.

Proposition 1.3 [36, Theorem 5.1] *Let $1 < q < p < \infty$. Then there exists a positive constant $C_{p,q}$ such that*

$$\int_Q |f(x)|dx \le C_{p,q}|Q|(1+|Q|)^{-\frac{1}{p}} \log\left(e + \frac{1}{|Q|}\right) ||(1-\Delta)^{d/2p} f||_{M^p_q}$$

holds for all $f \in M^p_q(\mathbf{R}^d)$ with $(1-\Delta)^{n/2p} f \in M^p_q(\mathbf{R}^d)$ and for all cubes Q.

In view of the integral kernel of $(1-\Delta)^{-\alpha/2}$ (see [37]) and the Adams theorem, we have

$$(1-\Delta)^{-\alpha/2} : M^p_q(\mathbf{R}^d) \to M^s_t(\mathbf{R}^d) \tag{1.5}$$

is bounded as long as

$$1 < q \le p < \infty,\ 1 < t \le s < \infty,\ \frac{1}{s} = \frac{1}{p} - \frac{\alpha}{d},\ \frac{t}{s} = \frac{q}{p}.$$

The operator norm of $(1-\Delta)^{\alpha/2} : M^p_q(\mathbf{R}^d) \to M^s_t(\mathbf{R}^d)$ blows up as $p \to \frac{d}{\alpha}$. Hence we can say that Proposition 1.3 substitutes (1.5). We refer to [36] for a counterexample showing that (1.5) is no longer true for $\alpha = \frac{d}{p}$.

Meanwhile, the function $q(\cdot)$ can be used to describe the Hardy–Littlewood maximal operator control in very subtle settings. To describe the situation, we place ourselves in the setting of the Euclidean space $\mathbf{R}^d$. We denote again by $B(x,r)$ the open ball centered at $x \in \mathbf{R}^d$ and of radius r. For a locally integrable function f on $\mathbf{R}^d$, we consider the Hardy–Littlewood maximal function

$$Mf(x) \equiv \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy \quad (x \in \mathbf{R}^d).$$

For the fundamental properties of the Hardy–Littlewood maximal function, see Duo andikotxea [5] and Stein [37]. It is known as Stein's theorem that there exists a universal constant $C > 0$ such that

$$\int_B Mf(x)\, dx \le C \inf\left\{\lambda > 0 : \int_B \frac{|f(x)|}{\lambda} \log\left(2 + \frac{|f(x)|}{\lambda}\right) dx \le 1\right\}$$

for all functions f supported on a ball B with radius 1.

Remark that, if $X = \mathbf{R}^d$, the parameter κ is not essential as long as $\kappa > 1$ as Proposition 1.4 below shows:

Proposition 1.4 *Let $\kappa_1, \kappa_2 > 1$ and $X = \mathbf{R}^d$ be the Euclidean space. Suppose that G is a bounded open set. Assume in addition that ν and β satisfy the log-Hölder continuity and the loglog-Hölder continuity, respectively. Then the spaces $L^{\Phi,\nu,\beta;\kappa_1}(G)$ and $L^{\Phi,\nu,\beta;\kappa_2}(G)$ coincide as sets and their norms are equivalent.*

We shall prove Proposition 1.4 in Sect. 3.

In the present paper, we consider a generalized and modified Hardy–Littlewood maximal function defined by

$$M_{16} f(x) \equiv \sup_{r>0} \frac{1}{\mu(B(x, 16r))} \int_{G \cap B(x,r)} |f(y)| d\mu(y) \tag{1.6}$$

for a locally integrable function f on G.

In the present paper, we shall also show that most of the results known as the limiting cases can be carried over to the non-doubling measure spaces. It counts that we take an attentive care of the parameters κ appearing in (1.4). For example, unlike the doubling measure spaces, we need delicate geometric observations [see the Proof of Lemma 4.2 and (6.17), for example]. Because we need careful geometric observations, we need to set everything up from the start. Section 4 is our actual starting point.

We organize the remaining part of the present paper as follows:

In Sect. 2, we intend to justify that the modification is necessary in the non-doubling setting by proving Theorem 1.1. To construct a counterexample, we shall refine the one in [32]. In Sect. 3, we see some more examples of this metric measure setting.

From Sect. 4, we are going to construct a general theory. Section 4 is devoted to the study of the modified centered Hardy–Littlewood maximal operator M_{16}.

We are going to obtain Sobolev's inequality for Riesz potentials $U_{\alpha(\cdot),32} f$ of functions in $L^{\Phi,\nu,\beta;2}(G)$ in Sect. 5. To this end, we apply Hedberg's trick [14] by the use of the boundedness of the Hardy–Littlewood maximal function M_{16} adapted to our setting. Our result (see Theorem 4.1 below) is given in Sect. 5, which extends the results in [15,21,22, 24,35].

A famous Trudinger inequality [39] insists that Sobolev functions in $W^{1,d}(\Omega)$ satisfy finite exponential integrability, where Ω is an open bounded set in $\mathbb{R}^d$. In Sect. 6, we are concerned with the Morrey counterpart of Trudinger's type exponential integrability for $U_{\alpha(\cdot),9} f$. Our result contains the result of Trudinger [39] as well as those in [21,22,34]. For related results, see [2,7–9,18–20,28,40].

In Sect. 7, we discuss the continuity of Riesz potentials of variable order, as an extension of [7,12,21,23]. For related results, see [8,19,20]. More precisely, in Sect. 7 we discuss the continuity of Riesz potentials $U_{\alpha(\cdot),4} f$, which can be considered as generalized variable smoothness. It seems of interest in other fields of mathematics such as PDEs that we investigate the continuity of functions according to each points. Indeed, the fundamental solution of $-\Delta u = f$ on $\mathbf{R}^d$ is continuous except on the origin. Therefore, we are interested in tools with which to investigate continuity differently according to the points. In view of the continuity we postulate on variable exponents, we can say that this is achieved to some extent.

Finally we explain some notations used in the present paper. The function χ_E denotes the characteristic function of E. Throughout the present paper, let C denote various constants independent of the variables in question.

2 Proof of Theorem 1.1

In Sect. 2, we prove Theorem 1.1. Here will be a series of definitions which are valid only in Sects. 2.1–2.2.

2.1 The space we work on

First, we define a set X on which we work.

Definition 2.1 (*Definition of* X) Define a set X as follows:

1. One writes $\Delta(z, r) \equiv \{w \in \mathbf{C} : |w - z| < r\}$.
2. Let $A_k \equiv \{z \in \mathbf{C} : |z| = 3^{-k}\}$ for k $\in \mathbb{N} \cup \{0\}$. (For the graph of A_0 we refer to the footnote 1.)
3. Define $X_0 \equiv \{0\} \cup \bigcup_{k=0}^{\infty} A_k \subset \mathbf{C}$. (For the graph of X_0 we refer to the footnote 1.)
4. Let $X \equiv X_0^{\mathbb{N}} \subset \mathbf{C}^{\mathbb{N}}$ be the cross product.
5. Let $\mathbf{O} \equiv (0, 0, \cdots) \in X$.

Remark 2.2 Here and below, we adopt the following rules in Sect. 2:

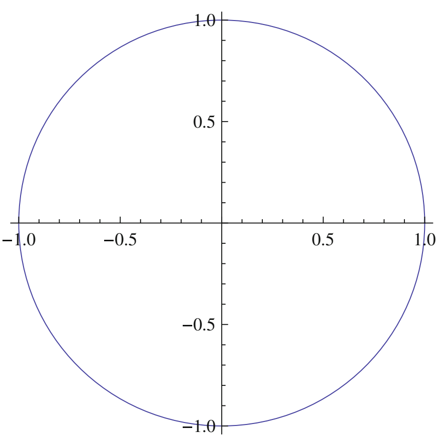
1. The letter z without subindex denotes the point in $\mathbf{C}$.
2. Points in X are written in the bold letters such as $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$.[1]
3. Symbols such as x_j, y_j, z_j and so on are complex numbers and they denote the jth component of elements in X.

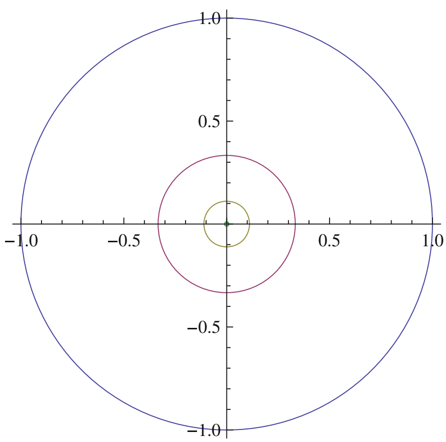
The point is that we give a "singular" metric on X. The precise definition is as follows:

Definition 2.3 (Definition of the metric)

1. The integer N_0 is chosen so that $\log_3 N_0$ is a big integer.
2. Denote by $[\cdot]$ a Gauss symbol and define $N(\delta) \equiv \max(1, [\log_{N_0} \delta^{-1}])$ for $\delta > 0$.

[1] We draw graphs of A_0 and X_0.





Note that A_0 is an annulus and X_0 is the union and X is a countable product of X_0.

3. Let $\mathbf{x} = \{x_j\}_{j=1}^{\infty}$ and $\mathbf{y} = \{y_j\}_{j=1}^{\infty}$ be points in X. Then define the distance $d(\mathbf{x}, \mathbf{y})$ of $\mathbf{x}$ and $\mathbf{y}$ by
$$d(\mathbf{x}, \mathbf{y}) \equiv \inf\{\delta > 0 : |x_j - y_j| \le \delta \text{ for all } j \le N(\delta)\}.$$
4. One defines a sphere S_k by
$$S_k \equiv (A_k)^{N(3^{-k})} \times X_0 \times X_0 \times \cdots \tag{2.1}$$
for each $k \in \mathbb{N}$.

At this moment, about the definition of the natural number N_0, it is only the fact that the function $N : (0, \infty) \to \mathbb{R}$ is a decreasing function that counts for the moment.

Before we start a (long) proof, let us outline it. Roughly speaking, sphere testing suffices. Based upon the metric space (X, d) given above, we shall show that (X, d) is in fact a separable metric measure space and that there does exist a Borel measure μ such that $L^{p,4,\nu}(\mu)$ and $L^{p,2,\nu}(\mu)$ do not coincide as sets: More precisely, we shall show that
$$\liminf_{k\to\infty} \frac{||\chi_{S_k}||_{L^{p,4,\nu}(\mu)}}{||\chi_{S_k}||_{L^{p,2,\nu}(\mu)}} = 0.$$
Choose an increasing sequence $\{k(n)\}_{n=1}^{\infty}$ such that $\frac{||\chi_{S_{k(n)}}||_{L^{p,4,\nu}(\mu)}}{||\chi_{S_{k(n)}}||_{L^{p,2,\nu}(\mu)}} \le 4^{-n}$. If we define
$$F = \sum_{n=1}^{\infty} \frac{2^n}{||\chi_{S_{k(n)}}||_{L^{p,2,\nu}(\mu)}} \chi_{S_{k(n)}},$$
then $\|F\|_{L^{p,2,\nu}(\mu)} \ge 2^n$ and $\|F\|_{L^{p,4,\nu}(\mu)} \le 1$ for all $n \in \mathbb{N}$. This implies $F \in L^{p,4,\nu}(\mu) \setminus L^{p,2,\nu}(\mu)$. The remainder of this subsection is devoted to some preparatory observation on this metric measure space and in the next subsection we get the conclusion.

Note that
$$d(\mathbf{x}, \mathbf{y}) = \inf\{\delta > 0 : |x_1 - y_1| \le \delta,\ |x_2 - y_2| \le \delta,\ \ldots, |x_{N(\delta)} - y_{N(\delta)}| \le \delta\}$$
and that
$$S_k = \{\mathbf{z} = (z_1, z_2, \ldots) \in {X_0}^{\mathbb{N}} : |z_1| = |z_2| = \cdots = |z_{N(3^{-k})}| = 3^{-k}\}.$$
Thus, $d_X = 2$. Let us check that d is a metric function and that χ_{S_k} is μ-measurable.

Lemma 2.4 *In Definition 2.3, d is a metric function, that is,*
$$0 \le d(\mathbf{x}, \mathbf{y}) < \infty \quad (\mathbf{x}, \mathbf{y} \in X), \tag{2.2}$$
$$d(\mathbf{x}, \mathbf{y}) = 0 \Longrightarrow \mathbf{x} = \mathbf{y} \quad (\mathbf{x}, \mathbf{y} \in X), \tag{2.3}$$
$$d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x}) \quad (\mathbf{x}, \mathbf{y} \in X), \tag{2.4}$$
$$d(\mathbf{x}, \mathbf{z}) \le d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) \quad (\mathbf{x}, \mathbf{y}, \mathbf{z} \in X). \tag{2.5}$$
Furthermore, the d-topology is exactly the product topology of X.

Proof Since $2 \in \{\delta > 0 : |x_j - y_j| \le \delta \text{ for all } j \le N(\delta)\}$, (2.2) is clear. If $\mathbf{x} \ne \mathbf{y}$, then $|x_{j_0} - y_{j_0}| > \delta$ for some $\delta > 0$ and $j_0 \in \mathbb{N}$. Therefore, if we choose $\delta^* > 0$ so that $N(\delta^*) > j_0$,
$$|x_{j_0} - y_{j_0}| > \min(\delta, \delta^*),\ j_0 < N(\min(\delta, \delta^*)).$$

This implies

$$\min(\delta, \delta^*) \notin \{r > 0 : |x_1 - y_1| \le r,\ |x_2 - y_2| \le r,\ \cdots,\ |x_{N(r)} - y_{N(r)}| \le r\}.$$

Hence, $d(\mathbf{x}, \mathbf{y}) \ge \min(\delta, \delta^*)$, which shows (2.3). Equality (2.4) follows immediately from the definition of d. Next, we check (2.5). To this end, we take $\varepsilon > 0$. Then by the definition of $d(\mathbf{x}, \mathbf{y})$ and $d(\mathbf{y}, \mathbf{z})$, we can find $\delta_1 \in (d(\mathbf{x}, \mathbf{y}), d(\mathbf{x}, \mathbf{y}) + \varepsilon)$ and $\delta_2 \in (d(\mathbf{y}, \mathbf{z}), d(\mathbf{y}, \mathbf{z}) + \varepsilon)$ so that

$$|x_j - y_j| \le \delta_1 \text{ for all } j \le N(\delta_1)$$

and that

$$|y_j - z_j| \le \delta_2 \text{ for all } j \le N(\delta_2).$$

Noting that $N(\delta_1 + \delta_2) \le \min(N(\delta_1), N(\delta_2))$, we have

$$|x_j - y_j| \le \delta_1,\ |y_j - z_j| \le \delta_2 \text{ for all } j \le N(\delta_1 + \delta_2)$$

and hence

$$|x_j - z_j| \le \delta_1 + \delta_2 \text{ for all } j \le N(\delta_1 + \delta_2).$$

Consequently, $d(\mathbf{x}, \mathbf{z}) \le \delta_1 + \delta_2 \le d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) + 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain (2.5).

Since any d-open set is open with respect to the product topology, the product topology is not weaker than the d-topology. However, we can express X as a union of d-open balls as long as X is given by

$$X = \Delta(z_1, r_1) \times \Delta(z_2, r_2) \times \cdots \times \Delta(z_{\tilde{N}}, r_{\tilde{N}}) \times X_0 \times X_0 \times \cdots$$

with some $\tilde{N} \in \mathbb{N}$, $(z_1, z_2, \cdots, z_{\tilde{N}}) \in X_0{}^{\tilde{N}}$ and $(r_1, r_2, \cdots, r_{\tilde{N}}) \in (0, \infty)^{\tilde{N}}$. Therefore, two topologies coincide. □

Lemma 2.5 *Let* $r > 0$ *and* $\mathbf{x} = (x_1, x_2, \cdots) \in X$. *Then*

$$B(\mathbf{x}, r) = \Delta(x_1, r) \times \Delta(x_2, r) \times \cdots \times \Delta(x_{N(r)}, r) \times X_0 \times X_0 \times \cdots. \tag{2.6}$$

Proof From the definition of the open ball, we have

$$\begin{aligned} B(\mathbf{x}, r) &= \{\mathbf{y} = \{y_j\}_{j=1}^\infty \in X : d(\{y_j\}_{j=1}^\infty, \{x_j\}_{j=1}^\infty) < r\} \\ &= \bigcup_{\delta \in (0,r)} \{\mathbf{y} = \{y_j\}_{j=1}^\infty \in X : |y_1 - x_1| \le \delta,\ |y_2 - x_2| \le \delta,\ \cdots,\ |y_{N(\delta)} - x_{N(\delta)}| \le \delta\}. \end{aligned}$$

Since $N : (0, \infty) \to \mathbb{R}$ is left-continuous and assumes its value in $\mathbb{Z}$, if δ is slightly less than r, $N(\delta) = N(r)$. Together with the monotonicity of the most-right hand side of the above equality, we conclude

$$B(\mathbf{x}, r) = \{\mathbf{y} = \{y_j\}_{j=1}^\infty \in X : |y_1 - x_1| < r,\ |y_2 - x_2| < r,\ \cdots,\ |y_{N(r)} - x_{N(r)}| < r\}.$$

Consequently, (2.6) was proved. □

The measure is given by way of product:

Definition 2.6 (*Definition of the measure*) Let $\mathcal{H}^1$ denote the 1 dimensional Hausdorff measure.

1. One defines a function $w_0 : X_0 \to [0, \infty)$ by $w_0 \equiv \gamma \sum_{k=0}^{\infty} \frac{1}{(k!)^k} \chi_{A_k}$, where γ is chosen so that $\int_{X_0} w_0(z)\, d\mathcal{H}^1(z) = 1$.
2. Define a measure on X_0 by $\mu_0 \equiv w_0\, d\mathcal{H}^1$.
3. One defines a measure μ on X by $\mu \equiv \mu_0 \times \mu_0 \times \cdots = w_0\, d\mathcal{H}^1 \times w_0\, d\mathcal{H}^1 \times \cdots$.

As for the measures μ and μ_0, we have the following relations.

Lemma 2.7 *For $k \in \mathbb{N} \cup \{0\}$, $\mu_0(A_k)$ and $\mu_0\left(\bigcup_{j=k}^{\infty} A_j\right)$ are comparable in the following sense:*

$$\frac{2\pi\gamma}{(3 \cdot k!)^k} = \mu_0(A_k) \le \mu_0\left(\bigcup_{j=k}^{\infty} A_j\right) \le \frac{2\pi\gamma}{(3 \cdot k!)^k}\left(1 + \frac{1}{\gamma \cdot (k+1)!}\right). \tag{2.7}$$

Proof From the definition of μ_0, we see

$$\mu_0(A_k) = \int_{A_k} w_0(z)\, d\mathcal{H}^1(z) = \frac{\gamma}{(k!)^k} \int_{|z|=3^{-k}} d\mathcal{H}^1(z) = \frac{2\pi\gamma}{(3 \cdot k!)^k}$$

and hence

$$\mu_0\left(\bigcup_{j=k}^{\infty} A_j\right) = \sum_{j=k}^{\infty} \frac{2\pi\gamma}{(3 \cdot j!)^j} = \frac{2\pi\gamma}{(3 \cdot k!)^k} \sum_{j=k}^{\infty} \frac{(3 \cdot k!)^k}{(3 \cdot j!)^j}. \tag{2.8}$$

This observation yields the lower bound for $\mu_0\left(\bigcup_{j=k}^{\infty} A_j\right)$. It remains to obtain the upper bound for $\mu_0\left(\bigcup_{j=k}^{\infty} A_j\right)$.

First of all, let us assume that $k = 0$. Note that $\mu_0(A_0) = 2\pi\gamma$. Hence, when $k = 0$, from (2.8), we deduce

$$\mu_0\left(\bigcup_{j=k}^{\infty} A_j\right) = \mu_0\left(\bigcup_{j=0}^{\infty} A_j\right) = 1 \le 2\pi\gamma\left(1 + \frac{1}{\gamma}\right).$$

Note that $\mu_0(A_1) = \frac{2\pi\gamma}{3}$. Hence, when $k = 1$, again from (2.8), we deduce

$$\mu_0\left(\bigcup_{j=k}^{\infty} A_j\right) \le \mu_0\left(\bigcup_{j=0}^{\infty} A_j\right) = 1 \le \frac{2\pi\gamma}{3}\left(1 + \frac{1}{2\gamma}\right). \tag{2.9}$$

Let $k \in \mathbb{N} \cap [2, \infty)$ below. We calculate

$$\mu_0\left(\bigcup_{j=k}^{\infty} A_j\right) = \frac{2\pi\gamma}{(3 \cdot k!)^k}\left(1 + \sum_{j=k+1}^{\infty} \frac{(3 \cdot k!)^k}{(3 \cdot j!)^j}\right) \le \frac{2\pi\gamma}{(3 \cdot k!)^k}\left(1 + \sum_{j=k+1}^{\infty} \frac{(3 \cdot k!)^k}{(3 \cdot j!)^k (3 \cdot j!)}\right).$$

If $j \ge k+1$, then $k! \times j \le j!$. Thus, we have

$$\mu_0\left(\bigcup_{j=k}^{\infty} A_j\right) \le \frac{2\pi\gamma}{(3 \cdot k!)^k}\left(1 + \sum_{j=k+1}^{\infty} \frac{1}{3 \cdot j^k (k+1)!}\right).$$

Note that $\gamma \in (0, 1)$ and that

$$\sum_{j=k+1}^{\infty} \frac{1}{j^k} \leq \sum_{j=2}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6} - 1 < 1$$

for all $k \in \mathbb{N} \cap [2, \infty)$. Thus, we obtain

$$\mu_0 \left(\bigcup_{j=k}^{\infty} A_j \right) \leq \frac{2\pi\gamma}{(3 \cdot k!)^k} \left(1 + \frac{1}{(k+1)!} \right) \leq \frac{2\pi\gamma}{(3 \cdot k!)^k} \left(1 + \frac{1}{\gamma \cdot (k+1)!} \right).$$

Thus, (2.7) was proved. □

Lemma 2.8 *Let* $r > 0$ *and* $\mathbf{x} = (x_1, x_2, \cdots, x_N, \cdots) \in X$. *Then*

$$\mu(B(\mathbf{x}, r)) = \prod_{j=1}^{N(r)} \mu_0(\Delta(x_j, r)). \tag{2.10}$$

Proof This is an easy consequence of (2.6). □

Lemma 2.9 *Let* $\nu > 0$ *be fixed. The sequence* $\{3^{\nu k} \mu(B(\mathbf{O}, 3^{-k}))\}_{k=1}^{\infty}$ *is almost decreasing, that is, there exists a constant* $C > 0$ *such that*

$$3^{\nu l} \mu(B(\mathbf{O}, 3^{-l})) \leq C 3^{\nu k} \mu(B(\mathbf{O}, 3^{-k}))$$

for all $k, l \in \mathbb{N}$ *with* $l \geq k$.

Proof In view of (2.6), we have

$$B(\mathbf{O}, 3^{-k}) \supset (A_{k+1})^{N(3^{-k})} \times X_0 \times X_0 \times \cdots$$

and hence, from (2.7) and Lemma 2.5, we deduce

$$\left(\frac{2\pi\gamma}{(3 \cdot (k+1)!)^{k+1}} \right)^{N(3^{-k})} \leq \mu(B(\mathbf{O}, 3^{-k})) = \left(\sum_{j=k+1}^{\infty} \frac{2\pi\gamma}{(3 \cdot j!)^j} \right)^{N(3^{-k})}. \tag{2.11}$$

Meanwhile, from the right inequality of (2.7), we have

$$\mu(B(\mathbf{O}, 3^{-k})) \leq \left(\frac{2\pi\gamma}{(3 \cdot (k+1)!)^{k+1}} \right)^{N(3^{-k})} \left(1 + \frac{1}{\gamma \cdot (k+2)!} \right)^{N(3^{-k})}.$$

Since $(k+2)!$ grows much faster than $N(3^{-k}) = \max(1, [\log_{N_0} 3^k])$, we see from (2.11) that

$$\left(\frac{2\pi\gamma}{(3 \cdot (k+1)!)^{k+1}} \right)^{N(3^{-k})} \leq \mu(B(\mathbf{O}, 3^{-k})) \leq C \left(\frac{2\pi\gamma}{(3 \cdot (k+1)!)^{k+1}} \right)^{N(3^{-k})}. \tag{2.12}$$

Therefore, from (2.12), instead of considering $\{3^{\nu k} \mu(B(\mathbf{O}, 3^{-k}))\}_{k=1}^{\infty}$ directly, we can deal with

$$\left\{ 3^{\nu k} \left(\frac{2\pi\gamma}{(3 \cdot (k+1)!)^{k+1}} \right)^{N(3^{-k})} \right\}_{k=1}^{\infty} = \left\{ 3^{\nu k} \left(\frac{2\pi\gamma}{(3 \cdot (k+1)!)^{k+1}} \right)^{\max(1, [(\log_{N_0} 3) k])} \right\}_{k=1}^{\infty}.$$

For this case, it is not so hard to see that this sequence is almost decreasing. □

The next lemma is an easy consequence of a simple geometric observation and the inequality $\sin^{-1} r \geq r$ for $r \in \left(0, \frac{\pi}{2}\right)$.

Lemma 2.10 *For all* $r \in (0, 2)$,[2]

$$\mathcal{H}^1(\{(x,y) \in \mathbf{R}^2 \,:\, x^2+y^2=1\} \cap \{(x,y) \in \mathbf{R}^2 \,:\, (x-1)^2+y^2 \leq r^2\}) = 4\sin^{-1}\frac{r}{2} \geq 2r. \tag{2.13}$$

Lemma 2.11 *For all* $\mathbf{x} \in X$ *and* $r > 0$, *we have*

$$\mu(B(\mathbf{O}, r)) \leq \mu(B(\mathbf{x}, 10r)).$$

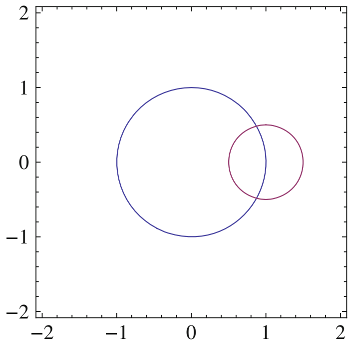
Proof Let us write $\mathbf{x} = (x_1, x_2, \cdots, x_N, \cdots) \in X$. In view of (2.10), we have

$$\mu(B(\mathbf{O}, r)) = \prod_{j=1}^{N(r)} \mu_0(\Delta(0, r)) \text{ and } \mu(B(\mathbf{x}, 10r)) = \prod_{j=1}^{N(10r)} \mu_0(\Delta(x_j, 10r)).$$

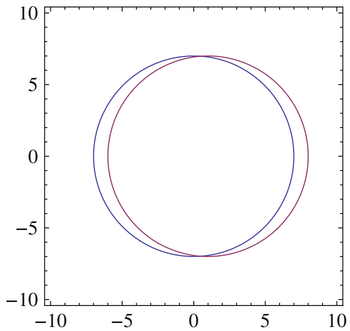
For the definition of $\Delta(z, r)$ see Definition 2.1. Now that $N(r) \geq N(10r)$ and μ_0 is a probability measure, it suffices to prove

$$\mu_0(\Delta(0, r)) \leq \mu_0(\Delta(z, 10r)) \tag{2.14}$$

[2] (a) The left circle is $x^2 + y^2 = 1$ and the right circle is $(x-1)^2 + y^2 = 1/4$.



Let us observe that the length of the set $\{x^2 + y^2 = 1,\ (x-1)^2 + y^2 < r^2\}$ grows linearly when r is small enough.
(b) The left circle is $x^2 + y^2 = 49$ and the right circle is $(x-1)^2 + y^2 = 49$.



Since the left circle is large enough, the intersection of the large disk and the small one is sufficiently large.

for all $z \in A_k$ for some $k = 0, 1, 2, \cdots$ and $r > 0$.

Assume first that $r > \frac{1}{9} \cdot 3^{-k}$. Then, since $|z| = 3^{-k}$, a geometric observation shows

$$\Delta(0, r) \subset \Delta(z, 10r). \tag{2.15}$$

This shows (2.14). Assume that $0 < r \leq \frac{1}{9} \cdot 3^{-k}$. Then, from the equality

$$\frac{\mu_0(\Delta(z, r) \cap A_k)}{\mu_0(A_k)} = \frac{\mu_0(\Delta(3^k z, 3^k r) \cap A_0)}{\mu_0(A_0)} = \frac{\mu_0(\Delta(1, 3^k r) \cap A_0)}{\mu_0(A_0)}$$

and Lemma 2.10 [see (2.13)], we deduce

$$\frac{\mu_0(\Delta(z, r) \cap A_k)}{\mu_0(A_k)} \geq \frac{2 \cdot 3^k r}{\mathcal{H}^1(A_0)} = \frac{3^k r}{\pi}.$$

Hence it follows from Lemma 2.7 that

$$\mu_0(\Delta(z, 10r)) \geq \mu_0(\Delta(z, r) \cap A_k) \geq \frac{3^k \mu_0(A_k)}{\pi} r = \frac{2 \cdot 3^k \gamma}{(3 \cdot k!)^k} r = \frac{2\gamma}{k!^k} r. \tag{2.16}$$

It follows from the definition of A_l that

$$\mu_0(\Delta(0, r)) = \mu_0 \left(\bigcup_{l > -\log_3 r} A_l \right).$$

Let $l \in \mathbf{N}$. Then $l > -\log_3 r$ if and only if $l \geq [1 - \log_3 r]$. Hence, from Lemma 2.7,

$$\mu_0(\Delta(0, r)) = \sum_{l > -\log_3 r} \frac{2\pi\gamma}{(3 \cdot l!)^l} \leq \sum_{l \geq [1 - \log_3 r]} \frac{2\pi\gamma}{3^l \cdot ([1 - \log_3 r]!)^{[1 - \log_3 r]}}.$$

If we calculate the geometric series, then we obtain

$$\mu_0(\Delta(0, r)) \leq \frac{3\pi\gamma}{3^{[1 - \log_3 r]} \cdot ([1 - \log_3 r]!)^{[1 - \log_3 r]}} < \frac{3\pi\gamma r}{([1 - \log_3 r]!)^{[1 - \log_3 r]}}. \tag{2.17}$$

Consequently, we deduce, from $0 < r \leq \frac{1}{9} \cdot 3^{-k}$, that is, $1 - \log_3 r \geq k + 1 + \log_3 9 = k + 3$, (2.16) and (2.17),

$$\mu_0(\Delta(z, 10r)) \geq \frac{2\gamma r}{k!^k} \geq \frac{((k+3)!)^{k+3}}{k!^k} \frac{2\gamma r}{([1 - \log_3 r]!)^{[1 - \log_3 r]}} > \mu_0(\Delta(0, r)). \tag{2.18}$$

Putting (2.15) and (2.18) together, we obtain (2.14) and the proof is complete. □

We specify the natural number N_0 in Definition 2.3 by

$$N_0 = 3^{2a} \tag{2.19}$$

for some $a \in \mathbb{N}$ large enough. As long as $b \in [1, 9]$ and k is an odd multiple of a, $\log_{N_0} b \cdot 3^{-k}$ and $\log_{N_0} 3^{-k}$ have the same integer part since we have (2.19).

Lemma 2.12 *There exists a constant $C > 0$ such that $\mu(B(\mathbf{O}, 2.2 \times 3^{-k})) \leq C\mu(S_k)$ for all $k \in \mathbb{N}$ such that k is an odd multiple of a.*

Proof Let us write $B(\mathbf{O}, 2.2 \times 3^{-k})$ out in full by using (2.6):

$$B(\mathbf{O}, 2.2 \times 3^{-k}) \\ = \{\mathbf{x}=\{x_j\}_{j=1}^{\infty} \in X \ : \ |x_1| < 2.2 \times 3^{-k}, \ |x_2| < 2.2\times 3^{-k}, \cdots, |x_{N(2.2\times 3^{-k})}| < 2.2 \times 3^{-k}\}.$$

Thus, from the definition of A_k, we have

$$\begin{aligned} B(\mathbf{O}, 2.2 \times 3^{-k}) &= \{\mathbf{x}=\{x_j\}_{j=1}^{\infty} \in X \ : \ |x_1| \le 3^{-k}, \ |x_2| \le 3^{-k}, \cdots, |x_{N(2.2\times 3^{-k})}| \le 3^{-k}\} \\ &= \left(\bigcup_{j=k}^{\infty} A_j\right)^{N(2.2\times 3^{-k})} \times X_0 \times X_0 \times \cdots . \end{aligned}$$

Hence, it follows from (2.7) that

$$\mu(B(\mathbf{O}, 2.2 \times 3^{-k})) \le \left(\frac{2\pi\gamma}{(3\cdot k!)^k}\right)^{N(2.2\times 3^{-k})} \left(1 + \frac{1}{\gamma\cdot(k+1)!}\right)^{N(2.2\times 3^{-k})}.$$

Now that $(k+1)!$ grows much faster than $N(2.2\times 3^{-k}) = \max(1, [(\log_{N_0} 3)k - \log_{N_0} 2.2])$, we have

$$\mu(B(\mathbf{O}, 2.2 \times 3^{-k})) \le C \left(\frac{2\pi\gamma}{(3\cdot k!)^k}\right)^{N(2.2\times 3^{-k})}. \tag{2.20}$$

Meanwhile, from (2.1) and (2.7), we deduce

$$\mu(S_k) = \prod_{j=1}^{N(3^{-k})} \left(\frac{2\pi\gamma}{(3\cdot k!)^k}\right) = \left(\frac{2\pi\gamma}{(3\cdot k!)^k}\right)^{N(3^{-k})}. \tag{2.21}$$

Since k/a is an odd integer, $N(3^{-k}) = N(2.2 \times 3^{-k})$. We thus deduce the desired result from (2.20) and (2.21). □

The next lemma concerns the norm estimates of χ_{S_k}.

Lemma 2.13 *Let k be an odd multiple of a. Then, equivalence*

$$\|\chi_{S_k}\|_{L^{p,2,\nu}(\mu)} = \sup_{\mathbf{x}\in X,\, r\in(0,2)} r^{\nu/p}\left(\frac{1}{\mu(B(\mathbf{x},2r))}\int_{B(\mathbf{x},r)} \chi_{S_k}(\mathbf{y})\,d\mu(\mathbf{y})\right)^{1/p} \sim 3^{-\nu k/p} \tag{2.22}$$

holds, where the implicit constant in $\sim$ *is independent of k.*

Proof The lower bound of $\|\chi_k\|_{L^{p,2,\nu}(\mu)}$ is a consequence of Lemma 2.12: It is easy, from Lemma 2.12, to see that

$$3^{-\nu k/p} \le C3^{-\nu k/p}\left(\frac{1}{\mu(B(\mathbf{O}, 2.2\times 3^{-k}))}\int_{B(\mathbf{O},1.1\times 3^{-k})} \chi_{S_k}(\mathbf{y})\,d\mu(\mathbf{y})\right)^{1/p}.$$

By using sup, we have

$$3^{-\nu k/p} \le C \sup_{\mathbf{x}\in X,\, r\in(0,2)} r^{\nu/p}\left(\frac{1}{\mu(B(\mathbf{x},2r))}\int_{B(\mathbf{x},r)} \chi_{S_k}(\mathbf{y})\,d\mu(\mathbf{y})\right)^{1/p}.$$

Consequently, (2.22) will have been established once we prove

$$\sup_{x \in X,\, r>0} r^{\nu/p} \left(\frac{1}{\mu(B(\mathbf{x},2r))} \int_{B(\mathbf{x},r)} \chi_{S_k}(\mathbf{y})\, d\mu(\mathbf{y}) \right)^{1/p} \leq C 3^{-\nu k/p}. \tag{2.23}$$

In order that S_k and $B(\mathbf{x}, r)$ intersect, we need to have $d(\mathbf{x}, \mathbf{O}) < r + 3^{-k}$. In this case we have either $d(\mathbf{x}, \mathbf{O}) < 2r$ or $d(\mathbf{x}, \mathbf{O}) < 2 \cdot 3^{-k}$. Actually, we distinguish two cases by setting

$$\mathrm{I} := r^{\nu/p} \left(\frac{1}{\mu(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} \chi_{S_k}(\mathbf{y})\, d\mu(\mathbf{y}) \right)^{1/p}.$$

Let us suppose $r \leq 3^{6-k}$. Then we have

$$\mathrm{I} \leq r^{\nu/p} \leq 729^{\nu/p} \cdot 3^{-\nu k/p}. \tag{2.24}$$

Let us suppose $r > 3^{6-k}$ instead. Then we have

$$\mathrm{I} \leq r^{\nu/p} \left(\frac{1}{\mu(B(\mathbf{O},r/10))} \int_{B(\mathbf{x},r)} \chi_{S_k}(\mathbf{y})\, d\mu(\mathbf{y}) \right)^{1/p} \leq r^{\nu/p} \left(\frac{\mu(S_k)}{\mu(B(\mathbf{O},r/10))} \right)^{1/p}$$

from Lemma 2.11. Thus, by choosing an integer $m \leq k$ so that $3^{6-m} < r \leq 3^{7-m}$, by virtue of Lemma 2.9 with (l, k) replaced by $(m-1, k-1)$, we obtain

$$\mathrm{I} \leq C (3^{1-m})^{\nu/p} \left(\frac{\mu(S_k)}{\mu(B(\mathbf{O},3^{1-m}))} \right)^{1/p} \leq C (3^{1-k})^{\nu/p} \left(\frac{\mu(B(\mathbf{O},3^{1-k}))}{\mu(B(\mathbf{O},3^{1-k}))} \right)^{1/p} = C 3^{-\nu k/p}. \tag{2.25}$$

In view of (2.24) and (2.25), we obtain (2.23). □

2.2 Conclusion of the proof of Theorem 1.1

As we announced before, we shall now show that there does exist a separable metric space (X, d, μ) such that $L^{p,4,\nu}(\mu)$ and $L^{p,2,\nu}(\mu)$ do not coincide as sets. Let us consider the norms of χ_{S_k}. It suffices to show that

$$\limsup_{k \to \infty} \frac{\|\chi_{S_k}\|_{L^{p,2,\nu}(\mu)}}{\|\chi_{S_k}\|_{L^{p,4,\nu}(\mu)}} = \infty. \tag{2.26}$$

More precisely

$$\lim_{k \to \infty} \frac{\|\chi_{S_{a+2ak}}\|_{L^{p,2,\nu}(\mu)}}{\|\chi_{S_{a+2ak}}\|_{L^{p,4,\nu}(\mu)}} = \infty. \tag{2.27}$$

Let $B(\mathbf{x}, r) = B(\{x_j\}_{j=1}^{\infty}, r)$ be a ball such that $B(\mathbf{x}, r)$ meets S_k at a point $\mathbf{y}$, that is, $\mathbf{y} \in S_k \cap B(\mathbf{x}, r)$. We distinguish three cases assuming that k is an odd multiple of a.

Case 1 Assume first that $3^{-k+2}/10 \leq r \leq 3^{-k+6}$. Then $d(\mathbf{x}, \mathbf{y}) < r$ implies

$$\frac{\mu(B(\mathbf{x},r) \cap S_k)}{\mu(B(\mathbf{x},4r))} \leq \frac{\mu(S_k)}{\mu(B(\mathbf{y},3r) \cap S_{k-1})} \leq \frac{\mu(S_k)}{\mu(B(\mathbf{y},2.7 \cdot 3^{-k}) \cap S_{k-1})}.$$

Meanwhile let us set

$$\theta \equiv \frac{\mathcal{H}^1(\{x^2+y^2=3^2,\ (x-1)^2+y^2 \le (2.7)^2\})}{6\pi} \in (0,1).$$

Since k is an odd multiple of a, we have $N(3^{-k}) = N(2.7\cdot 3^{-k}) = N(3^{1-k})$.

Recall that $\mathbf{y} \in S_k$. A geometric observation shows that

$$\begin{aligned}
&\mu(B(\mathbf{y}, 2.7\cdot 3^{-k}) \cap S_{k-1})\\
&\quad = \mu(\{\mathbf{x} = \{x_j\}_{j=1}^{\infty} \in S_{k-1} \,:\, |x_j - y_j| < 2.7\cdot 3^{-k} \text{ for all } j \le N(3^{-k})\})\\
&\quad = \prod_{j=1}^{N(3^{-k})} \mu_0(\{x_j \in A_{k-1} \,:\, |x_j - y_j| < 2.7\cdot 3^{-k}\})\\
&\quad = (\mu_0(\{x_1 \in A_{k-1} \,:\, |x_1 - y_1| < 2.7\cdot 3^{-k}\}))^{N(3^{-k})}\\
&\quad = (\theta\mu_0(A_{k-1}))^{N(3^{-k})}\\
&\quad = \theta^{N(3^{-k})}\mu(S_{k-1}).
\end{aligned}$$

Hence, from (2.21), we deduce

$$\frac{\mu(B(\mathbf{x},4r))}{\mu(B(\mathbf{x},r)\cap S_k)} \ge \frac{\theta^{N(3^{-k})}\mu(S_{k-1})}{\mu(S_k)} = \left(3\theta(k-1)!k^k\right)^{N(3^{-k})} \to \infty$$

as $k \to \infty$. Consequently, since $\|\chi_{S_k}\|_{L^{p,2,\nu}(\mu)} \sim 3^{-\nu k/p}$ from Lemma 2.13, we have

$$\begin{aligned}
r^{\nu/p}\left(\frac{\mu(B(\mathbf{x},r)\cap S_k)}{\mu(B(\mathbf{x},4r))}\right)^{1/p} &\le \frac{C}{\sqrt[p]{\left(3\theta(k-1)!k^k\right)^{N(3^{-k})}}}3^{-\nu k/p}\\
&\le \frac{C}{\sqrt[p]{\left(3\theta(k-1)!k^k\right)^{N(3^{-k})}}}\|\chi_{S_k}\|_{L^{p,2,\nu}(\mu)}.
\end{aligned} \tag{2.28}$$

Case 2 If $3^{-k+2}/10 \ge r$, then we use

$$\frac{\mu(B(\mathbf{x},r)\cap S_k)}{\mu(B(\mathbf{x},4r))} \le \frac{\mu(B(\mathbf{x},r)\cap S_k)}{\mu(B(\mathbf{x},4r)\cap S_k)} \le \frac{\mu(B(\mathbf{y},2r)\cap S_k)}{\mu(B(\mathbf{y},3r)\cap S_k)},$$

which follows from a geometric observation. Now we go into the structure of the measure μ; if we insert the definition of the measure μ, then we obtain

$$\begin{aligned}
&\frac{\mu(B(\mathbf{x},r)\cap S_k)}{\mu(B(\mathbf{x},4r))}\\
&= \left(\prod_{j=1}^{N(3^{-k})}\mu_0(\Delta(x_j,r)\cap A_k)\right)\left(\prod_{j=N(3^{-k})+1}^{N(r)}\mu_0(\Delta(x_j,r))\right)\left(\prod_{j=1}^{N(4r)}\mu_0(\Delta(x_j,4r))\right)^{-1}\\
&\le \left(\prod_{j=1}^{N(3^{-k})}\mu_0(\Delta(x_j,r)\cap A_k)\right)\left(\prod_{j=N(3^{-k})+1}^{N(r)}\mu_0(\Delta(x_j,r))\right)\left(\prod_{j=1}^{N(r)}\mu_0(\Delta(x_j,4r))\right)^{-1}\\
&\le \left(\prod_{j=1}^{N(3^{-k})}\mu_0(\Delta(x_j,r)\cap A_k)\right)\left(\prod_{j=1}^{N(3^{-k})}\mu_0(\Delta(x_j,4r))\right)^{-1}
\end{aligned}$$

$$
\begin{aligned}
&\le \left(\frac{\mu_0(\Delta(y_1,2r)\cap A_k)}{\mu_0(\Delta(y_1,3r)\cap A_k)}\right)^{N(3^{-k})}\\
&= \left(\frac{\mathcal{H}^1(\Delta(y_1,2r)\cap A_k)}{\mathcal{H}^1(\Delta(y_1,3r)\cap A_k)}\right)^{N(3^{-k})}.
\end{aligned}
$$

Now we consider a transform given by $z\in A_k \mapsto 3^k z\in A_k$ and we deduce

$$
\begin{aligned}
\frac{\mu(B(\mathbf{x},r)\cap S_k)}{\mu(B(\mathbf{x},4r))} &\le \left(\frac{\mathcal{H}^1(\Delta(3^k y_1, 2\cdot 3^k r)\cap A_0)}{\mathcal{H}^1(\Delta(3^k y_1, 3^{k+1} r)\cap A_0)}\right)^{N(3^{-k})}\\
&\le \left(\sup_{r\le 0.9}\left[\frac{\mathcal{H}^1(\Delta(1,2r)\cap A_0)}{\mathcal{H}^1(\Delta(1,3r)\cap A_0)}\right]\right)^{N(3^{-k})}.
\end{aligned}
$$

Consequently, since $\|\chi_{S_k}\|_{L^{p,2,\nu}(\mu)}\sim 3^{-\nu k/p}$ and $r\le 0.9\cdot 3^{-k}$, we have

$$
\begin{aligned}
&r^{\nu/p}\left(\frac{\mu(B(\mathbf{x},r)\cap S_k)}{\mu(B(\mathbf{x},4r))}\right)^{1/p}\\
&\le C\|\chi_{S_k}\|_{L^{p,2,\nu}(\mu)}\left(\sup_{r\le 0.9}\left[\frac{\mathcal{H}^1(\Delta(1,2r)\cap A_0)}{\mathcal{H}^1(\Delta(1,3r)\cap A_0)}\right]\right)^{N(3^{-k}))/p}. \qquad (2.29)
\end{aligned}
$$

Case 3 Finally, assume that $r\ge 3^{-k+6}$. Choose $l\le k$ so that $3^{6-l}\le r<3^{7-l}$. Then we have, from Lemma 2.11, we deduce

$$
\begin{aligned}
r^{\nu/p}\left(\frac{\mu(B(\mathbf{x},r)\cap S_k)}{\mu(B(\mathbf{x},4r))}\right)^{1/p} &\le r^{\nu/p}\left(\frac{\mu(S_k)}{\mu(B(\mathbf{O},4r/10))}\right)^{1/p}\\
&\le 3^{(7-l)\nu/p}\left(\frac{\mu(S_k)}{\mu(B(\mathbf{O},3^{-l+5}))}\right)^{1/p}.
\end{aligned}
$$

It follows from Lemma 2.9 and the fact that $\|\chi_{S_k}\|_{L^{p,2,\nu}(\mu)}\sim 3^{-k\nu/p}$ that

$$
\begin{aligned}
r^{\nu/p}\left(\frac{\mu(B(\mathbf{x},r)\cap S_k)}{\mu(B(\mathbf{x},4r))}\right)^{1/p} &\le C3^{-k\nu/p}\left(\frac{\mu(S_k)}{\mu(B(\mathbf{O},3^{2-k}))}\right)^{1/p}\\
&\le C\|\chi_{S_k}\|_{L^{p,2,\nu}(\mu)}\left(\frac{\mu(S_k)}{\mu(S_{k-1})}\right)^{1/p}.
\end{aligned}
$$

We have

$$
\begin{aligned}
r^{\nu/p}\left(\frac{\mu(B(\mathbf{x},r)\cap S_k)}{\mu(B(\mathbf{x},4r))}\right)^{1/p} &\le C\|\chi_{S_k}\|_{L^{p,2,\nu}(\mu)}\left(\frac{\theta^{N(3^{-k})}}{\left(3\theta(k-1)!k^k\right)^{N(3^{-k})}}\right)^{1/p}\\
&\le \frac{C}{\sqrt[p]{\left(3\theta(k-1)!k^k\right)^{N(3^{-k})}}}\|\chi_{S_k}\|_{L^{p,2,\nu}(\mu)}. \qquad (2.30)
\end{aligned}
$$

Inequalities (2.28) – (2.30) yield

$$
\begin{aligned}
&\|\chi_{S_k}\|_{L^{p,4,\nu}(\mu)}\\
&\le C\|\chi_{S_k}\|_{L^{p,2,\nu}(\mu)}\left\{\frac{1}{\sqrt[p]{\left(3\theta(k-1)!k^k\right)^{N(3^{-k})}}}+\left(\sup_{r\le 0.9}\left[\frac{\mathcal{H}^1(\Delta(1,2r)\cap A_0)}{\mathcal{H}^1(\Delta(1,3r)\cap A_0)}\right]\right)^{N(3^{-k}))/p}\right\}
\end{aligned}
$$

for all k such that k is an odd multiple of a. Thus, (2.26) follows.

3 Remarks and examples

3.1 Proof of Proposition 1.4

We follow the idea in [10], [33, Proposition 1.2], [34, Proposition 2.2] and [35, Proposition 1.1].

Before we start the proof of Proposition 1.4, we need some preparatory observations. By symmetry, we can assume that $\kappa_1 > \kappa_2$. Next, since $0 < \nu_- \le \nu_+ < \infty$ and $-\infty < \beta_- \le \beta_+ < \infty$, we can find a constant K independent of x such that

$$K^{-1} r^{\nu(x)} (\log(e+1/r))^{\beta(x)} \le (2r)^{\nu(x)} (\log(e+1/2r))^{\beta(x)} \le K r^{\nu(x)} (\log(e+1/r))^{\beta(x)} \tag{3.1}$$

for all $r > 0$. Based upon these observations, we prove Proposition 1.4. We need to compare the following two conditions (3.2) and (3.3):

$$\exists \lambda_1 > 0 \quad \sup_{x \in G, 0<r<d_G} \frac{r^{\nu(x)} (\log(e+1/r))^{\beta(x)}}{\mu(B(x, \kappa_1 r))} \int_{G \cap B(x,r)} \Phi(y, |f(y)|/\lambda_1)\, d\mu(y) \le 1. \tag{3.2}$$

$$\exists \lambda_2 > 0 \quad \sup_{x \in G, 0<r<d_G} \frac{r^{\nu(x)} (\log(e+1/r))^{\beta(x)}}{\mu(B(x, \kappa_2 r))} \int_{G \cap B(x,r)} \Phi(y, |f(y)|/\lambda_2)\, d\mu(y) \le 1. \tag{3.3}$$

If (3.3) holds, then (3.2) trivially holds with $\lambda_1 = \lambda_2$. So let us suppose (3.2). We need to show that, for $x \in G$,

$$\frac{r^{\nu(x)} (\log(e+1/r))^{\beta(x)}}{\mu(B(x, \kappa_2 r))} \int_{G \cap B(x,r)} \Phi(y, |f(y)|/\lambda_2)\, d\mu(y) \le 1$$

for some $\lambda_2 = N^* \lambda_1$, where N^* is independent of f, x and r. We decompose $B(x,r)$ into N small balls $B(x_1, s), B(x_2, s), \ldots, B(x_N, s)$, so that

$$s \le r, \; B(x,r) \subset \bigcup_{j=1}^{N} B(x_j, s), \; B(x, \kappa_2 r) \supset \bigcup_{j=1}^{N} B(x_j, \kappa_1 s), \tag{3.4}$$

where N depends only on κ_1 and κ_2. Observe that (3.4) shows that s and r satisfy

$$r \le 2^{m_0} s \tag{3.5}$$

for some constant m_0 depending only on κ_1 and κ_2. Then, from (3.1), (3.4) and (3.5), we have

$$\begin{aligned}
&\frac{r^{\nu(x)} (\log(e+1/r))^{\beta(x)}}{\mu(B(x, \kappa_2 r))} \int_{G \cap B(x,r)} \Phi(y, |f(y)|/\lambda_1)\, d\mu(y) \\
&\le \sum_{j=1}^{N} \frac{r^{\nu(x_j)} (\log(e+1/r))^{\beta(x_j)}}{\mu(B(x, \kappa_2 r))} \int_{G \cap B(x_j,s)} \Phi(y, |f(y)|/\lambda_1)\, d\mu(y) \\
&\le \sum_{j=1}^{N} \frac{r^{\nu(x_j)} (\log(e+1/r))^{\beta(x_j)}}{\mu(B(x_j, \kappa_1 s))} \int_{G \cap B(x_j,s)} \Phi(y, |f(y)|/\lambda_1)\, d\mu(y)
\end{aligned}$$

$$\leq K^{m_0} \sum_{j=1}^{N} \frac{s^{\nu(x_j)}(\log(e+1/s))^{\beta(x_j)}}{\mu(B(x_j,\kappa_1 s))} \int_{G\cap B(x_j,s)} \Phi(y,|f(y)|/\lambda_1)\,d\mu(y)$$
$$\leq K^{m_0} N. \tag{3.6}$$

By virtue of (3.6) and the convexity of $\Phi(y,\cdot)$, (3.3) holds with $\lambda_2 = K^{m_0} N \lambda_1$.

3.2 Other examples of non-doubling metric measure spaces

The doubling condition had been playing a key role in harmonic analysis. However, non-doubling measure spaces occur very naturally in many branches of mathematics. The typical examples we envisage are the following ones:

Example 3.1 Let $B(1) \equiv \{(x_1, x_2, \cdots, x_n) \in \mathbf{R}^n \,:\, x_1^2 + x_2^2 + \cdots + x_n^2 < 1\}$ be the unit ball in $\mathbf{R}^n$. Equip $B(1)$ with a metric given by

$$g \equiv 4\frac{dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + \cdots + dx_n \otimes dx_n}{1 - {x_1}^2 - {x_2}^2 - \cdots - {x_n}^2}.$$

Then $(B(1), g)$ is called the space with constant curvature -1 and if we denote by μ the induced measure, then $\mu(B(x,r))$ grows exponentially.

Example 3.2 Equip the Euclidean space $(\mathbf{R}^n, |\cdot_1 - \cdot_2|)$ with a measure $d\mu = \pi^{-n/2}\exp(-|x|^2)$. Then $(\mathbf{R}^n, |\cdot_1 - \cdot_2|, \mu)$ is called the Gauss measure space and the operator

$$L = -\Delta + (x \cdot \nabla)$$

is a self-adjoint operator on $L^2(\mu)$. Recently, the first author, Liguang Liu and Dachun Yang considered Morrey spaces in [16]. Let us set

$$\mathcal{B}_a \equiv \{B(x,r) \,:\, r \leq a \min(1, |x|^{-1})\}$$

be the set of locally doubling balls. Recently, in [16] the first author, Liguang Liu and Dachun Yang considered Morrey spaces given by

$$\|f\|_{\mathcal{M}^{p,q}_{\mathcal{B}_a}(\mu)} \equiv \sup_{B\in\mathcal{B}_a} \frac{1}{[\mu(B)]^{1/q-1/p}} \left\{ \int_B |f(y)|^q \, d\mu(y) \right\}^{1/q} < \infty.$$

In [16, Proposition 2.6], the space $\mathcal{M}^{p,q}_{\mathcal{B}_a}(\mu)$ is not depend upon the parameter $a > 0$. But unfortunately we cannot realize $\mathcal{M}^{p,q}_{\mathcal{B}_a}(\mu)$ by adjusting parameters.

Example 3.3 The attractors of a dynamical system can have non-doubling Hausdorff measures.

4 An estimate of the modified centered Hardy–Littlewood maximal operator M_{16}

In Sect. 4 we work on a bounded open set G and we write d_G for the diameter of G.

For a locally integrable function f on G, recall that in (1.6) we defined the centered and generalized Hardy–Littlewood maximal operator by

$$M_{16}f(x) = \sup_{r>0} \frac{1}{\mu(B(x,16r))} \int_{G\cap B(x,r)} |f(y)|d\mu(y) \tag{4.1}$$

for $x \in G$. Observe that

$$M_{16} f(x) = \sup_{r \in (0, d_G)} \frac{1}{\mu(B(x, 16r))} \int_{G \cap B(x,r)} |f(y)| d\mu(y), \tag{4.2}$$

since G is bounded.

In what follows, as we did in Sect. 1, if f is a function on G, then we assume that $f = 0$ outside G.

As a starting point of the present paper, we shall prove the following estimate of the centered Hardy–Littlewood maximal operator M_{16}. For the case $q = 0$, see Kokilashvili–Meskhi [15]. As a consequence of Theorem 4.1 the centered Hardy–Littlewood maximal operator M_{16} is bounded from $L^{\Phi,\nu,\beta,G;2}(G)$ to $L^{\Phi,\nu,\beta,G;4}(G)$.

Theorem 4.1 *Assume that $p(\cdot)$ and $q(\cdot)$ satisfy (P1), (P), (Q1) and (Q2) and that $\nu : X \to (0, \infty)$ and $\beta : X \to (-\infty, \infty)$ are bounded μ-measurable functions. Define Φ by (1.3). Suppose that $p_- > 1$ and that $\nu_- > 0$. Then there exists a constant $C > 0$ such that*

$$\frac{1}{\mu(B(z, 4r))} \int_{B(z,r)} M_{16} f(x)^{p(x)} (\log(e + M_{16} f(x)))^{q(x)} d\mu(x) \le C r^{-\nu(z)} (\log(e + 1/r))^{-\beta(z)}$$

for all $z \in G$, $r \in (0, d_G)$ and μ-measurable functions f with $\|f\|_{L^{\Phi,\nu,\beta;2}(G)} \le 1$.

To prove Theorem 4.1, we need several lemmas. Let us begin with the following result, which concerns an estimate for the case $p(x) \equiv p_0$ and $q(x) \equiv 0$ (cf. [21, Lemma 4.3] and [24, Lemma 2.2]).

Lemma 4.2 *Assume that $p(\cdot)$ and $\nu(\cdot)$ satisfy $p(\cdot) \equiv p_0 > 1$ and $\nu_- > 0$, respectively. Let f be a μ-measurable function on G satisfying*

$$\frac{1}{\mu(B(x, 2r))} \int_{B(x,r)} |f(y)|^{p_0} \, d\mu(y) \le r^{-\nu(x)} (\log(e + 1/r))^{-\beta(x)} \tag{4.3}$$

for all $x \in G$ and $0 < r < d_G$. Then there exists a constant $C > 0$ such that

$$\frac{1}{\mu(B(z, 4r))} \int_{B(z,r)} M_{16} f(x)^{p_0} \, d\mu(x) \le C r^{-\nu(z)} (\log(e + 1/r))^{-\beta(z)}$$

for all $z \in G$ and $0 < r < d_G$, where the constant C is independent of f satisfying (4.3).

Proof Let f satisfy (4.3), and fix $z \in G$ and $0 < r < d_G$. Write $A_0 \equiv B(z, 2r)$ and $A_j \equiv B(z, 2^{j+1}r) \setminus B(z, 2^j r)$ for each positive integer j. Based upon this partition $\{A_j\}_{j=1}^{\infty}$, we set

$$f_j \equiv f \chi_{A_j} \quad \text{for } j = 0, 1, 2, \cdots, \quad g_0 \equiv \sum_{j=1}^{\infty} |f_j|.$$

Let us set

$$\mathrm{I}_1 \equiv \int_{B(z,r)} M_{16} f_0(x)^{p_0} d\mu(x), \ \mathrm{I}_2 \equiv \int_{B(z,r)} M_{16} g_0(x)^{p_0} d\mu(x).$$

Then we have

$$\int_{B(z,r)} M_{16} f(x)^{p_0} d\mu(x) \le C(\mathrm{I}_1 + \mathrm{I}_2).$$

By virtue of (4.3), we have

$$\mathrm{I}_1 \le C \int_X |f_0(x)|^{p_0} d\mu(x) = C \int_{B(z,2r)} |f(x)|^{p_0} d\mu(x) \le C r^{-\nu(z)} (\log(e+1/r))^{-\beta(z)} \mu(B(z,4r)).$$

The estimate for I_1 is now valid.

Let us turn to I_2. In view of the definition of f_j and A_j, we have

$$M_{16} f_j(x) \le \sup_{t \in ((2^j-1)r, (2^{j+1}+1)r)} \frac{1}{\mu(B(x,16t))} \int_{B(x,t)} |f_j(y)| d\mu(y)$$

for $x \in B(z,r)$. For $x \in B(z,r)$, we estimate the right-hand side crudely:

$$\begin{aligned} M_{16} f_j(x) &\le \frac{1}{\mu(B(x, 16(2^j-1)r)} \int_{B(x,(2^{j+1}+1)r)} |f_j(y)| d\mu(y) \\ &\le \frac{1}{\mu(B(x, 16(2^j-1)r)} \int_{B(z,(2^{j+1}+2)r)} |f_j(y)| d\mu(y) \\ &\le \frac{1}{\mu(B(z, (2^{j+4}-17)r)} \int_{B(z,(2^{j+1}+2)r)} |f_j(y)| d\mu(y). \end{aligned}$$

By the Hölder inequality and the fact that $2^{j+4} - 17 \ge 2^{j+1} + 2$ for $j = 1, 2, \cdots$, we have

$$M_{16} f_j(x) \le \left(\frac{1}{\mu(B(z,(2^{j+4}-17)r))} \int_{B(z,(2^{j+1}+2)r)} |f(y)|^{p_0} d\mu(y) \right)^{1/p_0}.$$

Since $16(2^j-1) - 1 \ge 2(2^{j+1}+2)$ for any positive integer j, we see that for $x \in B(z,r)$

$$M_{16} f_j(x) \le \left((2^{j+1}r + 2r)^{-\nu(z)} (\log(e + 1/(2^{j+1}r + 2r)))^{-\beta(z)} \right)^{1/p_0}.$$

Finally, keeping in mind that $\beta(\cdot)$ and $\nu(\cdot)$ are both assumed to be bounded, we obtain

$$M_{16} f_j(x) \le C(2^j r)^{-\nu(z)/p_0} (\log(e + 1/(2^j r)))^{-\beta(z)/p_0},$$

so that, adding this estimate over j, we obtain a pointwise estimate: for all $x \in B(z,r)$,

$$\begin{aligned} M_{16} g_0(x) &\le \sum_{j=1}^{\infty} M_{16} f_j(x) \\ &\le C \sum_{j=1}^{\infty} (2^j r)^{-\nu(z)/p_0} (\log(e + 1/(2^j r)))^{-\beta(z)/p_0} \\ &\le C r^{-\nu(z)/p_0} (\log(e + 1/r))^{-\beta(z)/p_0}. \end{aligned}$$

Integrating the above estimate over $B(z,r)$, we obtain

$$\mathrm{I}_2 \le Cr^{-\nu(z)}(\log(e+1/r))^{-\beta(z)} \int_{B(z,r)} d\mu(x) = Cr^{-\nu(z)}(\log(e+1/r))^{-\beta(z)}\mu(B(z,r)).$$

Since $\mu(B(z,r)) \le \mu(B(z,4r))$, we deduce

$$\frac{1}{\mu(B(z,4r))}\int_{B(z,r)} M_{16}f(x)^{p_0}d\mu \le Cr^{-\nu(z)}(\log(e+1/r))^{-\beta(z)},$$

which proves Lemma 4.2. □

It is significant that $\nu(\cdot)$ and $\beta(\cdot)$ do not have to be continuous.

The next lemma concerns an estimate for x such that $|f(x)|$ is large. For convenience of the readers, we supply its proof the following key inequality (4.5) which is similar to the one dealt in [27].

Lemma 4.3 *Suppose $\nu_- > 0$. Let f be a non-negative μ-measurable function on G satisfying $\|f\|_{L^{\Phi,\nu,\beta;2}(G)} \le 1$ such that*

$$f(x) \ge 1 \text{ or } f(x) = 0 \tag{4.4}$$

for each $x \in G$. Define $g(y) \equiv f(y)^{p(y)}(\log(e+f(y)))^{q(y)}$ for $y \in X$. Then there exists a constant $C > 0$, independent of f, such that

$$M_{16}f(x)^{p(x)}(\log(e+M_{16}f(x)))^{q(x)} \le CM_{16}g(x) \tag{4.5}$$

for all $x \in G$.

Proof Let $x \in G$ and $r > 0$. We let

$$H \equiv H_{x,r} = \frac{1}{\mu(B(x,16r))}\int_{B(x,r)} g(y)\,d\mu(y). \tag{4.6}$$

To prove (4.5), it suffices to show that

$$\frac{1}{\mu(B(x,16r))}\int_{B(x,r)} f(y)\,d\mu(y) \le CH^{1/p(x)}(\log(e+H))^{-q(x)/p(x)} \tag{4.7}$$

for all $x \in G$ and $0 < r < d_G$ with the constant C independent of x and r. Indeed, once (4.7) is proved, if we insert (4.6) to (4.7) and consider the supremos over all admissible x and r, then we will have

$$M_{16}f(x) \le CM_{16}g(x)^{1/p(x)}(\log(e+M_{16}g(x)))^{-q(x)/p(x)}. \tag{4.8}$$

In view of the definition of g and the fact that the inverse function of $t \mapsto t^P(\log t)^Q$ with $P > 1$ and $Q \in \mathbf{R}$ is equivalent to the function $t \mapsto t^{1/P}(\log t)^{-Q/P}$, it follows that (4.8) implies the desired conclusion (4.5). So let us prove (4.7).

To show (4.7), first consider the case when $H \ge 1$. Set

$$k \equiv H^{1/p(x)}(\log(e+H))^{-q(x)/p(x)}.$$

Decompose the integral according to the set $\{f > k\}$;

$$\frac{1}{\mu(B(x,16r))}\int_{B(x,r)} f(y)\,d\mu(y)$$
$$= \frac{1}{\mu(B(x,16r))}\left(\int_{B(x,r)\cap\{0\le f\le k\}} f(y)\,d\mu(y) + \int_{B(x,r)\cap\{f>k\}} f(y)\,d\mu(y)\right). \quad (4.9)$$

Since we are assuming (Q1), we obtain

$$\frac{1}{\mu(B(x,16r))}\int_{B(x,r)} f(y)\,d\mu(y)$$
$$\le k + \frac{C}{\mu(B(x,16r))}\int_{B(x,r)} f(y)\left(\frac{f(y)}{k}\right)^{p(y)-1}\left(\frac{\log(e+f(y))}{\log(e+k)}\right)^{q(y)} d\mu(y)$$
$$= k + \frac{C}{\mu(B(x,16r))}\int_{B(x,r)} g(y)k^{-p(y)+1}(\log(e+k))^{-q(y)}d\mu(y).$$

Let $y \in B(x,r)$ be fixed. Since $\|f\|_{L^{\Phi,\nu,\beta;2}(G)} \le 1$, we have

$$H \le r^{-\nu(x)}(\log(e+1/r))^{-\beta(x)} \quad (4.10)$$

for all $x \in G$ and $0 < r < d_G$. Assuming that $d_G < \infty$ and that (P2) and (Q2) hold, we obtain by (4.10)

$$k^{-p(y)} \le Ck^{-p(x)} = CH^{-1}(\log(e+H))^{q(x)}$$

and

$$(\log(e+k))^{-q(y)} \le C(\log(e+k))^{-q(x)} \le C(\log(e+H))^{-q(x)}.$$

Consequently (4.7) follows in this case.

In the case $H \le 1$, we find

$$H \le CH^{1/p(x)}(\log(e+H))^{-q(x)/p(x)} \quad (4.11)$$

from (P1) and (Q1). In view of the assumption (4.4), we have

$$g(y) = f(y)\cdot f(y)^{p(y)-1}(\log(e+f(y)))^{q(y)} \ge Cf(y) \quad (y \in G)$$

for some $C > 0$ and hence

$$\frac{1}{\mu(B(x,16r))}\int_{B(x,r)} f(y)\,d\mu(y) \le C\frac{1}{\mu(B(x,16r))}\int_{B(x,r)} g(y)\,d\mu(y) = CH. \quad (4.12)$$

If we combine (4.11) and (4.12), we obtain (4.7) in the case $H \le 1$. □

Keeping Lemmas 4.2 and 4.3 in mind, we prove Theorem 4.1.

Proof We may assume that $f \ge 0$ by considering $|f|$ instead of f if necessary. Write

$$f = f\chi_{\{y\in X\,:\,f(y)\ge 1\}} + f\chi_{\{y\in X\,:\,f(y)<1\}} \equiv f_1 + f_2.$$

Take $p_0 \in (1, p_-)$ and define $g_1(y) \equiv f_1(y)^{p(y)/p_0}(\log(e + f_1(y)))^{q(y)/p_0}$. Set

$$\Phi^*(x, r) \equiv r^{p(x)/p_0}(\log(c_0 + r))^{q(x)/p_0}$$

for $x \in X$ and $r > 0$.

We claim that $\|f_1\|_{L^{\Phi^*,\nu,\beta;2}(G)} \le 1$. Indeed,

$$\begin{aligned}
&\frac{1}{\mu(B(x, 2r))} \int\limits_{B(x,r)} g_1(y)\, d\mu(y) \\
&= \frac{1}{\mu(B(x, 2r))} \int\limits_{B(x,r)} f_1(y)^{p(y)/p_0}(\log(e + f_1(y)))^{q(y)/p_0} d\mu(y) \\
&\le C \frac{1}{\mu(B(x, 2r))} \int\limits_{B(x,r)} f_1(y)^{p(y)}(\log(e + f_1(y)))^{q(y)}\, d\mu(y) \\
&\le C r^{-\nu(x)}(\log(e + 1/r))^{-\beta(x)}
\end{aligned}$$

for all $x \in G$ and $0 < r < d_G$. Applying Lemma 4.3 with $p(x)$ replaced by $p(x)/p_0$, we obtain

$$M_{16} f_1(x)^{p(x)/p_0}(\log(e + M_{16} f_1(x)))^{q(x)/p_0} \le C M_{16} g_1(x). \tag{4.13}$$

Since $M_{16} f_2(x) \le 1$ for all $x \in G$, it follows from (4.13) that

$$\begin{aligned}
&M_{16} f(x)^{p(x)}(\log(e + M_{16} f(x)))^{q(x)} \\
&\quad \le C \left\{ M_{16} f_1(x)^{p(x)}(\log(e + M_{16} f_1(x)))^{q(x)} + M_{16} f_2(x)^{p(x)}(\log(e + M_{16} f_2(x)))^{q(x)} \right\} \\
&\quad \le C(1 + M_{16} g_1(x)^{p_0}).
\end{aligned}$$

By Lemma 4.2 with f replaced by g_1, from the fact that $\nu_- > 0$, we see that

$$\begin{aligned}
&\frac{1}{\mu(B(z, 4r))} \int\limits_{B(z,r)} M_{16} f(x)^{p(x)}(\log(e + M_{16} f(x)))^{q(x)}\, d\mu(x) \\
&\quad \le C \frac{1}{\mu(B(z, 4r))} \int\limits_{B(z,r)} (1 + M_{16} g_1(x)^{p_0})\, d\mu(x) \\
&\quad \le C + C r^{-\nu(z)}(\log(e + 1/r))^{-\beta(z)}.
\end{aligned}$$

Assuming that $0 < r < d_G$, we obtain

$$\frac{1}{\mu(B(z, 4r))} \int\limits_{B(z,r)} M_{16} f(x)^{p(x)}(\log(e+M_{16} f(x)))^{q(x)}\, d\mu(x) \le C r^{-\nu(z)}(\log(e + 1/r))^{-\beta(z)}$$

for all $z \in G$ and $0 < r < d_G$, as required. □

5 Sobolev's inequality

In Sect. 5, we deal with the Hardy–Littlewood–Sobolev inequality for the operator defined in Sect. 1 by

$$U_{\alpha(\cdot),32} f(x) \equiv \int_G \frac{d(x,y)^{\alpha(x)} f(y)}{\mu(B(x,32d(x,y)))} d\mu(y).$$

Recall that $\alpha : X \to (0,\infty)$ and $\nu : X \to (0,\infty)$ are both bounded μ-measurable functions and that $\alpha_- > 0$ [see (1.2) above]. Throughout Sect. 5, we assume in addition that

$$\inf_{x\in X} (1/p(x) - \alpha(x)/\nu(x)) > 0. \tag{5.1}$$

In this case we have $\nu_- \geq \alpha_- > 0$ automatically.

We consider the Sobolev exponent $p^\sharp(\cdot)$ given by

$$1/p^\sharp(x) \equiv 1/p(x) - \alpha(x)/\nu(x) \quad (x \in X) \tag{5.2}$$

and the new modular function

$$\Psi(x,t) \equiv t^{p^\sharp(x)} (\log(e+t))^{p^\sharp(x)(q(x)/p(x)+\alpha(x)\beta(x)/\nu(x))} \quad [x \in X,\ t \in (0,\infty)]. \tag{5.3}$$

In Sect. 5 we shall prove the following result asserting that $U_{\alpha(\cdot),32}$ is bounded from $L^{\Phi,\nu,\beta;2}(G)$ to $L^{\Psi,\nu,\beta;4}(G)$ postulating only (1.1) on μ:

Theorem 5.1 *Assume (P1), (P2), (Q1) and (Q2) and define Ψ by (5.3) and an exponent $p^\sharp$ by (5.2). Assume in addition that $\nu : X \to (0,\infty)$ and $\beta : X \to (-\infty,\infty)$ are bounded μ-measurable functions. Then, there exists a positive constant c such that*

$$\frac{1}{\mu(B(z,4r))} \int_{B(z,r)} \Psi(x, U_{\alpha(\cdot),32} f(x)) d\mu(x) \leq c r^{-\nu(z)} (\log(e+1/r))^{-\beta(z)}$$

for all $z \in G$ and $0 < r < d_G$, whenever f is a non-negative μ-measurable function on G satisfying $\|f\|_{L^{\Phi,\nu,\beta;2}(G)} \leq 1$.

We plan to prove Theorem 5.1 by using three auxiliary estimates, keeping in mind the original proof of Hedberg [14].

The first lemma concerns the embedding property of Morrey spaces. If we let $\Theta(x,t) \equiv t$ for $x \in X$ and $t \geq 0$, then $L^{\Phi,\nu,\beta;2}(G)$ is embedded into $L^{\Theta,\nu/p,(q+\beta)/p;2}(G)$.

Lemma 5.2 (cf. [21, Lemma 2.7]) *There exists a constant $C > 0$ such that*

$$\frac{1}{\mu(B(x,2r))} \int_{B(x,r)} f(y) d\mu(y) \leq C r^{-\nu(x)/p(x)} (\log(e+1/r))^{-(q(x)+\beta(x))/p(x)}$$

for all $x \in G$, $r \in (0,d_G)$ and non-negative μ-measurable functions f satisfying

$$\|f\|_{L^{\Phi,\nu,\beta;2}(G)} \leq 1. \tag{5.4}$$

Proof Let us write $g(y) \equiv f(y)^{p(y)} (\log(e+f(y)))^{q(y)}$ as usual. We fix $x \in G$ and $r \in (0,d_G)$. For $k = r^{-\nu(x)/p(x)} (\log(e+1/r))^{-(q(x)+\beta(x))/p(x)} > 0$, as we did in (4.9), we have

$$
\begin{aligned}
&\frac{1}{\mu(B(x,2r))}\int_{B(x,r)} f(y)d\mu(y)\\
&= \frac{1}{\mu(B(x,2r))}\int_{B(x,r)\cap\{f\le k\}} f(y)d\mu(y) + \frac{1}{\mu(B(x,2r))}\int_{B(x,r)\cap\{f>k\}} f(y)d\mu(y)\\
&\le k + \frac{1}{\mu(B(x,2r))}\int_{B(x,r)} f(y)\left(\frac{f(y)}{k}\right)^{p(y)-1}\left(\frac{\log(e+f(y))}{\log(e+k)}\right)^{q(y)} d\mu(y)\\
&= k + \frac{1}{\mu(B(x,2r))}\int_{B(x,r)} g(y)k^{-p(y)+1}(\log(e+k))^{-q(y)}d\mu(y).
\end{aligned}
$$

We find by (P2) and (Q2)

$$
\begin{aligned}
&\frac{1}{\mu(B(x,2r))}\int_{B(x,r)} f(y)d\mu(y)\\
&\le k + C\frac{1}{\mu(B(x,2r))}\int_{B(x,r)} g(y)k^{-p(x)+1}(\log(e+k))^{-q(x)}d\mu(y)\\
&\le k + Ckr^{\nu(x)}(\log(e+1/r))^{\beta(x)}\frac{1}{\mu(B(x,2r))}\int_{B(x,r)} g(y)\,d\mu(y).
\end{aligned}
$$

In view of (5.4), we obtain

$$
\frac{1}{\mu(B(x,2r))}\int_{B(x,r)} f(y)d\mu(y) \le Ck = Cr^{-\nu(x)/p(x)}(\log(e+1/r))^{-(q(x)+\beta(x))/p(x)},
$$

as required. □

The next lemma concerns an estimate inside balls.

Lemma 5.3 (c.f. [24]) *If f is a non-negative μ-measurable function on G, then*

$$
\int_{B(x,\delta)} \frac{d(x,y)^{\alpha(x)}f(y)}{\mu(B(x,32d(x,y)))}d\mu(y) \le C\delta^{\alpha(x)}M_{16}f(x) \tag{5.5}
$$

for $x \in G$ and $\delta > 0$.

Proof The proof is similar to the one in [24, Lemma 2.3]. Assuming that μ does not charge a point $\{x\}$, we have

$$
\begin{aligned}
&\int_{B(x,\delta)} \frac{d(x,y)^{\alpha(x)}f(y)}{\mu(B(x,32d(x,y)))}d\mu(y)\\
&= \sum_{j=1}^{\infty}\int_{B(x,2^{-j+1}\delta)\setminus B(x,2^{-j}\delta)} \frac{d(x,y)^{\alpha(x)}f(y)}{\mu(B(x,32d(x,y)))}d\mu(y)\\
&\le \sum_{j=1}^{\infty}\int_{B(x,2^{-j+1}\delta)} \frac{(2^{-j+1}\delta)^{\alpha(x)}f(y)}{\mu(B(x,2^{-j+5}\delta))}d\mu(y)\\
&\le \sum_{j=1}^{\infty}(2^{-j+1}\delta)^{\alpha(x)}M_{16}f(x).
\end{aligned}
$$

If we use (1.2), then we see the geometric series of the most right-hand side converges and, with a constant C independent of x, we have

$$\int_{B(x,\delta)} \frac{d(x,y)^{\alpha(x)} f(y)}{\mu(B(x,32d(x,y)))} d\mu(y) \le C\delta^{\alpha(x)} M_{16} f(x).$$

Thus, Lemma 5.3 is proved. □

We get information outside a fixed ball by using Lemma 5.4 below.

Lemma 5.4 *Let f be a non-negative μ-measurable function on G such that*

$$\|f\|_{L^{\Phi,\nu,\beta;2}(G)} \le 1. \tag{5.6}$$

Then

$$\int_{G\setminus B(x,\delta)} \frac{d(x,y)^{\alpha(x)} f(y)}{\mu(B(x,32d(x,y)))} d\mu(y) \le C\delta^{\alpha(x)-\nu(x)/p(x)} (\log(e+1/\delta))^{-(q(x)+\beta(x))/p(x)} \tag{5.7}$$

for $x \in G$ and small $\delta > 0$.

Proof Let d_G be the diameter of G as before and let j_0 be the smallest integer such that $2^{j_0}\delta \ge d_G$. By Lemma 5.2 and our convention that f is 0 outside G, we have

$$\begin{aligned}
&\int_{G\setminus B(x,\delta)} \frac{d(x,y)^{\alpha(x)} f(y)}{\mu(B(x,32d(x,y)))} d\mu(y) \\
&= \sum_{j=0}^{j_0} \int_{B(x,2^{j+1}\delta)\setminus B(x,2^j\delta)} \frac{d(x,y)^{\alpha(x)} f(y)}{\mu(B(x,32d(x,y)))} d\mu(y) \\
&\le \sum_{j=0}^{j_0} (2^{j+1}\delta)^{\alpha(x)} \frac{1}{\mu(B(x,2^{j+5}\delta))} \int_{B(x,2^{j+1}\delta)} f(y)d\mu(y) \\
&\le \sum_{j=0}^{j_0} (2^{j+1}\delta)^{\alpha(x)-\nu(x)/p(x)} (\log(e+1/(2^{j+1}\delta)))^{-(q(x)+\beta(x))/p(x)}.
\end{aligned}$$

Let $\eta \equiv \inf_{x\in G}(\nu(x)/p(x) - \alpha(x))$. Then $\eta > 0$ by (5.1). Since the functions $\log\alpha, q, \beta$ are all bounded, $p_- > 1$ and the function ν is positive, we obtain

$$\begin{aligned}
&\int_{G\setminus B(x,\delta)} \frac{d(x,y)^{\alpha(x)} f(y)}{\mu(B(x,32d(x,y)))} d\mu(y) \\
&\le C \sum_{j=0}^{j_0} (2^j\delta)^{\alpha(x)-\nu(x)/p(x)} (\log(e+1/(2^j\delta)))^{-(q(x)+\beta(x))/p(x)} \\
&\le C \int_{\delta}^{2d_G} t^{\alpha(x)-\nu(x)/p(x)} (\log(e+1/t))^{-(q(x)+\beta(x))/p(x)} \frac{dt}{t}.
\end{aligned}$$

So we need to consider the integral

$$\mathrm{I}(\delta) = \int_{\delta}^{2d_G} t^{\alpha(x)-\nu(x)/p(x)} (\log(e+1/t))^{-(q(x)+\beta(x))/p(x)} \frac{dt}{t}. \tag{5.8}$$

Recalling that $\eta > 0$, we have

$$\begin{aligned}
&\int_{G\setminus B(x,\delta)} \frac{d(x,y)^{\alpha(x)} f(y)}{\mu(B(x,32d(x,y)))} d\mu(y) \\
&\quad \le C\delta^{\alpha(x)-\nu(x)/p(x)+\eta/2} (\log(e+1/\delta))^{-(q(x)+\beta(x))/p(x)} \int_{\delta}^{2d_G} t^{-\eta/2} \frac{dt}{t} \\
&\quad \le C\delta^{\alpha(x)-\nu(x)/p(x)} (\log(e+1/\delta))^{-(q(x)+\beta(x))/p(x)},
\end{aligned}$$

which completes the proof. □

With the aid of Theorem 4.1, Lemma 5.3 and Lemma 5.4, we can apply Hedberg's trick (see [14]) to obtain a Sobolev type inequality for Riesz potentials, as an extension of Adams [1, Theorem 3.1], Chiarenza and Frasca [4, Theorem 2], Sawano–Tanaka [35, Theorem 3.3] and Mizuta–Shimomura–Sobukawa [24, Theorem 2.5], Mizuta–Nakai–Ohno–Shimomura [21, Theorem 4.5] and Kokilashvili–Meskhi [15, Theorem 4.4].

Proof of Theorem 5.1 We see from Lemmas 5.3 and 5.4 that

$$\begin{aligned}
U_{\alpha(\cdot),32} f(x) &= \int_{X\setminus B(x,\delta)} \frac{d(x,y)^{\alpha(x)} f(y)}{\mu(B(x,32d(x,y)))} d\mu(y) + \int_{B(x,\delta)} \frac{d(x,y)^{\alpha(x)} f(y)}{\mu(B(x,32d(x,y)))} d\mu(y) \\
&\le C\delta^{\alpha(x)} M_{16} f(x) + C\delta^{\alpha(x)-\nu(x)/p(x)} (\log(e+1/\delta))^{-(q(x)+\beta(x))/p(x)}
\end{aligned}$$

for all $\delta > 0$. Here, we optimize the above estimate by letting

$$\delta \equiv \min\left\{d_G, M_{16} f(x)^{-p(x)/\nu(x)} (\log(e+M_{16} f(x)))^{-(q(x)+\beta(x))/\nu(x)}\right\},$$

and we have

$$\begin{aligned}
U_{\alpha(\cdot),32} f(x) &\le C\left\{1 + M_{16} f(x)^{1-\alpha(x)p(x)/\nu(x)} (\log(e+M_{16} f(x)))^{-\alpha(x)(q(x)+\beta(x))/\nu(x)}\right\} \\
&= C\left\{1 + M_{16} f(x)^{p(x)/p^*(x)} (\log(e+M_{16} f(x)))^{-\alpha(x)(q(x)+\beta(x))/\nu(x)}\right\}.
\end{aligned}$$

Then from (5.3) we find

$$\Psi(x, U_{\alpha(\cdot),32} f(x)) \le C\left\{1 + M_{16} f(x)^{p(x)} (\log(e+M_{16} f(x)))^{q(x)}\right\}$$

for all $x \in G$. It follows from Theorem 4.1 that

$$\frac{1}{\mu(B(z,4r))} \int_{B(z,r)} \Psi(x, U_{\alpha(\cdot),32} f(x))\, d\mu(x) \le Cr^{-\nu(z)} (\log(e+1/r))^{-\beta(z)}$$

for all $z \in G$ and $0 < r < d_G$, as required. □

6 Trudinger exponential integrability

Based on what we have culminated in the present paper, we shall obtain Trudinger exponential integrabilities for $U_{\alpha(\cdot),9}f$. We seek to discuss the exponential integrability in Sect. 6, assuming that

$$\inf_{x\in G}(\alpha(x)-\nu(x)/p(x))\geq 0, \tag{6.1}$$

or equivalently,

$$\sup_{x\in G}(1/p(x)-\alpha(x)/\nu(x))\leq 0.$$

Set

$$\Gamma(x,r)\equiv c_0\frac{\min(2,r)}{2}\int_1^{\max(2,r)}(\log(e+t))^{-(q(x)+\beta(x))/p(x)}\frac{dt}{t}$$

for $x\in G$ and $r>0$, where we choose a normalization constant c_0 so that $\inf_{x\in G}\Gamma(x,2)=2$. Note that $\sup_{x\in G,\,r\geq 2}\frac{\Gamma(x,r^2)}{\Gamma(x,r)}<\infty$, since $-(q+\beta)/p$ is bounded. Let

$$s_x\equiv\sup_{r\geq 2}\Gamma(x,r)=c_0\int_1^{\infty}(\log(e+t))^{-(q(x)+\beta(x))/p(x)}\frac{dt}{t}\quad (x\in G).$$

Then $2\leq s_x\leq\infty$ and $\Gamma(x,\cdot)$ is bijective from $[0,\infty)$ to $[0,s_x)$. We denote by $\Gamma^{-1}(x,\cdot)$ the inverse function of $\Gamma(x,\cdot)$. If $s_x<\infty$, we set $\Gamma^{-1}(x,r)\equiv\infty$ for $r\geq s_x$. So we always have

$$\Gamma^{-1}(x,r)=\inf(\{s>0\,:\,\Gamma(x,s)\geq r\}\cup\{\infty\})\quad (x\in G,\ r>0).$$

Theorem 6.1 (Trudinger type inequality) *Suppose that $\nu_->0$ and assume in addition that the functions $\alpha(\cdot)$, $p(\cdot)$, $\nu(\cdot)$ satisfy (6.1). Let ε be a μ-measurable function on G such that*

$$\inf_{x\in G}(\nu(x)/p(x)-\varepsilon(x))>0 \text{ and } 0<\varepsilon_-\leq\varepsilon_+<\alpha_-. \tag{6.2}$$

Then there exist constants $c_1,c_2>0$ such that

$$\frac{1}{\mu(B(z,4r))}\int_{B(z,r)}\Gamma^{-1}\left(x,\frac{|U_{\alpha(\cdot),9}f(x)|}{c_1}\right)d\mu(x)\leq c_2\,r^{\varepsilon(z)-\nu(z)/p(z)} \tag{6.3}$$

for all $z\in G$, $0<r<d_G$ and μ-measurable functions f satisfying $\|f\|_{L^{\Phi,\nu,\beta;2}(G)}\leq 1$. In the above $|U_{\alpha(\cdot),9}f(x)|/c_1<s_x$ for a.e. $x\in B(z,r)$.

The aim of this section is to prove Theorem 6.1. Before we go into the detail, let us explain why Theorem 6.1 deserves its name.

Remark 6.2 Let p,q,β be all constants. Define

$$c_0\equiv 2\left(\int_1^2(\log(e+t))^{-(q+\beta)/p}\frac{dt}{t}\right)^{-1}$$

and

$$\Gamma(r) \equiv c_0 \int_1^r (\log(e+t))^{-(q+\beta)/p} \frac{dt}{t} \quad (r \geq 2).$$

(1) If $q+\beta < p$, then, for $r \geq 2$,

$$C^{-1}\Gamma(r) \leq (\log(e+r))^{1-(q+\beta)/p} \leq C\Gamma(r) \tag{6.4}$$

and hence

$$\Gamma^{-1}(C^{-1}r) \leq \exp(r^{p/(p-q-\beta)}) \leq \Gamma^{-1}(Cr).$$

Indeed, just observe that, when $r \geq 2$,

$$C^{-1}\Gamma(r) \leq \int_1^r (\log(e+t))^{-(q+\beta)/p} \frac{dt}{t} < \frac{1}{1-(q+\beta)/p} (\log r)^{1-(q+\beta)/p} \leq C\Gamma(r),$$

which proves (6.4).

(2) If $q+\beta = p$, then, for $r \geq 2$,

$$C^{-1}\Gamma(r) \leq \log(\log(e+r)) \leq C\Gamma(r) \tag{6.5}$$

and

$$\Gamma^{-1}(C^{-1}r) \leq \exp(\exp(r)) \leq \Gamma^{-1}(Cr).$$

Indeed, the proof of (6.5) is a minor modification of (6.4), when $r \geq 4$,

$$C^{-1}\Gamma(r) \leq \int_1^r (\log(e+t))^{-(q+\beta)/p} \frac{dt}{t} < (e+1)\log\log(e+r) \leq C\Gamma(r),$$

which proves (6.5).

If we combine Theorem 6.1 and Remark 6.2, then we obtain the following result, which was called the Trudinger inequality.

Corollary 6.3 *Let G be bounded. Suppose $\nu_- > 0$ and (6.1) holds. Let ε be a μ-measurable function on G such that*

$$\inf_{x\in G} (\nu(x)/p(x) - \varepsilon(x)) > 0 \text{ and that } 0 < \varepsilon_- \leq \varepsilon_+ < \alpha_-. \tag{6.6}$$

Then there exist constants $c_1, c_2 > 0$ such that

(1) *in case* $\sup_{x\in G} (q(x)+\beta(x))/p(x) < 1$,

$$\frac{1}{\mu(B(z,4r))} \int_{B(z,r)} \exp\left(\frac{|U_{\alpha(\cdot),9} f(x)|^{p(x)/(p(x)-q(x)-\beta(x))}}{c_1} \right) d\mu(x) \leq c_2 r^{\varepsilon(z)-\nu/p(z)}; \tag{6.7}$$

(2) *in case* $\inf_{x\in G} (q(x)+\beta(x))/p(x) \geq 1$,

$$\frac{1}{\mu(B(z,4r))} \int_{B(z,r)} \exp\left(\exp\left(\frac{|U_{\alpha(\cdot),9} f(x)|}{c_1} \right) \right) d\mu(x) \leq c_2 r^{\varepsilon(z)-\nu/p(z)} \tag{6.8}$$

for all $z \in G$ and $0 < r \le d_G$, whenever f is a μ-measurable function on G satisfying

$$\|f\|_{L^{\Phi,\nu,\beta;2}(G)} \le 1. \tag{6.9}$$

Remark 6.4 When $X = \mathbf{R}^d$, see [21, Corollary 5.3].

To prove Theorem 6.1, we use the following lemmas. The first lemma can be proved with minor changes of the proof of Lemma 5.4. We begin with investigating the functions from outside the balls.

Lemma 6.5 *Suppose that $\nu_- > 0$ and (6.1) holds. Then there exists a constant $C > 0$ such that*

$$\int_{G\setminus B(x,\delta)} \frac{d(x,y)^{\alpha(x)} f(y)}{\mu(B(x,9d(x,y)))} d\mu(y) \le C\Gamma(x,1/\delta)$$

for all $x \in G$, $0 < \delta < d_G$ and non-negative μ-measurable functions f satisfying $\|f\|_{L^{\Phi,\nu,\beta;2}(G)} \le 1$.

Proof Since

$$\int_{G\setminus B(x,\delta)} \frac{d(x,y)^{\alpha(x)} f(y)}{\mu(B(x,9d(x,y)))} d\mu(y) \le \int_{G\setminus B(x,\delta)} \frac{d(x,y)^{\alpha(x)} f(y)}{\mu(B(x,4d(x,y)))} d\mu(y),$$

we can reexamine and modify the proof of Lemma 5.4. Indeed, we need to estimate $\mathrm{I}(\delta)$, where $\mathrm{I}(\delta)$ is given by (5.8). Assuming (6.1), we have

$$\begin{aligned}
\mathrm{I}(\delta) &= \int_\delta^{2d_G} t^{\alpha(x)-\nu(x)/p(x)} (\log(e+1/t))^{-(q(x)+\beta(x))/p(x)} \frac{dt}{t} \\
&\le C \int_\delta^{2d_G} (\log(e+1/t))^{-(q(x)+\beta(x))/p(x)} \frac{dt}{t} \\
&\le C \int_{(2d_G)^{-1}}^{\delta^{-1}} (\log(e+t))^{-(q(x)+\beta(x))/p(x)} \frac{dt}{t} \\
&\le C\Gamma(x,1/\delta).
\end{aligned}$$

The proof of Lemma 6.5 is thus complete. □

To prove Theorem 6.1, we need another lemma. By generalizing the integral kernel, we are going to prove Theorem 6.1, as is seen from the beginning of the proof. This is where the number "9" comes into play in Theorem 6.1 [see (6.17) below].

Lemma 6.6 *Let $\varepsilon : G \to (0,\infty)$ be a μ-measurable function satisfying (6.2) and let z be a fixed point in G. Also we write*

$$\rho(z,r) \equiv r^{\varepsilon(z)} (\log(e+1/r))^{(q(z)+\beta(z))/p(z)}. \tag{6.10}$$

Define $I_{\rho(z)} f(x)$ by

$$I_{\rho(z)} f(x) \equiv \int_G \frac{\rho(z,d(x,y))}{\mu(B(x,9d(x,y)))} f(y)\, d\mu(y). \tag{6.11}$$

Then there exists a constant $C > 0$ *such that*

$$\frac{1}{\mu(B(z,4r))}\int_{B(z,r)} I_{\rho(z)} f(x) d\mu(x) \le C r^{\varepsilon(z)-\nu(z)/p(z)} \tag{6.12}$$

for all $z \in G$, $0 < r < d_G$ *and non-negative* μ*-measurable functions* f *satisfying*

$$\|f\|_{L^{\Phi,\nu,\beta;2}(G)} \le 1. \tag{6.13}$$

Proof Let $x \in X$ be fixed. Write

$$\begin{aligned} & I_{\rho(z)} f(x) \\ & = \int_{B(z,2r)} \frac{\rho(z,d(x,y))}{\mu(B(x,9d(x,y)))} f(y)\, d\mu(y) + \int_{G\setminus B(z,2r)} \frac{\rho(z,d(x,y))}{\mu(B(x,9d(x,y)))} f(y)\, d\mu(y) \\ & =: \mathrm{I}_1(x) + \mathrm{I}_2(x). \end{aligned}$$

As for I_1, we integrate I_1 over $B(z,r)$ to conclude

$$\begin{aligned} \int_{B(z,r)} \mathrm{I}_1(x)\, d\mu(x) &= \int_{B(z,2r)} \left(\int_{B(z,r)} \frac{\rho(z,d(x,y))}{\mu(B(x,9d(x,y)))}\, d\mu(x) \right) f(y)\, d\mu(y) \\ &\le \int_{B(z,2r)} \left(\int_{B(y,3r)} \frac{\rho(z,d(x,y))}{\mu(B(x,9d(x,y)))}\, d\mu(x) \right) f(y)\, d\mu(y). \end{aligned}$$

By Fubini's theorem, we obtain a crude estimate. The result is

$$\begin{aligned} & \int_{B(z,r)} \mathrm{I}_1(x)\, d\mu(x) \\ & \le \int_{B(z,2r)} \left(\sum_{j=0}^{\infty} \int_{B(y,2^{-j+2}r)\setminus B(y,2^{-j+1}r)} \frac{\rho(z,d(x,y))}{\mu(B(x,9d(x,y)))}\, d\mu(x) \right) f(y)\, d\mu(y) \\ & \le \int_{B(z,2r)} \left(\sum_{j=0}^{\infty} \int_{B(y,2^{-j+2}r)\setminus B(y,2^{-j+1}r)} \frac{\rho(z,2^{-j+2}r)}{\mu(B(x,2^{-j+4}r))}\, d\mu(x) \right) f(y)\, d\mu(y) \\ & \le \int_{B(z,2r)} \left(\sum_{j=0}^{\infty} \int_{B(y,2^{-j+2}r)\setminus B(y,2^{-j+1}r)} \frac{\rho(z,2^{-j+2}r)}{\mu(B(y,2^{-j+2}r))}\, d\mu(x) \right) f(y)\, d\mu(y) \\ & \le \int_{B(z,2r)} \left(\sum_{j=0}^{\infty} \rho(z,2^{-j+2}r) \right) f(y)\, d\mu(y). \end{aligned}$$

Since $\varepsilon_+ < \infty$, the functions q, β are bounded, $p_- \ge 1$ and the function ν is positive, we have

$$\int_{B(z,r)} \mathrm{I}_1(x)\,d\mu(x) \le C \int_{B(z,2r)} \left(\sum_{j=1}^{\infty} \rho(z, 2^{-j}r)\right) f(y)\,d\mu(y)$$
$$\le C \int_{B(z,2r)} \left(\sum_{j=1}^{\infty} \int_{2^{-j}r}^{2^{-j+1}r} \rho(z,t)\,\frac{dt}{t}\right) f(y)\,d\mu(y)$$
$$\le C \int_{B(z,2r)} \left(\int_0^r \rho(z,t)\,\frac{dt}{t}\right) f(y)\,d\mu(y).$$

Since we are assuming (6.6), it follows that

$$\int_{B(z,r)} \mathrm{I}_1(x)\,d\mu(x) \le C\rho(z,r) \int_{B(z,2r)} f(y)\,d\mu(y)$$
$$\le C\rho(z,r)\mu(B(z,4r))(2r)^{-\nu(z)/p(z)}(\log(e+1/(2r)))^{-(q(z)+\beta(z))/p(z)}$$
$$\le Cr^{\varepsilon(z)-\nu(z)/p(z)}\mu(B(z,4r)).$$

In summary, we obtain

$$\int_{B(z,r)} \mathrm{I}_1(x)\,d\mu(x) \le Cr^{\varepsilon(z)-\nu(z)/p(z)}\mu(B(z,4r)). \tag{6.14}$$

For I_2, note first that there exists a constant $C > 0$ such that

$$C^{-1} \le \frac{\rho(z,r)}{\rho(z,s)} \le C \tag{6.15}$$

for $z \in G$, $\frac{1}{2} \le \frac{r}{s} \le 2$ in view of the definition of ρ [see (6.10) above].

Next, we claim that $x \in B(z,r)$ and $y \notin B(z,2r)$ imply that

$$\frac{2}{3}d(x,y) \le d(y,z) \le 2d(x,y) \tag{6.16}$$

and that

$$B(x, 9d(x,y)) \supset B(z, 4d(z,y)). \tag{6.17}$$

Indeed, we have $d(x,z) \le r$ and $d(y,z) > 2r$. Hence, it follows that

$$d(x,y) \le d(y,z) + d(x,z) \le d(y,z) + \frac{1}{2}d(y,z) = \frac{3}{2}d(y,z)$$

and that

$$d(y,z) \le d(x,y) + d(x,z) \le d(x,y) + \frac{1}{2}d(y,z),$$

which yields (6.16). Also observe that when $w \in B(z, 4d(z,y))$, we have

$$d(w,x) \le d(z,x) + d(w,z) \le d(z,x) + 4d(z,y) \le \frac{1}{2}d(y,z) + 4d(z,y) = \frac{9}{2}d(z,y) \le 9d(y,x).$$

Consequently it follows from (6.15) through (6.17) that

$$\begin{aligned} \mathrm{I}_2(x) &= \int_{G\setminus B(z,2r)} \frac{\rho(z, d(x,y))}{\mu(B(x, 9d(x,y)))} f(y)\, d\mu(y) \\ &\le C \int_{G\setminus B(z,2r)} \frac{\rho(z, d(z,y))}{\mu(B(z, 4d(z,y)))} f(y)\, d\mu(y) \end{aligned} \tag{6.18}$$

for $x \in B(z, r)$. Now we proceed in the same way as the proof of Lemma 5.4. We decompose (6.18) diadically:

$$\mathrm{I}_2(x) \le C \sum_{j=1}^{\infty} \int_{B(z,2^{j+1}r)\setminus B(z,2^j r)} \frac{\rho(z, d(z,y))}{\mu(B(z, 4d(z,y)))} f(y)\, d\mu(y),$$

where we used a tacit understanding that f vanishes outside G. Hence, we obtain by Lemma 5.2 and (6.6)

$$\begin{aligned} \mathrm{I}_2(x) &\le C \sum_{j=1}^{\infty} \rho(z, 2^{j+1}r) \frac{1}{\mu(B(z, 2^{j+2}r))} \int_{B(z,2^{j+1}r)} f(y)\, d\mu(y) \\ &\le C \sum_{j=1}^{\infty} (2^{j+1}r)^{\varepsilon(z)-\nu(z)/p(z)} \\ &\le C r^{\varepsilon(z)-\nu(z)/p(z)}. \end{aligned} \tag{6.19}$$

Thus, from (6.14) and (6.19), Lemma 6.6 is proved. □

Proof of Theorem 6.1 If necessary, by replacing f with $|f|$, we have only to deal with non-negative f such that $\|f\|_{L^{\Phi,\nu,\beta;2}(G)} \le 1$. By Lemma 6.5 we find

$$\begin{aligned} &U_{\alpha(\cdot),9} f(x) \\ &= \int_{G\cap B(x,\delta)} \frac{d(x,y)^{\alpha(x)} f(y)}{\mu(B(x, 9d(x,y)))} d\mu(y) + \int_{G\setminus B(x,\delta)} \frac{d(x,y)^{\alpha(x)} f(y)}{\mu(B(x, 9d(x,y)))} d\mu(y) \\ &= \int_{G\cap B(x,\delta)} d(x,y)^{\alpha(x)-\varepsilon(z)} (\log(e + 1/d(x,y)))^{-(q(z)+\beta(z))/p(z)} \frac{\rho(z, d(x,y))}{\mu(B(x, 9d(x,y)))} f(y)\, d\mu(y) \\ &\quad + C\Gamma(x, 1/\delta) \\ &\le C \left\{ \delta^{\alpha(x)-\varepsilon(z)} (\log(e + 1/\delta))^{-(q(z)+\beta(z))/p(z)} I_{\rho(z)} f(x) + \Gamma(x, 1/\delta) \right\} \end{aligned}$$

for any $\delta > 0$. We now specify δ by

$$\delta \equiv \min \left\{ d_G, \left(\frac{\Gamma(x, I_{\rho(z)} f(x)) (\log(e + I_{\rho(z)} f(x)))^{(q(z)+\beta(z))/p(z)}}{I_{\rho(z)} f(x)} \right)^{1/(\alpha(x)-\varepsilon(z))} \right\}$$

and we have the inequality

$$U_{\alpha(\cdot),9} f(x) \le c_1 \max \left\{ 1, \Gamma(x, I_{\rho(z)} f(x)) \right\},$$

for some constant $c_1 > 0$. We denote by $\Gamma^{-1}(x,\cdot)$ the inverse function of $\Gamma(x,\cdot)$. Since $1 \le \Gamma(x,1) = \Gamma(x,2)/2$, we have $\Gamma^{-1}(x,1) \le 1$. Then

$$\frac{1}{\mu(B(z,4r))}\int_{B(z,r)} \Gamma^{-1}\left(x, \frac{U_{\alpha(\cdot),9}f(x)}{c_1}\right) d\mu(y) \le \frac{1}{\mu(B(z,4r))}\int_{B(z,r)} \left\{1 + I_{\rho(z)}f(x)\right\} d\mu(y)$$

for all $z \in G$ and $0 < r < d_G$. Hence, Lemma 6.6 gives the conclusion. □

7 Continuity of potential functions

In Sect. 7, we are concerned with continuity for Riesz potentials $U_{\alpha(\cdot),4}f$ when

$$0 \le \inf_{x\in G}(\alpha(x) - \nu(x)/p(x)) \le \sup_{x\in G}(\alpha(x) - \nu(x)/p(x)) < 1 \tag{7.1}$$

and the following condition holds: For $x \in G$ and $r > 0$, define

$$\omega(x,r) \equiv \int_0^r t^{\alpha(x)-\nu(x)/p(x)}(\log(e+1/t))^{-(q(x)+\beta(x))/p(x)}\frac{dt}{t} < \infty.$$

In this case

$$\omega(x,r) \le \omega(x,2r) \le C\omega(x,r) \tag{7.2}$$

for some constant $C > 0$ independent of $x \in G$ and $0 < r < \infty$. We use ω to measure the continuity of functions. See [26] for the Lebesgue measure case.

Further, we assume the following Hölmander type condition: There are $0 < \theta \le 1$ and $C > 0$ such that

$$\left|\frac{d(x,y)^{\alpha(x)}}{\mu(B(x,4d(x,y)))} - \frac{d(z,y)^{\alpha(z)}}{\mu(B(z,4d(z,y)))}\right| \le C\left(\frac{d(x,z)}{d(x,y)}\right)^{\theta}\frac{d(x,y)^{\alpha(x)}}{\mu(B(x,4d(x,y)))} \tag{7.3}$$

whenever $d(x,z) \le d(x,y)/2$, and

$$\sup_{x\in G}(\alpha(x) - \nu(x)/p(x)) < \theta \le 1.$$

Concerning the continuity of $U_{\alpha(\cdot),4}f$, we have the following result. Lemmas 7.2 and 7.3 justify that the integral defining $U_{\alpha(\cdot),4}f(x)$ converges absolutely.

Theorem 7.1 *Assume that $\alpha(\cdot)$, $\nu(\cdot)$ and $p(\cdot)$ satisfy (7.1)–(7.3) and that $\beta(\cdot)$ is a bounded μ-measurable function. Define $\Phi = \Phi_{p(\cdot),q(\cdot)}$ by using $p(\cdot)$ and $q(\cdot)$ satisfying (P1), (P2), (Q1) and (Q2) through (1.3). There exists a constant $C > 0$ such that*

$$|U_{\alpha(\cdot),4}f(x) - U_{\alpha(\cdot),4}f(z)| \le C\{\omega(x,d(x,z)) + \omega(z,d(x,z))\} \quad (x,z \in G),$$

whenever f is a non-negative μ-measurable function on G satisfying $\|f\|_{L^{\Phi,\nu,\beta;2}(G)} \le 1$.

To prove Theorem 7.1, we need Lemmas 7.2 and 7.3. Lemmas 7.2 and 7.3 concern an estimate inside the ball and that outside the ball, respectively.

Lemma 7.2 *Assume $p_- > 0$. Let f be a non-negative μ-measurable function on G such that $\|f\|_{L^{\Phi,\nu,\beta;2}(G)} \le 1$. Then there exists a constant $C > 0$ such that*

$$\int_{B(x,\delta)} \frac{d(x,y)^{\alpha(x)}}{\mu(B(x,4d(x,y)))} f(y)\,d\mu(y) \le C\omega(x,\delta)$$

for all $x \in G$ and $\delta > 0$.

Proof As usual we start by decomposing $B(x,\delta)$ dyadically:

$$\begin{aligned}\int_{B(x,\delta)} \frac{d(x,y)^{\alpha(x)} f(y)}{\mu(B(x,4d(x,y)))} d\mu(y) &= \sum_{j=1}^{\infty} \int_{B(x,2^{-j+1}\delta)\setminus B(x,2^{-j}\delta)} \frac{d(x,y)^{\alpha(x)} f(y)}{\mu(B(x,4d(x,y)))} d\mu(y) \\ &\le \sum_{j=1}^{\infty} (2^{-j+1}\delta)^{\alpha(x)} \frac{1}{\mu(B(x,2^{-j+2}\delta))} \int_{B(x,2^{-j+1}\delta)} f(y)d\mu(y).\end{aligned}$$

Recall that the functions q, α, β are all bounded, $p_- > 1$ and the function ν is positive. By Lemma 5.2, we have

$$\begin{aligned}\int_{B(x,\delta)} \frac{d(x,y)^{\alpha(x)} f(y)}{\mu(B(x,4d(x,y)))} d\mu(y) &\le C \sum_{j=1}^{\infty} (2^{-j}\delta)^{\alpha(x)-\nu(x)/p(x)} (\log(e+1/(2^{-j}\delta)))^{-(q(x)+\beta(x))/p(x)} \\ &\le C \int_0^{\delta} t^{\alpha(x)-\nu(x)/p(x)} (\log(e+1/t))^{-(q(x)+\beta(x))/p(x)} \frac{dt}{t} \\ &= C\omega(x,\delta).\end{aligned}$$

□

The following lemma can be proved in the same manner as Lemma 5.4, whose proof we omit.

Lemma 7.3 *Let f be a non-negative μ-measurable function on G such that $\|f\|_{L^{\Phi,\nu,\beta;2}(G)} \le 1$. Then there exists a constant $C > 0$ such that*

$$\int_{G\setminus B(x,\delta)} \frac{d(x,y)^{\alpha(x)-\theta}}{\mu(B(x,4d(x,y)))} f(y)\,d\mu(y) \le C\delta^{\alpha(x)-\theta-\nu(x)/p(x)} (\log(e+1/\delta))^{-(q(x)+\beta(x))/p(x)}$$

for all $x \in G$ and $\delta > 0$. Here the constant θ is from (7.3).

Proof of Theorem 7.1 Let f be a non-negative μ-measurable and $\|f\|_{L^{\Phi,\nu,\beta;2}(G)} \le 1$. Write

$$\begin{aligned}&U_{\alpha(\cdot),4} f(x) - U_{\alpha(\cdot),4} f(z) \\ &\quad = \int_{B(x,2d(x,z))} \frac{d(x,y)^{\alpha(x)}}{\mu(B(x,4d(x,y)))} f(y)\,d\mu(y) - \int_{B(x,2d(x,z))} \frac{d(z,y)^{\alpha(z)}}{\mu(B(z,4d(z,y)))} f(y)\,d\mu(y) \\ &\qquad + \int_{G\setminus B(x,2d(x,z))} \left(\frac{d(x,y)^{\alpha(x)}}{\mu(B(x,4d(x,y)))} - \frac{d(z,y)^{\alpha(z)}}{\mu(B(z,4d(z,y)))} \right) f(y)\,d\mu(y)\end{aligned}$$

for $x, z \in G$. Using Lemma 7.2 and (7.2), we have

$$\int_{B(x,2d(x,z))} \frac{d(x,y)^{\alpha(x)}}{\mu(B(x,4d(x,y)))} f(y)\,d\mu(y) \le C\omega(x,2d(x,z)) \le C\omega(x,d(x,z))$$

and

$$\begin{aligned}\int_{B(x,2d(x,z))} \frac{d(z,y)^{\alpha(z)}}{\mu(B(z,4d(z,y)))} f(y)\,d\mu(y) &\le \int_{B(z,3d(x,z))} \frac{d(z,y)^{\alpha(z)}}{\mu(B(z,4d(z,y)))} f(y)\,d\mu(y)\\ &\le C\omega(z,3d(x,z)) \le C\omega(z,d(x,z)).\end{aligned}$$

On the other hand, by (7.2), (7.3) and Lemma 7.2, we have

$$\begin{aligned}&\int_{G\setminus B(x,2d(x,z))} \left| \frac{d(x,y)^{\alpha(x)}}{\mu(B(x,4d(x,y)))} - \frac{d(z,y)^{\alpha(z)}}{\mu(B(z,4d(z,y)))} \right| f(y)\,d\mu(y)\\ &\le Cd(x,z)^{\theta} \int_{G\setminus B(x,2d(x,z))} \frac{d(x,y)^{\alpha(x)-\theta}}{\mu(B(x,4d(x,y)))} f(y)\,dy\\ &\le Cd(x,z)^{\theta} d(x,z)^{\alpha(x)-\theta-\nu(x)/p(x)} (\log(e+1/d(x,z)))^{-(q(x)+\beta(x))/p(x)}\\ &\le Cd(x,z)^{\alpha(x)-\nu(x)/p(x)} (\log(e+1/d(x,z)))^{-(q(x)+\beta(x))/p(x)}\\ &\le C\omega(x,d(x,z)).\end{aligned}$$

Then we have the conclusion. □

Theorem 7.1 asserts that $U_{\alpha(\cdot),4} f$ is continuous. In view of the result in [21, Section 6], this can be quantified more precisely.

Remark 7.4 (cf. [21, Section 6]) As $r \downarrow 0$, $\omega(x,r) \to 0$ uniformly in $x \in E$ ($E \subset G$) if either

$$\inf_{x\in E} \left(\alpha(x) - \frac{\nu(x)}{p(x)} \right) > 0 \tag{7.4}$$

or

$$\inf_{x\in E} \left(\alpha(x) - \frac{\nu(x)}{p(x)} \right) = 0 \quad \text{and} \quad \inf_{x\in E} \frac{q(x)+\beta(x)}{p(x)} > 1. \tag{7.5}$$

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