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On the Banach lattice structure of L_w^1 of a vector measure on a δ-ring

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Abstract We study some Banach lattice properties of the space $L_w^1(\nu)$ of weakly integrable functions with respect to a vector measure ν defined on a δ-ring. Namely, we analyze order continuity, order density and Fatou type properties. We will see that the behavior of $L_w^1(\nu)$ differs from the case in which ν is defined on a σ-algebra whenever ν does not satisfy certain local σ-finiteness property.

Keywords Banach lattice · δ-ring · Fatou property · Order density · Order continuity · Integration with respect to vector measures

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1 Introduction

The space of integrable functions with respect to a vector measure finds applications in important problems as, for instance, the representation of abstract Banach lattices as spaces of functions and the study of the optimal domain of linear operators. Classical vector measures $\nu\colon \Sigma \to X$ are considered to be defined on a σ-algebra and with values in a Banach space. The spaces $L^1(\nu)$ and $L^1_w(\nu)$ of integrable and weakly integrable functions respectively have been studied in depth by many authors and their behavior is well understood, (see [7] and [25, Chapter 3]) and the references therein. However, this framework is not enough, for instance, for applications to operators on spaces which do not contain the characteristic functions of sets (see [2,10,11]) or Banach lattices without weak unit (see [12]). These cases require ν to be defined on a weaker structure than σ-algebra, namely, a δ-ring. Bear in mind the spaces $\ell^p(\Gamma)$, $1 \le p \le \infty$, for an uncountable set Γ. So, vector measures defined on a δ-ring also play an important role and deserve to be studied together with their spaces of integrable functions. The integration theory with respect to these vector measures ν goes back to the late sixties (see [14,18,21–24]). In [9], there is an analysis of the space $L^1(\nu)$ which gives evidence of how large the difference can be between the δ-ring and σ-algebra cases. Indeed, for the general case, bounded functions may be not integrable and this fact is crucial.

The aim of this paper is the study of the Banach lattice properties of the space $L^1_w(\nu)$. The case when these spaces contain c_0 becomes specially relevant. This research is a part of a general project of analysis of these abstract integration structures that has already shown to be useful in applications. For instance, a general version of Komlós Theorem on the pointwise convergence of the Cesàro sums of functions have been recently obtained using spaces of vector measure integrable functions on a δ-ring as main tool (see [17]). More applications in the setting of the theory of operators on Banach function spaces can be found in [2,3]. The relevant case of the Hardy operator has been studied in [11].

More precisely, we study some properties related to order continuity (Sect. 3) and order density (Sect. 4), and some Fatou type properties (Sect. 5). We will see that many properties satisfied for this space when ν is defined on a σ-algebra remain true in general only in the case when ν satisfies certain local σ-finiteness property, which guarantees that every function in $L^1_w(\nu)$ is the ν-a.e. pointwise limit of a sequence of functions in $L^1(\nu)$. Also we revisit the representation theorems for abstract Banach lattices (Sect. 6), and we finish with an illustrative example (Sect. 7).

2 Preliminaries

2.1 Banach lattices

Let E be a Banach lattice with norm $\|\cdot\|$ and order $\le$. A closed subspace F of E is an *ideal* of E if $y \in E$ with $|y| \le |x|$ for some $x \in F$ implies $y \in F$. We say that E is *order continuous* if for every $(x_\tau) \subset E$ downwards directed system $x_\tau \downarrow 0$ it follows that $\|x_\tau\| \downarrow 0$ and E is σ*-order continuous* if for every $(x_n) \subset E$ decreasing sequence $x_n \downarrow 0$ it follows that $\|x_n\| \downarrow 0$. We denote by E_{an} the *order continuous part* of E, that is, the largest order continuous ideal in E. It can be described as

$$E_{an} = \{x \in E : |x| \ge x_\tau \downarrow 0 \text{ implies } \|x_\tau\| \downarrow 0\}.$$

Similarly, E_a will denote the *σ-order continuous part* of E, that is, the largest σ-order continuous ideal in E, which can be described as

$$E_a = \{x \in E : |x| \geq x_n \downarrow 0 \text{ implies } \|x_n\| \downarrow 0\}.$$

The Banach lattice E is *Dedekind complete* if every non empty subset which is bounded from above has a supremum and is *Dedekind σ-complete* if every non empty countable subset which is bounded from above has a supremum. We say that E has the *Fatou property* if for every $(x_\tau) \subset E$ upwards directed system $0 \leq x_\tau \uparrow$ such that $\sup \|x_\tau\| < \infty$ it follows that there exists $x = \sup x_\tau$ in E and $\|x\| = \sup \|x_\tau\|$, and E has the *σ-Fatou property* if for every $(x_n) \subset E$ increasing sequence $0 \leq x_n \uparrow$ such that $\sup \|x_n\| < \infty$ it follows that there exists $x = \sup x_n$ in E and $\|x\| = \sup \|x_n\|$. An ideal F in E is said to be *order dense* if for every $0 \leq x \in E$ there exists an upwards directed system $0 \leq x_\tau \uparrow x$ such that $(x_\tau) \subset F$ and is said to be *super order dense* if for every $0 \leq x \in E$ there exists an increasing sequence $0 \leq x_n \uparrow x$ such that $(x_n) \subset F$. A *weak unit* of E is an element $0 \leq e \in E$ such that $x \wedge e = 0$ implies $x = 0$. Every *positive* linear operator $T : E \to F$ between Banach lattices (i.e. $Tx \geq 0$ whenever $0 \leq x \in E$) is continuous, see [19, p. 2]. An operator $T : E \to F$ between Banach lattices is said to be an *order isometry* if it is a linear isometry which is also an order isomorphism, that is, T is linear, one to one, onto, $\|Tx\|_F = \|x\|_E$ for all $x \in E$ and $T(x \wedge y) = Tx \wedge Ty$ for all $x, y \in E$.

Let (Ω, Σ, μ) be a measure space (without assumptions of finiteness on μ) and $L^0(\mu)$ be the space of all measurable real functions on Ω, where functions which are equal μ-a.e. are identified. Considering the μ-a.e. pointwise order, we have that $L^0(\mu)$ is an Archimedean vector lattice. Note that for $f, f_n \in L^0(\mu)$, it follows that $0 \leq f_n \uparrow f$ μ-a.e. if and only if $0 \leq f_n \uparrow f$ in $L^0(\mu)$, that is, the μ-a.e. pointwise supremum coincides with the lattice supremum. We will simple write $f \leq g$ for $f \leq g$ μ-a.e. By *Banach function space* (briefly, B.f.s.) related to μ we mean a Banach space $X \subset L^0(\mu)$ satisfying that if $|f| \leq |g|$ μ-a.e. with $f \in L^0(\mu)$ and $g \in X$ then $f \in X$ and $\|f\|_X \leq \|g\|_X$. Every B.f.s. is a Banach lattice with the μ-a.e. pointwise order, in which convergence in norm of a sequence implies μ-a.e. convergence for some subsequence. Note that for $f, f_n \in X$, it follows that $0 \leq f_n \uparrow f$ μ-a.e. if and only if $0 \leq f_n \uparrow f$ in X.

These and other issues related to Banach lattices can be found in [20] and [26].

2.2 Integration with respect to vector measures on δ-rings.

Let $\mathcal{R}$ be a *δ-ring* of subsets of an abstract set Ω, that is, a ring closed under countable intersections. We write $\mathcal{R}^{loc}$ for the σ-algebra of all subsets A of Ω such that $A \cap B \in \mathcal{R}$ for all $B \in \mathcal{R}$. Note that if $\mathcal{R}$ is a σ-algebra then $\mathcal{R}^{loc} = \mathcal{R}$. Denote by $\mathcal{M}(\mathcal{R}^{loc})$ the space of all measurable real functions on $(\Omega, \mathcal{R}^{loc})$, by $\mathcal{S}(\mathcal{R}^{loc})$ the space of all simple functions and by $\mathcal{S}(\mathcal{R})$ the space of all *$\mathcal{R}$-simple functions* (i.e. simple functions supported in $\mathcal{R}$).

Let $\lambda : \mathcal{R} \to \mathbb{R}$ be a countably additive measure, that is, $\sum \lambda(A_n)$ converges to $\lambda(\cup A_n)$ whenever (A_n) is a sequence of pairwise disjoint sets in $\mathcal{R}$ with $\cup A_n \in \mathcal{R}$. The *variation* of λ is the countably additive measure $|\lambda| : \mathcal{R}^{loc} \to [0, \infty]$ given by

$$|\lambda|(A) = \sup \left\{ \sum |\lambda(A_i)| : (A_i) \text{ finite disjoint sequence in } \mathcal{R} \cap 2^A \right\}.$$

For every $A \in \mathcal{R}$ we have that $|\lambda|(A) < \infty$. The space $L^1(\lambda)$ of integrable functions with respect to λ is defined as the space $L^1(|\lambda|)$ with the usual norm. Every $\mathcal{R}$-simple function $\varphi = \sum_{i=1}^n \alpha_i \chi_{A_i}$ is in $L^1(\lambda)$ and the integral of φ with respect to λ is defined as usual by $\int \varphi \, d\lambda = \sum_{i=1}^n \alpha_i \lambda(A_i)$. Moreover, the space $\mathcal{S}(\mathcal{R})$ is dense in $L^1(\lambda)$. For every

$f \in L^1(\lambda)$, the integral of f with respect to λ is defined as $\int f\,d\lambda = \lim \int \varphi_n\,d\lambda$ for any sequence $(\varphi_n) \subset \mathcal{S}(\mathcal{R})$ converging to f in $L^1(\lambda)$.

Let $\nu\colon \mathcal{R} \to X$ be a *vector measure* with values in a real Banach space X, that is, $\sum \nu(A_n)$ converges to $\nu(\cup A_n)$ in X whenever (A_n) is a sequence of pairwise disjoint sets in $\mathcal{R}$ with $\cup A_n \in \mathcal{R}$. Denoting by X^* the topological dual of X and by B_{X^*} the unit ball of X^*, the *semivariation* of ν is the map $\|\nu\|\colon \mathcal{R}^{loc} \to [0,\infty]$ given by $\|\nu\|(A) = \sup\{|x^*\nu|(A) : x^* \in B_{X^*}\}$ for all $A \in \mathcal{R}^{loc}$, where $|x^*\nu|$ is the variation of the measure $x^*\nu\colon \mathcal{R} \to \mathbb{R}$. A set $A \in \mathcal{R}^{loc}$ is ν-*null* if $\|\nu\|(A) = 0$, or equivalently, $\nu(B) = 0$ for all $B \in \mathcal{R} \cap 2^A$. A property holds ν-*almost everywhere* (briefly, ν-a.e.) if it holds except on a ν-null set. For every $\mathcal{R}^{loc}$-measurable function $f\colon \Omega \to \mathbb{R} \cup \{\pm\infty\}$ we can define

$$\|f\|_\nu = \sup_{x^* \in B_{X^*}} \int |f|\,d|x^*\nu| \le \infty.$$

Note that if $\|f\|_\nu < \infty$ then $|f| < \infty$ ν-a.e. Let $L^1_w(\nu)$ denote the space of functions in $\mathcal{M}(\mathcal{R}^{loc})$ which are integrable with respect to $|x^*\nu|$ for all $x^* \in X^*$, where functions which are equal ν-a.e. are identified. The space $L^1_w(\nu)$ is a Banach space with the norm $\|\cdot\|_\nu$. A function $f \in L^1_w(\nu)$ is *integrable with respect to* ν if for each $A \in \mathcal{R}^{loc}$ there exists a vector denoted by $\int_A f\,d\nu \in X$, such that

$$x^*\left(\int_A f\,d\nu\right) = \int_A f\,dx^*\nu \text{ for all } x^* \in X^*.$$

Let $L^1(\nu)$ denote the space of all integrable functions with respect to ν. Then, $L^1(\nu)$ is a closed subspace of $L^1_w(\nu)$ and so it is a Banach space with the norm $\|\cdot\|_\nu$. Moreover, $\mathcal{S}(\mathcal{R})$ is dense in $L^1(\nu)$. Note that for every $\mathcal{R}$-simple function $\varphi = \sum_{i=1}^n \alpha_i \chi_{A_i}$, we have that $\int \varphi\,d\nu = \sum_{i=1}^n \alpha_i \nu(A_i)$. From [1, Theorem 3.2], there always exists a measure $\lambda\colon \mathcal{R} \to [0,\infty]$ with the same null sets as ν. Then, $L^1(\nu)$ and $L^1_w(\nu)$ are B.f.s. related to $|\lambda|$. Moreover, $L^1(\nu)$ is order continuous and $L^1_w(\nu)$ has the σ-Fatou property.

For any measure $\mu\colon \mathcal{R}^{loc} \to [0,\infty]$ with the same null sets as ν, since the μ-a.e. pointwise order coincides with the ν-a.e. one, we will denote $L^0(\nu) = L^0(\mu)$ and say B.f.s. related to ν for B.f.s. related to μ.

For these and other issues related to integration with respect to vector measures defined on a δ-ring, see [18,21,22,9].

3 Order continuous part of $L^1_w(\nu)$

All along in this paper $\nu\colon \mathcal{R} \to X$ will be a vector measure defined on a δ-ring $\mathcal{R}$ of subsets of an abstract set Ω, with values in a real Banach space X. Recall that measurable functions are referred to the σ-algebra $\mathcal{R}^{loc}$.

Let us begin by noting that the σ-order continuous and the order continuous parts of $L^1_w(\nu)$ coincide. Indeed, $L^1_w(\nu)$ is Dedekind σ-complete as it has the σ-Fatou property (see [26, Theorem 113.1]), and so, since $\left(L^1_w(\nu)\right)_a$ is an ideal in $L^1_w(\nu)$, it is also Dedekind σ-complete. Then, from [26, Theorem 103.6], $\left(L^1_w(\nu)\right)_a$ is order continuous and thus $\left(L^1_w(\nu)\right)_a = \left(L^1_w(\nu)\right)_{an}$.

It was noted in [6, p. 192], that in the case when $\mathcal{R}$ is a σ-algebra, the order continuous part of $L^1_w(\nu)$ is just $L^1(\nu)$. This follows from the facts that $L^1(\nu)$ is order continuous and $\mathcal{S}(\mathcal{R}^{loc}) = \mathcal{S}(\mathcal{R}) \subset L^1(\nu)$. In the general case, $\mathcal{S}(\mathcal{R}^{loc})$ may not be in $L^1(\nu)$, even so, we will see that $\left(L^1_w(\nu)\right)_a = L^1(\nu)$ remains true. First, let us characterize when a characteristic function of a measurable set is in $L^1(\nu)$.

Lemma 3.1 *The following statements are equivalent for any $A \in \mathcal{R}^{loc}$.*

(a) $\chi_A \in L^1(\nu)$.
(b) $\|\nu\|(A_n) \to 0$ *for all decreasing sequences* $(A_n) \subset \mathcal{R}^{loc} \cap 2^A$ *with* $\cap A_n$ *ν-null.*
(c) $\nu(A_n) \to 0$ *for all disjoint sequences* $(A_n) \subset \mathcal{R} \cap 2^A$.

Proof Suppose that $\chi_A \in L^1(\nu)$ and let $(A_n) \subset \mathcal{R}^{loc} \cap 2^A$ be a decreasing sequence with $\cap A_n$ ν-null. Since $L^1(\nu)$ is order continuous and $\chi_A \geq \chi_{A_n} \downarrow 0$, then $\|\nu\|(A_n) = \|\chi_{A_n}\|_\nu \to 0$. So, (a) implies (b).

Let $(A_n) \subset \mathcal{R} \cap 2^A$ be a disjoint sequence. Taking $B_n = \cup_{j \geq n} A_j$ we have a decreasing sequence $(B_n) \subset \mathcal{R}^{loc} \cap 2^A$ with $\cap B_n = \emptyset$ and $\|\nu(A_n)\| \leq \|\nu\|(B_n)$. So, (b) implies (c).

Suppose that (c) holds and consider the vector measure $\nu_A : \mathcal{R} \to X$ defined by $\nu_A(B) = \nu(A \cap B)$ for all $B \in \mathcal{R}$. Noting that $|x^*\nu_A|(B) = |x^*\nu|(A \cap B)$ for every $B \in \mathcal{R}^{loc}$ and $x^* \in X^*$, it can be checked that $\int |f|\, d|x^*\nu_A| = \int |f|\chi_A\, d|x^*\nu|$. Indeed, this is trivial for simple functions, and for all measurable functions it is consequence of the monotone convergence theorem. Thus, $\|f\|_{\nu_A} = \|f\chi_A\|_\nu$ for every $f \in \mathcal{M}(\mathcal{R}^{loc})$. Then, $f \in L^1_w(\nu_A)$ if and only if $f\chi_A \in L^1_w(\nu)$. Since $\mathcal{S}(\mathcal{R})$ is dense in both $L^1(\nu)$ and $L^1(\nu_A)$, it follows that $f \in L^1(\nu_A)$ if and only if $f\chi_A \in L^1(\nu)$. By hypothesis ν_A is strongly additive, so, from [9, Corollary 3.2.b)], we have that $\chi_\Omega \in L^1(\nu_A)$ and thus $\chi_A \in L^1(\nu)$. □

Let us prove now the announced result.

Theorem 3.2 *The equality* $\left(L^1_w(\nu)\right)_a = L^1(\nu)$ *holds.*

Proof Obviously $L^1(\nu) \subset \left(L^1_w(\nu)\right)_a$ as $L^1(\nu)$ is order continuous. For the converse inclusion, consider first a set $A \in \mathcal{R}^{loc}$ such that $\chi_A \in \left(L^1_w(\nu)\right)_a$. For every decreasing sequence $(A_n) \subset \mathcal{R}^{loc} \cap 2^A$ with $\cap A_n$ ν-null it follows that $\chi_A \geq \chi_{A_n} \downarrow 0$ and so $\|\nu\|(A_n) = \|\chi_{A_n}\|_\nu \to 0$. Then we get $\chi_A \in L^1(\nu)$, from Lemma 3.1.

Consider now $\varphi \in \mathcal{S}(\mathcal{R}^{loc})$ such that $\varphi \in \left(L^1_w(\nu)\right)_a$. Write $\varphi = \sum_{j=1}^n \alpha_j \chi_{A_j}$ with $(A_j) \subset \mathcal{R}^{loc}$ being a disjoint sequence and $\alpha_j \neq 0$. Since $\chi_{A_j} \leq |\frac{\varphi}{\alpha_j}|$ and $\left(L^1_w(\nu)\right)_a$ is an ideal, $\chi_{A_j} \in \left(L^1_w(\nu)\right)_a$. Then, $\chi_{A_j} \in L^1(\nu)$ and so $\varphi \in L^1(\nu)$.

Finally, let $f \in \left(L^1_w(\nu)\right)_a$ and take a sequence $(\varphi_n) \subset \mathcal{S}(\mathcal{R}^{loc})$ satisfying that $0 \leq \varphi_n \uparrow |f|$ ν-a.e. Note that $\varphi_n \in \left(L^1_w(\nu)\right)_a$ as $\varphi_n \leq |f|$, and so $\varphi_n \in L^1(\nu)$. Since $|f| \geq |f| - \varphi_n \downarrow 0$, we have that $\|\,|f| - \varphi_n\|_\nu \to 0$. Then, as $L^1(\nu)$ is closed in $L^1_w(\nu)$, we have that $|f|$, and so also f, is in $L^1(\nu)$. □

4 Order density of $L^1(\nu)$ in $L^1_w(\nu)$

The topic of this section is trivial for the case when $\mathcal{R}$ is a σ-algebra. Indeed, for each $0 \leq f \in L^0(\nu)$ there exists $(\varphi_n) \subset \mathcal{S}(\mathcal{R}^{loc})$ such that $0 \leq \varphi_n \uparrow f$ ν-a.e. Since, in this case $\mathcal{R}^{loc} = \mathcal{R}$ and $\mathcal{S}(\mathcal{R}) \subset L^1(\nu)$, obviously we have that $L^1(\nu)$ is super order dense (and so order dense) in $L^0(\nu)$ (and so in $L^1_w(\nu)$). However, this argument fails for the general case as $\mathcal{S}(\mathcal{R}^{loc})$ may not be contained in $L^1(\nu)$.

Example 4.1 Let Γ be an uncountable abstract set, $\mathcal{R}$ the δ-ring of finite subsets of Γ and $\nu : \mathcal{R} \to c_0(\Gamma)$ the vector measure defined by $\nu(A) = \chi_A$ (see [9, Example 2.2]). Then, $\chi_\Gamma \in L^1_w(\nu) = \ell^\infty(\Gamma)$, but there is no sequence $(f_n) \subset L^1(\nu) = c_0(\Gamma)$ such that $0 \leq f_n \uparrow \chi_\Gamma$. Indeed, in this case, since the only ν-null set is the empty set, $\Gamma = \cup_n \mathrm{supp}(f_n)$ is countable.

Therefore, in general $L^1(\nu)$ is not super order dense in $L^1_w(\nu)$, but order dense.

Theorem 4.2 *The space $L^1(\nu)$ is order dense in $L^1_w(\nu)$.*

Proof Since every Banach lattice is Archimedean, by [20, Ch. 3, Theorem 22.3] it is enough to prove that $L^1(\nu)$ is *quasi order dense* in $L^1_w(\nu)$, i.e. for every $0 \neq f \in L^1_w(\nu)$ there exists $0 \neq g \in L^1(\nu)$ such that $|g| \leq |f|$.

Let $f \in L^1_w(\nu)$ with $\|\nu\|(\mathrm{supp}(f)) > 0$. For $A_n = \{\omega \in \Omega : |f(\omega)| > \frac{1}{n}\}$, we have that $A_n \uparrow \mathrm{supp}(f)$ and so $\|\nu\|(\mathrm{supp}(f)) = \lim_n \|\nu\|(A_n)$ (see [22, Corollary 3.5.(e)]). Take n large enough such that $\|\nu\|(A_n) > 0$. Since $\|\nu\|(A_n) = \sup_{B \in \mathcal{R} \cap 2^{A_n}} \|\nu\|(B)$ (see [22, Lemma 3.4.(g)]), there exists $B_n \in \mathcal{R} \cap 2^{A_n}$ such that $\|\nu\|(B_n) > 0$.

On the other hand, take a sequence $(\psi_j)_j \subset \mathcal{S}(\mathcal{R}^{loc})$ such that $0 \leq \psi_j \uparrow |f|$ ν-a.e. Then, there exists a ν-null set $Z \in \mathcal{R}^{loc}$ such that $0 \leq \psi_j(\omega) \uparrow f(\omega)$ for all $\omega \in \Omega \backslash Z$. Let us consider $B_n = (\cup_j B_n \cap \mathrm{supp}(\psi_j) \backslash Z) \cup (B_n \cap Z)$. Since $B_n \cap \mathrm{supp}(\psi_j) \backslash Z \uparrow$, it follows that $\|\nu\|(B_n) = \|\nu\|(\cup_j B_n \cap \mathrm{supp}(\psi_j) \backslash Z) = \lim_j \|\nu\|(B_n \cap \mathrm{supp}(\psi_j) \backslash Z)$. Take j_n large enough such that $\|\nu\|(B_n \cap \mathrm{supp}(\psi_{j_n}) \backslash Z) > 0$ and consider the function $g = \psi_{j_n} \chi_{B_n} \in \mathcal{S}(\mathcal{R}) \subset L^1(\nu)$. Then, $g \neq 0$ and $0 \leq g \leq |f|$. □

Remark 4.3 Since $L^0(\nu)$ with the ν-a.e. pointwise order is an Archimedean vector lattice, actually in Theorem 4.2 we have proved that $L^1(\nu)$ is order dense in $L^0(\nu)$.

Now, the natural question is when $L^1(\nu)$ is super order dense in $L^1_w(\nu)$. It is easy to see that this happens if ν is *σ-finite*, that is, $\Omega = (\cup A_n) \cup N$ with $N \in \mathcal{R}^{loc}$ ν-null and (A_n) a sequence in $\mathcal{R}$. In this case, if $0 \leq f \in L^0(\nu)$ and $(\psi_n) \subset \mathcal{S}(\mathcal{R}^{loc})$ is such that $0 \leq \psi_n \uparrow f$ ν-a.e., taking $\varphi_n = \psi_n \chi_{\cup_{j=1}^n A_j} \in \mathcal{S}(\mathcal{R})$ we have that $0 \leq \varphi_n \uparrow f$ ν-a.e. Then, $L^1(\nu)$ is super order dense in $L^0(\nu)$ and so in $L^1_w(\nu)$. However, $L^1(\nu)$ being super order dense in $L^1_w(\nu)$ does not imply that ν is σ-finite.

Example 4.4 The vector measure ν in Example 4.1 considered with values in $\ell^1(\Gamma)$ instead of $c_0(\Gamma)$, satisfies that $L^1(\nu) = L^1_w(\nu) = \ell^1(\Gamma)$. Then, obviously $L^1(\nu)$ is super order dense in $L^1_w(\nu)$ but ν is not σ-finite.

We will characterize the super order density of $L^1(\nu)$ in $L^1_w(\nu)$ by a weaker condition on ν than σ-finiteness. Namely, ν will be said to be *locally σ-finite* if every set $A \in \mathcal{R}^{loc}$ with $\|\nu\|(A) < \infty$, can be written as $A = (\cup A_n) \cup N$, with $N \in \mathcal{R}^{loc}$ ν-null and (A_n) a sequence in $\mathcal{R}$.

Remark 4.5 If ν is such that $L^1(\nu) = L^1_w(\nu)$ (e.g. if X does not contain any copy of c_0, see [18, Theorem 5.1]), then for every $A \in \mathcal{R}^{loc}$ with $\|\nu\|(A) < \infty$, we have that $\chi_A \in L^1_w(\nu) = L^1(\nu)$ and so, from [22, Theorem 4.9.(a)], ν is locally σ-finite.

Let us see that there are plenty of locally σ-finite vector measures which are not σ-finite.

Lemma 4.6 *Suppose that ν is discrete, that is, for every $\omega \in \Omega$ it follows that $\{\omega\} \in \mathcal{R}$ and $\nu(\{\omega\}) \neq 0$. Then,*

(a) *$N \in \mathcal{R}^{loc}$ is ν-null if and only if $N = \emptyset$.*
(b) *$\{A \subset \Omega : A$ is finite$\} \subset \mathcal{R} \subset \{A \subset \Omega : A$ is countable$\}$.*
(c) *$\mathcal{R}^{loc} = 2^\Omega$.*
(d) *ν is σ-finite if and only if Ω is countable.*

Proof (a) Suppose $N \in \mathcal{R}^{loc}$ is ν-null. If $\gamma \in N$, then $\{\gamma\} \in \mathcal{R} \cap 2^N$ and so $\|\nu(\{\gamma\})\| \leq \|\nu\|(N) = 0$ which contradicts $\nu(\{\gamma\}) \neq 0$. Hence, $N = \emptyset$. The converse is obvious.

(b) If $A \subset \Omega$ is finite then $A = \cup_{\gamma\in A}\{\gamma\}$ is a finite union of sets in $\mathcal{R}$, so the first containment holds. For the second one, consider $A \in \mathcal{R}$ and the vector measure $\nu_A : \mathcal{R}^{loc} \to X$ defined by $\nu_A(B) = \nu(A \cap B)$ for all $B \in \mathcal{R}^{loc}$. Note that $B \in \mathcal{R}^{loc}$ is ν_A-null if and only if $A \cap B$ is ν-null, that is, $A \cap B = \emptyset$. Since ν_A is defined on a σ-algebra we can take $x^*_A \in B_{X^*}$ such that $|x^*_A\nu_A|$ has the same null sets as ν_A (see [13, Theorem IX.2.2]). For every finite set $J \subset \Omega$ it follows that

$$\sum_{\gamma\in J} |x^*_A\nu_A|(\{\gamma\}) = |x^*_A\nu_A|(J) \le \|\nu_A\|(J) \le \|\nu_A\|(\Omega) < \infty.$$

Then, there exists a countable set $I \subset \Omega$ such that $|x^*_A\nu_A|(\{\gamma\}) = 0$ for all $\gamma \in \Omega\backslash I$, that is, $A \cap \{\gamma\} = \emptyset$ for all $\gamma \in \Omega\backslash I$. So, $A \subset I$ is countable.

(c) Note that $\{A \subset \Omega : A \text{ is countable}\} \subset \mathcal{R}^{loc}$, since if $A \subset \Omega$ is countable then $A = \cup_{\gamma\in A}\{\gamma\}$ is a countable union of sets in $\mathcal{R}$. Given $A \in 2^\Omega$, from (b) we have that $A \cap B$ is countable, and so it is in $\mathcal{R}^{loc}$ for every $B \in \mathcal{R}$. Hence, $A \cap B = B \cap (A \cap B) \in \mathcal{R}$ for every $B \in \mathcal{R}$, that is, $A \in \mathcal{R}^{loc}$.

(d) It follows from (a) and (b). □

From Remark 4.5 and Lemma 4.6, every discrete vector measure on a δ-ring of subsets of an uncountable set with values in a Banach space without any copy of c_0 is locally σ-finite, but not σ-finite. Also, there are locally σ-finite vector measures which are not σ-finite with values in a Banach space containing a copy of c_0.

Example 4.7 Consider the δ-ring $\mathcal{R} = \{A \subset [0,\infty) : A \text{ is finite}\}$ of subsets of $[0,\infty)$ and the vector measure $\nu : \mathcal{R} \to c_0$ defined by $\nu(A) = \sum_n \frac{\sharp(A\cap[n-1,n))}{2^n} e_n$, where $(e_n)_n$ is the canonical basis of c_0 and $\sharp$ denotes the cardinal of a set. Note that ν is discrete, so ν is not σ-finite. It can be proved that $L^1_w(\nu)$ is the space of functions $f : [0,\infty) \to \mathbb{R}$ such that

$$f\chi_{[n-1,n)} \in \ell^1([0,\infty)) \text{ for all } n \text{ and } \sup_n \frac{1}{2^n}\left\| |f|\chi_{[n-1,n)}\right\|_{\ell^1([0,\infty))} < \infty,$$

and $\|f\|_\nu = \sup_n \frac{1}{2^n}\left\||f|\chi_{[n-1,n)}\right\|_{\ell^1([0,\infty))}$ for all $f \in L^1_w(\nu)$. Moreover, $L^1(\nu)$ is the space of functions $f : [0,\infty) \to \mathbb{R}$ such that

$$f\chi_{[n-1,n)} \in \ell^1([0,\infty)) \text{ for all } n \text{ and } \lim_n \frac{1}{2^n}\left\||f|\chi_{[n-1,n)}\right\|_{\ell^1([0,\infty))} = 0.$$

Note that every $f \in L^1_w(\nu)$ has countable support as $\mathrm{supp}(f) \cap [n-1,n)$ is countable for all n. If $B \in \mathcal{R}^{loc}$ is such that $\|\nu\|(B) < \infty$, that is $\chi_B \in L^1_w(\nu)$, then B is countable. Hence, ν is locally σ-finite.

Let us prove now that the super order density of $L^1(\nu)$ in $L^1_w(\nu)$ is characterized by the local σ-finiteness of ν.

Theorem 4.8 *The space $L^1(\nu)$ is super order dense in $L^1_w(\nu)$ if and only if ν is locally σ-finite.*

Proof Suppose that $L^1(\nu)$ is super order dense in $L^1_w(\nu)$. Take $A \in \mathcal{R}^{loc}$ with $\|\nu\|(A) < \infty$. Since $0 \le \chi_A \in L^1_w(\nu)$, there exists a sequence $(f_n) \subset L^1(\nu)$ such that $0 \le f_n \uparrow \chi_A$ ν-a.e. Then, there exists $Z \in \mathcal{R}^{loc}$ ν-null such that $0 \le f_n(\omega) \uparrow \chi_A(\omega)$ for all $\omega \in \Omega\backslash Z$. Thus, $A\backslash Z = \cup_n \mathrm{supp}(f_n)\backslash Z$.

On the other hand, since each $f_n \in L^1(\nu)$, from [22, Theorem 4.9.(a)] , there exist $(A^n_j)_j \subset \mathcal{R}$ and a ν-null set $N_n \in \mathcal{R}^{loc}$ such that $\mathrm{supp}(f_n) = (\cup_j A^n_j) \cup N_n$. Then,

$$A = (\cup_n \cup_j A^n_j\backslash Z) \cup (\cup_n N_n\backslash Z) \cup (A \cap Z)$$

where $A_j^n \backslash Z \in \mathcal{R}$ and $(\cup_n N_n \backslash Z) \cup (A \cap Z)$ is ν-null.

Conversely, suppose that ν is locally σ-finite and let $0 \le f \in L_w^1(\nu)$. There exists a sequence $(\psi_n) \subset \mathcal{S}(\mathcal{R}^{loc})$ such that $0 \le \psi_n \uparrow f$ ν-a.e. For each n, we can write $\psi_n = \sum_{j=1}^{k_n} \alpha_j^n \chi_{B_j^n}$ with $(B_j^n)_j$ pairwise disjoint and $\alpha_j^n > 0$. Then, taking $\beta_n = \min\{\alpha_1^n, ..., \alpha_{k_n}^n\}$, it follows

$$\|\nu\|(\mathrm{supp}(\psi_n)) = \|\chi_{\mathrm{supp}(\psi_n)}\|_\nu \le \frac{1}{\beta_n}\|\psi_n\|_\nu \le \frac{1}{\beta_n}\|f\|_\nu < \infty.$$

So, there exist $(A_j^n)_j \subset \mathcal{R}$ and Z_n ν-null such that $\mathrm{supp}(\psi_n) = (\cup_j A_j^n) \cup Z_n$. Denote $\varphi_n = \psi_n \chi_{\cup_{i=1}^n \cup_{j=1}^n A_j^i} \in \mathcal{S}(\mathcal{R})$. For $\omega \notin \cup_n Z_n$ we have that $\omega \in \Omega \backslash (\cup_n \mathrm{supp}(\psi_n))$ or $\omega \in \cup_n \cup_j A_j^n$. In any case, $\varphi_n(\omega) = \psi_n(\omega)$ for all n large enough. Then, $\varphi_n \uparrow f$ ν-a.e. □

We have seen just before Example 4.4 that if ν is σ-finite then $L^1(\nu)$ is super order dense in $L^0(\nu)$. The converse also holds, indeed taking Ω instead of A in the proof of the local σ-finiteness of ν in Theorem 4.8, the same argument works to show $\Omega = (\cup A_n) \cup N$, with $N \in \mathcal{R}^{loc}$ ν-null and $(A_n) \subset \mathcal{R}$.

We know from [22, Theorem 4.9.(a)] that for each $f \in L^1(\nu)$ there are $(A_n) \subset \mathcal{R}$ and a ν-null set $N \in \mathcal{R}^{loc}$ such that $\mathrm{supp}(f) = (\cup A_n) \cup N$. Does the same hold for functions in $L_w^1(\nu)$?

Proposition 4.9 *For each $f \in L_w^1(\nu)$ there exist $N \in \mathcal{R}^{loc}$ ν-null and $(A_n) \subset \mathcal{R}$ such that supp$(f) = (\cup A_n) \cup N$ if and only if ν is locally σ-finite.*

Proof Suppose that ν is locally σ-finite and take $f \in L_w^1(\nu)$. From the proof of Theorem 4.8, there exists a sequence $(\varphi_n) \subset \mathcal{S}(\mathcal{R})$ such that $0 \le \varphi_n \uparrow |f|$ ν-a.e. Let $Z \in \mathcal{R}^{loc}$ be a ν-null set such that $0 \le \varphi_n(\omega) \uparrow |f(\omega)|$ for all $\omega \in \Omega \backslash Z$. Then,

$$\mathrm{supp}(f) = (\cup\, \mathrm{supp}(\varphi_n) \backslash Z) \cup (\mathrm{supp}(f) \cap Z)$$

where $\mathrm{supp}(\varphi_n) \backslash Z \in \mathcal{R}$ and $\mathrm{supp}(f) \cap Z$ is ν-null. For the converse only note that if $B \in \mathcal{R}^{loc}$ is such that $\|\nu\|(B) < \infty$, then $\chi_B \in L_w^1(\nu)$. □

Let $\{\Omega_\alpha : \alpha \in \Delta\}$ be a maximal family of non ν-null sets in $\mathcal{R}$ with $\Omega_\alpha \cap \Omega_\beta$ ν-null for $\alpha \ne \beta$ (see the proof of [1, Theorem 3.1] for the existence of such a family). Then, $L^1(\nu)$ is the unconditional direct sum of the spaces $L^1(\nu_\alpha)$ where $\nu_\alpha : \Sigma_\alpha \to X$ is the restriction of ν to the σ-algebra $\Sigma_\alpha = \{A \in \mathcal{R} : A \subset \Omega_\alpha\}$. More precisely, for each $f \in L^1(\nu)$ there exists a countable set $I \subset \Delta$ such that $f = \sum_{\alpha \in I} f \chi_{\Omega_\alpha}$ ν-a.e. and the sum converges unconditionally in $L^1(\nu)$, see [9, Theorem 3.6]. Does a similar result hold for the space $L_w^1(\nu)$? The ν-a.e. pointwise convergence of the sum for functions in $L_w^1(\nu)$ is again characterized by the local σ-finiteness of ν.

Proposition 4.10 *For each $f \in L_w^1(\nu)$ there exists a countable $I \subset \Delta$ such that $f = \sum_{\alpha \in I} f \chi_{\Omega_\alpha}$ ν-a.e. pointwise if and only if ν is locally σ-finite.*

Proof Suppose that for every $f \in L_w^1(\nu)$ there exists a countable $I \subset \Delta$ such that $f = \sum_{\alpha \in I} f \chi_{\Omega_\alpha}$ ν-a.e. pointwise. Then, given $B \in \mathcal{R}^{loc}$ with $\|\nu\|(B) < \infty$, since $\chi_B \in L_w^1(\nu)$, we can write $\chi_B = \sum_{\alpha \in I} \chi_{B \cap \Omega_\alpha}$ pointwise except on a ν-null set Z, for some countable $I \subset \Delta$. So, $B = (\cup_{\alpha \in I} B \cap \Omega_\alpha) \cup (B \cap Z)$, where $B \cap \Omega_\alpha \in \mathcal{R}$ and $B \cap Z$ is ν-null.

Conversely, suppose that ν is locally σ-finite and take $f \in L_w^1(\nu)$. From Proposition 4.9, there exist $(A_n) \subset \mathcal{R}$ and a ν-null set $N \in \mathcal{R}^{loc}$ such that $\mathrm{supp}(f) = (\cup A_n) \cup N$. Since each

$A_n \in \mathcal{R}$, there exists a countable set $I_n \subset \Delta$ such that $A_n \cap \Omega_\beta$ is ν-null for all $\beta \in \Delta\backslash I_n$ (see the proof of [1, Theorem 3.1]). Take $I = \cup I_n$ and $Z = \text{supp}(f)\backslash \cup_{\alpha\in I} \Omega_\alpha$. Let us see that Z is a ν-null set. Given $B \in \mathcal{R} \cap 2^Z$, if $\beta \in I$ we have that $B \cap \Omega_\beta = \emptyset$. On the other hand, if $\beta \notin I$, since $B \cap \Omega_\beta \subset \text{supp}(f) \cap \Omega_\beta = (\cup A_n \cap \Omega_\beta) \cup (N \cap \Omega_\beta)$ where each $A_n \cap \Omega_\beta$ is ν-null, we have that $B \cap \Omega_\beta$ is ν-null. From the maximality of the family $\{\Omega_\alpha : \alpha \in \Delta\}$ it follows that B is ν-null. Then, $f = \sum_{\alpha\in I} f\chi_{\Omega_\alpha}$ pointwise except on $Z \cup (\cup_{\beta\in I} \cup_{\alpha\in I\backslash\{\beta\}} \Omega_\alpha \cap \Omega_\beta)$ which is a ν-null set. □

Since $f\chi_{\Omega_\alpha} \in L^1_w(\nu_\alpha)$ for all $\alpha \in \Delta$ whenever $f \in L^1_w(\nu)$, in the case of ν being locally σ-finite, we can say that the space $L^1_w(\nu)$ is the ν-a.e. pointwise direct sum of the spaces $L^1_w(\nu_\alpha)$. We cannot expect that $\sum_{\alpha\in I} f\chi_{\Omega_\alpha}$ converges unconditionally to f in $L^1_w(\nu)$ for a countable set $I \subset \Delta$. Indeed, unconditional convergence of the sum in $L^1(\nu)$ is due to the order continuity of $L^1(\nu)$. For instance, assume that ν is a discrete vector measure. Note that the maximal family $\{\{\gamma\} : \gamma \in \Gamma\}$ of non ν-null sets in $\mathcal{R}$ satisfies that $\{\alpha\} \cap \{\beta\}$ ν-null for $\alpha \neq \beta$. We have that if $f \in L^1_w(\nu)$ is such that $\sum_n f\chi_{\{\gamma_n\}}$ converges to f in norm $\|\cdot\|_\nu$, then $f \in L^1(\nu)$. This is due to the fact that $\sum_{k=1}^n f\chi_{\{\gamma_k\}} = \sum_{k=1}^n f(\gamma_k)\chi_{\{\gamma_k\}} \in \mathcal{S}(\mathcal{R}) \subset L^1(\nu)$ and $L^1(\nu)$ is closed in $L^1_w(\nu)$.

5 Fatou property for $L^1_w(\nu)$

The space $L^1_w(\nu)$ always has the σ-Fatou property. Indeed, take $(f_n) \subset L^1_w(\nu)$ such that $0 \le f_n \uparrow$ and $\sup \|f_n\|_\nu < \infty$. Then there exists a ν-null set $Z \in \mathcal{R}^{loc}$ such that $0 \le f_n(\omega) \uparrow$ for all $\omega \in \Omega\backslash Z$. Taking the measurable function $g\colon \Omega \to [0, \infty]$ defined by $g(\omega) = \sup f_n(\omega)$ if $\omega \in \Omega\backslash Z$ and $g(\omega) = 0$ if $\omega \in Z$, we have that $0 \le f_n\chi_{\Omega\backslash Z} \uparrow g$ pointwise. Hence, the monotone convergence theorem, gives

$$\int g\, d|x^*\nu| = \lim_n \int f_n\chi_{\Omega\backslash Z}\, d|x^*\nu| \le \|x^*\| \sup \|f_n\|_\nu,$$

for every $x^* \in X^*$. So, $\|g\|_\nu \le \sup \|f_n\|_\nu < \infty$, and then $g < \infty$ ν-a.e. (except on a ν-null set N). Taking $f = g\chi_{\Omega\backslash N}$ we have that $f\colon \Omega \to [0, \infty)$ and $\|f\|_\nu = \|g\|_\nu < \infty$, so $f \in L^1_w(\nu)$. Moreover, $0 \le f_n \uparrow f$ ν-a.e. with $\|f\|_\nu = \sup \|f_n\|_\nu$, as $\|f_n\|_\nu \le \|f\|_\nu \le \sup \|f_n\|_\nu$ for all n. Therefore $L^1_w(\nu)$ always has the σ-Fatou property.

In the case when ν is defined on a σ-algebra, it was noted in [6, p. 191] that $L^1_w(\nu)$ is the *σ-Fatou completion* of $L^1(\nu)$, that is, the minimal B.f.s. related to ν with the σ-Fatou property and containing $L^1(\nu)$. This fact does not hold for the general case. For instance, if ν is the vector measure defined in Example 4.1 and $\ell_0^\infty(\Gamma)$ denotes the Banach lattice of all real bounded functions on Γ with countable support, then $L^1(\nu) \subsetneq \ell_0^\infty(\Gamma) \subsetneq L^1_w(\nu)$ where $\ell_0^\infty(\Gamma)$ has the σ-Fatou property. Note that in this case ν is not locally σ-finite, as $\chi_\Gamma \in L^1_w(\nu)$. This is the reason for which $L^1_w(\nu)$ fails to be the σ-Fatou completion of $L^1(\nu)$. Let us denote by $[L^1(\nu)]_{\sigma-\text{F}}$ the σ-Fatou completion of $L^1(\nu)$. In general we have that $[L^1(\nu)]_{\sigma-\text{F}} \subset L^1_w(\nu)$.

Theorem 5.1 *The σ-Fatou completion of $L^1(\nu)$ can be described as*

$$[L^1(\nu)]_{\sigma-\text{F}} = \left\{ f \in L^1_w(\nu) : \textit{supp}(f) = (\cup A_n) \cup N \textit{ with } (A_n) \subset \mathcal{R} \textit{ and } N\ \nu\textit{-null} \right\}.$$

Consequently, the space $L^1_w(\nu) = [L^1(\nu)]_{\sigma-\text{F}}$ if and only if ν is locally σ-finite.

Proof Denote by F the space of functions $f \in L^1_w(\nu)$ for which there exist $(A_n) \subset \mathcal{R}$ and a ν-null set $N \in \mathcal{R}^{loc}$ such that $\text{supp}(f) = (\cup A_n) \cup N$. Let us see that F is a closed subspace of $L^1_w(\nu)$. Given $f \in L^1_w(\nu)$ and $(f_n) \subset F$ such that $\|f - f_n\|_\nu \to 0$, we can take a subsequence such that $f_{n_k} \to f$ ν-a.e. That is, there exists a ν-null set $Z \in \mathcal{R}^{loc}$ such that $f_{n_k}(\omega) \to f(\omega)$ for all $\omega \in \Omega\backslash Z$. Then, $\mathrm{supp}(f)\backslash Z \subset \cup_k \mathrm{supp}(f_{n_k})$. On the other hand, each f_{n_k} satisfies that $\mathrm{supp}(f_{n_k}) = (\cup_j A^k_j) \cup N_k$ for some $(A^k_j)_j \subset \mathcal{R}$ and $N_k \in \mathcal{R}^{loc}$ ν-null. So, $\mathrm{supp}(f) = \cup_k \cup_j B^k_j \cup N$ where $B^k_j = A^k_j \cap \mathrm{supp}(f)\backslash Z \in \mathcal{R}$ and $N = (\cup_k N_k \cap \mathrm{supp}(f)\backslash Z) \cup (\mathrm{supp}(f) \cap Z)$ is ν-null, that is, $f \in F$. Note that if $|f| \le |g|$ ν-a.e. with $f \in L^0(\nu)$ and $g \in F$, then $f \in F$ since $\mathrm{supp}(f)\backslash Z = (\mathrm{supp}(f)\backslash Z) \cap \mathrm{supp}(g)$ for some ν-null set Z. Therefore, F endowed with the norm $\|\cdot\|_\nu$, is a B.f.s. related to ν, which, by [22, Theorem 4.9.(a)], contains $L^1(\nu)$. Let us see now that F has the σ-Fatou property. Given $(f_n) \subset F$ such that $0 \le f_n \uparrow$ and $\sup \|f_n\|_\nu < \infty$, since $L^1_w(\nu)$ has the σ-Fatou property, there exists $f = \sup f_n \in L^1_w(\nu)$ with $\|f\|_\nu = \sup \|f_n\|_\nu$. Moreover, since $0 \le f_n \uparrow f$ ν-a.e., $\mathrm{supp}(f) = (\cup\, \mathrm{supp}(f_n)\backslash Z) \cup (\mathrm{supp}(f) \cap Z)$ for some ν-null set $Z \in \mathcal{R}^{loc}$. Then, it follows that $f \in F$, as each $f_n \in F$.

Suppose that E is a B.f.s. related to ν, with the σ-Fatou property and containing $L^1(\nu)$. Let $f \in F$ and take a sequence $(A_n) \subset \mathcal{R}$ and a ν-null set $N \in \mathcal{R}^{loc}$ such that supp $(f) = (\cup A_n) \cup N$. On the other hand, take a sequence $(\psi_n) \subset \mathcal{S}(\mathcal{R}^{loc})$ such that $0 \le \psi_n \uparrow |f|$ ν-a.e. Denoting $\varphi_n = \psi_n \chi_{\cup_{j=1}^n A_j} \in \mathcal{S}(\mathcal{R}) \subset L^1(\nu)$ we have that $0 \le \varphi_n \uparrow |f|$ ν-a.e. Since $L^1(\nu) \subset E$ continuously (bear in mind that the inclusion is a positive operator) we have that $\sup \|\varphi_n\|_E \le C \sup \|\varphi_n\|_\nu \le C\|f\|_\nu < \infty$ for some positive constant C. It follows that there exists $g = \sup \varphi_n \in E$. Then, since $0 \le \varphi_n \uparrow g$ ν-a.e., we have that $|f| = g \in E$ and so $f \in E$.

The consequence follows from Proposition 4.9. □

Consider now the *Fatou completion* $[L^1(\nu)]_{\mathrm{F}}$ of $L^1(\nu)$, namely, the minimal B.f.s. related to ν with the Fatou property and containing $L^1(\nu)$. The σ-Fatou completion $[L^1(\nu)]_{\sigma-\mathrm{F}}$ always exists since $L^1_w(\nu)$ has always the σ-Fatou property. However, we do not know if in general $L^1_w(\nu)$ has the Fatou property, so $[L^1(\nu)]_{\mathrm{F}}$ could not exist.

Remark 5.2 In the case when $[L^1(\nu)]_{\mathrm{F}}$ exists, we have that

$$L^1(\nu) \subset [L^1(\nu)]_{\sigma-\mathrm{F}} \subset L^1_w(\nu) \subset [L^1(\nu)]_{\mathrm{F}}.$$

Indeed, given $f \in L^1_w(\nu)$, from Remark 4.3, there exists $(f_\tau) \subset L^1(\nu)$ such that $0 \le f_\tau \uparrow |f|$ in $L^0(\nu)$. Since $L^1(\nu) \subset [L^1(\nu)]_{\mathrm{F}}$ continuously, it follows that $\sup \|f_\tau\|_{[L^1(\nu)]_{\mathrm{F}}} \le C \sup \|f_\tau\|_\nu \le C\|f\|_\nu < \infty$ for some constant $C > 0$. Then, there exists $g = \sup f_\tau$ in $[L^1(\nu)]_{\mathrm{F}}$. Noting that $f_\tau \le g \in L^0(\nu)$ for all τ, we have that $|f| \le g$ and so $|f| \in [L^1(\nu)]_{\mathrm{F}}$. Hence, $f \in [L^1(\nu)]_{\mathrm{F}}$. Note that actually $|f| = g$, since $f_\tau \le |f| \in [L^1(\nu)]_{\mathrm{F}}$ for all τ and so $g \le |f|$.

Remark 5.3 If $L^1_w(\nu)$ has the Fatou property, then $[L^1(\nu)]_{\mathrm{F}}$ exists and, from Remark 5.2, we have that $L^1_w(\nu) = [L^1(\nu)]_{\mathrm{F}}$.

In the following result we give conditions under which $L^1_w(\nu)$ has the Fatou property. These conditions are satisfied for instance if ν takes values in a Banach space without any copy of c_0.

Proposition 5.4 *The following statements are equivalent:*

(a) $L^1(\nu) = L^1_w(\nu)$.
(b) $L^1_w(\nu)$ *is order continuous.*
(c) $L^1(\nu)$ *has the σ-Fatou property.*

If (a)–(c) *hold, then* $L^1_w(\nu)$ *has the Fatou property and*

$$L^1(\nu) = [L^1(\nu)]_{\sigma-\mathrm{F}} = L^1_w(\nu) = [L^1(\nu)]_{\mathrm{F}}.$$

Proof The equivalence between (a) and (b) follows from Theorem 3.2. Condition (a) implies (c) as $L^1_w(\nu)$ always has the σ-Fatou property. Conversely, if $L^1(\nu)$ has the σ-Fatou property, from [26, Theorem 113.4], it follows that it actually has the Fatou property. Then, $[L^1(\nu)]_{\mathrm{F}} = L^1(\nu)$ and, from Remark 5.2, we have that $L^1(\nu) = L^1_w(\nu)$. So, (c) implies (a) and the last part of the theorem holds. □

It is an open question if in general $L^1_w(\nu)$ has the Fatou property. The problem is that for an upwards directed system $0 \leq f_\tau \uparrow$ such that $(f_\tau) \subset L^1_w(\nu)$ with $\sup \|f_\tau\|_\nu < \infty$ the pointwise supremum $f = \sup f_\tau$ may not be measurable. Moreover, even if $f \in L^1_w(\nu)$ it can happen that $f_\tau \uparrow f$ does not hold, that is, f may be not the lattice supremum of (f_τ).

Remark 5.5 If ν is σ-finite, we can take a measure of the type $|x_0^*\nu|$ (with $x_0^* \in B_{X^*}$) having the same null sets as ν, see [9, Remark 3.4]. Then, since $L^1_w(\nu) \subset L^1(|x_0^*\nu|)$ and $L^1(|x_0^*\nu|)$ has the Fatou property, there exists $f = \sup f_\tau$ in $L^1(|x_0^*\nu|)$. By using the fact that $L^1(|x_0^*\nu|)$ is order separable (see [26, Theorem 113.4]), we can take a sequence $f_{\tau_n} \uparrow f$ in $L^1(|x_0^*\nu|)$ and prove that $f \in L^1_w(\nu)$. Then, $L^1_w(\nu)$ has the Fatou property, see [12, Proposition 1]. Moreover, it follows that $[L^1(\nu)]_{\sigma-\mathrm{F}} = L^1_w(\nu) = [L^1(\nu)]_{\mathrm{F}}$ from Theorem 5.1 and Remark 5.3.

We will give a more general condition than the σ-finiteness of ν under which $L^1_w(\nu)$ has the Fatou property. This new condition is inspired by the particular vector measure ν constructed in [12, Theorem 9] to prove that a Banach lattice E with the Fatou property and such that E_a is order dense in E, is order isometric to a $L^1_w(\nu)$. In this case, $L^1_w(\nu)$ has the Fatou property due to a good decomposition property satisfied by ν.

Definition 5.6 A vector measure ν will be said to be $\mathcal{R}$-*decomposable* if we can write $\Omega = (\cup_{\alpha\in\Delta}\Omega_\alpha) \cup N$ where $N \in \mathcal{R}^{loc}$ is a ν-null set and $\{\Omega_\alpha : \alpha \in \Delta\}$ is a family of pairwise disjoint sets in $\mathcal{R}$ satisfying that

(i) if $A_\alpha \in \mathcal{R} \cap 2^{\Omega_\alpha}$ for all $\alpha \in \Delta$, then $\cup_{\alpha\in\Delta} A_\alpha \in \mathcal{R}^{loc}$, and
(ii) for each $x^* \in X^*$, if $Z_\alpha \in \mathcal{R} \cap 2^{\Omega_\alpha}$ is $|x^*\nu|$-null for all $\alpha \in \Delta$, then $\cup_{\alpha\in\Delta} Z_\alpha$ is $|x^*\nu|$-null.

Note that condition (ii) implies that if $Z_\alpha \in \mathcal{R} \cap 2^{\Omega_\alpha}$ is ν-null for all $\alpha \in \Delta$, then $\cup_{\alpha\in\Delta} Z_\alpha$ is ν-null. Also note that N can be taken to be disjoint with $\cup_{\alpha\in\Delta}\Omega_\alpha$.

Remark 5.7 There always exists a maximal family $\{\widetilde{\Omega}_\alpha : \alpha \in \Delta\}$ of non ν-null sets in $\mathcal{R}$ with $\widetilde{\Omega}_\alpha \cap \widetilde{\Omega}_\beta$ ν-null for $\alpha \neq \beta$ (see the proof of [1, Theorem 3.1]). If this family satisfies (i) and (ii) of Definition 5.6, then by taking $\Omega_\alpha = \widetilde{\Omega}_\alpha \backslash (\cup_{\beta\in\Delta\backslash\{\alpha\}}\widetilde{\Omega}_\beta)$ we obtain a disjoint decomposition of Ω as in Definition 5.6.

There are plenty of $\mathcal{R}$-decomposable vector measures, for instance σ-finite vector measures and discrete vector measures are so.

Theorem 5.8 *If* ν *is* $\mathcal{R}$-*decomposable, then* $L^1_w(\nu)$ *has the Fatou property.*

Proof Suppose that ν is $\mathcal{R}$-decomposable and take a ν-null set $N \in \mathcal{R}^{loc}$ and a family $\{\Omega_\alpha : \alpha \in \Delta\}$ of pairwise disjoint sets in $\mathcal{R}$ satisfying conditions (i) and (ii) in Definition 5.6. So we have $\Omega = (\cup_{\alpha\in\Delta}\Omega_\alpha) \cup N$ with disjoint union. For every finite set $I \subset \Delta$, consider $\Omega_I = \cup_{\alpha\in I}\Omega_\alpha \in \mathcal{R}$ and the vector measure $\nu_I : \mathcal{R}^{loc} \to X$ defined by $\nu(A \cap \Omega_I)$ for all

$A \in \mathcal{R}^{loc}$. Given $f \in \mathcal{M}(\mathcal{R}^{loc})$, by using a similar argument as in the proof of (c) implies (a) in Lemma 3.1, it follows that $f \in L^1_w(\nu_I)$ if and only if $f\chi_{\Omega_I} \in L^1_w(\nu)$, and in this case $\|f\|_{\nu_I} = \|f\chi_{\Omega_I}\|_\nu$. Note that, if $f \in L^1_w(\nu)$ then $f\chi_{\Omega_I} \in L^1_w(\nu)$ and so $f \in L^1_w(\nu_I)$. Also note that $L^1_w(\nu_I)$ has the Fatou property as ν_I is defined on a σ-algebra, see Remark 5.5.

Let $(f_\tau) \subset L^1_w(\nu)$ be such that $0 \le f_\tau \uparrow$ and $\sup \|f_\tau\|_\nu < \infty$. Since $L^1_w(\nu) \subset L^1_w(\nu_I)$ and every $Z \in \mathcal{R}^{loc}$ ν-null is ν_I-null (as $\|\nu_I\|(Z) = \|\nu\|(Z \cap \Omega_I)$), then $0 \le f_\tau \uparrow$ in $L^1_w(\nu_I)$. Moreover, $\sup \|f_\tau\|_{\nu_I} = \sup \|f_\tau\chi_{\Omega_I}\|_\nu \le \sup \|f_\tau\|_\nu < \infty$. By the Fatou property of $L^1_w(\nu_I)$, there exists $f^I = \sup f_\tau$ in $L^1_w(\nu_I)$ and $\|f^I\|_{\nu_I} = \sup \|f_\tau\|_{\nu_I}$.

Now we consider $I = \{\alpha\}$ for each $\alpha \in \Delta$ and construct the function $f : \Omega \to \mathbb{R}$ as $f(\omega) = f^{\{\alpha\}}(\omega)$ when $\omega \in \Omega_\alpha$ and $f(\omega) = 0$ when $\omega \in N$, which is well defined since Ω is a disjoint union of $(\Omega_\alpha)_{\alpha\in\Delta}$ and N. By (i), we have that $f^{-1}(B) = \cup_{\alpha\in\Delta}(f^{\{\alpha\}})^{-1}(B) \cap \Omega_\alpha \in \mathcal{R}^{loc}$ for every Borel subset B of $\mathbb{R}$ such that $0 \notin B$. If $0 \in B$, we put also in the union the set N to get $f^{-1}(B)$. So, $f \in \mathcal{M}(\mathcal{R}^{loc})$.

Let us see that $f \in L^1_w(\nu)$. First note that for each finite set $I \subset \Delta$ and $\alpha \in I$, it follows that $f^{\{\alpha\}}\chi_{\Omega_\alpha} \le f^I\chi_{\Omega_\alpha}$ ν-a.e. Indeed, $f_\tau\chi_{\Omega_\alpha} \uparrow f^{\{\alpha\}}\chi_{\Omega_\alpha}$ in $L^1_w(\nu_{\{\alpha\}})$ as $f_\tau \uparrow f^{\{\alpha\}}$ in $L^1_w(\nu_{\{\alpha\}})$. Since $f_\tau\chi_{\Omega_\alpha} \le f^I\chi_{\Omega_\alpha}$ ν_I-a.e. (and so also $\nu_{\{\alpha\}}$-a.e. and $f^I\chi_{\Omega_\alpha} \in L^1_w(\nu_{\{\alpha\}})$ as $f^I\chi_{\Omega_\alpha} \le f^I\chi_{\Omega_I} \in L^1_w(\nu)$) we have that $f^{\{\alpha\}}\chi_{\Omega_\alpha} \le f^I\chi_{\Omega_\alpha}$ $\nu_{\{\alpha\}}$-a.e. (except on a $\nu_{\{\alpha\}}$-null set Z) and so ν-a.e. (except on the ν-null set $Z \cap \Omega_\alpha$). Then, $f\chi_{\Omega_I} = \sum_{\alpha\in I} f^{\{\alpha\}}\chi_{\Omega_\alpha} \le f^I\chi_{\Omega_I}$ ν-a.e.

Fix $x^* \in X^*$. For every finite set $I \subset \Delta$, it follows

$$\begin{aligned}\sum_{\alpha\in I}\int |f|\chi_{\Omega_\alpha}\, d|x^*\nu| &= \int |f|\chi_{\Omega_I}\, d|x^*\nu| \le \int |f^I|\chi_{\Omega_I}\, d|x^*\nu| \\ &\le \|x^*\| \cdot \|f^I\chi_{\Omega_I}\|_\nu = \|x^*\| \cdot \|f^I\|_{\nu_I} \\ &= \|x^*\| \cdot \sup \|f_\tau\|_{\nu_I} \le \|x^*\| \cdot \sup \|f_\tau\|_\nu < \infty.\end{aligned}$$

Then, there exists a countable set $J \subset \Delta$ such that $\int |f|\chi_{\Omega_\alpha}\, d|x^*\nu| = 0$ for all $\alpha \in \Delta\backslash J$ and so $f\chi_{\Omega_\alpha} = 0$ $|x^*\nu|$-a.e. (except on a $|x^*\nu|$-null set $Z_\alpha \in \mathcal{R}^{loc}$ which can be taken such that $Z \subset \Omega_\alpha$) for all $\alpha \in \Delta\backslash J$. Hence, $f = \sum_{\alpha\in J} f\chi_{\Omega_\alpha}$ $|x^*\nu|$-a.e. (except on the $|x^*\nu|$-null set $\cup_{\alpha\in\Delta\backslash J} Z_\alpha \cup N \in \mathcal{R}^{loc}$). By the monotone convergence theorem we have that

$$\int |f|\, d|x^*\nu| = \sum_{\alpha\in J}\int |f|\chi_{\Omega_\alpha}\, d|x^*\nu| \le \|x^*\| \cdot \sup \|f_\tau\|_\nu < \infty.$$

So $f \in L^1_w(\nu)$ and $\|f\|_\nu \le \sup \|f_\tau\|_\nu$.

Let us see now that $f_\tau \uparrow f$ in $L^1_w(\nu)$. Fixing τ, for each $\alpha \in \Delta$, there exists a $\nu_{\{\alpha\}}$-null set $Z_\alpha \in \mathcal{R}^{loc}$ such that $f_\tau(\omega) \le f^{\{\alpha\}}(\omega)$ for all $\omega \in \Omega_\alpha\backslash Z_\alpha$. Then, $Z = \cup_{\alpha\in\Delta} Z_\alpha \cap \Omega_\alpha$ is ν-null and $f_\tau(\omega) \le f(\omega)$ for all $\omega \in \Omega\backslash(Z \cup N)$, that is, $f_\tau \le f$ ν-a.e. Suppose that $h \in L^1_w(\nu)$ is such that $f_\tau \le h$ ν-a.e. (except on a ν-null set $Z \in \mathcal{R}^{loc}$) and so $\nu_{\{\alpha\}}$-a.e. (except Z which also is $\nu_{\{\alpha\}}$-null) for each τ. Since $h \in L^1_w(\nu_{\{\alpha\}})$, we have that $f^{\{\alpha\}} \le h$ $\nu_{\{\alpha\}}$-a.e. (except on a $\nu_{\{\alpha\}}$-null set $Z_\alpha \in \mathcal{R}^{loc}$). Therefore, $f \le h$ ν-a.e. (except on the ν-null set $(\cup_{\alpha\in\Delta} Z_\alpha \cap \Omega_\alpha) \cup N \in \mathcal{R}^{loc}$). So, $f_\tau \uparrow f$ and $\|f\|_\nu = \sup \|f_\tau\|_\nu$. □

The converse of Theorem 5.8 does not hold as the next example shows.

Example 5.9 Following [16, p. 12, Definition 211E], a measure space (X, Σ, μ) is *decomposable* (or *strictly localizable*) if there exists a disjoint family $\{X_\alpha : \alpha \in \Delta\}$ of measurable sets of finite measure such that $X = \cup_{\alpha\in\Delta} X_\alpha$ and

$$\Sigma = \{E \subset X : E \cap X_\alpha \in \Sigma \text{ for all } \alpha \in \Delta\}$$

with $\mu(E) = \sum_{\alpha\in\Delta} \mu(E \cap X_\alpha)$ for every $E \in \Sigma$. In [16, p. 50, 216E], Fremlin constructs a measure space which is not decomposable as follows.

Let C be an abstract set of cardinal greater than the cardinal of the continuum, $\mathcal{K} = \{K \subset 2^C : K \text{ is countable}\}$ and X the set of all functions $f : 2^C \to \{0, 1\}$. For each $\gamma \in C$, write f_γ for the function in X defined by $f_\gamma(A) = \chi_A(\gamma)$ for all $A \in 2^C$ and $F_{\gamma,K} = \{f \in X : f_{|K} = f_{\gamma|K}\}$ for every $K \in \mathcal{K}$. Consider the σ-algebra $\Sigma = \cap_{\gamma\in C}\Sigma_\gamma$, where

$$\Sigma_\gamma = \{E \subset X : \exists K \in \mathcal{K} \text{ with } F_{\gamma,K} \subset E \text{ or } \exists K \in \mathcal{K} \text{ with } F_{\gamma,K} \subset X\backslash E\},$$

and the measure $\mu : \Sigma \to [0, \infty]$ defined by $\mu(E) = \sharp(\{\gamma \in C : f_\gamma \in E\})$ for all $E \in \Sigma$, where $\sharp$ denotes the cardinal of a set. Then, (X, Σ, μ) is not decomposable.

Taking the δ-ring $\mathcal{R} = \{E \in \Sigma : \mu(E) < \infty\}$, we will show that the measure $\widetilde{\mu} : \mathcal{R} \to [0, \infty)$ given by the restriction of μ to $\mathcal{R}$ is not $\mathcal{R}$-decomposable. Let us see first that

$$\mathcal{R}^{loc} = \Sigma. \tag{1}$$

If $A \in \Sigma$, then obviously $A \cap E \in \mathcal{R}$ for every $E \in \mathcal{R}$, that is $A \in \mathcal{R}^{loc}$. Conversely, suppose that $A \in \mathcal{R}^{loc}$. For a fixed $\gamma \in C$, the set $G_{\{\gamma\}} = \{f \in X : f(\{\gamma\}) = 1\}$ is in Σ and $\mu(G_{\{\gamma\}}) = \sharp(\{\gamma\}) = 1$ (see [16, 216E.(c)]). So, $G_{\{\gamma\}} \in \mathcal{R}$ and thus $A \cap G_{\{\gamma\}} \in \mathcal{R} \subset \Sigma \subset \Sigma_\gamma$. If there exists $K \in \mathcal{K}$ such that $F_{\gamma,K} \subset A \cap G_{\{\gamma\}} \subset A$, then $A \in \Sigma_\gamma$. If there exists $K \in \mathcal{K}$ such that $F_{\gamma,K} \subset X\backslash(A \cap G_{\{\gamma\}})$, then, since $F_{\gamma,K\cup\{\gamma\}} \subset F_{\gamma,K}$ and $F_{\gamma,K\cup\{\gamma\}} \subset G_{\{\gamma\}}$, it follows that $F_{\gamma,K\cup\{\gamma\}} \subset X\backslash A$ and so $A \in \Sigma_\gamma$. Therefore, $A \in \Sigma$ and (1) holds. Moreover, for $N \in \mathcal{R}^{loc}$ we have that

$$N \text{ is } \widetilde{\mu} - \text{null if} \quad \text{and} \quad \text{only if } N \text{ is } \mu - \text{null}. \tag{2}$$

Indeed, if N is μ-null, for every $E \in \mathcal{R}\cap 2^N$ we have that $\widetilde{\mu}(E) = \mu(E) \le \mu(N) = 0$ and so N is $\widetilde{\mu}$-null. Conversely, suppose that N is $\widetilde{\mu}$-null. If $\mu(N) > 0$, then there exists $\gamma \in C$ such that $\mu(N \cap G_{\{\gamma\}}) = 1$ (see [16, 216E.(h)]), this is a contradiction as $N \cap G_{\{\gamma\}} \in \mathcal{R}\cap 2^N$ and so $\mu(N \cap G_{\{\gamma\}}) = \widetilde{\mu}(N \cap G_{\{\gamma\}}) = 0$.

Suppose that $\widetilde{\mu}$ is $\mathcal{R}$-decomposable, that is, we can write $X = (\cup_{\alpha\in\Delta} X_\alpha) \cup N$ where $\{X_\alpha : \alpha \in \Delta\}$ is a family of pairwise disjoint sets in $\mathcal{R}$ satisfying that

(i) if $A_\alpha \in \mathcal{R} \cap 2^{X_\alpha}$ for all $\alpha \in \Delta$, then $\cup_{\alpha\in\Delta} A_\alpha \in \mathcal{R}^{loc}$,
(ii) if $Z_\alpha \in \mathcal{R} \cap 2^{X_\alpha}$ is $\widetilde{\mu}$-null for all $\alpha \in \Delta$, then $\cup_{\alpha\in\Delta} Z_\alpha$ is $\widetilde{\mu}$-null,

and $N \in \mathcal{R}^{loc}$ is a $\widetilde{\mu}$-null set disjoint with each X_α. Then, $\{X_\alpha : \alpha \in \Delta\} \cup \{N\}$ is a disjoint family of sets in Σ with $\mu(N), \mu(X_\alpha) < \infty$. Let us see that

$$\Sigma = \{E \subset X : E \cap N \in \Sigma \text{ and } E \cap X_\alpha \in \Sigma \text{ for all } \alpha \in \Delta\}.$$

If $E \in \Sigma$, then obviously $E \cap X_\alpha \in \Sigma$ for all $\alpha \in \Delta$ and, by (1), $E \cap N \in \Sigma$. Conversely, if $E \subset X$ is such that $E \cap N \in \Sigma$ and $E \cap X_\alpha \in \Sigma$ for all $\alpha \in \Delta$, since $E \cap X_\alpha \in \mathcal{R}\cap 2^{X_\alpha}$, by (i) and (1), we have that $\cup_{\alpha\in\Delta} E \cap X_\alpha \in \Sigma$. So, $E = E \cap X = (\cup_{\alpha\in\Delta} E \cap X_\alpha) \cup (E \cap N) \in \Sigma$. Moreover, $\mu(E) = \sum_{\alpha\in\Delta} \mu(E \cap X_\alpha)$ for every $E \in \Sigma$. Indeed, if $\sum_{\alpha\in\Delta} \mu(E \cap X_\alpha) < \infty$, then $\mu(E \cap X_\alpha) = 0$ for all $\alpha \in \Delta\backslash\Gamma$ for some countable $\Gamma \subset \Delta$. Since, by (ii) and (2), $\cup_{\alpha\in\Delta\backslash\Gamma} E \cap X_\alpha$ is μ-null,

$$\mu(E) = \mu(\cup_{\alpha\in\Gamma} E \cap X_\alpha) = \sum_{\alpha\in\Gamma} \mu(E \cap X_\alpha) = \sum_{\alpha\in\Delta} \mu(E \cap X_\alpha).$$

If $\sum_{\alpha\in\Delta} \mu(E \cap X_\alpha) = \infty$ then $\mu(E) = \infty$, as $\sup_{\substack{J\subset\Delta \\ \text{finite}}} \sum_{\alpha\in J} \mu(E \cap X_\alpha) \le \mu(E)$. Therefore (X, Σ, μ) is decomposable which is a contradiction.

So, $\widetilde{\mu}$ is not $\mathcal{R}$-decomposable. However, since $L^1(\widetilde{\mu}) = L^1_w(\widetilde{\mu})$ as $\widetilde{\mu}$ takes values in $\mathbb{R}$, we have that $L^1_w(\widetilde{\mu})$ has the Fatou property (see Proposition 5.4).

Now we can say that there is no relation between the main properties used in this paper, $\mathcal{R}$-decomposability and local σ-finiteness. Indeed, the vector measure given in the example above is locally σ-finite (see Remark 4.5) but not $\mathcal{R}$-decomposable. However, the vector measure given in Example 4.1 is $\mathcal{R}$-decomposable, since it is discrete but not locally σ-finite.

6 Representation theorems for Banach lattices

It is always interesting to know when a Banach lattice is order isometric to some Banach function space. This problem has been studied using vector measures by several authors. It was proved in [5, Theorem 8] that every order continuous Banach lattice with a weak unit is order isometric to an space $L^1(\nu)$ for a vector measure ν defined on a σ-algebra. This result allows to represent any Banach lattice E with the σ-Fatou property with a weak unit belonging to E_a as an space $L^1_w(\nu)$ with ν defined on a σ-algebra, since in this case the order isometry between E_a and $L^1(\nu)$ can be extended to E and turns out to be an order isometry between E and $L^1_w(\nu)$, see [6, Theorem 2.5]. So, we have the following equivalences between classes of spaces:

$$\left\{\begin{array}{l} E \text{ order continuous Banach} \\ \text{lattice with a weak unit} \end{array}\right\} \equiv \left\{ L^1(\nu) \text{ with } \nu \text{ on a } \sigma\text{-algebra} \right\}$$

and

$$\left\{\begin{array}{l} E \text{ Banach lattice with the} \\ \sigma\text{-Fatou property such that} \\ E_a \text{has a weak unit} \end{array}\right\} \equiv \left\{ L^1_w(\nu) \text{ with } \nu \text{ on a } \sigma\text{-algebra} \right\}. \tag{3}$$

For versions with E being p-convex see [15, Proposition 2.4] and [8, Theorem 4]. If we forget about the weak unit, it was stated in [4, pp. 22–23] and proved in detail in [12, Theorem 5] that

$$\left\{ E \text{ order continuous Banach lattice } \right\} \equiv \left\{ L^1(\nu) \text{ with } \nu \text{ on a } \delta\text{-ring} \right\}.$$

Moreover, from [12, Theorem 9] and Theorems 3.2, 4.2, 5.8, we have that

$$\left\{\begin{array}{l} E \text{ Banach lattice with the Fatou property} \\ \text{such that } E_a \text{ is order dense in } E \end{array}\right\} \equiv \left\{\begin{array}{l} L^1_w(\nu) \text{ with } \nu \text{ on a } \delta\text{-ring} \\ \text{being } \mathcal{R}- \text{ decomposable} \end{array}\right\}.$$

Note that although the converse of Theorem 5.8 does not hold, if $L^1_w(\nu)$ has the Fatou property, by Theorems 3.2 and 4.2, there exists an $\mathcal{R}$-decomposable vector measure $\widetilde{\nu}$ such that $L^1_w(\nu)$ is order isometric to $L^1_w(\widetilde{\nu})$.

Now, we add another equivalence:

$$\left\{\begin{array}{l} E \text{ Banach lattice with the} \\ \sigma\text{-Fatou property such that} \\ E_a \text{ is super order dense in } E \end{array}\right\} \equiv \left\{ [L^1(\nu)]_{\sigma-\mathrm{F}} \text{ with } \nu \text{ on a } \delta\text{-ring} \right\}. \tag{4}$$

Indeed, since $L^1(\nu) \subset [L^1(\nu)]_{\sigma-\mathrm{F}} \subset L^1_w(\nu)$, then $\left([L^1(\nu)]_{\sigma-\mathrm{F}}\right)_a \subset \left(L^1_w(\nu)\right)_a$ and so, from Theorem 3.2, we have that $\left([L^1(\nu)]_{\sigma-\mathrm{F}}\right)_a = L^1(\nu)$ which is super order dense in $[L^1(\nu)]_{\sigma-\mathrm{F}}$, see the last part of the proof of Theorem 5.1. Let us prove the converse containment.

Proposition 6.1 *Every Banach lattice E with the σ-Fatou property such that E_a is super order dense in E is order isometric to $[L^1(\nu)]_{\sigma-\mathrm{F}}$ for some vector measure ν defined on a δ-ring.*

Proof Let E be a Banach lattice with the σ-Fatou property such that E_a is super order dense in E and consider the vector measure ν defined on a δ-ring such that the integration operator $I_\nu\colon L^1(\nu) \to E_a$ given by $I_\nu(f) = \int f\, d\nu$ for all $f \in L^1(\nu)$, is an order isometry, see [12, Theorem 5]. Let us extend I_ν to $[L^1(\nu)]_{\sigma-\mathrm{F}}$. First, consider $0 \le f \in [L^1(\nu)]_{\sigma-\mathrm{F}}$ and take $(f_n) \subset L^1(\nu)$ such that $0 \le f_n \uparrow f$. This is always possible since $L^1(\nu)$ is super order dense in $[L^1(\nu)]_{\sigma-\mathrm{F}}$ as we have noted above. Since I_ν is an order isometry, the sequence $(I_\nu(f_n)) \subset E_a \subset E$ satisfies that $0 \le I_\nu(f_n) \uparrow$ and $\sup \|I_\nu(f_n)\|_E = \sup \|f_n\|_\nu \le \|f\|_\nu < \infty$. Then, as E has the σ-Fatou property, there exists $e = \sup I_\nu(f_n)$ in E and $\|e\|_E = \sup \|I_\nu(f_n)\|_E$. We define $T(f) = e$.

A similar argument to the one in [6, Theorem 2.5], shows that T is well defined. To be precise, take another sequence $(g_n) \subset L^1(\nu)$ such that $0 \le g_n \uparrow f$ and denote $z = \sup I_\nu(g_n)$. Let $0 \le x^* \in E^*$ be fixed. Then, $x^*(e) \ge x^*(I_\nu(f_n)) = \int f_n\, dx^*\nu$ for all n. Since $0 \le f_n \uparrow f$ ν-a.e. and so $x^*\nu$-a.e., by using the monotone convergence theorem, we have that $x^*(e) \ge \int f\, dx^*\nu \ge x^*(I_\nu(f_n))$ for all n. In a similar way, $x^*(z) \ge \int f\, dx^*\nu \ge x^*(I_\nu(g_n))$ for all n. Thus, it follows that $x^*(e) \ge x^*(I_\nu(g_n))$ and $x^*(z) \ge x^*(I_\nu(f_n))$ for all n. Since this holds for all $0 \le x^* \in E^*$, we have that $e \ge I_\nu(g_n)$ and $z \ge I_\nu(f_n)$ for all n. Then, $e \ge z$ and $z \ge e$, and so $e = z$. So, T is well defined. Moreover,

$$\|T(f)\|_E = \|e\|_E = \sup \|I_\nu(f_n)\|_E = \sup \|f_n\|_\nu = \|f\|_\nu,$$

where in the last equality we have used that $[L^1(\nu)]_{\sigma-\mathrm{F}}$ has the σ-Fatou property. Let us see now that T preserves the lattice structure, that is $T(f \wedge g) = Tf \wedge Tg$ for every $0 \le f, g \in [L^1(\nu)]_{\sigma-\mathrm{F}}$. Consider sequences $(f_n), (g_n) \subset L^1(\nu)$ satisfying that $0 \le f_n \uparrow f$ and $0 \le g_n \uparrow g$. Then, $Tf = \sup I_\nu(f_n)$ and $Tg = \sup I_\nu(g_n)$. Note that if $x_n \uparrow x$ and $y_n \uparrow y$ in a Banach lattice then $x_n \wedge y_n \uparrow x \wedge y$, see for instance [20, Theorem 15.3]. Then, since $0 \le f_n \wedge g_n \uparrow f \wedge g$ with $(f_n \wedge g_n) \subset L^1(\nu)$ and I_ν is an order isometry, we have that

$$T(f \wedge g) = \sup I_\nu(f_n \wedge g_n) = \sup I_\nu(f_n) \wedge I_\nu(g_n) = Tf \wedge Tg.$$

For a general $f \in [L^1(\nu)]_{\sigma-\mathrm{F}}$, we define $Tf = Tf^+ - Tf^-$ where f^+ and f^- are the positive and negative parts of f respectively. So, $T\colon [L^1(\nu)]_{\sigma-\mathrm{F}} \to E$ is a positive linear operator extending I_ν. For the linearity, see for instance [20, Theorem 15.2]. Moreover T is an isometry. Indeed, $Tf^+ \wedge Tf^- = T(f^+ \wedge f^-) = 0$ as $f^+ \wedge f^- = 0$, and so $|Tf| = |Tf^+ - Tf^-| = Tf^+ + Tf^- = T|f|$, see [20, Theorem 14.4]. Then, $\|T(f)\|_E = \|T(|f|)\|_E = \|f\|_\nu$ for all $f \in [L^1(\nu)]_{\sigma-\mathrm{F}}$.

Let us prove that T is onto. Let $0 \le e \in E$. Since E_a is super order dense in E, there exists $(e_n) \subset E_a$ such that $0 \le e_n \uparrow e$. Let $(f_n) \subset L^1(\nu) \subset [L^1(\nu)]_{\sigma-\mathrm{F}}$ be such that $e_n = I_\nu(f_n)$. Since I_ν^{-1} is an order isometry, we have that $0 \le f_n \uparrow$ and $\sup \|f_n\|_\nu = \sup \|e_n\|_E \le \|e\|_E < \infty$. Then, by the σ-Fatou property of $[L^1(\nu)]_{\sigma-\mathrm{F}}$, there exists $f = \sup f_n$ in $[L^1(\nu)]_{\sigma-\mathrm{F}}$. From the definition of T, we have that $Tf = \sup I_\nu(f_n) = \sup e_n = e$. For a general $e \in E$, consider e^+ and e^- the positive and negative parts of e. Let $g, h \in [L^1(\nu)]_{\sigma-\mathrm{F}}$ be such that $Tg = e^+$ and $Th = e^-$. Then, taking $f = g - h \in [L^1(\nu)]_{\sigma-\mathrm{F}}$ we have that $Tf = e$. Note that T^{-1} is positive. So, T is positive, linear, one to one and onto with inverse being positive, then T is an order isomorphism (see [19, p. 2]). □

Note that the class of spaces in (3) is contained in the one in (4). Indeed, take a weak unit $0 \le u \in E_a$. Then $0 \le e \wedge n\, u \uparrow e$ for each $0 \le e \in E$ where $e \wedge n\, u \in E_a$, and so E_a is super order dense in E. In this case we obtain that $[L^1(\nu)]_{\sigma-\mathrm{F}} = L^1_w(\nu)$, since ν is defined on a σ-algebra.

7 Example

We end by showing that there exist $\mathcal{R}$-decomposable vector measures ν which are not σ-finite nor discrete.

Let Γ be an abstract set. For each $\gamma \in \Gamma$, consider a non null vector measure $\nu_\gamma \colon \Sigma_\gamma \to X_\gamma$ defined on a σ-algebra Σ_γ of subsets of a set Ω_γ and with values in a Banach space X_γ. Take the set $\Omega = \cup_{\gamma\in\Gamma}\{\gamma\} \times \Omega_\gamma$ and the δ-ring $\mathcal{R}$ of subsets of Ω given by the sets $\cup_{\gamma\in\Gamma}\{\gamma\}\times A_\gamma$ with $A_\gamma \in \Sigma_\gamma$ for all $\gamma \in \Gamma$, for which there exists a finite set $J \subset \Gamma$ such that A_γ is ν_γ-null for all $\gamma \in \Gamma\backslash J$, see [12, p. 5]. Then,

$$\mathcal{R}^{loc} = \left\{\cup_{\gamma\in\Gamma}\{\gamma\} \times A_\gamma \,:\, A_\gamma \in \Sigma_\gamma \text{ for all } \gamma \in \Gamma\right\}.$$

Note that a function $f \colon \Omega \to \mathbb{R}$ is $\mathcal{R}^{loc}$-measurable if and only if $f(\gamma, \cdot) \colon \Omega_\gamma \to \mathbb{R}$ is Σ_γ-measurable for all $\gamma \in \Gamma$.

Denote by $c_0\left(\Gamma, (X_\gamma)_{\gamma\in\Gamma}\right)$ the Banach space of all families $(x_\gamma)_{\gamma\in\Gamma}$ such that $x_\gamma \in X_\gamma$ for every $\gamma \in \Gamma$ and $\left(\|x_\gamma\|_{X_\gamma}\right)_{\gamma\in\Gamma} \in c_0(\Gamma)$, endowed with the norm $\|(x_\gamma)_{\gamma\in\Gamma}\| = \sup_{\gamma\in\Gamma}\|x_\gamma\|_{X_\gamma}$. Note that the topological dual $c_0\left(\Gamma, (X_\gamma)_{\gamma\in\Gamma}\right)^*$ can be identified with the Banach space $\ell^1\left(\Gamma, (X^*_\gamma)_{\gamma\in\Gamma}\right)$ of families $(x^*_\gamma)_{\gamma\in\Gamma}$ such that $x^*_\gamma \in X^*_\gamma$ for every $\gamma \in \Gamma$ and $\left(\|x^*_\gamma\|_{X^*_\gamma}\right)_{\gamma\in\Gamma} \in \ell^1(\Gamma)$, endowed with the norm $\|(x^*_\gamma)_{\gamma\in\Gamma}\| = \sum_{\gamma\in\Gamma}\|x^*_\gamma\|_{X_\gamma}$. The action of any $x^* = (x^*_\gamma)_{\gamma\in\Gamma} \in \ell^1\left(\Gamma, (X^*_\gamma)_{\gamma\in\Gamma}\right)$ on $x = (x_\gamma)_{\gamma\in\Gamma} \in c_0\left(\Gamma, (X_\gamma)_{\gamma\in\Gamma}\right)$ is given by $x^*(x) = \sum_{\gamma\in\Gamma} x^*_\gamma(x_\gamma)$.

Consider the vector measure $\nu \colon \mathcal{R} \to c_0\left(\Gamma, (X_\gamma)_{\gamma\in\Gamma}\right)$ given by

$$\nu\left(\cup_{\gamma\in\Gamma}\{\gamma\} \times A_\gamma\right) = \left(\nu_\gamma(A_\gamma)\right)_{\gamma\in\Gamma}.$$

Note that a set $A = \cup_{\gamma\in\Gamma}\{\gamma\}\times A_\gamma \in \mathcal{R}^{loc}$ is ν-null if and only if A_γ is ν_γ-null for all $\gamma \in \Gamma$. Then, it is direct to check that:

(a) ν is $\mathcal{R}$-decomposable.
(b) ν is σ-finite if and only if Γ is countable.
(c) ν is discrete if and only if ν_γ is discrete for all $\gamma \in \Gamma$.

Let us prove that $L^1_w(\nu)$ can be described as the space of functions $f \in \mathcal{M}(\mathcal{R}^{loc})$ such that $f(\gamma, \cdot) \in L^1_w(\nu_\gamma)$ for all $\gamma \in \Gamma$ with $(\|f(\gamma,\cdot)\|_{\nu_\gamma})_{\gamma\in\Gamma} \in \ell^\infty(\Gamma)$, and moreover, $\|f\|_\nu = \sup_{\gamma\in\Gamma}\|f(\gamma,\cdot)\|_{\nu_\gamma}$ for all $f \in L^1_w(\nu)$, that is,

$$L^1_w(\nu) = \ell^\infty\left(\Gamma, (L^1_w(\nu_\gamma))_{\gamma\in\Gamma}\right).$$

Given $x^* = (x^*_\gamma)_{\gamma\in\Gamma} \in \ell^1\left(\Gamma, (X^*_\gamma)_{\gamma\in\Gamma}\right)$, since $|x^*\nu|(A) = \sum_{\gamma\in\Gamma}|x^*_\gamma\nu_\gamma|(A_\gamma) \le \infty$ for every $A = \cup_{\gamma\in\Gamma}\{\gamma\} \times A_\gamma \in \mathcal{R}^{loc}$, we have that

$$\int |f|\, d|x^*\nu| = \sum_{\gamma\in\Gamma}\int |f(\gamma,\cdot)|\, d|x^*_\gamma\nu_\gamma| \le \infty, \text{ for all } f \in \mathcal{M}(\mathcal{R}^{loc}). \tag{5}$$

Indeed, (5) holds for $\mathcal{R}^{loc}$-simple functions, and so for a general f by using the monotone convergence theorem. Let us see that if $f \in L^1(x^*\nu)$, then

$$\int_A f\,dx^*\nu = \sum_{\gamma\in\Gamma} \int_{A_\gamma} f(\gamma,\cdot)\,dx^*_\gamma\nu_\gamma. \tag{6}$$

In this case, by (5), $f(\gamma,\cdot) \in L^1(x^*_\gamma\nu_\gamma)$ for every $\gamma \in \Gamma$ and $\int |f(\gamma,\cdot)|\,d|x^*_\gamma\nu_\gamma| = 0$ (and so $f(\gamma,\cdot) = 0$ except on a $x^*_\gamma\nu_\gamma$-null set Z_γ) for all $\gamma \in \Gamma\backslash J$ with J being some countable subset of Γ. Then, $f\chi_A = f\chi_{\cup_{\gamma\in J}\{\gamma\}\times A_\gamma}$ ν-a.e. (except on the ν-null set $\cup_{\gamma\in\Gamma\backslash J}\{\gamma\}\times A_\gamma \cap Z_\gamma$). By using the dominated convergence theorem, we have that

$$\int_A f\,dx^*\nu = \sum_{\gamma\in J} \int_{\{\gamma\}\times A_\gamma} f\,dx^*\nu.$$

Noting that $\int_{\{\gamma\}\times A_\gamma} f\,dx^*\nu = \int_{A_\gamma} f(\gamma,\cdot)\,dx^*_\gamma\nu_\gamma$ holds for $\mathcal{R}^{loc}$-simple functions and so for any $f \in L^1(x^*\nu)$ by density of the $\mathcal{R}^{loc}$-simple functions in $L^1(x^*\nu)$, we conclude that (6) holds.

Let $f \in L^1_w(\nu)$ and fix $\beta \in \Gamma$. Given $x^*_\beta \in X^*_\beta$, define the element $x^* = (x^*_\gamma)_{\gamma\in\Gamma}$ in $\ell^1\left(\Gamma, (X^*_\gamma)_{\gamma\in\Gamma}\right)$ by $x^*_\gamma = x^*_\beta$ if $\gamma = \beta$ and $x^*_\gamma = 0$ in other case. Then, from (5), we have that $\int |f(\beta,\cdot)|\,d|x^*_\beta\nu_\beta| = \int |f|\,d|x^*\nu| < \infty$ and so $f(\beta,\cdot) \in L^1_w(\nu_\beta)$ with $\|f(\beta,\cdot)\|_{\nu_\beta} \le \|f\|_\nu$. Thus, $(\|f(\gamma,\cdot)\|_{\nu_\gamma})_{\gamma\in\Gamma} \in \ell^\infty(\Gamma)$ and $\sup_{\gamma\in\Gamma}\|f(\gamma,\cdot)\|_{\nu_\gamma} \le \|f\|_\nu$.

Let now $f \in \mathcal{M}(\mathcal{R}^{loc})$ satisfying that $f(\gamma,\cdot) \in L^1_w(\nu_\gamma)$ for every $\gamma \in \Gamma$ and $(\|f(\gamma,\cdot)\|_{\nu_\gamma})_{\gamma\in\Gamma} \in \ell^\infty(\Gamma)$. Given $x^* = (x^*_\gamma)_{\gamma\in\Gamma} \in \ell^1\left(\Gamma, (X^*_\gamma)_{\gamma\in\Gamma}\right)$, from (5), we have that

$$\begin{aligned}\int |f|\,d|x^*\nu| &= \sum_{\gamma\in\Gamma}\int |f(\gamma,\cdot)|\,d|x^*_\gamma\nu_\gamma| \le \sum_{\gamma\in\Gamma}\|x^*_\gamma\|_{X^*_\gamma}\,\|f(\gamma,\cdot)\|_{\nu_\gamma}\\ &\le \sup_{\gamma\in\Gamma}\|f(\gamma,\cdot)\|_{\nu_\gamma}\sum_{\gamma\in\Gamma}\|x^*_\gamma\|_{X^*_\gamma} < \infty.\end{aligned}$$

Then, $f \in L^1_w(\nu)$ and $\|f\|_\nu \le \sup_{\gamma\in\Gamma}\|f(\gamma,\cdot)\|_{\nu_\gamma}$.

Moreover, $L^1(\nu)$ can be described as the space of functions $f \in \mathcal{M}(\mathcal{R}^{loc})$ such that $f(\gamma,\cdot) \in L^1(\nu_\gamma)$ for every $\gamma \in \Gamma$ with $(\|f(\gamma,\cdot)\|_{\nu_\gamma})_{\gamma\in\Gamma} \in c_0(\Gamma)$, that is,

$$L^1(\nu) = c_0\left(\Gamma, (L^1_w(\nu_\gamma))_{\gamma\in\Gamma}\right).$$

Indeed, if $f \in L^1(\nu)$ we can take $(\varphi_n) \subset \mathcal{S}(\mathcal{R})$ such that $\varphi_n \to f$ in $L^1(\nu)$. For each $\gamma \in \Gamma$, we have that $f(\gamma,\cdot) \in L^1_w(\nu_\gamma)$ (as $f \in L^1_w(\nu)$) and $(\varphi_n(\gamma,\cdot)) \subset \mathcal{S}(\Sigma_\gamma) \subset L^1(\nu_\gamma)$. Then, since $\|f(\gamma,\cdot) - \varphi_n(\gamma,\cdot)\|_{\nu_\gamma} \le \|f - \varphi_n\|_\nu$ and $L^1(\nu_\gamma)$ is closed in $L^1_w(\nu_\gamma)$, it follows that $f(\gamma,\cdot) \in L^1(\nu_\gamma)$. On the other hand, for each n we can write $\varphi_n = \sum_{j=1}^m \alpha_j\chi_{A_j}$ where $\alpha_j \in \mathbb{R}$ and $A_j = \cup_{\gamma\in\Gamma}\{\gamma\}\times A^j_\gamma$. Here, $A^j_\gamma \in \Sigma_\gamma$ for all $\gamma \in \Gamma$ and satisfies that A^j_γ is ν_γ-null for all $\gamma \in \Gamma\backslash J_j$ for some finite set $J_j \subset \Gamma$. Then, $\varphi_n(\gamma,\cdot) = \sum_{j=1}^m \alpha_j\chi_{A^j_\gamma} = 0$ ν_γ-a.e. for all $\gamma \in \Gamma\backslash\cup_{j=1}^m J_j$ where $\cup_{j=1}^m J_j$ is a finite set, and so $(\|\varphi_n(\gamma,\cdot)\|_{\nu_\gamma})_{\gamma\in\Gamma} \in c_0(\Gamma)$. Since $(\|f(\gamma,\cdot)\|_{\nu_\gamma})_{\gamma\in\Gamma} \in \ell^\infty(\Gamma)$ and

$$\sup_{\gamma\in\Gamma}\left|\|f(\gamma,\cdot)\|_{\nu_\gamma} - \|\varphi_n(\gamma,\cdot)\|_{\nu_\gamma}\right| \le \sup_{\gamma\in\Gamma}\|f(\gamma,\cdot) - \varphi_n(\gamma,\cdot)\|_{\nu_\gamma} = \|f - \varphi_n\|_\nu,$$

it follows that $(\|f(\gamma,\cdot)\|_{\nu_\gamma})_{\gamma\in\Gamma} \in c_0(\Gamma)$.

Conversely, suppose that $f \in \mathcal{M}(\mathcal{R}^{loc})$ is such that $f(\gamma,\cdot) \in L^1(\nu_\gamma)$ for all $\gamma \in \Gamma$ and $(\|f(\gamma,\cdot)\|_{\nu_\gamma})_{\gamma\in\Gamma} \in c_0(\Gamma)$. In particular, $f \in L^1_w(\nu)$. Given an element $x^* = (x^*_\gamma)_{\gamma\in\Gamma} \in \ell^1\left(\Gamma, (X^*_\gamma)_{\gamma\in\Gamma}\right)$ and $A = \cup_{\gamma\in\Gamma}\{\gamma\} \times A_\gamma \in \mathcal{R}^{loc}$, we note that $\left(\int_{A_\gamma} f(\gamma,\cdot)\,d\nu_\gamma\right)_{\gamma\in\Gamma} \in c_0\left(\Gamma, (X_\gamma)_{\gamma\in\Gamma}\right)$ as $\|\int_{A_\gamma} f(\gamma,\cdot)\,d\nu_\gamma\|_{X_\gamma} \le \|f(\gamma,\cdot)\|_{\nu_\gamma}$ for each $\gamma \in \Gamma$. Moreover, by (6),

$$x^*\left(\left(\int_{A_\gamma} f(\gamma,\cdot)\,d\nu_\gamma\right)_{\gamma\in\Gamma}\right) = \sum_{\gamma\in\Gamma} x^*_\gamma\left(\int_{A_\gamma} f(\gamma,\cdot)\,d\nu_\gamma\right)$$

$$= \sum_{\gamma\in\Gamma}\int_{A_\gamma} f(\gamma,\cdot)\,dx^*_\gamma\nu_\gamma = \int_A f\,dx^*\nu.$$

So, $f \in L^1(\nu)$ and $\int_A f\,d\nu = \left(\int_{A_\gamma} f(\gamma,\cdot)\,d\nu_\gamma\right)_{\gamma\in\Gamma}$.

Note that if ν is locally σ-finite, since $h = \sum_{\gamma\in\Gamma} \frac{1}{\|\nu_\gamma\|(\Omega_\gamma)}\chi_{\{\gamma\}\times\Omega_\gamma} \in L^1_w(\nu)$ and $\mathrm{supp}(h) = \Omega$, from Proposition 4.9, it follows that ν is σ-finite. So, in this case ν is locally σ-finite if and only if ν is σ-finite if and only if Γ is countable.

In particular, consider a non atomic measure space (Θ, Σ, μ) and an order continuous B.f.s. X related to μ which does not contain any copy of c_0 and such that $\chi_\Theta \in X$, for instance $X = L^p[0,1]$ related to the Lebesgue measure for $p \ge 1$. Then, $\eta\colon \Sigma \to X$ given by $\eta(A) = \chi_A$ for all $A \in \Sigma$, is a non discrete vector measure such that $L^1_w(\nu) = L^1(\nu) = X$. Taking Γ uncountable and $\nu_\gamma = \eta$ for all $\gamma \in \Gamma$, we obtain an $\mathcal{R}$-decomposable vector measure ν which is not σ-finite nor discrete. In this case, $L^1_w(\nu) = \ell^\infty(\Gamma, X)$ and $L^1(\nu) = c_0(\Gamma, X)$.

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