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Capacitary function spaces

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Abstract These notes are devoted to the analysis on a capacity space, with capacities as substitutes of measures in the study of function spaces. The goal is to extend to the associated function lattices some aspects of the theory of Banach function spaces, to show how the general theory can be applied to classical function spaces such as Lorentz spaces, and to complete the real interpolation theory for these spaces included in Cerdà (J Math Anal Appl 304:269–295, 2005) and Cerdà et al. (AMS Contemp Math 445:49–55, 2007).

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1 Introducction

The purpose of this paper is to present some basic developments connected with properties of function spaces defined on capacity spaces, instead of measure spaces. It is our feeling that these developments, because of their relations with important aspects of mathematical analysis on one hand and their simple and basic character on the other, deserve to be widely known.

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The emphasis of our exposition is placed upon the study of the essential functional analytic elements such that a satisfactory theory can be developed in the context of quasi-Banach spaces. One of the main problems is that we are forced to work with a non-additive integral, the Choquet integral, so that the dual spaces are not easily identifiable and some basic properties, such as the dominated convergence theorem, are not longer available.

In the literature, a capacity on a space Ω is usually supposed to be an increasing set function $C : \Sigma \to [0, \infty]$, with Σ a family of subsets in Ω, with different properties depending on the context, and the Choquet integral is defined as

$$\int f\, dC := \int_0^\infty C\{f > t\}\, dt$$

if $f \geq 0$ is a measurable function in the sense that $\{f > t\} \in \Sigma$ for every $t > 0$.

In many important examples of capacities the domain Σ of C is a σ-algebra. This is the case of the variational capacities, and of the Fuglede [18] and Meyers [21] capacities of nonlinear potential theory. They are countably subadditive set functions on all subsets of $\mathbf{R}^n$ which include the Riesz and the Bessel capacities. Although they are not Caratheodory metric outer measures, they satisfy a Fatou type condition and, by a general theorem due to G. Choquet (cf. [16, Chapter VI]), every Borel set $B \subset \mathbf{R}^n$ is capacitable, this meaning that

$$\sup\{C(K);\ K \subset B,\ K \text{ compact}\} = C(B) = \inf\{C(G);\ G \supset B,\ G \text{ open}\}.$$

Then the class of all Borel sets turns out to be a convenient domain for all of them. We refer to [2] and [20] for an extended overview of these capacities.

Another well known class of capacities are the Hausdorff contents. If h is a continuous increasing function on $[0, \infty)$ vanishing only at 0, which is called a measure function in [11], denote μ_h the corresponding Hausdorff measure on $\mathbf{R}^n$, and let I or I_k represent a general cube in $\mathbf{R}^n$ with its sides parallel to the axes. The use of the corresponding Hausdorff capacity or Hausdorff content,

$$E_h(A) := \inf_{A \subset \bigcup_{k=1}^\infty I_k} \left\{ \sum_{k=1}^\infty h(|I_k|) \right\},$$

is often more convenient than μ_h, and $E_h(A) = 0$ if and only if $\mu_h(A) = 0$.

If $h(t) = t^\alpha$ ($\alpha > 0$), it is customary to write H_α^∞ instead of E_h, and this capacity is called the α-dimensional Hausdorff content. The case $h(x) := x \log(1/x)$ on $[0, 1/e]$ corresponds to the Shannon entropy considered in [17].

New examples appear when studying interpolation properties of function spaces as in [15]. If E is a quasi-Banach function space on the measure space (Ω, Σ, μ), then

$$C_E(A) := \|\chi_A\|_E \qquad (A \in \Sigma) \tag{1}$$

defines a capacity and, as in the case of Hausdorff capacities, there is a measure μ such that $C_E(A) = 0$ if and only if $\mu(A) = 0$.

The goal of these notes is to clearly set the basic properties of the capacity spaces (Ω, Σ, C) and their associated Lebesgue spaces $L^p(C)$ and $L^{p,q}(C)$, to show how the general theory can be applied to function spaces such as classical Lorentz spaces, and to complete the real interpolation theory for these spaces started in [15] and [14].

Further applications of the use of these capacities will appear in forthcoming work. In [3] it will be shown how they are a useful tool to extend the Riesz-Herz estimates concerning the Hardy-Littlewood operator.

The notation $A \lesssim B$ means that $A \leq \gamma B$ for some absolute constant $\gamma \geq 1$, and $A \simeq B$ if $A \lesssim B \lesssim A$. We refer to [7] for general facts concerning function spaces.

2 Capacitary function spaces

Let (Ω, Σ) be a measurable space. Sets will always be assumed to be in the σ-algebra Σ and functions will be real mesurable functions on (Ω, Σ).

From now on, by a capacity C we mean a set function defined on Σ satisfying at least the following properties:

(a) $C(\emptyset) = 0$,
(b) $0 \leq C(A) \leq \infty$,
(c) $C(A) \leq C(B)$ if $A \subset B$, and
(d) Quasi-subadditivity: $C(A \cup B) \leq c(C(A) + C(B))$, where $c \geq 1$ is a constant.

If $c = 1$, we say that the capacity is subadditive.

If C is a capacity on Σ, we will say that (Ω, Σ, C) is a capacity space. It will play the role of a measure space (Ω, Σ, μ) in the theory of Banach function spaces. We are going to check which of the properties for measure spaces are still satisfied by capacity spaces.

The distribution function C_f and the nonincreasing rearrangement $f_C^\star$ are defined as in the case of measures by

$$C_f(t) := C\{|f| > t\},$$

and

$$f_C^\star(x) := \inf\{t;\ C\{|f| > t\} \leq x\} = \int_0^\infty \chi_{[0, C\{|f|>t\})}(x)\, dt,$$

since $\{t;\ C\{|f| > t\} \leq x\}$ is the interval $[f_C^\star(x), \infty]$.

Many of the basic properties remain true in this capacitary setting. The following ones are easily proved:

(a) $(\chi_A)_C^\star = \chi_{[0, C(A))}$.
(b) If $s = \sum_{k=1}^N a_k \chi_{A_k}$, $A_k \cap A_j = \emptyset$ if $k \neq j$ and $a_1 > a_2 > \cdots > a_N > 0 = a_{N+1}$, then $s_C^\star = \sum_{k=1}^N (a_k - a_{k-1}) \chi_{[0, C(A_1 \cup \cdots \cup A_k))}$.
(c) If $\psi : [0, \infty) \to [0, \infty)$ is nondecreasing and right-continuous, then $\psi(|f|)_C^\star = \psi(f_C^\star)$. For instance, $(|f|^p)_C^\star = (f_C^\star)^p$ $(p > 0)$.

Note that

$$(f + g)_C^\star(x) \leq f_C^\star\left(\frac{x}{2c}\right) + g_C^\star\left(\frac{x}{2c}\right). \tag{2}$$

Indeed, let $\lambda := f_C^\star(x_1) + g_C^\star(x_2) < \infty$ and $x_1, x_2 \geq 0$. Then

$$C\{|f + g| > f_C^\star(x_1) + g_C^\star(x_2)\} \leq cC_f(f_C^\star(x_1)) + cC_g(g_C^\star(x_2)) \leq cx_1 + cx_2,$$

so that

$$(f + g)_C^\star(cx_1 + cx_2) \leq (f + g)_C^\star(x) \leq \lambda = f_C^\star(x_1) + g_C^\star(x_2).$$

In particular, $(f + g)_C^\star(x) \leq f_C^\star(x/2c) + g_C^\star(x/2c)$ as announced.

A property is said to hold quasi-everywhere (C-q.e. for short) if the exceptional set has zero capacity.

Pointwise convergence $f_n \to f$ will mean $C(\{f_n \not\to f\}) = 0$. Similarly, $f_n \uparrow f$ that $f_n \to f$ and $C(\{f_n > f_{n+1}\}) = 0$. Also, we write $A_n \uparrow A$ or $A_n \downarrow A$ when $\chi_{A_n} \uparrow \chi_A$ or $\chi_{A_n} \downarrow \chi_A$ in the above sense, respectively.

If $f \geq 0$, the Choquet integral

$$\int f\, dC := \int_0^\infty C\{f > t\}\, dt \in [0, \infty]$$

satisfies $\int f\, dC = 0$ if and only if $f = 0$ C-q.e. and it is positive-homogeneous,

$$\int \alpha f\, dC = \alpha \int f\, dC \qquad (\alpha > 0).$$

Moreover, by Fubini's theorem,

$$\int_0^\infty f_C^\star(x)\, dx = \int f\, dC. \tag{3}$$

The relation $\{f + g > t\} \subset \{f > t/2\} \cup \{g > t/2\}$ shows that this integral, defined on nonnegative functions, is quasi-subadditive with constant $2c$,

$$\int (f + g)\, dC \leq 2c \left(\int f\, dC + \int g\, dC \right). \tag{4}$$

Observe that, if $f = g$ C-q.e. and C is subadditive, then $\int f\, dC = \int g\, dC$ since, if $A = \{f \neq g\}$, then

$$C\{f > t\} \leq C\left((\{f > t\} \cap A^c) \cup (\{f > t\} \cap A) \right) \leq C\{g > t\}.$$

This will be also true if $C(A_n) \to C(A)$ whenever $A_n \uparrow A$. In this case we say that C has the Fatou property (or that it is a Fatou capacity).

If C is a Fatou capacity, the countable unions of $C-$null sets are also $C-$null. Indeed, $C(A_1 \cup \cdots \cup A_n) \leq c^n(C(A_1) + \cdots + C(A_n)) = 0$ if $C(A_k) = 0$ $(k \in \mathbf{N})$, and then $C(\bigcup_{k=1}^\infty A_k) = \lim_{n\to\infty} C(A_1 \cup \cdots \cup A_n) = 0$.

If $\chi_A = \chi_B$ C-q.e., then $C(A) = C(B)$ by the Fatou property, since $f_n := \chi_A \to \chi_B$ C-q.e. and $C(A) \leq C(B)$. Similarly, $C(B) \leq C(A)$.

We consider equivalent two functions, f and g, if they are equal C-q.e. In this case $\int |f|\, dC = \int |g|\, dC$, since $C\{|f| > t\} = C\{|g| > t\}$ for every $t \geq 0$. Thus, $\int |f|\, dC = 0$ if and only if $f = 0$ C-q.e.

Note that if a Fatou capacity is subadditive, then it is σ-subadditive.

The Fatou property can be presented in several equivalent ways:

Theorem 1 *The following properties are equivalent:*

(a) C *is a Fatou capacity.*
(b) $|f| \leq \liminf_n |f_n| \implies f_C^\star \leq \liminf_n (f_n)_C^\star$.
(c) $\int (\liminf_n |f_n|)\, dC \leq \liminf_n \int |f_n|\, dC$.
(d) $0 \leq f_n \uparrow f \implies (f_n)_C^\star \uparrow f_C^\star$.

Proof (c) follows from (b) and (3), and (a) follows from (c) by taking $f_n = \chi_{A_n}$.

Suppose now that C satisfies (a) and that $|f| \leq \liminf_n |f_n|$. Let $A^t := \{|f| > t\}$ and $A_n^t := \{|f_n| > t\}$. Then,

$$A^t \subset \liminf_n A_n^t = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} A_n^t$$

and, by (a),

$$C(A^t) \leq \lim_m C\left(\bigcap_{n=m}^{\infty} A_n^t\right) \leq \liminf_n C(A_n^t),$$

so that $\chi_{[0,C(A^t))} \leq \liminf_n \chi_{[0,C(A_n^t))}$ and (b) follows:

$$f_C^\star(x) = \int_0^\infty \chi_{[0,C(A^t))}(x)\,dt \leq \liminf_n \int_0^\infty \chi_{[0,C(A_n^t))}(x)\,dt = \liminf_n (f_n)_C^\star(x).$$

Moreover, (d) implies (a) by taking $f_n = \chi_{A_n}$ and $f = \chi_A$.

Finally, suppose now that C satisfies (a) and that $0 \leq f_n \uparrow f$. Then $(f_n)_C^\star \leq f_C^\star$ and hence $\lim_{n\to\infty}(f_n)_C^\star(x) \leq f_C^\star(x)$. Let $A_n^t := \{|f_n| > t\}$ and $A^t := \{|f| > t\}$. From (a) we obtain that $C(A^t) = \lim_{n\to\infty} C(A_n^t)$ and

$$f_C^\star(x) = \int_0^\infty \chi_{[0,C(A^t))}(x)\,dt = \int_0^\infty \lim_{n\to\infty} \chi_{[0,C(A_n^t))}(x)\,dt \leq \lim_{n\to\infty} (f_n)_C^\star(x).$$

□

Theorem 2 *If* $1 \leq p \leq \infty$ *and* $p' = p/(p-1)$, *then the following versions of Hölder and Minkowski inequalities hold:*

$$\int_\Omega |fg|dC \leq 2c\left(\int_\Omega |f|^p dC\right)^{1/p}\left(\int_\Omega |g|^{p'} dC\right)^{1/p'}$$

and

$$\left(\int_\Omega |f+g|^p dC\right)^{1/p} \leq 4c^2\left[\left(\int_\Omega |f|^p dC\right)^{1/p} + \left(\int_\Omega |g|^p dC\right)^{1/p}\right].$$

If the Choquet integral is subadditive (cf. Sect. 4*), then the Hölder and the Minkowski inequalities are satisfied with constant* 1*:*

$$\int_\Omega |fg|dC \leq \left(\int_\Omega |f|^p dC\right)^{1/p}\left(\int_\Omega |g|^{p'} dC\right)^{1/p'} \tag{5}$$

$$\left(\int_\Omega |f+g|^p dC\right)^{1/p} \leq \left(\int_\Omega |f|^p dC\right)^{1/p} + \left(\int_\Omega |g|^p dC\right)^{1/p} \tag{6}$$

Proof We write $|fg| = (|f|^p)^{1/p}(|g|^{p'})^{1/p'}$. Since the inequality $a \le b^\theta c^{1-\theta}$ holds if and only if $a \le \theta\epsilon^{\theta-1}b + (1-\theta)\epsilon^\theta c$ for all $\epsilon > 0$, by taking $\theta = 1/p$, $a = |fg|$, $b = |f|^p$, $c = |g|^{p'}$ we obtain $|fg| \le \theta\epsilon^{\theta-1}|f|^p + (1-\theta)\epsilon^\theta|g|^{p'}$.

Hence, in the subadditive case (in the general case the proof is the same but the constant $2c$ from (4) has to be included), we have

$$\int_\Omega |fg|\,dC \le \theta\epsilon^{\theta-1}\int_\Omega |f|^p\,dC + (1-\theta)\epsilon^\theta\int_\Omega |g|^{p'}\,dC.$$

Denote $A = \int_\Omega |f|^p\,dC$, $B = \int_\Omega |g|^{p'}\,dC$ and $\gamma(\epsilon) = \theta\epsilon^{\theta-1}A + (1-\theta)\epsilon^\theta B$ we have that $\int_\Omega |fg|\,dC \le \gamma(\epsilon)$, and $\gamma(\epsilon) \ge \gamma(\epsilon_0)$ with $\epsilon_0 = A/B$. Hence

$$\int_\Omega |fg|\,dC \le \gamma(\epsilon_0) = \frac{A^\theta}{B^{\theta-1}} = \left(\int_\Omega |f|^p\,dC\right)^{1/p}\left(\int_\Omega |g|^{p'}\,dC\right)^{1/p'}.$$

The Minkowski inequality (6) follows from (5) in the usual way. □

One could wonder if these estimates are always true with constant 1. We will see in Sect. 4 that subadditivity holds only if C is concave. It is easily checked that Hölder's inequality is always true for sets, since

$$\int \chi_A\chi_B\,dC = C(A\cap B) \le C(A)^{1/p}C(B)^{1/p'},$$

but the following example shows that it is not longer true for functions:

Example 1 Consider the "Lorentz-type" capacity $C(A) := \int_0^{|A|} w(t)\,dt$ on $(0, 1)$ with $w(t) = t\chi_{(0,1)}(t)$, and the functions

$$f(x) = x^{-1/2}, \quad g(x) = x^{1/2} \qquad (\text{on } (0,1)).$$

Then

$$\int f^2\,dC\int g^2\,dC < \left(\int fg\,dC\right)^2.$$

Just note that $\int fg\,dC = C((0,1)) = 1/2$, $\int f^2\,dC = \int_0^1 f(x)^2x\,dx = 1$ and $\int g^2\,dC = \int_0^1 (1-x)^2x\,dx = 1/6$.

Hence, there is no hope to obtain the Hölder and Minkowski inequalities with constant 1 in the general case. We do not know whether the subadditivity of the Choquet integral is a necessary condition to get Hölder's estimate with constant 1.

3 Lebesgue capacitary spaces

From now on, C will represent a Fatou capacity on (Ω, Σ) and $c \ge 1$ its subadditivity constant.

In this section we study the completeness of the spaces $L^{p,q}(C)$ $(p, q > 0)$ defined by the condition

$$\|f\|_{L^{p,q}(C)} := \left(q\int_0^\infty t^{q-1}C\{|f| > t\}^{q/p}dt\right)^{1/q} < \infty$$

if $q < \infty$. If $q = \infty$, $\|f\|_{L^{p,\infty}(C)} := \sup_{t>0} tC\{|f| > t\}^{1/p}$.

Observe that $\|f\|_{L^{p,q}(C)} = 0$ if and only if $f = 0$ C-q.e. and equivalent functions (in the sense of C-q.e. identity) have the same $\|\cdot\|_{L^{p,q}(C)}$-norm. Moreover $\|\lambda f\|_{L^{p,q}(C)} = |\lambda|\|f\|_{L^{p,q}(C)}$ and $\|f+g\|_{L^{p,q}(C)} \leq 2c(\|f\|_{L^{p,q}(C)} + \|g\|_{L^{p,q}(C)})$.

We write $L^p(C) = L^{p,p}(C)$ if $p < \infty$ with $\|f\|_{L^p(C)} = \left(\int_\Omega |f|^p dC\right)^{1/p}$. $L^\infty(C)$ is defined as usually by the condition

$$\|f\|_\infty := \inf\{M > 0;\ |f| \leq M\, C - \text{q.e.}\} < \infty.$$

As for function spaces, there are several descriptions of these "norms":

Theorem 3 $\|f\|_{L^p(C)} = \|f_C^\star\|_p = \||f|^p\|^{1/p} = \left(p\int_0^\infty t^{p-1} C\{|f| > t\}\, dt\right)^{1/p}$.

Proof Let $\psi(t) = t^p$. Then $\int_0^\infty \psi(f_C^\star(t))\, dt = \int_0^\infty \psi(|f|)_C^\star(t)\, dt$ and, if we denote $g = \psi(|f|)$, an application of Fubini theorem gives

$$\begin{aligned}\int_0^\infty g_C^\star(t)\, dt &= \int_0^\infty\int_0^\infty \chi_{[0,C\{g>x\})}(t)\, dx\, dt \\ &= \int_0^\infty\int_0^\infty \chi_{[0,C\{g>x\})}(t)\, dt\, dx = \int_0^\infty C\{g > x\}\, dx,\end{aligned}$$

this is, $\int_0^\infty \psi(f)_C^\star(t)\, dt = \int_0^\infty C\{\psi(|f|) > t\}\, dt$.

Also, if $x = \psi(t)$, then

$$\int_0^\infty C\{|f| > t\}\, d\psi(t) = \int_0^\infty C\{|f| > \psi^{-1}(x)\}\, dx = \int_0^\infty C\{\psi(|f|) > x\}\, dx$$

and $\int_0^\infty C\{|f| > t\}\, d\psi(t) = \int_0^\infty C\{\psi(|f|) > t\}\, dt$. □

Theorem 4 *$\|\cdot\| := \|\cdot\|_{L^p(C)}$ is quasi-subadditive, with constant $c_p = (2c)^{1/p}$ if $1 \leq p < \infty$ and $c_p = c^{1/p}2^{(2-p)/p}$ if $0 < p < 1$.*

Proof Suppose $1 \leq p < \infty$. By (2),

$$\|f+g\|^p \leq \int_0^\infty \left(f_C^\star\left(\frac{x}{2c}\right) + g_C^\star\left(\frac{x}{2c}\right)\right)^p\, dx = 2c\|f_C^\star + g_C^\star\|_p^p$$

and the results follow from the estimates for $L^p(\mathbf{R}^+)$.

If $p < 1$, then, since $a^p + b^p \leq 2^{1-p}(a+b)^p$ $(a, b \geq 0)$, we conclude that

$$\|f_C^\star + g_C^\star\|_p^p \leq \int_0^\infty f_C^\star(y)^p\, dy + \int_0^\infty g_C^\star(y)^p\, dy \leq 2^{1-p}(\|f_C^*\|_p + \|g_C^*\|_p)^p,$$

and $\|f+g\| \leq (2c)^{1/p}2^{(1-p)/p}(\|f\| + \|g\|)$. □

Now, recall that if $\|\cdot\|$ is a quasi-seminorm with constant $c \geq 1$ and $(2c)^\varrho = 2$ then, by Aoki's theorem (cf. Section 3.10 of [8]),

$$\|f\|^* := \inf\left\{\sum_{j=1}^n \|f_j\|^\varrho : n \geq 1,\ \sum_{j=1}^n f_j = f\right\},$$

is a 1-norm $\|\cdot\|^*$ such that

$$\|f\|^* \le \|f\|^\varrho \le 2\|f\|^*, \tag{7}$$

and it follows that

$$\left\|\sum_i |f_i|\right\| \le 2^{1/\varrho}\left(\sum_i \|f_i\|^*\right)^{1/\varrho} \le 2^{1/\varrho}\left(\sum_i \|f_i\|^\varrho\right)^{1/\varrho}. \tag{8}$$

In the special case $f_i = \chi_{A_i}$ and $p = 1$ we obtain

$$C\left(\bigcup_{i=1}^\infty A_i\right)^\varrho \le 2\sum_{i=1}^\infty C(A_i)^\varrho. \tag{9}$$

We say that $\{f_n\}$ converges to f in capacity if $C\{|f_n - f| > \epsilon\} \to 0$ as $n \to \infty$, for every $\epsilon > 0$.

Note that if the sequence $\{f_n\}$ converges in capacity, then it is a Cauchy sequence in capacity, that is, for every $\epsilon > 0$, $C\{|f_p - f_q| > \epsilon\} \to 0$ as $p, q \to \infty$. The converse is also true:

Theorem 5 *A sequence $\{f_n\}$ is convergent in capacity to a function f if and only if it is a Cauchy sequence in capacity. In this case, the sequence has a subsequence which is C-q.e. convergent to f.*

Proof If $\{f_n\}$ is a Cauchy sequence in capacity, then there exists $n_k \in N$ so that

$$C\{|f_p - f_q| > 2^{-k}\} < 2^{-k} \qquad (p, q \ge n_k > n_{k-1}).$$

Denote $A_k := \{|f_{n_k} - f_{n_{k+1}}| > 1/2^k\}$ and $F_m := \bigcup_{k\ge m} A_k$. If $j \ge i \ge m$, then $|f_{n_i} - f_{n_j}| \le 1/2^{m-1}$ on $\Omega \setminus F_m$. So $\{f_{n_k}\}$ is uniformly Cauchy on $\Omega \setminus F_m$ and it is simply convergent to a function f on $E := \bigcup_{m=1}^\infty (\Omega \setminus F_m)$. By (8) and the Fatou property,

$$\begin{aligned} C(\Omega \setminus E) &\le \lim_{m\to\infty} C(F_m) = \lim_{m\to\infty} \|\chi_{F_m}\|_{L^1(C)} = \lim_{m\to\infty} \|\chi_{\bigcup_{k\ge m} A_k}\|_{L^1(C)} \\ &\le \lim_{m\to\infty} \|\sum_{k\ge m} \chi_{A_k}\|_{L^1(C)} \le \lim_{m\to\infty} 2^{1/\varrho}\left(\sum_{k\ge m} \|\chi_{A_k}\|^\varrho_{L^1(C)}\right)^{1/\varrho} \\ &= \lim_{m\to\infty} 2^{1/\varrho}\left(\sum_{k\ge m} C(A_k)^\varrho\right)^{1/\varrho} = 0. \end{aligned}$$

By the Fatou property

$$C\{|f_{n_k} - f| > \eta\} = C\{\lim_{j\to\infty} |f_{n_k} - f_{n_j}| > \eta\} \le \lim_{j\to\infty} C\{|f_{n_k} - f_{n_j}| > \eta\} < \epsilon.$$

Since $\{f_n\}$ is a Cauchy sequence in capacity which has a subsequence which is convergent in capacity to f, $\{f_n\}$ converges also to f in capacity. □

The topology and the uniform structure of $L^p(C)$ are given by the metric $d(f, g) := \|f - g\|^*$, where $\|\cdot\|^*$ is associated to $\|\cdot\|_{L^p(C)}$ as in (7).

Theorem 6 *$L^p(C)$ $(0 < p < \infty)$ is complete.*

Proof We follow some usual arguments of measure theory combined with (9):

Let $\{f_n\} \subset L^p(C)$ be a Cauchy sequence. For each $k \in \mathbf{N}$, pick $n_k > n_{k-1}$ so that

$$\|f_m - f_n\|^p = \int |f_m - f_n|^p \, dC < \frac{1}{3^k} \quad (m, n \geq n_k).$$

If $A_k = \{|f_{n_{k+1}} - f_{n_k}|^p > 1/2^k\}$, then $C(A_k) < 2^k/3^k$ since

$$\frac{C(A_k)}{2^k} \leq \int_{A_k} |f_{n_{k+1}} - f_{n_k}|^p \, dC < \frac{1}{3^k}.$$

Note that

$$\sum_{k=1}^{\infty} |f_{n_{k+1}}(t) - f_{n_k}(t)| < \infty \quad \forall t \notin \bigcup_{k>N} A_k$$

because $|f_{n_{k+1}}(t) - f_{n_k}(t)| \leq 1/2^{k/p}$ if $k > N$. Therefore, there exists

$$f(t) := f_{n_1}(t) + \sum_{k=1}^{\infty} \big(f_{n_{k+1}}(t) - f_{n_k}(t)\big) = \lim_k f_{n_k}(t) \quad \forall t \notin A = \bigcap_{N=1}^{\infty} \bigcup_{k>N} A_k$$

and $C(A) = 0$ since, by (9),

$$C(A)^\varrho \leq C\left(\bigcup_{k>N} A_k\right)^\varrho \leq 2 \sum_{k>N} \left(\frac{2}{3}\right)^{\varrho k}$$

and $\sum_{k>N} (2/3)^{\varrho k} < \infty$. Put $f(t) := 0$ if $t \in A$.

As $n_k \to \infty$, $|f_{n_k}(t) - f_n(t)|^p \to |f(t) - f_n(t)|^p$ C-q.e. and

$$\int |f - f_n|^p \, dC \leq \liminf_k \int |f_{n_k} - f_n|^p \, dC \leq \varepsilon$$

for n large enough. □

The proof of completeness of $L^p(C)$ can be easily adapted to show that all $L^{p,q}(C)$-spaces are also complete.

Remark 1 The absence of additivity for the Choquet integral makes it difficult to give a description of the dual of $L^p(C)$. See for instance [1, Section 4], where duality in the case of Hausdorff and Bessel capacities is studied.

If p' is the conjugate exponent of $p \in [1, \infty]$, Hölder's inequality shows that every $g \in L^{p'}(C)^+$ defines a functional $u_g(f) := \int fg \, dC$ which is homogeneous and bounded on $L^p(C)^+$,

$$u_g(f) \leq 2c \left(\int g^{p'} \, dC\right)^{1/p'} \left(\int f^p \, dC\right)^{1/p},$$

but in general u_g is not additive.

4 Subadditivity

The Choquet integral is subadditive on sets,

$$\int (\chi_A + \chi_B)\, dC \leq \int \chi_A\, dC + \int \chi_B\, dC,$$

if and only if

$$C(A \cup B) + C(A \cap B) \leq C(A) + C(B).$$

Then the Choquet integral is also subadditive on nonnegative simple functions. These facts were proved by Choquet in [16] (see also [15] or [14] for a direct elementary proof).

In this case C is said to be strongly subadditive or concave.

Variational capacities and those of Fuglede and Meyers are examples of concave capacities. Shannon entropy is concave if $n = 1$, but not if $n > 1$ (see [17]). In the case of the entropies C_E associated to Banach function spaces, examples and counterexamples of concave capacities are given in [15].

Concave capacities give rise to normed L^p-spaces, since the Minkowski inequality holds with constant 1, and a natural question is to determine when, for a non-concave capacity C, $L^p(C)$ is normable, this meaning that there exists in $L^p(C)$ a norm which is equivalent to $\|\cdot\|_{L^p(C)}$.

As for usual Lorentz spaces, one could try to substitute $f_C^\star$ by

$$f^{\star\star}(t) = \frac{1}{t} \int_0^t f_C^\star(s)\, ds,$$

but unfortunately this average function is subadditive precisely when $L^p(C)$ $(p \geq 1)$ are normed spaces:

Theorem 7 *$f^{\star\star}$ is subadditive with respect to f if and only if C is concave.*

Proof It is clear that $C_t(A) := \min(C(A), t)$ is a Fatou capacity. For a fixed $t > 0$, $f^{\star\star}(t)$ is subadditive in f if and only if C_t is concave, since

$$\int_0^t f_C^\star(s)\, ds = \int_0^\infty dy \int_0^t \chi_{[0, C\{f>y\})}(s)\, ds = \int_0^\infty \min(C\{f > y\}, t)\, dy,$$

and the theorem follows. □

We do not have a satisfactory sufficient normability condition. Let us see a restrictive one, which extends a known result for classical Lorentz spaces.

In the rest of the section μ represents a measure on (Ω, Σ) such that $\mu(\Sigma) = [0, \mu(\Omega)] \subset [0, \infty]$, and we will suppose that C is **μ-invariant**, this meaning that $C(A) = C(B)$ if $\mu(A) = \mu(B)$.

A capacity C on (Ω, Σ) will be said to be **quasi-concave** with respect to μ if there exists a constant $\gamma \geq 1$ such that, whenever $\mu(A) \leq \mu(B)$, the following two conditions are satisfied:

- $C(A) \leq \gamma\, C(B)$, and
- $\frac{C(B)}{\mu(B)} \leq \gamma \frac{C(A)}{\mu(A)}$,

that is, for all $A, B \in \Sigma$,

$$C(B) \leq \gamma \max \left(1, \frac{\mu(B)}{\mu(A)}\right) C(A).$$

Example 2 If $J : [0, \mu(\Omega)] \to \mathbf{R}$ is an increasing function such that $J(t)/t$ is decreasing, then it is readily seen that $C(A) := J(\mu(A))$ defines a μ-invariant and quasi-concave capacity with respect to μ. For instance, $C(A) := \varphi_X(\mu(A))$ when φ_X is the fundamental function of an r.i. space. Note that φ_X is a quasi-concave function.

Theorem 8 *If the capacity C is μ-invariant and quasi-concave with respect to μ, then*

$$\widetilde{C}(A) := \sup \left\{ \sum_{i=1}^{n} \lambda_i C(A_i) : n \in \mathbf{N}, \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i \mu(A_i) \leq \mu(A) \right\}$$

defines a concave capacity and

$$\bar{C}(A) := \inf_{A_n \uparrow A,\, A_n \in \Sigma} \left\{ \lim_{n \to \infty} \widetilde{C}(A_n) \right\}$$

a concave Fatou capacity. Both $\widetilde{C}$ and $\bar{C}$ are equivalent to C.

Proof It is clear that $\tilde{C}(A) \geq 0$ and it is readily seen that $\tilde{C}$ is increasing.

Let us show that

$$C(A) \leq \tilde{C}(A) \leq 2\gamma C(A) \tag{10}$$

Obviously $C(A) \leq \tilde{C}(A)$. On the other hand, if $\varepsilon > 0$ is given, we can find $\sum_{i=1}^{n} \lambda_i \mu(A_i) \leq \mu(A)$ with $\sum_{i=1}^{n} \lambda_i = 1$ and $\lambda_i \geq 0$ such that

$$\tilde{C}(A) - \varepsilon \leq \sum_{i=1}^{n} \lambda_i C(A_i) \leq \gamma \sum_{i=1}^{n} \lambda_i \max \left(1, \frac{\mu(A_i)}{\mu(A)}\right) C(A) \leq 2\gamma C(A)$$

and (10) follows.

To prove that $\tilde{C}$ is concave, let $0 < \theta < 1$ and $\varepsilon > 0$. If $A, B \in \Sigma$, we can find $\sum_{i=1}^{n} \lambda_i \mu(A_i) \leq \mu(A)$ with $\sum_{i=1}^{n} \lambda_i = 1$ and $\lambda_i \geq 0$ such that

$$(1-\theta)\widetilde{C}(A) - \frac{\varepsilon}{2} \leq (1-\theta) \sum_{i=1}^{n} \lambda_i C(A_i)$$

and, similarly,

$$\theta \widetilde{C}(B) - \frac{\varepsilon}{2} \leq \theta \sum_{j=1}^{m} \lambda'_j C(B_j).$$

with $\sum_{j=1}^{m} \lambda'_j \mu(B_j) \leq \mu(B)$ with $\sum_{j=1}^{m} \lambda'_j = 1$ and $\lambda'_i \geq 0$.

Then $(1-\theta)\mu(A) + \theta\mu(B) \geq \sum_{i=1}^{n} (1-\theta)\lambda_i \mu(A_i) + \sum_{j=1}^{m} \theta \lambda'_j \mu(B_j)$ and $\sum_{i=1}^{n} (1-\theta)\lambda_i + \sum_{j=1}^{m} \theta \lambda'_j = 1$. We can choose $D \in \Sigma$ such that $\mu(D) = (1-\theta)\mu(A) + \theta\mu(B)$, and then

$$(1-\theta)\widetilde{C}(A) + \theta\widetilde{C}(B) - \varepsilon \leq \sum_{i=1}^{n} (1-\theta)\lambda_i C(A_i) + \sum_{j=1}^{m} \theta \lambda'_j C(B_j) \leq \widetilde{C}(D),$$

so that

$$(1-\theta)\widetilde{C}(A)+\theta\widetilde{C}(B)\leq\widetilde{C}(D). \tag{11}$$

Since C is μ-invariant, the same happens with $\widetilde{C}$ and we may define $\varphi(s):=\widetilde{C}(A)$ if $s=\mu(A)$, which is a concave function on $[0,\mu(\Omega)]$, by (11).

We claim that, if $x, y\geq t>0$, then

$$\varphi(x+y-t)+\varphi(t)\leq\varphi(x)+\varphi(y), \tag{12}$$

and the concavity of $\tilde{C}$ follows by taking $t=\mu(A\cap B)$, $x=\mu(A)$ and $y=\mu(B)$, since then $\varphi(t)=\widetilde{C}(A\cap B)$, $\varphi(x+y-t)=\varphi(\mu(A)+\mu(B)-\mu(A\cap B))=\varphi(\mu(A\cup B))=\widetilde{C}(A\cup B)$, and $\varphi(x)+\varphi(y)=\widetilde{C}(A)+\widetilde{C}(B)$.

To prove the claim, we may assume that $0<t<x\leq y$ and write

$$x=(1-\tau)t+\tau(x+y-t),\quad y=(1-\tau')t+\tau'(x+y-t)\qquad(\tau,\tau'\in(0,1)).$$

Since φ is concave,

$$\left(1-\frac{x-t}{x+y-2t}\right)\varphi(t)+\frac{x-t}{x+y-2t}\varphi(x+y-t)\leq\varphi(x)$$

and

$$\left(1-\frac{y-t}{x+y-2t}\right)\varphi(t)+\frac{y-t}{x+y-2t}\varphi(x+y-t)\leq\varphi(y).$$

Finally, by addition we obtain (12).

Since $\widetilde{C}$ is concave, it is also subadditive.

Let $\epsilon>0$ and $A, B\in\Sigma$. There exists $\{A_n\}$ with $A_n\uparrow A$ such that $\lim_{n\to\infty}\widetilde{C}(A_n)\leq\bar{C}(A)+\epsilon$. Since $\widetilde{C}\simeq C$ then there exists $c'>0$ such that for all $C(A)\leq c'\widetilde{C}(A)$ for every set A; and hence,

$$\frac{1}{c'}\lim_{n\to\infty}C(A_n)\leq\lim_{n\to\infty}\widetilde{C}(A_n)\leq\bar{C}(A)+\epsilon$$

and $\lim_{n\to\infty}C(A_n)=C(A)$, since C has the Fatou property. We get the equivalence of the capacities with the equivalence of $\widetilde{C}$ to C.

Moreover there exist increasing sequences $\{A_n\}$ and $\{B_n\}$ such that

$$\lim_n\widetilde{C}(A_n)\leq\bar{C}(A)+\epsilon/2,\quad\lim_n\widetilde{C}(B_n)\leq\bar{C}(B)+\epsilon/2.$$

Assume that $\bar{C}(A)+\bar{C}(B)<\infty$. By the concavity of $\widetilde{C}$,

$$\begin{aligned}\lim_{n\to\infty}\left[\widetilde{C}(A_n\cup B_n)+\widetilde{C}(A_n\cap B_n)\right]&\leq\lim_{n\to\infty}\widetilde{C}(A_n)+\widetilde{C}(B_n)\\&=\lim_{n\to\infty}\widetilde{C}(A_n)+\lim_{n\to\infty}\widetilde{C}(B_n)\leq\bar{C}(A)+\bar{C}(B)+\epsilon\end{aligned}$$

and then, from the definition of $\bar{C}$, since $A_n\cup B_n\uparrow A\cup B$ and $A_n\cap B_n\uparrow A\cap B$, we obtain

$$\lim_{n\to\infty}\left[\widetilde{C}(A_n\cup B_n)+\widetilde{C}(A_n\cap B_n)\right]\geq\bar{C}(A\cup B)+\bar{C}(A\cap B)$$

and the concavity of $\bar{C}$ follows.

To prove that $\bar{C}$ has the Fatou property, if $\{A_n\}$ with $A_n\uparrow A$, we need to prove $\bar{C}(A)\leq\lim_{n\to\infty}\bar{C}(A_n)$ if this limit is finite. For each n, there exist $A_{n,m}\uparrow A_n$ so that $\lim_{m\to\infty}\widetilde{C}(A_{n,m})\leq\bar{C}(A_n)+\epsilon$. Since $A_{n,n}\uparrow A$ as $n\to\infty$,

$$\bar{C}(A)\leq\lim_{n\to\infty}\widetilde{C}(B_n)=\lim_{n\to\infty}\widetilde{C}(A_{n,n})\leq\lim_{n\to\infty}\bar{C}(A_n)+\epsilon\qquad\square$$

Remark 2 $\widetilde{C}$ has the Fatou property if and only if $\bar{C} = \widetilde{C}$. Hence, $L^p(C) = L^p(\bar{C})$ and $L^p(C)$ $(1 \le p \le \infty)$ is normable.

Indeed, suppose that $\widetilde{C}$ has the Fatou property. If $A_n \uparrow A$ then

$$\widetilde{C}(A) = \lim_{n\to\infty} \widetilde{C}(A_n) \ge \bar{C}(A)$$

and there exist $A_n \uparrow A$ such that $\widetilde{C}(A) = \lim_{n\to\infty} \widetilde{C}(A_n) < \bar{C}(A) + \epsilon$.

Remark 3 Suppose $L^1(C) = L^1(\widehat{C})$ with equivalent quasi-norms, $\|\cdot\|$ and $\|\cdot\|_*$, and suppose that $\widehat{C}$ is concave. If $\|\cdot\|_*$ is μ-invariant ($\|\chi_A\|_* = \|\chi_B\|_*$ when $\mu(A) = \mu(B)$), then C is quasi-concave with respect to μ.

Indeed, in this case $\widehat{C}(A) := \|\chi_A\|_*$ is a μ-invariant capacity, $\widehat{C}(A) \simeq C(A)$ and $\widehat{C}$ is concave:

$$\widehat{C}(A \cup B) + \widehat{C}(A \cap B) \le \widehat{C}(A) + \widehat{C}(B).$$

We can suppose $0 \le \mu(A \cap B) < \mu(A) \le \mu(B)$ and define $\varphi(\mu(A)) := \widehat{C}(A)$.

Let $t = \mu(A \cap B)$, $x = \mu(A)$ and $y = \mu(B)$, so that $0 < t < x \le y$ and $\varphi(x + y - t) + \varphi(t) \le \varphi(x) + \varphi(y)$. In particular, if $m \in N$ and $r > 0$, then $\varphi(mr) \le m\varphi(r)$. Moreover, if $a \le b$, then there exists $m \ge 2$ such that $(m-1)a \le b \le ma$ and

$$\frac{\varphi(b)}{b} \le \frac{\varphi(ma)}{b} \le \frac{ma}{b}\frac{\varphi(a)}{a} \le \frac{m}{m-1}\frac{\varphi(a)}{a}.$$

But $x \le y$ and there is some $m \in N$ such that

$$\frac{\varphi(y)}{y} \le \frac{m}{m-1}\frac{\varphi(x)}{x},$$

which means that $\widehat{C}$ is quasi-concave respect to the measure μ, with constant $\gamma = 2$. Since $\widehat{C} \simeq C$, C is also quasi-concave with respect to μ. □

5 Interpolation

If $\bar{A} = (A_0, A_1)$ is a couple of quasi-Banach spaces, $0 < \theta < 1$ and $0 < q \le \infty$, the interpolation space $\bar{A}_{\theta,q}$ is the quasi-Banach space of all $f \in A_0 + A_1$ such that

$$\|f\|_{\theta,q} := \left(\int_0^\infty (t^{-\theta} K(t, f, \bar{A}))^q \frac{dt}{t} \right)^{1/q} < \infty$$

where $K(t, f, \bar{A})$ is the K-functional,

$$K(t, f; \bar{A}) := \inf \left\{ \|f_0\|_{A_0} + t\|f_1\|_{A_1};\ f = f_0 + f_1 \right\}$$

We refer to [8] and [10] for general facts concerning interpolation theory.

Here we are wishing to extend the results on real interpolation of capacitary $L^p(C)$-spaces included in [15] and [14], where the capacities were supposed to be concave and $p \ge 1$, since only Banach couples were allowed.

The capacities will be still supposed to be Fatou but the Choquet integral will not be necessarily subadditive anymore, and $0 < p < 1$ is also allowed.

For the sake of completeness we include the details of the proof of the estimates of $K(t, f) = K(t, f; L^p(C), L^\infty(C))$ similar to those of [15] and [14].

Theorem 9 *If $p > 0$, then*

$$K(t, f; L^p(C), L^\infty(C)) \approx \left(\int_0^\infty y^{p-1} \min(C\{f > y\}, t^p) dy \right)^{1/p}$$
$$\approx \left(\int_0^{t^p} f_C^\star(y)^p dy \right)^{1/p}.$$

Proof Let $0 \le f \in L^p(C) + L^\infty(C)$. For $t > 0$ given, let

$$y^* := \inf\{y > 0 : C\{f > y\} \le t^p\} = f_C^\star(t^p),$$

and consider

$$g_0(x) := \int_{y^*}^\infty \chi_{\{f>y\}}(x) dy, \quad g_1(x) := \int_0^{y^*} \chi_{\{f>y\}}(x) dy.$$

Then $f = g_0 + g_1$ and $\{g_0 > y\} = \{f > y + y^*\}$. So

$$\|g_0\|_{L^p(C)}^p = \int_0^{y^*} y^{p-1} C\{f > y + y^*\} dy + \int_{y^*}^\infty y^{p-1} C\{f > y + y^*\} dy$$
$$\le \int_0^{y^*} y^{p-1} C\{f > y^*\} dy + \int_{y^*}^\infty y^{p-1} C\{f > y\} dy$$
$$\lesssim t^p (y^*)^p + \int_{y^*}^\infty y^{p-1} C\{f > y\} dy.$$

Hence,

$$K(t, f) \le \|g_0\|_{L^p(C)} + t\|g_1\|_{L^\infty(C)}$$
$$\le \left(t^p (y^*)^p + \int_{y^*}^\infty y^{p-1} C\{f > y\} dy \right)^{1/p} + t y^*$$
$$\lesssim \left(t^p (y^*)^p + \int_{y^*}^\infty y^{p-1} C\{f > y\} dy \right)^{1/p} + \left(t^p \int_0^{y^*} y^{p-1} dy \right)^{1/p}$$
$$\lesssim \left(\int_{y^*}^\infty y^{p-1} C\{f > y\} dy + t^p \int_0^{y^*} y^{p-1} dy \right)^{1/p}$$
$$\lesssim \left(\int_0^\infty y^{p-1} \min(C\{f > y\}, t^p) dy \right)^{1/p}.$$

For the reverse estimate we use that there exists $\Omega_f(t) \subset \Omega$ such that

$$K(t, f) \simeq \|f\chi_{\Omega_f(t)}\|_{L^p(C)} + t\|f\chi_{\Omega\setminus\Omega_f(t)}\|_{L^\infty(C)} = \|f_0\|_{L^p(C)} + t\|f_1\|_{L^\infty(C)},$$

with $f_0 := f\chi_{\Omega_f(t)}$, $f_1 := f\chi_{\Omega\setminus\Omega_f(t)}$ (just consider $f = f_0 + f_1$ such that $\|f_0\|_{L^p(C)} + t\|f_1\|_{L^\infty(C)} \leq 2K(t, f)$ and take $\Omega_f(t) = \{|f_0| \geq |f_1|\}$).

If $f = \chi_A$, then

$$K(t, \chi_A) \simeq \inf\{C(A_0) + tC(A_1);\ A = A_0 \cup A_1,\ A_0 \cap A_1 = \emptyset\} \simeq \min(C(A), t).$$

Since f_0, f_1 are disjointly supported, $\chi_{\{f>y\}} = \chi_{\{f_0>y\}} + \chi_{\{f_1>y\}}$, and

$$\min(C\{f > y\}, t) \simeq K(t, \chi_{\{f>y\}}) \lesssim C\{f_0 > y\} + t\|\chi_{\{f_1>y\}}\|_{L^\infty(C)}$$

and $\|f_1\|_{L^\infty(C)} = p^{1/p}\left(\int_0^\infty y^{p-1}\|\chi_{\{f_1>y\}}\|_{L^\infty(C)}dy\right)^{1/p}$. So

$$\begin{aligned}
K(t, f) &\simeq \left(\int_0^\infty y^{p-1}C\{f_0 > y\}dy\right)^{1/p} + \left(t^p\int_0^\infty y^{p-1}\|\chi_{\{f_1>y\}}\|_{L^\infty(C)}dy\right)^{1/p} \\
&\simeq \left(\int_0^\infty y^{p-1}(C\{f_0 > y\} + t^p\|\chi_{\{f_1>y\}}\|_{L^\infty(C)})dy\right)^{1/p} \\
&\geq \left(\int_0^\infty y^{p-1}\min(C\{f > y\}, t^p)dy\right)^{1/p}.
\end{aligned}$$

To prove that also $K(t, f) \simeq \left(\int_0^{t^p} f_C^\star(y)^p dy\right)^{1/p}$, let

$$f_0(x) := \begin{cases} f(x) - f_C^\star(t^p)\frac{f(x)}{|f(x)|}, & \text{if } |f(x)| > f_C^\star(t^p) \\ 0, & \text{otherwise,} \end{cases}$$

$f_1 := f - f_0$ and $E := \{x : f_0(x) \neq 0\} = \{x : |f(x)| > f_C^\star(t^p)\}$. Then $C(E) \leq t^p$ and since $f_C^\star$ is constant on $[C(E), t^p]$,

$$\begin{aligned}
K(t, f) &\leq \|f_0\|_{L^p(C)} + t\|f_1\|_{L^\infty(C)} \\
&= \left(\int_E (|f(x)| - f_C^\star(t^p))^p dC\right)^{1/p} + tf_C^\star(t^p) \\
&= \left(\int_0^{C(E)} (f_C^\star(s) - f_C^\star(t^p))^p ds\right)^{1/p} + \left(\int_0^{t^p} f_C^\star(t^p)^p ds\right)^{1/p} \\
&\leq \left(\int_0^{t^p} (f_C^\star(s) - f_C^\star(t^p))^p ds\right)^{1/p} + \left(\int_0^{t^p} f_C^\star(t^p)^p ds\right)^{1/p}
\end{aligned}$$

$$\lesssim \left\{ \int_0^{t^p} (f_C^\star(s) - f_C^\star(t^p))^p ds + \int_0^{t^p} f_C^\star(t^p)^p ds \right\}^{1/p}$$

$$\lesssim \left(\int_0^{t^p} f_C^\star(s)^p ds \right)^{1/p}.$$

Conversely, consider $f = g + h$ with $g \in L^p(C)$ and $h \in L^\infty(C)$. Then

$$\begin{aligned}
\int_0^{t^p} f_C^\star(s)^p ds &= \int_0^{t^p} (|f|^p(s))_C^\star ds \lesssim \int_0^{t^p} (|g|^p + |h|^p)_C^\star(s) ds \\
&\lesssim \int_0^{t^p} [(|g|^p)_C^\star(s) + (|h|^p)_C^\star(s)] ds \\
&= \int_0^{t^p} (g_C^\star(s))^p ds + \int_0^{t^p} (h_C^\star(s))^p ds \\
&\lesssim \int_0^{t^p} (g_C^\star(s))^p ds + t^p h_C^\star(0)^p \lesssim \|g\|_{L^p(C)}^p + t^p \|h\|_{L^\infty(C)}^p \\
&\lesssim (\|g\|_{L^p(C)} + t\|h\|_{L^\infty(C)})^p
\end{aligned}$$

and then $\left(\int_0^{t^p} f_C^\star(s)^p ds\right)^{1/p} \lesssim K(t, f)$. □

Once we have the description of $K(t, f)$, real interpolation follows easily as in [8], Theorem 5.2.1:

Theorem 10 *Suppose* $0 < \theta < 1$, $0 < p_0 < q \leq \infty$ *or* $0 < p_0 \leq q < \infty$, *and* $\frac{1}{p} = \frac{1-\theta}{p_0}$. *Then*

$$(L^{p_0}(C), L^\infty(C))_{\theta,q} = L^{p,q}(C)$$

Proof We know from Theorem 9 that

$$\int_0^\infty y^{p-1} \min(C\{f > y\}, t^p) dy \simeq \int_0^{t^p} f_C^\star(y)^p dy$$

so that, using the integral Minkowski inequality ($q/p_0 \geq 1$),

$$\begin{aligned}
\|f\|_{\theta,q} &= \left(\int_0^\infty t^{-\theta q} K(t, f)^q \frac{dt}{t} \right)^{1/q} \\
&\simeq \left(\int_0^\infty t^{-\theta q} \left(\int_0^{t^{p_0}} f_C^\star(s)^{p_0} ds \right)^{q/p_0} \frac{dt}{t} \right)^{1/q}
\end{aligned}$$

$$= \left(\int_0^\infty \left(t^{-\theta p_0 + p_0} \int_0^1 f_C^\star (y t^{p_0})^{p_0} y \frac{dy}{y} \right)^{q/p_0} \frac{dt}{t} \right)^{1/q}$$

$$\lesssim \int_0^1 \left(y^{q/p_0} \int_0^\infty t^{(1-\theta)q} (f_C^\star (y t^{p_0}))^q \frac{dt}{t} \right)^{p_0/q} \frac{dy}{y}$$

$$\lesssim \left(\int_0^\infty \left(s^{\frac{1-\theta}{p_0}} f_C^\star(s) \right)^q \frac{ds}{s} \right)^{1/q} = \left(\int_0^\infty \left(s^{1/p} f_C^\star(s) \right)^q \frac{ds}{s} \right)^{1/q}.$$

Then $\|f\|_{\theta,q} \lesssim \|f\|_{L^{p,q}(C)}$ since $\|f\|_{L^{p,q}(C)} \simeq \left(\int_0^\infty \left(s^{1/p} f_C^\star(s) \right)^q \frac{ds}{s} \right)^{1/q}$.

Conversely,

$$\|f\|_{L^{p,q}(C)} \simeq \left(\int_0^\infty \left(s^{1/p} f_C^\star(s) \right)^q \frac{ds}{s} \right)^{1/q} = \left(\int_0^\infty \left(s^{\frac{1-\theta}{p_0}} f_C^\star(s) \right)^q \frac{ds}{s} \right)^{1/q}$$

$$\simeq \left(\int_0^\infty \left(t^{1-\theta} f_C^\star(t^{p_0}) \right)^q \frac{dt}{t} \right)^{1/q} = \left(\int_0^\infty \left(t^{(1-\theta)p_0} f_C^\star(t^{p_0})^{p_0} \right)^{q/p_0} \frac{dt}{t} \right)^{1/q}$$

$$\lesssim \left(\int_0^\infty \left(t^{-\theta} \left(\int_0^{t^{p_0}} f_C^\star(s)^{p_0} ds \right)^{1/p_0} \right)^q \frac{dt}{t} \right)^{1/q} = \|f\|_{\theta,q},$$

where we have used that $f_C^\star$ is decreasing. □

In the case of a single quasi-subadditive Fatou capacity, Theorem 10 is extended by reiteration:

Corollary 1 *Let* $0 < p_0, p_1, q_0, q_1 < \infty$ *and* $0 < \eta < 1$. *Then*

$$(L^{p_0,q_0}(C), L^{p_1,q_1}(C))_{\eta,q} = L^{p,q}(C)$$

with $1/p = (1-\eta)/p_0 + \eta/p_1$.

Proof Let $r < \min(p_0, p_1, q_0, q_1)$ and choose $(1-\theta_i) := r/p_i$ $(i = 0, 1)$. Then, if $\theta := (1-\eta)\theta_0 + \eta\theta_1$, since $1/p = (1-\theta)/r$, Theorem 10 gives

$$(L^{p_0,q_0}(C), L^{p_1,q_1}(C))_{\eta,q} = ((L^r(C), L^\infty(C))_{\theta_0,q_0}, (L^r(C), L^\infty(C))_{\theta_1,q_1})_{\eta,q}.$$

Then by reiteration (cf. [8], Theorem 3.11.5) we obtain

$$((L^r(C), L^\infty(C))_{\theta_0,q_0}, (L^r(C), L^\infty(C))_{\theta_1,q_1})_{\eta,q} = (L^r(C), L^\infty(C))_{\theta,q}.$$

and, again from Theorem 10, $(L^{p_0,q_0}(C), L^{p_1,q_1}(C))_{\eta,q} = L^{p,q}(C)$. □

We also want to consider interpolation with change of capacities. Let (C_0, C_1) be a couple of capacities on the same measurable space with the same null sets. Examples are given by Hausdorff capacities and the corresponding Hausdorff measures, or by couples of capacities associated to quasi-Banach spaces, as in (1), on the same measure space.

We will denote

$$[C_0 + tC_1](A) := K(t, \chi_A; L^1(C_0), L^1(C_1)),$$

which is a quasi-subadditive capacity.

In the proof of the following theorem we use the reiteration properties for the triple $(L^r(C_0), L^r(C_1), L^\infty)$ that we describe in the appendix.

Theorem 11 *Let C_0, C_1 be a couple of quasi-subadditive Fatou capacities with the same null sets and $0 < \eta < 1$. If $0 < p_0, p_1 < \infty$, $0 < q_0, q_1 \le \infty$ and $\frac{1}{p} := \frac{1-\eta}{p_0} + \frac{\eta}{p_1}$, then*

$$(L^{p_0,q_0}(C_0), L^{p_1,q_1}(C_1))_{\eta,q} = L^{p,q}(C_{\eta p/p_1, q/p}),$$

where $C_{\theta,q}(A) := \|\chi_A\|_{(L^1(C_0),L^1(C_1))_{\theta,q}}$.

Proof Choose $0 < r < \min(p_0, p_1, q_0, q_1)$. Then, by Theorem 10, for $(1-\alpha_0)p_0 = r$ and $(1-\alpha_1)p_1 = r$ we have

$$(L^{p_0,q_0}(C_0), L^{p_1,q_1}(C_1))_{\eta,q} = ((L^r(C_0), L^\infty)_{\alpha_0,q_0}, (L^r(C_1), L^\infty)_{\alpha_1,q_1})_{\eta,q}$$

and, if $\bar{X} = (L^r(C_0), L^r(C_1), L^\infty)$, by reiteration (cf. Theorem 13),

$$((L^r(C_0), L^\infty)_{\alpha_0,q_0}, (L^r(C_1), L^\infty)_{\alpha_1,q_1})_{\eta,q} = \bar{X}_{(\theta_0,\theta_1),q}$$

with $\theta_0 = (1-\alpha_1)\eta$, $\theta_1 = \alpha_0(1-\eta) + \alpha_1\eta$.

Since

$$K(t_1, t_2, f; \bar{X}) \simeq K(t_2, f; L^r([C_0 + t_1^r C_1]), L^\infty)$$

and $K(t, f; L^r(C), L^\infty(C)) = (\int_0^\infty y^{r-1}\min(C\{|f| > y\}, t^r)dy)^{1/r}$ by Theorem 9, we get

$$K(t_1, t_2, f; \bar{X}) \simeq \left(\int_0^\infty y^{r-1}\min([C_0 + t_1^r C_1]\{|f| > y\}, t_2^r)dy\right)^{1/r}.$$

So, the value of $\|f\|^q_{\bar{X}_{(\theta_0,\theta_1),q}}$ is

$$\int_{\mathbf{R}_+^2}\left[t_1^{-\theta_0}t_2^{-\theta_1}\left(\int_0^\infty y^{r-1}\min([C_0 + t_1^r C_1]\{|f| > y\}, t_2^r)dy\right)^{1/r}\right]^q \frac{dt_2}{t_2}\frac{dt_1}{t_1},$$

where

$$\begin{aligned}
&\int_0^\infty\left[t_2^{-\theta_1}\left(\int_0^\infty y^{r-1}\min([C_0 + t_1^r C_1]\{|f| > y\}, t_2^r)dy\right)^{1/r}\right]^q \frac{dt_2}{t_2}\\
&= \int_0^\infty t_2^{-\theta_1 q} K(t_2, f; L^r([C_0 + t_1^r C_1]), L^\infty)^q \frac{dt_2}{t_2}\\
&\simeq \|f\|^q_{(L^r([C_0+t_1^r C_1]), L^\infty)_{\theta_1,q}} = \|f\|_{L^{r/(1-\theta_1),q}([C_0+t_1^r C_1])}\\
&\simeq \int_0^\infty y^{q-1}[C_0 + t_1^r C_1]\{|f| > y\}^{\frac{(1-\theta_1)q}{r}}dy,
\end{aligned}$$

by Theorem 10.

Thus

$$\begin{aligned}\|f\|^q_{\bar{X}_{(\theta_0,\theta_1),q}} &\simeq \int_0^\infty t_1^{-\theta_0 q}\int_0^\infty y^{q-1}[C_0+t_1^r C_1]\{|f|>y\}^{\frac{q(1-\theta_1)}{r}}dy\frac{dt_1}{t_1}\\ &= \int_0^\infty\int_0^\infty \tau^{\frac{-\theta_0 q}{r}} y^{q-1}[C_0+\tau C_1]\{|f|>y\}^{\frac{q(1-\theta_1)}{r}}dy\frac{1}{r}\frac{d\tau}{\tau}\\ &= \frac{1}{r}\int_0^\infty y^{q-1}\int_0^\infty \tau^{\frac{-\eta q}{p_1}}[C_0+\tau C_1]\{|f|>y\}^{q/p}\frac{d\tau}{\tau}dy\\ &= \frac{1}{r}\int_0^\infty y^{q-1}\int_0^\infty \left(\tau^{\frac{-\eta p}{p_1}}[C_0+\tau C_1]\{|f|>y\}\right)^{q/p}\frac{d\tau}{\tau}dy\end{aligned}$$

where

$$\|\chi_{\{|f|>y\}}\|^{q/p}_{(L^1(C_0),L^1(C_1))_{\frac{\eta p}{p_1},q/p}} = \int_0^\infty \left(\tau^{\frac{-\eta p}{p_1}}K(\tau,\chi_{\{|f|>y\}};L^1(C_0),L^1(C_1))\right)^{q/p}\frac{d\tau}{\tau}$$

Hence, from the definition $C_{\theta,q}(A) := \|\chi_A\|_{(L^1(C_0),L^1(C_1))_{\theta,q}}$,

$$\|f\|^q_{\bar{X}_{(\theta_0,\theta_1),q}} \simeq \int_0^\infty y^{q-1}\|\chi_{\{|f|>y\}}\|^{\frac{q}{p}}_{(L^1(C_0),L^1(C_1))_{\frac{\eta p}{p_1},\frac{q}{p}}}dy \simeq \|f\|_{L^{p,q}(C_{\frac{\eta p}{p_1},\frac{q}{p}})}$$

□

6 Some applications to classical Lorentz spaces

Let $p,q>0$, μ a measure or weight on $\mathbf{R}^n$, and w a weight on $\mathbf{R}^+$. The Lorentz spaces $\Lambda^{p,q}_\mu(w)$ are defined by the condition

$$\|f\|_{\Lambda^{p,q}_\mu(w)} = \left(\int_0^\infty s^{q/p}f^*_\mu(s)^q w(s)\frac{ds}{s}\right)^{1/q} < \infty.$$

Here f^*_μ is the decreasing rearrangement of f with respect to μ,

$$f^*_\mu(s) := \inf\{t;\ \mu\{|f|>t\}\le s\}.$$

If $p=q$, $\Lambda^p_\mu(w)=\Lambda^{p,p}_\mu(w)$ and

$$\|f\|_{\Lambda^p_\mu(w)} = \left(\int_0^\infty f^*_\mu(s)^p w(s)\,ds\right)^{1/p}.$$

If $w=1$, $\Lambda^{p,q}_\mu(1)=L^{p,q}(\mu)$ and $\Lambda^{p,p}_\mu(1)=L^p(\mu)$.

Some basic questions are to determine whether they are normed or quasi-normed function spaces, to prove when $\Lambda^{p,q_0}_\mu(w)\subset\Lambda^{p,q_1}_\mu(w)$ for $0<q_0<q_1\le\infty$, and to find the weights

for which classical operators, such as the Hardy operator $Sf(x) = \int_0^x f(x)dx$, are bounded from $\Lambda^{p_0}_{\mu_0}(w_0)$ to $\Lambda^{p_1}_{\mu_1}(w_1)$.

Two good reference for these topics are [13] and [12].

Here we are going to show that, in fact, Lorentz spaces Λ^p are Lebesgue capacity spaces. Denote $V(A) = \int_A v(t)\,dt$. Then we can write

$$\|f\|_{\Lambda^p_\mu(v)} = \left(\int_0^\infty f^*_\mu(s)^p v(s)\,ds\right)^{1/p} = \left(\int_0^\infty pt^{p-1} V([0,\mu\{|f|>t\}))\,dt\right)^{1/p},$$

so $\Lambda^p_\mu(v) = L^p(C)$ with $C(A) = V[0,\mu(A))$. It follows from our results that $\Lambda^p_\mu(v)$ is a normed space when C is concave, which means that V is concave.

But such a remark can be also applied to new Lorentz spaces obtained from some other well known symmetrization method of analysis:

- Spherical symmetrization: $f^*_S(y) := f^*_\mu(\sigma_n|y|^n) = \int_0^\infty \chi_{\{|f|>s\}^*}$ if $\chi_{A^*} = \chi^*_A$. Also for Steiner of order k $(1 < k \le n)$.
- Multidimensional symmetrization f^*_2 is defined in [6] as follows: For a set $A \subset \mathbf{R}^2$, $A^*_2 = \{(s,t);\ 0 < t < \chi^*_{E(s)}\}$, where $E(s)$ is the s-section $\{y \in \mathbf{R};\ (s,y) \in E\}$. Then s^*_2 is defined for a simple function s, and finally $f^*_2 := \lim_k (s_k)^*_2$.

$$\|f\|_{\Lambda^p_2(v)} := \|f^*_2\|_{L^p(v)}.$$

- Discrete rearrangements on trees as in [19].

In [9], S. Boza and J. Soria consider increasing transformations $A \mapsto \mathcal{R}(A)$ on measure spaces with the Fatou property, $A_n \uparrow A \Rightarrow \mathcal{R}(A_n) \uparrow \mathcal{R}(A)$, that allow to define the corresponding rearrangements of functions

$$f^*_{\mathcal{R}}(y) := \int_0^\infty \chi_{\mathcal{R}\{|f|>t\}}(y)\,dt$$

that allow to unify various Lorentz spaces found in the literature, included all the mentioned above:

$$\|f\|_{\Lambda^p_{\mathcal{R}}(v)} := \|f^*_{\mathcal{R}}\|_{L^p(v)} = \left(\int_0^\infty pt^{p-1} V(\mathcal{R}\{|f|>t\})\,dt\right)^{1/p}$$

Obviously $\Lambda^p_{\mathcal{R}}(v) = L^p(C_{V,\mathcal{R}})$ if we define the capacity $C_{V,\mathcal{R}}$ as

$$C_{V,\mathcal{R}}(A) = V(\mathcal{R}(A)),$$

and our results on capacities apply to this special case.

As a final example, let us show how interpolation of capacitary Lebesgue spaces can be used in interpolation of Lorentz spaces, $(\Lambda^{p_0}(v_0), \Lambda^{p_1}(v_1))_{\eta,p}$.

We know from Theorem 11 that if C_0 and C_1 have the same null sets, $0 < p_0, p_1 < \infty$, and $0 < \eta < 1$, then

$$\big(L^{p_0}(C_0), L^{p_1}(C_1)\big)_{\eta,p} = L^p(C_{\eta p/p_1,1}) \qquad (1/p = (1-\eta)/p_0 + \eta/p_1),$$

where

$$C_{\theta,q}(A) := \|\chi_A\|_{(\Lambda(C_0),\Lambda(C_1))_{\theta,q}}.$$

We start from the identity $(\Lambda^1(v_0), \Lambda^1(v_1))_{\theta,1} = \Lambda^1(v)$, where $V = V_0^{1-\theta} V_1^{\theta}$, and we consider $\Lambda^{p_j}(v_j) = L^{p_j}(C_j)$ with $C_j = V_j \circ \mathcal{R}$ $(j = 0, 1)$. Then

$$(\Lambda^{p_0}(v_0), \Lambda^{p_1}(v_1))_{\eta,p} = (L^{p_0}(C_0), L^{p_1}(C_1))_{\eta,p} = L^p(C_{\theta,1})$$

with $\theta = \eta p/p_1$.

Recall that

$$C_{\theta,1}(A) = \|\chi_A\|_{(L^1(C_0),L^1(C_1))_{\theta,1}} = \|\chi_A\|_{\Lambda^1(v)} = V \circ \mathcal{R}(A)$$

and $L^p(C_{\theta,1}) = \Lambda^p(v)$, so that

$$(\Lambda^{p_0}(v_0), \Lambda^{p_1}(v_1))_{\eta,p} = \Lambda^p(v)$$

with

$$V = V_0^{1-\theta} V_1^{\theta} = V_0^{(1-\eta)p/p_0} V_1^{\eta p/p_1}.$$

Appendix: interpolation of triples

Let C_0 and C_1 be a couple of Fatou capacities on (Ω, Σ) having the same null sets and $\bar{X} = (L^r(C_0), L^r(C_1), L^\infty)$ with $0 < r < \infty$, a triple of quasi-Banach spaces. Here $L^\infty = L^\infty(C_0) = L^\infty(C_1)$ and we write X_i $(i = 0, 1, 2)$ for the components of $\bar{X}$, which are continuously contained in the sum space $\Sigma(\bar{X})$ endowed with

$$\|f\|_\Sigma := \inf\{\|f_0\|_0 + \|f_1\|_1 + \|f_2\|_2;\ f = f_0 + f_1 + f_2\}.$$

If $\|f\|_\Sigma = 0$, it is readily seen that $f = 0$ C-q.e. and $\|\cdot\|_\Sigma$ is a quasi-norm.

For every $f \in \Sigma(\bar{X})$, the K-functional $K(\mathbf{t}, f) = K(\mathbf{t}, f; \bar{X})$ is defined as

$$K(\mathbf{t}, f) := \inf\{\|f_0\|_0 + t_1\|f_1\|_1 + t_2\|f_2\|_2;\ f = f_0 + f_1 + f_2\} \quad (\mathbf{t} = (t_1, t_2) \in \mathbf{R}_+^2).$$

If $\Theta = (\theta_0, \theta_1)$ with $\theta_0, \theta_1 > 0$ and $\theta_0 + \theta_1 < 1$, the interpolation space $\bar{X}_{\Theta,q;K}$ is defined for every $q > 0$ by the condition

$$\|f\|_{\Theta,q;K} := \|K(\cdot, f)\|_{(\Theta,q)} < \infty,$$

where

$$\|g\|_{(\Theta,q)} := \left(\int_{\mathbf{R}_+^2} (t_1^{-\theta_0} t_2^{-\theta_1} g(\mathbf{t}))^q \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right)^{1/q}.$$

Also, the J-space $\bar{X}_{\Theta,q;J}$ is defined by the condition

$$\|f\|_{\Theta,q;J} := \inf \left\{ \|J(\cdot, u(\cdot))\|_{(\Theta,q)};\ f = \int_{\mathbf{R}_+^2} u(\mathbf{t}) \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right\} < \infty,$$

where $u : \mathbf{R}_+^2 \to \Delta(\bar{X}) = X_0 \cap X_1 \cap X_2$ is any measurable function such that, in $\Sigma(\bar{X})$,

$$f = \int_{\mathbf{R}_+^2} u(\mathbf{t}) \frac{dt_1}{t_1} \frac{dt_2}{t_2},$$

and $J(\mathbf{t}, x) = \max(\|x\|_0, t_1\|x\|_1, t_2\|x\|_2)$ if $x \in \Delta(\bar{X})$.

To show that one can apply to $\bar{X}$ the methods of [4] and [5], let $\varrho \in (0, 1]$ be the parameter in Aoki's theorem corresponding to a common constant c in the triangle inequality for the quasi-Banach spaces in $\bar{X}$.

Denote

$$(S_\rho f)(\mathbf{t})^\rho := \int_{\mathbf{R}_+^2} \left[\min\left(1, \frac{t_1}{s_1}, \frac{t_2}{s_2}\right) f(\mathbf{s})\right]^\rho \frac{ds_1}{s_1} \frac{ds_2}{s_2} \quad (\mathbf{t} \in \mathbf{R}_+^2),$$

a modified Calderón operator, which satisfies $K(\cdot, f) \lesssim S_\rho K(\cdot, f)$, and consider the space

$$\sigma_\varrho(\bar{X}) := \left\{f \in \Sigma(\bar{X});\ (S_\rho^2 K(., f))(\mathbf{1}) < \infty\right\},$$

which allows to adapt the construction in [4] to our quasi-Banach triple, since $\bar{X}_{\Theta,q;K} \hookrightarrow X_{\Theta,\infty;K} \hookrightarrow \sigma_\varrho(\bar{X})$.

In our situation, the fundamental lemma with S_ρ holds for $\bar{X}$:

Lemma 1 *Every $f \in \sigma_\varrho(\bar{X})$ admits a representation as a sum in $\Sigma(\bar{X})$,*

$$f = \sum_{\mathbf{k}\in\mathbf{Z}^2} f_{\mathbf{k}} \quad (f_{\mathbf{k}} \in \Delta(\bar{X})),$$

where $\sum_{\mathbf{k}\in\mathbf{Z}^2} \|f_{\mathbf{k}}\|^\rho_{\Sigma(\bar{X})} < \infty$ and

$$J(2^{\mathbf{k}}, f_{\mathbf{k}}) \le C(S_\rho K(\cdot, f))(2^{\mathbf{k}}).$$

Proof Exactly as in [4], one can find a family of sets $B(\mathbf{k})$ ($\mathbf{k} = (k_1, k_2) \in \mathbf{Z}^2$) such that the functions $g_{\mathbf{k}} := |f|\chi_{B(\mathbf{k})}$ satisfy

$$2^{k_j}\|g_{\mathbf{k}}\|_j \lesssim (S_\rho K(\cdot, f))(2^{\mathbf{k}}) \quad (j = 0, 1, 2 \text{ with } k_0 := 0)$$

and, using Aoki's theorem, a straightforward but lengthy argument, similar to the computations in [4] (which corresponds to the case $\rho = 1$), shows that

$$\sum_{\mathbf{k}\in\mathbf{Z}^2} \|g_{\mathbf{k}}\|^\rho_{\Sigma(\bar{X})} \lesssim (S_\rho[S_\rho K(\cdot, f)])^\rho(\mathbf{1}) < \infty,$$

so $\sum_{\mathbf{k}\in\mathbf{Z}^2} g_{\mathbf{k}} = g$ exists in $\Sigma(\bar{X})$ and $|f| \le g$, and the decomposition is then obtained by choosing $f_{\mathbf{k}} = f g_{\mathbf{k}}/g$. □

Now the following equivalence theorem is proved as in the case $\rho = 1$ for Banach function spaces:

Theorem 12 *Let $\Theta = (\theta_0, \theta_1)$ with $\theta_0, \theta_1 > 0$ and $\theta_0 + \theta_1 < 1$, and consider $q \ge 1$. Then for the triple $\bar{X} = (L^r(C_0), L^r(C_1), L^\infty)$ the identity*

$$\bar{X}_{\Theta,q;J} = \bar{X}_{\Theta,q;K}$$

holds with equivalent quasi-norms and we may write $\bar{X}_{\Theta,q;J} = \bar{X}_{\Theta,q;K} = X_{\Theta,q}$.

To obtain Theorem 13 we will use the following simple result concerning triples $\bar{X} = (X_0, X_1, X_2)$ of quasi-Banach spaces (cf. [5]):

Lemma 2 *If* $0 < \alpha_0, \alpha_1, \eta < 1$, *and* $\theta_1 = \eta(1-\alpha_1)$ *and* $\theta_2 = \eta\alpha_1 + (1-\eta)\alpha_0$, *then*

$$\big((X_0, X_2)_{\alpha_0,1;K}, (X_1, X_2)_{\alpha_1,1;K}\big)_{\eta,1;K} \subset \bar{X}_{\Theta,1;K}$$

and

$$\bar{X}_{\Theta,1;J} \subset \big((X_0, X_2)_{\alpha_0,1;J}, (X_1, X_2)_{\alpha_1,1;J}\big)_{\eta,1;J}.$$

Finally, a reiteration theorem follows from the power theorem as in [5]:

Theorem 13 *Let* $\bar{X} = (X_0, X_1, X_2) = (L^r(C_0), L^r(C_1), L^\infty)$ *where* $0 < r < \infty$ *and* $0 < q, q_0, q_1 < \infty$, *and suppose that* $0 < \alpha_0, \alpha_1, \mu < 1$. *Then*

$$\bar{X}_{(\theta_0,\theta_1),q} = ((X_0, X_2)_{\alpha_0,q_0}, (X_1, X_2)_{\alpha_1,q_1})_{\mu,q},$$

with $\theta_0 = (1-\alpha_1)\eta$ *and* $\theta_1 = \alpha_0(1-\eta) + \alpha_1\eta$.

Proof Observe that, by the power theorem,

$$((X_0, X_2)_{\alpha_0,q_0}, (X_1, X_2)_{\alpha_1,q_1})^q_{\mu,q} = ((X_0, X_2)^{q_0}_{\alpha_0,q_0}, (X_1, X_2)^{q_1}_{\alpha_1,q_1})_{\eta,1},$$

if $\eta = \mu q/q_1$. We choose $0 < \beta_0, \beta_1 < 1$ so that $\alpha_0 q_0/\beta_0 = \alpha_1 q_1/\beta_1$, and if $s_0 = q_0(1-\alpha_0)/(1-\beta_0)$, $s_1 = q_1(1-\alpha_1)/(1-\beta_1)$ and $s_2 = q_0\alpha_0/\beta_0 = q_1\alpha_1/\beta_1$, then

$$((X_0, X_2)^{q_0}_{\alpha_0,q_0}, (X_1, X_2)^{q_1}_{\alpha_1,q_1})_{\eta,1} = ((X_0^{s_0}, X_2^{s_2})_{\beta_0,1}, (X_1^{s_1}, X_2^{s_2})_{\beta_1,1})_{\eta,1}.$$

From Theorem 12 and Lemma 2, it follows that

$$((X_0^{s_0}, X_2^{s_2})_{\beta_0,1}, (X_1^{s_1}, X_2^{s_2})_{\beta_1,1})_{\eta,1} = (X_0^{s_0}, X_1^{s_1}, X_2^{s_2})_{(\lambda_1,\lambda_2),1}$$

with $\lambda_1 = \eta(1-\beta_1)$ and $\lambda_2 = (1-\eta)\beta_0 + \eta\beta_1$.

Now an application of the power theorem for triples of quasi-Banach spaces (cf. [22]) gives

$$(X_0^{s_0}, X_1^{s_1}, X_2^{s_2})_{(\lambda_1,\lambda_2),1} = \bar{X}^q_{(\theta_0,\theta_1),q}$$

with $\theta_0 = \eta(1-\alpha_1)$ and $\theta_1 = (1-\eta)\alpha_0 + \eta\alpha_1$. Thus

$$((X_0, X_2)_{\alpha_0,q_0}, (X_1, X_2)_{\alpha_1,q_1})_{\eta,q} = \bar{X}_{(\theta_0,\theta_1),q}$$

as announced. □

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