China Ocean Eng., 2021, Vol. 35, No. 4, P. 518–530
DOI: https://doi.org/10.1007/s13344-021-0047-7, ISSN 0890-5487
http://www.chinaoceanengin.cn/ E-mail: coe@nhri.cn

In the Atmosphere and Oceanic Fluids: Scaling Transformations, Bilinear Forms, Bäcklund Transformations and Solitons for A Generalized Variable-Coefficient Korteweg-de Vries-Modified Korteweg-de Vries Equation

GAO Xin-yi, GUO Yong-jiang*, SHAN Wen-rui*, ZHOU Tian-yu, WANG Meng, YANG Dan-yu

State Key Laboratory of Information Photonics and Optical Communications, and School of Science, Beijing University of Posts and Telecommunications, Beijing 100876, China

Received February 1, 2021; revised March 17, 2021; accepted April 28, 2021

Abstract

The atmosphere is an evolutionary agent essential to the shaping of a planet, while in oceanic science and daily life, liquids are commonly seen. In this paper, we investigate a generalized variable-coefficient Korteweg-de Vries-modified Korteweg-de Vries equation for the atmosphere, oceanic fluids and plasmas. With symbolic computation, beginning with a presumption, we work out certain scaling transformations, bilinear forms through the binary Bell polynomials and our scaling transformations, N solitons (with N being a positive integer) via the aforementioned bilinear forms and bilinear auto-Bäcklund transformations through the Hirota method with some solitons. In addition, Painlevé-type auto-Bäcklund transformations with some solitons are symbolically computed out. Respective dependences and constraints on the variable/constant coefficients are discussed, while those coefficients correspond to the quadratic-nonlinear, cubic-nonlinear, dispersive, dissipative and line-damping effects in the atmosphere, oceanic fluids and plasmas.

Key words: atmosphere, oceanic fluids, plasmas, generalized variable-coefficient Korteweg-de Vries-modified Korteweg-de Vries equation, scaling transformations, bilinear forms, N solitons, auto-Bäcklund transformations

Citation: Gao, X. Y., Guo, Y. J., Shan, W. R., Zhou, T. Y., Wang, M., Yang, D. Y., 2021. In the atmosphere and oceanic fluids: Scaling transformations, bilinear forms, Bäcklund transformations and solitons for a generalized variable-coefficient Korteweg-de Vries-modified Korteweg-de Vries equation. China Ocean Eng., 35(4): 518–530, doi: https://doi.org/10.1007/s13344-021-0047-7

1 Introduction

The atmosphere has been said to be an evolutionary agent essential to the shaping of a planet (Farlex, 2021; Li et al., 2020a; Chen et al., 2020a; Feng et al., 2019; Su et al., 2019a; Su et al., 2019b). In oceanic science, natural science, engineering, medical science and daily life, liquids have been commonly seen (Khorram, 2020; Hu et al., 2019a; Hu et al., 2019b; Hu et al., 2021; Jia et al., 2019; Deng et al., 2020a; Liu et al., 2021a, 2021c; Shen et al., 2021a, 2021b, 2021e, 2021f; Ding et al., 2019; Gao et al., 2020a, 2020b, 2021a, 2021b, 2021d; Grave et al., 2020; Wang et al., 2019a, 2019b, 2020a, 2020b, 2020e, 2021b). Plasmas have been believed to be possibly the most abundant form of ordinary matter in the Universe, which are mostly associated with stars, extending to the rarefied intracluster media and intergalactic regions (Plasma, 2021; Gao et al., 2020c; Feng et al., 2020; Deng et al., 2020b; Liu et al., 2020b; Shen et al., 2021c; Shen et al., 2021d; Ding et al., 2020; Wang et al., 2020c; Du et al., 2020; Zhao et al., 2021; Chen et al., 2020b; Liu et al., 2021b).

For certain atmospheric blocking phenomenon, Wang et al. (2012) have considered a generalized variable-coefficient Korteweg-de Vries (KdV)-modified Korteweg-de Vries (mKdV) equation, as follows:

$$u_t-6\mu_0(t)u\,u_x-6\mu_1(t)u^2u_x+\mu_2(t)u_{xxx}-\mu_3(t)u_x+\mu_4(t)(Au+xu_x)=0, \quad (1)$$

where $u(x,t)$ is a real function of the variables x and t, $\mu_0(t)$, $\mu_1(t)$, $\mu_2(t)$, $\mu_3(t)$ and $\mu_4(t)$ are all the smooth functions of the variable t, and $A\neq 0$ is a constant (Wang et al., 2012). Eq. (1) has also been seen in Meng et al. (2012); Triki et al. (2010); Djoudi and Zerarka (2016) and Tang et al. (2016), and known to arise in many wave problems in fluid mechanics, nonlinear optics and plasma physics, where t is the

Foundation item: This work was financially supported by the National Natural Science Foundation of China (Grant No. 11871116), the Fundamental Research Funds for the Central Universities of China (Grant No. 2019XD-A11), the BUPT Innovation and Entrepreneurship Support Program, Beijing University of Posts and Telecommunications, and the National Scholarship for Doctoral Students of China .
*Corresponding authors. E-mail: yongerguo@bupt.edu.cn; shwr@bupt.edu.cn

time variable (Djoudi and Zerarka, 2016). In Eq. (1), $\mu_0(t)$, $\mu_1(t)$ and $\mu_2(t)$ represent the quadratic-nonlinear, cubic-nonlinear and dispersive coefficients, respectively, while $-\mu_3(t)+x\mu_4(t)$ and $\mu_4(t)A$ correspond to the dissipative and line-damping terms (Meng et al., 2012).

In oceanic/atmospheric fluid mechanics, plasma dynamics and other fields, there have appeared some special cases of Eq. (1):

(1) When $\mu_1(t)=0$, variable-coefficient KdV equation for the oceanic shallow water waves, or dust-ion-acoustic or ion-acoustic or dust-acoustic waves in a dusty plasma (possibly magnetized), or small-amplitude waves in a fluid-filled elastic or viscoelastic tube, or electromagnetic waves in a ferromagnetic medium, or nonlinear waves in a mixture of liquid and gas bubbles (Triki et al., 2010; Tang et al., 2016; Meng et al., 2012; and references therein).

(2) When $\mu_0(t)$ is proportional to $\mu_2(t)$, $\mu_1(t)=0$ and $A=2$, nonisospectral variable-coefficient KdV equation modelling the shallow water waves, dust-acoustic structures in a magnetized dusty plasma, and ion-acoustic waves in a plasma, as well as appearing in a relaxation cylindrical plasma to describe, e.g., the radially ingoing acoustic waves (Triki et al., 2010; Wang et al., 2012; Tang et al., 2016; and references therein).

(3) When $\mu_0(t)=0$, variable-coefficient mKdV equation for the oceanic shallow water waves, or dust-acoustic waves in a magnetized dusty plasma, or ion-acoustic waves in an inhomogeneous magnetized plasma, or interfacial waves in a two-layer liquid with gradually varying depth, or weakly nonlinear waves in a fluid-filled elastic tube, or electromagnetic waves in a size-quantized film, or phonons in an anharmonic lattice (Triki et al., 2010; Wang et al., 2012; Meng et al., 2012; Tang et al., 2016; and references therein).

(4) When $\mu_4(t)=0$, variable-coefficient Gardner equation for the internal solitary waves in such coastal zones as the north-west shelf of Australia and that of the Baltic, or for a variety of the waves in plasma physics, quantum field theory and solid state physics (Meng et al., 2012; Tang et al., 2016; and references therein).

(5) In addition, when $\mu_0(t)=$ constant, $\mu_1(t)=$ constant, $\mu_2(t)=$ constant, $\mu_3(t)=\mu_4(t)=0$, Gardner equation for the internal/ interfacial waves in a shallow sea or atmosphere, or dust-acoustic waves or ion-acoustic waves in a plasma with negative ions (Wang et al., 2012; Meng et al., 2012; Triki et al., 2010; and references therein).

Wang et al. (2012), Meng et al. (2012); Triki et al. (2010) and Tang et al. (2016) have said that Eq. (1) can model the weakly nonlinear long waves in a KdV-type medium that is characterized by the varying dispersive and nonlinear coefficients. For Eq. (1) itself, certain Lax pair, infinitely-many conservation laws and soliton/breather/ double-pole solutions have been obtained (Wang et al., 2012); some Painlevé-integrability consideration, Lax pair, bilinear forms, multi-soliton/breather solutions and Bäcklund transformations have been derived (Meng et al., 2012); types of the solitary-wave solutions have been constructed along with the formation conditions (Triki et al., 2010); kinds of the traveling-wave solutions have been found (Djoudi and Zerarka, 2016); certain soliton solutions, triangular periodic solutions and Jacobi-type solutions have been obtained via an auxiliary equation (Tang et al., 2016).

To our knowledge, however, the following issues for Eq. (1) have not been investigated in the existing literatures as yet, which we will discuss next: In Section 2 of this paper, with symbolic computation (Jia et al., 2021; Li et al., 2020b; Ma et al., 2021a, 2021b, 2021c; Wang et al., 2020d; Wang et al., 2021a; Du et al., 2019; Liu et al., 2020a; Zhang et al., 2019, 2020; Chen et al., 2019, 2020c; Tian et al., 2021a, 2021b; Yang et al., 2021a, 2021b; Zhao et al., 2020; Wang et al., 2021c; Yang et al., 2020, 2021c; Zhou et al., 2021), we will begin with a presumption, which is different from those presented in the existing literatures, and work out the corresponding scaling transformations, bilinear forms with the binary Bell polynomials and one/two/three/N solitons for Eq. (1), where N is a positive integer. In the Appendix, Bell-polynomial preliminary will be given. In Section 3, the aforementioned presumption will also lead to some bilinear auto-Bäcklund transformations with the Hirota method for Eq. (1). In Section 4, Painlevé-type auto-Bäcklund transformations will be symbolically computed out with some solitons for Eq. (1), which have not been obtained in the existing literatures, either. Conclusions will be given in Section 5, with respect to the atmosphere, oceanic fluids and plasmas.

2 Scaling transformations and bilinear forms with the binary Bell polynomials and N solitons for Eq. (1)

Bell-polynomial preliminary can be seen in the Appendix, with the definitions of the relevant symbols there (Bell, 1934; Lambert et al., 1994; Matveev and Salle, 1991; Wadati, 1975; Cariello and Tabor, 1989; Wang et al., 2017).

Motivated by the work in Lambert and Springael (2001) as well as Lambert and Springael (2008), we can now present the scaling transformations,

$$\begin{aligned}&x\to\varphi^1 x,\ t\to\varphi^K t,\ u\to\varphi^{-1}u,\\&\mu_0(t)\to\varphi^2\mu_0(t),\ \mu_1(t)\to\varphi^3\mu_1(t),\\&\mu_2(t)\to\varphi^3\mu_2(t),\ \mu_3(t)\to\varphi^1\mu_3(t),\\&\mu_4(t)\to\varphi^0\mu_4(t),\end{aligned}\tag{2}$$

which lead to our assumption

$$u(x,t)=\Upsilon(t)p_x(x,t),\tag{3}$$

where K is an integer, φ is a positive constant, $\Upsilon(t)$ is a differentiable function, while $p(x,t)$ can be seen in the Appendix.

Under the variable/constant-coefficient constraints,

$$A=1;\tag{4a}$$

$$\mu_1(t)=-\frac{\mu_2(t)}{\lambda},\tag{4b}$$

by virtue of the substitution of Assumption (3) back into Eq. (1), Bell-polynomial procedure and symbolic computation help us to obtain

$$\Upsilon(t)=\pm\mathrm{i}\sqrt{\lambda},$$

and correspondingly, to reduce Assumption (3) to

$$u(x,t)=\pm\mathrm{i}\sqrt{\lambda}p_x(x,t), \tag{5}$$

where λ is a positive constant, while $\mathrm{i}=\sqrt{-1}$.

It is noted that Presumption (5) is different from those presented in Wang et al. (2012) and Meng et al. (2012).

For the Bell-polynomial format, we hereby introduce $q(x,t)$, $\mathcal{Y}$, etc., from the Appendix, and integrate Eq. (1) with respect to x with the integration constant vanishing, so that

$$\mathcal{Y}_t(p)+\mu_2(t)\mathcal{Y}_{3x}(p,q)-\mu_3(t)\mathcal{Y}_x(p)+x\mu_4(t)\mathcal{Y}_x(p)- 3\mathcal{Y}_x(p)\left[\mu_2(t)\mathcal{Y}_{2x}(p,q)\pm\mathrm{i}\sqrt{\lambda}\mu_0(t)\mathcal{Y}_x(p)\right]=0,$$

in which we may further assume that

$$\mathcal{Y}_t(p)+\mu_2(t)\mathcal{Y}_{3x}(p,q)-\mu_3(t)\mathcal{Y}_x(p)+x\mu_4(t)\mathcal{Y}_x(p)=0; \tag{6a}$$

$$\mu_2(t)\mathcal{Y}_{2x}(p,q)\pm\mathrm{i}\sqrt{\lambda}\mu_0(t)\mathcal{Y}_x(p)=0. \tag{6b}$$

Next, in line with Eq. (63) in the Appendix, we make use of

$$p(x,t)=\ln\left[\frac{g(x,t)}{f(x,t)}\right]; \tag{7a}$$

$$q(x,t)=\ln\left[f(x,t)g(x,t)\right], \tag{7b}$$

and reduce Eq. (1), through System (6), into the following two branches of the bilinear forms with the binary Bell polynomials:

$$\left[D_t+\mu_2(t)D_x^3-\mu_3(t)D_x+x\mu_4(t)D_x\right]g\cdot f=0; \tag{8a}$$

$$\left[\mu_2(t)D_x^2\pm\mathrm{i}\sqrt{\lambda}\mu_0(t)D_x\right]g\cdot f=0, \tag{8b}$$

where $f(x,t)$ and $g(x,t)$ can also be seen in the Appendix.

In terms of the real differentiable functions $\alpha(x,t)$ and $\beta(x,t)$, i.e.,

$$g(x,t)=\alpha(x,t)+\mathrm{i}\beta(x,t)\quad\text{and}\quad f(x,t)=\alpha(x,t)-\mathrm{i}\beta(x,t),$$

Bilinear Forms (8) become

$$\left[D_t+\mu_2(t)D_x^3-\mu_3(t)D_x+x\mu_4(t)D_x\right]\alpha\cdot\beta=0; \tag{9a}$$

$$\mu_2(t)D_x^2(\alpha\cdot\alpha+\beta\cdot\beta)\pm2\sqrt{\lambda}\mu_0(t)D_x\,\alpha\cdot\beta=0, \tag{9b}$$

while

$$u(x,t)=\mp2\sqrt{\lambda}\left\{\arctan\left[\frac{\beta(x,t)}{\alpha(x,t)}\right]\right\}_x. \tag{10}$$

The reason for the existence of two branches of Bilinear Forms (8) or (9) with the binary Bell polynomials is that there appear the “±” signs. We also call the attention that each branch of Bilinear Forms (8) or (9) through the binary Bell polynomials is

(1) under Variable/Constant-Coefficient Constraints (4);

(2) dependent on $\mu_0(t)$, $\mu_2(t)$, $\mu_3(t)$ and $\mu_4(t)$.

Expanding $\alpha(x,t)$ and $\beta(x,t)$ in Bilinear Forms (9) with respect to a formal expansion parameter ϵ as

$$\alpha(x,t)=1+\sum_{\varrho=1}^{N}\epsilon^{\varrho}\alpha_{\varrho}(x,t); \tag{11a}$$

$$\beta(x,t)=1+\sum_{\varpi=1}^{N}\epsilon^{\varpi}\beta_{\varpi}(x,t), \tag{11b}$$

and then setting $\epsilon=1$, under the variable-coefficient constraint

$$\mu_0(t)=m_0\mu_2(t)\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t}, \tag{12}$$

we obtain the N-soliton solutions for Eq. (1) according to Bilinear Forms (9) as follows:

$u(x,t)=$ Expression (10),

with

$$\alpha(x,t)=\sum_{\rho_i,\rho_j=0,1}\exp\left\{\sum_{i=1}^{N}\rho_i[\lambda_i(t)x+\tau_i(t)]+\sum_{1\leqslant i<j}^{(N)}\rho_i\rho_j\omega_{ij}(t)+\sum_{1\leqslant i<j<l}^{(N)}\rho_i\rho_j\rho_l\sigma_{ijl}(t)+\cdots\right\}; \tag{13a}$$

$$\beta(x,t)=\sum_{\rho_i,\rho_j=0,1}\exp\left\{\sum_{i=1}^{N}\rho_i[\lambda_i(t)x+\tau_i(t)+\kappa_i(t)]+\sum_{1\leqslant i<j}^{(N)}\rho_i\rho_j\delta_{ij}(t)+\sum_{1\leqslant i<j<l}^{(N)}\rho_i\rho_j\rho_l\vartheta_{ijl}(t)+\cdots\right\}; \tag{13b}$$

$$\mathrm{e}^{\kappa_i(t)}=\frac{\pm\sqrt{\lambda}m_0+m_i}{\pm\sqrt{\lambda}m_0-m_i}; \tag{13c}$$

$$\tau_i(t)=\int\left[m_i\mu_3(t)\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t}-m_i^3\mu_2(t)\mathrm{e}^{-3\int\mu_4(t)\mathrm{d}t}\right]\mathrm{d}t; \tag{13d}$$

$$\lambda_i(t)=m_i\mathrm{e}^{-\int f_4(t)\mathrm{d}t}; \tag{13e}$$

$$\mathrm{e}^{\omega_{ij}(t)}=\frac{\left(m_i-m_j\right)^2\left(\lambda m_0^2\mp\sqrt{\lambda}m_im_0\mp\sqrt{\lambda}m_jm_0-m_im_j\right)}{\left(m_i+m_j\right)^2\left(\pm\sqrt{\lambda}m_0-m_i\right)\left(\pm\sqrt{\lambda}m_0-m_j\right)}; \tag{13f}$$

$$\mathrm{e}^{\delta_{ij}(t)}=\frac{\left(m_i-m_j\right)^2\left(\lambda m_0^2\pm\sqrt{\lambda}m_im_0\pm\sqrt{\lambda}m_jm_0-m_im_j\right)}{\left(m_i+m_j\right)^2\left(\pm\sqrt{\lambda}m_0+m_i\right)\left(\pm\sqrt{\lambda}m_0+m_j\right)}; \tag{13g}$$

$$\mathrm{e}^{\sigma_{ijl}(t)}=\frac{\left(\pm\sqrt{\lambda}m_0-m_i\right)}{\left(\lambda m_0^2\mp\sqrt{\lambda}m_im_0\mp\sqrt{\lambda}m_jm_0-m_im_j\right)}\times\frac{\left(\pm\sqrt{\lambda}m_0-m_j\right)}{\left(\lambda m_0^2\mp\sqrt{\lambda}m_im_0\mp\sqrt{\lambda}m_lm_0-m_im_l\right)}\times\frac{\left(\pm\sqrt{\lambda}m_0-m_l\right)}{\left(\lambda m_0^2\mp\sqrt{\lambda}m_jm_0\mp\sqrt{\lambda}m_lm_0-m_jm_l\right)}\times\left(\pm\lambda^{3/2}m_0^3-\lambda m_im_0^2-\lambda m_jm_0^2-\lambda m_lm_0^2\mp\sqrt{\lambda}m_im_jm_0\mp\sqrt{\lambda}m_im_lm_0\mp\sqrt{\lambda}m_jm_lm_0+m_im_jm_l\right); \tag{13h}$$

$$e^{\vartheta_{ijl}(t)}=\frac{\left(\pm\sqrt{\lambda}m_0+m_i\right)}{\left(\lambda m_0^2\pm\sqrt{\lambda}m_im_0\pm\sqrt{\lambda}m_jm_0-m_im_j\right)}\times\frac{\left(\pm\sqrt{\lambda}m_0+m_j\right)}{\left(\lambda m_0^2\pm\sqrt{\lambda}m_im_0\pm\sqrt{\lambda}m_lm_0-m_im_l\right)}\times\frac{\left(\pm\sqrt{\lambda}m_0+m_l\right)}{\left(\lambda m_0^2\pm\sqrt{\lambda}m_jm_0\pm\sqrt{\lambda}m_lm_0-m_jm_l\right)}\times\left(\pm\lambda^{3/2}m_0^3+\lambda m_im_0^2+\lambda m_jm_0^2+\lambda m_lm_0^2\mp\sqrt{\lambda}m_im_jm_0\mp\sqrt{\lambda}m_im_lm_0\mp\sqrt{\lambda}m_jm_lm_0-m_im_jm_l\right),\tag{13i}$$

m_0 representing a real constant, ϱ, ϖ, i, j and l being the positive integers with $\varrho\leqslant N$, $\varpi\leqslant N$, $i\leqslant N$, $j\leqslant N$ and $l\leqslant N$, m_i's denoting the real constants with $m_i\neq 0$, $m_i+m_j\neq 0$ and $m_i\neq\pm\sqrt{\lambda}m_0$, $\alpha_\varrho(x,t)$'s and $\beta_\varpi(x,t)$'s being all the real differentiable functions of x and t, m_i and the integration constant from $\tau_i(t)$ via Eq. (13d) representing the parameters characterizing the i-th soliton, the sum $\sum_{\rho_i,\rho_j=0,1}$ taken over all the possible combinations of $\rho_j=0,1$, while $\sum_{1\leqslant i<j}^{(N)}$ being the summation over all the possible pairs chosen from the N elements under the condition $i<j$.

It is noted that there exist two branches of N-Soliton Solutions (13) for Eq. (1) because of the "±" signs. Each branch is

(1) under Variable/Constant-Coefficient Constraints (4) and (12);

(2) dependent on $\mu_2(t)$, $\mu_3(t)$ and $\mu_4(t)$.

For $N=1$, two branches of the one-soliton solutions: $u(x,t)=$ Expression (10), with

$$\alpha(x,t)=\frac{\pm\sqrt{\lambda}m_0-m_1}{2m_1}e^{m_1e^{-\int\mu_4(t)dt}x+\int\left[m_1\mu_3(t)e^{-\int\mu_4(t)dt}-m_1^3\mu_2(t)e^{-3\int\mu_4(t)dt}\right]dt}+1;\tag{14a}$$

$$\beta(x,t)=\frac{\pm\sqrt{\lambda}m_0+m_1}{2m_1}e^{m_1e^{-\int\mu_4(t)dt}x+\int\left[m_1\mu_3(t)e^{-\int\mu_4(t)dt}-m_1^3\mu_2(t)e^{-3\int\mu_4(t)dt}\right]dt}+1.\tag{14b}$$

For $N=2$, two branches of the two-soliton solutions:

$u(x,t)=$ Expression (10), with

$$\begin{aligned}\alpha(x,t)=&1-\frac{m_2\mp\sqrt{\lambda}m_0}{2m_2}e^{m_2e^{-\int\mu_4(t)dt}x+\int\left[m_2\mu_3(t)e^{-\int\mu_4(t)dt}-m_2^3\mu_2(t)e^{-3\int\mu_4(t)dt}\right]dt}-\\&\frac{m_1\mp\sqrt{\lambda}m_0}{2m_1}e^{m_1e^{-\int\mu_4(t)dt}x+\int\left[m_1\mu_3(t)e^{-\int\mu_4(t)dt}-m_1^3\mu_2(t)e^{-3\int\mu_4(t)dt}\right]dt}-\\&\frac{(m_1-m_2)^2\left[m_1m_2-\lambda m_0^2\pm\sqrt{\lambda}(m_1+m_2)m_0\right]}{4m_1m_2(m_1+m_2)^2}\times\\&e^{(m_1+m_2)e^{-\int\mu_4(t)dt}x+\int\left[(m_1+m_2)\mu_3(t)e^{-\int\mu_4(t)dt}-\left(m_1^3+m_2^3\right)\mu_2(t)e^{-3\int\mu_4(t)dt}\right]dt};\end{aligned}\tag{15a}$$

$$\begin{aligned}\beta(x,t)=&1+\frac{m_2\pm\sqrt{\lambda}m_0}{2m_2}e^{m_2e^{-\int\mu_4(t)dt}x+\int\left[m_2\mu_3(t)e^{-\int\mu_4(t)dt}-m_2^3\mu_2(t)e^{-3\int\mu_4(t)dt}\right]dt}+\\&\frac{m_1\pm\sqrt{\lambda}m_0}{2m_1}e^{m_1e^{-\int\mu_4(t)dt}x+\int\left[m_1\mu_3(t)e^{-\int\mu_4(t)dt}-m_1^3\mu_2(t)e^{-3\int\mu_4(t)dt}\right]dt}-\\&\frac{(m_1-m_2)^2\left[m_1m_2-\lambda m_0^2\mp\sqrt{\lambda}(m_1+m_2)m_0\right]}{4m_1m_2(m_1+m_2)^2}\times\\&e^{(m_1+m_2)e^{-\int\mu_4(t)dt}x+\int\left[(m_1+m_2)\mu_3(t)e^{-\int\mu_4(t)dt}-\left(m_1^3+m_2^3\right)\mu_2(t)e^{-3\int\mu_4(t)dt}\right]dt}.\end{aligned}\tag{15b}$$

Via Solutions (10) and (15), Fig. 1a shows the interaction between the two dark solitons, and Fig. 1b displays the interaction between the two bright solitons.

For $N=3$, two branches of the three-soliton solutions:

$u(x,t)=$ Expression (10), with

$$\begin{aligned}\alpha(x,t)=&1+\sum_{i=1}^{3}e^{\int\left[e^{-\int\mu_4(t)dt}m_i\mu_3(t)-e^{-3\int\mu_4(t)dt}m_i^3\mu_2(t)\right]dt+e^{-\int\mu_4(t)dt}m_ix}+\\&\prod_{1\leqslant i<j\leqslant 3}e^{\int\left[e^{-\int\mu_4(t)dt}(m_i+m_j)\mu_3(t)-e^{-3\int\mu_4(t)dt}\left(m_i^3+m_j^3\right)\mu_2(t)\right]dt+e^{-\int\mu_4(t)dt}(m_i+m_j)x}\times\end{aligned}$$

$$\frac{\left(\lambda m_0^2\mp\sqrt{\lambda}m_im_0\mp\sqrt{\lambda}m_jm_0-m_im_j\right)\left(m_i-m_j\right)^2}{\left(\pm\sqrt{\lambda}m_0-m_i\right)\left(\pm\sqrt{\lambda}m_0-m_j\right)\left(m_i+m_j\right)^2}+$$
$$\mathrm{e}^{\int\left[\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t}(m_1+m_2+m_3)\mu_3(t)-\mathrm{e}^{-3\int\mu_4(t)\mathrm{d}t}\left(m_1^3+m_2^3+m_3^3\right)\mu_2(t)\right]\mathrm{d}t+\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t}(m_1+m_2+m_3)x}\times$$
$$\frac{\left[m_1m_2m_3\pm\lambda^{3/2}m_0^3-\lambda(m_1+m_2+m_3)m_0^2\mp\sqrt{\lambda}(m_1m_2+m_3m_2+m_1m_3)m_0\right]}{\left(\pm\sqrt{\lambda}m_0-m_1\right)\left(\pm\sqrt{\lambda}m_0-m_3\right)\left(\pm\sqrt{\lambda}m_0-m_2\right)}\times\frac{(m_1-m_2)^2(m_1-m_3)^2(m_2-m_3)^2}{(m_1+m_2)^2(m_1+m_3)^2(m_2+m_3)^2},\tag{16a}$$

$$\beta(x,t)=1+\sum_{i=1}^{3}\mathrm{e}^{\int\left[\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t}m_i\mu_3(t)-\mathrm{e}^{-3\int\mu_4(t)\mathrm{d}t}m_i^3\mu_2(t)\right]\mathrm{d}t+\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t}m_ix}\frac{\pm\sqrt{\lambda}m_0+m_i}{\pm\sqrt{\lambda}m_0-m_i}+$$
$$\prod_{1\leqslant i<j\leqslant 3}\mathrm{e}^{\int\left[\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t}(m_i+m_j)\mu_3(t)-\mathrm{e}^{-3\int\mu_4(t)\mathrm{d}t}\left(m_i^3+m_j^3\right)\mu_2(t)\right]\mathrm{d}t+\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t}(m_i+m_j)x}\times$$
$$\frac{\left(\lambda m_0^2\pm\sqrt{\lambda}m_im_0\pm\sqrt{\lambda}m_jm_0-m_im_j\right)\left(m_i-m_j\right)^2}{\left(\pm\sqrt{\lambda}m_0+m_i\right)\left(\pm\sqrt{\lambda}m_0+m_j\right)\left(m_i+m_j\right)^2}+$$
$$\mathrm{e}^{\int\left[\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t}(m_1+m_2+m_3)\mu_3(t)-\mathrm{e}^{-3\int\mu_4(t)\mathrm{d}t}\left(m_1^3+m_2^3+m_3^3\right)\mu_2(t)\right]\mathrm{d}t+\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t}(m_1+m_2+m_3)x}\times$$
$$\frac{\left[-m_1m_2m_3\pm\lambda^{3/2}m_0^3+\lambda(m_1+m_2+m_3)m_0^2\mp\sqrt{\lambda}(m_1m_2+m_3m_2+m_1m_3)m_0\right]}{\left(\pm\sqrt{\lambda}m_0-m_1\right)\left(\pm\sqrt{\lambda}m_0-m_3\right)\left(\pm\sqrt{\lambda}m_0-m_2\right)}\times$$
$$\frac{(m_1-m_2)^2(m_1-m_3)^2(m_2-m_3)^2}{(m_1+m_2)^2(m_1+m_3)^2(m_2+m_3)^2}.\tag{16b}$$

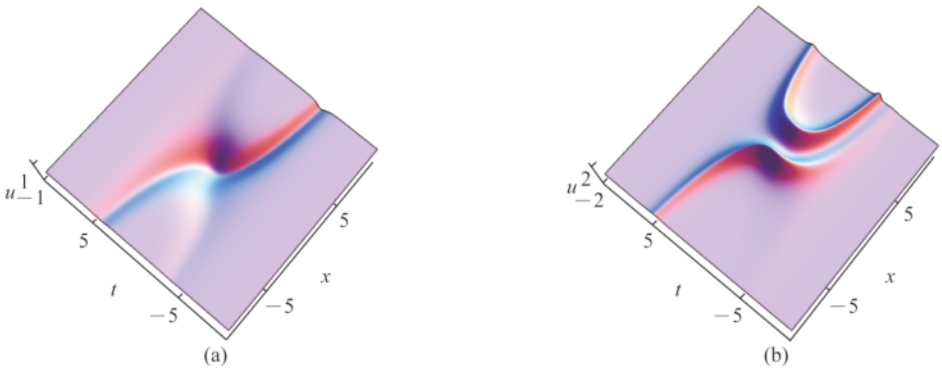

Fig. 1. Two dark/bright solitons via Solutions (10) and (15) with (a) $m_0=m_1=1$, $m_2=2$, $\mu_2(t)=\frac{t^3}{81}-\frac{t^2}{9}$, $\mu_4(t)=\frac{t}{5}$, $\mu_3(t)=\frac{t^2}{100}$ and $\lambda=100$ with the "_" sign in Expression (10), (b) the same as (a) except for the "+" sign in Expression (10).

3 Bilinear auto-Bäcklund transformations with the Hirota method and solitons for Eq. (1)

Beginning with Presumption (5), which is different from those presented in Wang et al. (2012) and Meng et al. (2012), and according to Bilinear Forms (8), we will hereby construct some bilinear auto-Bäcklund transformations with the Hirota method (Hirota, 1980).

Firstly, according to Eq. (8b), we consider the expression

$$gf\left[\pm\mathrm{i}\sqrt{\lambda}\mu_0(t)D_x\tilde{g}\cdot\tilde{f}+\mu_2(t)D_x^2\tilde{g}\cdot\tilde{f}\right]-\tilde{g}\tilde{f}\left[\pm\mathrm{i}\sqrt{\lambda}\mu_0(t)D_xg\cdot f+\mu_2(t)D_x^2g\cdot f\right]=0,\tag{17}$$

where $\tilde{f}(x,t)$ and $\tilde{g}(x,t)$ are another set of the solutions of Bilinear Forms (8).

Exchange formulae (Matsuno, 1984),

$$(D_x^2\tilde{g}\cdot\tilde{f})gf-\tilde{g}\tilde{f}(D_x^2g\cdot f)=D_x[(D_x\tilde{g}\cdot f)\cdot(g\tilde{f})+(\tilde{g}f)\cdot(D_xg\cdot\tilde{f})];\tag{18a}$$

$$(D_x\tilde{g}\cdot\tilde{f})gf-\tilde{g}\tilde{f}(D_xg\cdot f)=D_x(\tilde{g}f)\cdot(\tilde{f}g),\tag{18b}$$

lead to

$$0 =\mu_2(t)D_x[(D_x\tilde{g}\cdot f)\cdot(g\tilde{f})+(\tilde{g}f)\cdot(D_xg\cdot\tilde{f})]\pm \mathrm{i}\sqrt{\lambda}\mu_0(t)D_x(\tilde{g}f)\cdot(\tilde{f}g) = \\ D_x\left[\mu_2(t)D_x\tilde{g}\cdot f\pm \mathrm{i}\frac{\sqrt{\lambda}}{2}\mu_0(t)\tilde{g}f\right]\cdot(g\tilde{f})+D_x(\tilde{g}f)\cdot\left[\mu_2(t)D_xg\cdot\tilde{f}\pm \mathrm{i}\frac{\sqrt{\lambda}}{2}\mu_0(t)g\tilde{f}\right], \tag{19}$$

from which we can assume that

$$D_x\left[\mu_2(t)D_x\tilde{g}\cdot f\pm \mathrm{i}\frac{\sqrt{\lambda}}{2}\mu_0(t)\tilde{g}f\right]\cdot(g\tilde{f})=0; \tag{20a}$$

$$D_x(\tilde{g}f)\cdot\left[\mu_2(t)D_xg\cdot\tilde{f}\pm \mathrm{i}\frac{\sqrt{\lambda}}{2}\mu_0(t)g\tilde{f}\right]=0, \tag{20b}$$

so as to further obtain

$$\mu_2(t)\neq 0, \tag{21}$$

and

$$\mu_2(t)D_x\tilde{g}\cdot f\pm \mathrm{i}\frac{\sqrt{\lambda}}{2}\mu_0(t)\tilde{g}f=\pm \mathrm{i}\frac{\sqrt{\lambda}}{2}\zeta(t)g\tilde{f}; \tag{22a}$$

$$\mu_2(t)D_xg\cdot\tilde{f}\pm \mathrm{i}\frac{\sqrt{\lambda}}{2}\mu_0(t)g\tilde{f}=\pm \mathrm{i}\frac{\sqrt{\lambda}}{2}\zeta(t)\tilde{g}f, \tag{22b}$$

with $\zeta(t)$ being a real function.

Secondly, according to Eq. (8a), we consider another expression, which is

$$0 =gf\left[D_t\tilde{g}\cdot\tilde{f}+\mu_2(t)D_x^3\tilde{g}\cdot\tilde{f}-\mu_3(t)D_x\tilde{g}\cdot\tilde{f}+x\mu_4(t)D_x\tilde{g}\cdot\tilde{f}\right]- \\ \tilde{g}\tilde{f}\left[D_tg\cdot f+\mu_2(t)D_x^3g\cdot f-\mu_3(t)D_xg\cdot f+x\mu_4(t)D_xg\cdot f\right]. \tag{23}$$

According to the exchange formulae (Matsuno, 1984),

$$(D_x\tilde{g}\cdot\tilde{f})gf-\tilde{g}\tilde{f}(D_xg\cdot f)=(D_x\tilde{g}\cdot g)\tilde{f}f-\tilde{g}g(D_x\tilde{f}\cdot f); \tag{24a}$$

$$(D_t\tilde{g}\cdot\tilde{f})gf-\tilde{g}\tilde{f}(D_tg\cdot f)=(D_t\tilde{g}\cdot g)\tilde{f}f-\tilde{g}g(D_t\tilde{f}\cdot f); \tag{24b}$$

$$(D_x^3\tilde{g}\cdot\tilde{f})gf-\tilde{g}\tilde{f}(D_x^3g\cdot f)=(D_x^3\tilde{g}\cdot g)\tilde{f}f-\tilde{g}g(D_x^3\tilde{f}\cdot f)+3D_x(D_x\tilde{g}\cdot f)\cdot(D_xg\cdot\tilde{f}), \tag{24c}$$

one can see that

$$[(D_t\tilde{g}\cdot g)\tilde{f}f-\tilde{g}g(D_t\tilde{f}\cdot f)]+\mu_2(t)[(D_x^3\tilde{g}\cdot g)\tilde{f}f-\tilde{g}g(D_x^3\tilde{f}\cdot f)]+3\mu_2(t)D_x(D_x\tilde{g}\cdot f)\cdot(D_xg\cdot\tilde{f})- \\ \mu_3(t)[(D_x\tilde{g}\cdot g)\tilde{f}f-\tilde{g}g(D_x\tilde{f}\cdot f)]+x\mu_4(t)[(D_x\tilde{g}\cdot g)\tilde{f}f-\tilde{g}g(D_x\tilde{f}\cdot f)]=0. \tag{25}$$

Then, Eqs. (22a) and (22b) result in

$$0=\tilde{f}f\left[(D_t\tilde{g}\cdot g)+\mu_2(t)(D_x^3\tilde{g}\cdot g)-\mu_3(t)(D_x\tilde{g}\cdot g)+x\mu_4(t)(D_x\tilde{g}\cdot g)\right]- \\ \tilde{g}g\left[(D_t\tilde{f}\cdot f)+\mu_2(t)(D_x^3\tilde{f}\cdot f)-\mu_3(t)(D_x\tilde{f}\cdot f)+x\mu_4(t)(D_x\tilde{f}\cdot f)\right]+ \\ \frac{3}{\mu_2(t)}D_x\left[\mp \mathrm{i}\frac{\sqrt{\lambda}}{2}\mu_0(t)\tilde{g}f\pm \mathrm{i}\frac{\sqrt{\lambda}}{2}\zeta(t)g\tilde{f}\right]\cdot\left[\mp \mathrm{i}\frac{\sqrt{\lambda}}{2}\mu_0(t)g\tilde{f}\pm \mathrm{i}\frac{\sqrt{\lambda}}{2}\zeta(t)\tilde{g}f\right]= \\ \tilde{f}f\left[(D_t\tilde{g}\cdot g)+\mu_2(t)(D_x^3\tilde{g}\cdot g)-\mu_3(t)(D_x\tilde{g}\cdot g)+x\mu_4(t)(D_x\tilde{g}\cdot g)\right]- \\ \tilde{g}g\left[(D_t\tilde{f}\cdot f)+\mu_2(t)(D_x^3\tilde{f}\cdot f)-\mu_3(t)(D_x\tilde{f}\cdot f)+x\mu_4(t)(D_x\tilde{f}\cdot f)\right]+\frac{3\lambda[\zeta^2(t)-\mu_0^2(t)]}{4\mu_2(t)}D_x(\tilde{g}f)\cdot(g\tilde{f}). \tag{26}$$

Exchange formula (Matsuno, 1984),

$$D_x(\tilde{g}f)\cdot(\tilde{f}g)=(D_x\tilde{g}\cdot g)\tilde{f}f-\tilde{g}g(D_x\tilde{f}\cdot f), \tag{27}$$

comes to

$$\begin{aligned}&\tilde{f}f\left\{(D_t\tilde{g}\cdot g)+\mu_2(t)(D_x^3\tilde{g}\cdot g)+\frac{3\lambda[\zeta^2(t)-\mu_0^2(t)]}{4\mu_2(t)}(D_x\tilde{g}\cdot g)-\mu_3(t)(D_x\tilde{g}\cdot g)+x\mu_4(t)(D_x\tilde{g}\cdot g)\right\}-\\&\quad\tilde{g}g\left\{(D_t\tilde{f}\cdot f)+\mu_2(t)(D_x^3\tilde{f}\cdot f)+\frac{3\lambda[\zeta^2(t)-\mu_0^2(t)]}{4\mu_2(t)}(D_x\tilde{f}\cdot f)-\mu_3(t)(D_x\tilde{f}\cdot f)+x\mu_4(t)(D_x\tilde{f}\cdot f)\right\}=0,\end{aligned}\tag{28}$$

from which we may assume that

$$\left\{D_t+\mu_2(t)D_x^3+\frac{3\lambda\left[\zeta^2(t)-\mu_0^2(t)\right]}{4\mu_2(t)}D_x-\mu_3(t)D_x+x\mu_4(t)D_x\right\}\tilde{g}\cdot g=0;\tag{29a}$$

$$\left\{D_t+\mu_2(t)D_x^3+\frac{3\lambda\left[\zeta^2(t)-\mu_0^2(t)\right]}{4\mu_2(t)}D_x-\mu_3(t)D_x+x\mu_4(t)D_x\right\}\tilde{f}\cdot f=0.\tag{29b}$$

Thirdly, it can be noted that Eqs. (5), (7a), (22) and (29) constitute two branches of the bilinear auto-Bäcklund transformations with the Hirota method for Eq. (1), with respect to the "±" signs. Each branch of Bilinear Auto-Bäcklund Transformations (5), (7a), (22) and (29) with the Hirota method is

(1) mutually consistent, or, explicitly solvable with respect to $f(x,t)$, $g(x,t)$, $\tilde{f}(x,t)$, $\tilde{g}(x,t)$, $\mu_0(t)$, $\mu_2(t)$, $\mu_3(t)$ and $\mu_4(t)$, to be seen right below;

(2) under Variable/Constant-Coefficient Constraints (4) and (21).

Finally, we symbolically compute out an explicitly-solvable solitonic example. With a set of the seed solutions, $\tilde{f}=1$ and $\tilde{g}=1$, Eqs. (22) and (29) can be simplified as

$$f_x\mp\frac{\mathrm{i}\sqrt{\lambda}}{\mu_2(t)}[\mu_0(t)f-\zeta(t)g]=0;\tag{30a}$$

$$g_x\pm\frac{\mathrm{i}\sqrt{\lambda}}{\mu_2(t)}[\mu_0(t)g-\zeta(t)f]=0;\tag{30b}$$

$$f_t+\mu_2(t)f_{xxx}+\left\{\frac{3\lambda\left[\zeta^2(t)-\mu_0^2(t)\right]}{4\mu_2(t)}-\mu_3(t)+x\mu_4(t)\right\}f_x=0;\tag{30c}$$

$u(x,t)=$ Expressions (5) and (7a),

with

$$g_t+\mu_2(t)g_{xxx}+\left\{\frac{3\lambda\left[\zeta^2(t)-\mu_0^2(t)\right]}{4\mu_2(t)}-\mu_3(t)+x\mu_4(t)\right\}g_x=0,\tag{30d}$$

into which we can substitute the assumptions

$$\begin{aligned}f(x,t)&=\{c_1\cosh[\chi_1(t)x+\chi_2(t)]+c_2\sinh[\chi_1(t)x+\chi_2(t)]\}+\\&\quad\mathrm{i}\,c_0\{c_1\sinh[\chi_1(t)x+\chi_2(t)]+c_2\cosh[\chi_1(t)x+\chi_2(t)]\};\\g(x,t)&=\{c_1\cosh[\chi_1(t)x+\chi_2(t)]+c_2\sinh[\chi_1(t)x+\chi_2(t)]\}-\\&\quad\mathrm{i}\,c_0\{c_1\sinh[\chi_1(t)x+\chi_2(t)]+c_2\cosh[\chi_1(t)x+\chi_2(t)]\},\end{aligned}\tag{31}$$

under the variable-coefficient constraint

$$\mu_0(t)=\mp\frac{1-c_0^2}{c_0\sqrt{\lambda}}c_3\mu_2(t)\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t},\tag{32}$$

so as to obtain

$$\begin{aligned}&\chi_1(t)=c_3\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t};\\&\zeta(t)=\mp\frac{1+c_0^2}{c_0\sqrt{\lambda}}c_3\mu_2(t)\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t};\\&\chi_2(t)=-\int\left[4\mu_2(t)\chi_1^3(t)-\mu_3(t)\chi_1(t)\right]\mathrm{d}t,\end{aligned}\tag{33}$$

where $\chi_1(t)$ and $\chi_2(t)$ denote the real differentiable functions of t, while c_0, c_1, c_2 and c_3 are the real constants with $c_0\neq 0,\pm1$. Therefore, we obtain the following one-soliton solutions for Eq. (1):

$$\begin{aligned}f(x,t)=&\left\{c_1\cosh\left\{c_3\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t}x-\int\left[\frac{7}{4}c_3^3\mu_2(t)\mathrm{e}^{-3\int\mu_4(t)\mathrm{d}t}-c_3\mu_3(t)\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t}\right]\mathrm{d}t\right\}+\right.\\&\left.c_2\sinh\left\{c_3\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t}x-\int\left[\frac{7}{4}c_3^3\mu_2(t)\mathrm{e}^{-3\int\mu_4(t)\mathrm{d}t}-c_3\mu_3(t)\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t}\right]\mathrm{d}t\right\}\right\}+\\&\mathrm{i}c_0\left\{c_1\sinh\left\{c_3\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t}x-\int\left[\frac{7}{4}c_3^3\mu_2(t)\mathrm{e}^{-3\int\mu_4(t)\mathrm{d}t}-c_3\mu_3(t)\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t}\right]\mathrm{d}t\right\}+\right.\\&\left.c_2\cosh\left\{c_3\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t}x-\int\left[\frac{7}{4}c_3^3\mu_2(t)\mathrm{e}^{-3\int\mu_4(t)\mathrm{d}t}-c_3\mu_3(t)\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t}\right]\mathrm{d}t\right\}\right\};\end{aligned}\tag{34a}$$

$$\begin{aligned}g(x,t)=&\left\{c_1\cosh\left\{c_3\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t}x-\int\left[\frac{7}{4}c_3^3\mu_2(t)\mathrm{e}^{-3\int\mu_4(t)\mathrm{d}t}-c_3\mu_3(t)\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t}\right]\mathrm{d}t\right\}+\right.\\&\left.c_2\sinh\left\{c_3\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t}x-\int\left[\frac{7}{4}c_3^3\mu_2(t)\mathrm{e}^{-3\int\mu_4(t)\mathrm{d}t}-c_3\mu_3(t)\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t}\right]\mathrm{d}t\right\}\right\}-\\&\mathrm{i}c_0\left\{c_1\sinh\left\{c_3\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t}x-\int\left[\frac{7}{4}c_3^3\mu_2(t)\mathrm{e}^{-3\int\mu_4(t)\mathrm{d}t}-c_3\mu_3(t)\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t}\right]\mathrm{d}t\right\}+\right.\\&\left.c_2\cosh\left\{c_3\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t}x-\int\left[\frac{7}{4}c_3^3\mu_2(t)\mathrm{e}^{-3\int\mu_4(t)\mathrm{d}t}-c_3\mu_3(t)\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t}\right]\mathrm{d}t\right\}\right\},\end{aligned}\tag{34b}$$

under Variable/Constant-Coefficient Constraints (4), (21) and (32).

4 Painlevé-type auto-Bäcklund transformations and relevant soliton features for Eq. (1)

We in this section carry out the investigation on Eq. (1) by seeking a Painlevé expansion in the form of the generalized Laurent series (Gao, 2019; Gao et al., 2020b; Gao et al., 2020d; Gao et al., 2020e; Gao et al., 2021c; and references therein),

$$u(x,t)=\phi^{-\Xi}(x,t)\sum_{\xi=0}^{\infty}u_\xi(x,t)\phi^{\xi}(x,t),\tag{35}$$

in the neighborhood of the non-characteristic singular manifold $\phi(x,t)=0$, where Ξ is a positive integer, $u_\xi(x,t)$'s and $\phi(x,t)$ are all the analytic functions with $u_0(x,t)\neq 0$ and $\phi_x(x,t)\neq 0$. Balancing the powers of ϕ at the lowest orders yields $\Xi=1$, we truncate Painlevé Expansion (35) at the constant level terms (Gao, 2019; Gao et al., 2020b; Gao et al., 2020d; Gao et al., 2020e; Gao et al., 2021c), as

$$u(x,t)=\frac{u_0(x,t)}{\phi(x,t)}+u_1(x,t),\tag{36}$$

which is substituted back into Eq. (1). With symbolic computation, the way for the coefficients of like powers of ϕ to vanish leads to the Painlevé-Bäcklund equations:

$$\phi^{-4}:u_0=\pm\frac{\sqrt{\mu_2(t)}}{\sqrt{\mu_1(t)}}\phi_x;\tag{37}$$

$$\mu_1(t)\neq 0;\tag{38}$$

$$\mu_2(t)\neq 0;\tag{39}$$

$$\phi^{-3}:2\mu_1(t)\phi_x u_1+\mu_0(t)\phi_x\pm\sqrt{\mu_1(t)}\sqrt{\mu_2(t)}\phi_{xx}=0;\tag{40}$$

$$\begin{aligned}\phi^{-2}:&\pm 6\mu_1(t)^{3/2}\phi_x^2u_1^2\pm 6\mu_0(t)\sqrt{\mu_1(t)}\phi_x^2u_1\pm\sqrt{\mu_1(t)}\mu_3(t)\phi_x^2\mp\\&x\sqrt{\mu_1(t)}\mu_4(t)\phi_x^2\mp\sqrt{\mu_1(t)}\phi_t\phi_x\mp 4\sqrt{\mu_1(t)}\mu_2(t)\phi_{xxx}\phi_x\mp\\&3\sqrt{\mu_1(t)}\mu_2(t)\phi_{xx}^2-6\mu_1(t)\sqrt{\mu_2(t)}\phi_x^2u_{1,x}-\\&12\mu_1(t)\sqrt{\mu_2(t)}\phi_{xx}\phi_xu_1-6\mu_0(t)\sqrt{\mu_2(t)}\phi_{xx}\phi_x=0;\end{aligned}\tag{41}$$

$$\begin{aligned}\phi^{-1}:&2A\mu_2(t)\mu_4(t)\mu_1(t)\phi_x-24\mu_2(t)\mu_1(t)^2\phi_xu_1u_{1,x}-\\&12\mu_0(t)\mu_2(t)\mu_1(t)\phi_xu_{1,x}-\mu_2(t)\mu_1'(t)\phi_x-\\&12\mu_2(t)\mu_1(t)^2\phi_{xx}u_1^2-12\mu_0(t)\mu_2(t)\mu_1(t)\phi_{xx}u_1+\\&\mu_1(t)\mu_2'(t)\phi_x+2\mu_2(t)\mu_1(t)\phi_{xt}-\\&2\mu_2(t)\mu_3(t)\mu_1(t)\phi_{xx}+2x\mu_2(t)\mu_4(t)\mu_1(t)\phi_{xx}+\\&2\mu_2(t)^2\mu_1(t)\phi_{xxxx}=0;\end{aligned}\tag{42}$$

$$\begin{aligned}\phi^{0}:\ &u_{1,t}-6\mu_0(t)u_1u_{1,x}-6\mu_1(t)u_1^2u_{1,x}+\mu_2(t)u_{1,xxx}-\\&\mu_3(t)u_{1,x}+\mu_4(t)(Au_1+xu_{1,x})=0,\end{aligned}\tag{43}$$

where $u_1(x,t)$ can be treated as the seed solutions for Eq. (1) (Gao, 2019; Gao et al., 2020b; Gao et al., 2020d; Gao et al., 2020e; Gao et al., 2021c).

The sets of Eqs. (36), (37) and (40)–(43) constitute two branches of the Painlevé-type auto-Bäcklund transformations, due to the "±" signs[1] and based on the fact that, to be seen right below, each of those two sets is

(1) mutually consistent, or, explicitly solvable with respect to $\phi(x,t)$, $u_0(x,t)$, $u_1(x,t)$, A, $\mu_0(t)$, $\mu_1(t)$, $\mu_2(t)$, $\mu_3(t)$ and $\mu_4(t)$;

(2) under Variable-Coefficient Constraints (38) and (39).

Next, let us find out some explicitly-solvable solitons with symbolic computation. Assumptions hereby are

$$\phi(x,t)=\mathrm{e}^{\eta_1(t)x+\eta_2(t)}+1,\ \ u_1(x,t)=\eta_3(t),\tag{44}$$

which need to be substituted into Eqs. (40)-(43), resulting in

$$\phi^0:\eta_3(t)=0\ \text{ or }\ \eta_3(t)\neq 0,\tag{45}$$

where $\eta_1(t)$, $\eta_2(t)$ and $\eta_3(t)$ are the real differentiable functions, with $\eta_1(t)\neq 0$ since $\phi_x\neq 0$.

Case I: $\eta_3(t)=0$

$$\begin{aligned}\phi^{-3}:\ \ &\eta_1(t)=\mp\frac{\mu_0(t)}{\sqrt{\mu_1(t)}\sqrt{\mu_2(t)}};\\&\mu_0(t)\neq 0\ \text{ since }\ \eta_1(t)\neq 0;\end{aligned}\tag{46}$$

$$\begin{aligned}&\phi^{-2}x^0:\eta_2(t)=\pm\int\frac{\mu_0(t)^3}{\mu_1(t)^{3/2}\sqrt{\mu_2(t)}}\mathrm{d}t\mp\int\frac{\mu_0(t)\mu_3(t)}{\sqrt{\mu_1(t)}\sqrt{\mu_2(t)}}\mathrm{d}t+\eta_4;\\&\phi^{-2}x^1:\mu_4(t)=-\frac{\mu_0'(t)}{\mu_0(t)}+\frac{\mu_1'(t)}{2\mu_1(t)}+\frac{\mu_2'(t)}{2\mu_2(t)};\end{aligned}\tag{47}$$

1 Each of Auto-Bäcklund Transformations (36), (37) and (40)–(43) works as a system of the equations relating a set of the solutions of Eq. (1), e.g., Solutions (49), to another set of the solutions of Eq. (1) itself. Therefore, we could, by and large at least, be able to increasingly construct more and more complicated solutions of Eq. (1).

$$\phi^{-1}:\quad \mu_2(t)=\eta_5\mu_0(t)^{\frac{2(A-1)}{A}}\mu_1(t)^{\frac{2-A}{A}},\tag{48}$$

where η_4 and $\eta_5\neq 0$ are the real constants. Computing with Expression (36), we obtain the following variable/constant-coefficient-dependent solitonic solutions of Eq. (1):

$$u^{(I)}(x,t)=-\frac{\mu_0(t)}{2\mu_1(t)}\tanh\left[\frac{\eta_4}{2}\mp\frac{\mu_0(t)x}{2\sqrt{\mu_1(t)}\sqrt{\eta_5\mu_0(t)^{2-\frac{2}{A}}\mu_1(t)^{\frac{2}{A}-1}}}\pm\frac{1}{2\eta_5}\int\mu_0(t)^{\frac{A+2}{A}}\mu_1(t)^{-\frac{A+4}{2A}}\sqrt{\eta_5\mu_0(t)^{2-\frac{2}{A}}\mu_1(t)^{\frac{2}{A}-1}}\,\mathrm{d}t\mp\frac{1}{2}\int\frac{\mu_0(t)\mu_3(t)}{\sqrt{\mu_1(t)}\sqrt{\eta_5\mu_0(t)^{2-\frac{2}{A}}\mu_1(t)^{\frac{2}{A}-1}}}\mathrm{d}t\right]-\frac{\mu_0(t)}{2\mu_1(t)},\tag{49}$$

under Variable/Constant-Coefficient Constraints (38), (39), (46), (47) and (48).

There exist two branches of Solitonic Solutions (49) because of the "±" signs. Fig. 2 displays them.

Case II: $\eta_3(t)\neq 0$

$$\phi^0:\quad \eta_3(t)=\eta_4\mathrm{e}^{-A\int\mu_4(t)\mathrm{d}t};$$

$$\phi^{-2}x^1:\quad \eta_1'(t)+\eta_1(t)\mu_4(t)=0,$$

i.e., $\eta_1=\text{constant}\neq 0,\ \mu_4(t)=0$ or $\eta_1=\eta_1(t)\neq 0$,

where $\eta_4\neq 0$ (only in Case II).

Case IIa: $\eta_1=\text{constant}\neq 0$

$$\phi^{-2}x^1:\mu_4(t)=0;\tag{50}$$

$$\phi^{-3}:\quad \mu_0(t)=\mp\eta_1\sqrt{\mu_1(t)}\sqrt{\mu_2(t)}-2\eta_4\mu_1(t);\tag{51}$$

$$\phi^{-1}:\quad \eta_2(t)=\mp6\eta_1^2\eta_4\int\sqrt{\mu_1(t)}\sqrt{\mu_2(t)}\mathrm{d}t-\eta_1^3\int\mu_2(t)\mathrm{d}t-6\eta_1\eta_4^2\int\mu_1(t)\mathrm{d}t+\eta_1\int\mu_3(t)\mathrm{d}t+\eta_5+\frac{1}{2}\ln[\mu_1(t)]-\frac{1}{2}\ln[\mu_2(t)];$$

$$\phi^{-2}:\quad \mu_1(t)=\eta_6\mu_2(t),\tag{52}$$

where η_5 could be zero (only in Case IIa) and $\eta_6\neq 0$ is a real constant.

Computing with Expression (36), we obtain the following variable-coefficient-dependent solitonic solutions of Eq. (1):

$$u^{(\mathrm{IIa})}(x,t)=\pm\frac{\eta_1}{2\sqrt{\eta_6}}\tanh\left[\frac{\eta_1x}{2}-\frac{\eta_1}{2}\left(\eta_1^2\pm6\eta_1\eta_4\sqrt{\eta_6}+6\eta_4^2\eta_6\right)\int\mu_2(t)\mathrm{d}t+\frac{\eta_1}{2}\int\mu_3(t)\mathrm{d}t+\frac{\eta_5}{2}+\frac{\ln\left(\sqrt{\eta_6}\right)}{2}\right]\pm\frac{\eta_1}{2\sqrt{\eta_6}}+\eta_4,\tag{53}$$

under Variable-Coefficient Constraints (38), (39), (50), (51) and (52). Solitonic Solutions (53) are independent of A owing to Variable-Coefficient Constraint (50). There exist two branches of Solitonic Solutions (53) because of the "±" signs.

Case IIb: $\eta_1=\eta_1(t)\neq 0$

$$\phi^{-2}x^1:\eta_1(t)=\eta_5\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t};$$

$$\phi^{-3}:\quad \mu_0(t)=-2\eta_4\mu_1(t)\mathrm{e}^{-A\int\mu_4(t)\mathrm{d}t}\mp\eta_5\sqrt{\mu_1(t)}\sqrt{\mu_2(t)}\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t};\tag{54}$$

$$\phi^{-2}x^0:\eta_2(t)=\eta_5\int\mu_3(t)\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t}\mathrm{d}t\mp6\eta_4\eta_5^2\int\sqrt{\mu_1(t)}\sqrt{\mu_2(t)}\mathrm{e}^{-A\int\mu_4(t)\mathrm{d}t-2\int\mu_4(t)\mathrm{d}t}\mathrm{d}t-6\eta_4^2\eta_5\int\mu_1(t)\mathrm{e}^{-2A\int\mu_4(t)\mathrm{d}t-\int\mu_4(t)\mathrm{d}t}\mathrm{d}t-\eta_5^3\int\mu_2(t)\mathrm{e}^{-3\int\mu_4(t)\mathrm{d}t}\mathrm{d}t;$$

$$\phi^{-1}:\quad \mu_1(t)=\eta_6\mu_2(t)\mathrm{e}^{\int[2A\mu_4(t)-2\mu_4(t)]\mathrm{d}t},\tag{55}$$

where $\eta_5\neq 0$ and $\eta_6\neq 0$ (only in Case IIb).

Computing with Eq. (36), we obtain the following variable/constant-coefficient-dependent solitonic solutions of Eq. (1):

$$u^{(\mathrm{IIb})}(x,t)=\eta_4\mathrm{e}^{-A\int\mu_4(t)\mathrm{d}t}\pm\frac{\eta_5\mathrm{e}^{-(2A-1)\int\mu_4(t)\mathrm{d}t}\sqrt{\mathrm{e}^{2(A-1)\int\mu_4(t)\mathrm{d}t}}}{2\sqrt{\eta_6}}\pm\frac{\eta_5\mathrm{e}^{(1-2A)\int\mu_4(t)\mathrm{d}t}\sqrt{\mathrm{e}^{2(A-1)\int\mu_4(t)\mathrm{d}t}}}{2\sqrt{\eta_6}}\times\tanh\left\{\frac{\eta_5}{2}\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t}x+\frac{\eta_5}{2}\int\mu_3(t)\mathrm{e}^{-\int\mu_4(t)\mathrm{d}t}\mathrm{d}t-\frac{\eta_5}{2}\left[\pm6\eta_4\eta_5\sqrt{\eta_6}\int\mu_2(t)\mathrm{e}^{-(A+2)\int\mu_4(t)\mathrm{d}t}\sqrt{\mathrm{e}^{2(A-1)\int\mu_4(t)\mathrm{d}t}}\mathrm{d}t+\left(6\eta_4^2\eta_6+\eta_5^2\right)\int\mu_2(t)\mathrm{e}^{-3\int\mu_4(t)\mathrm{d}t}\mathrm{d}t\right]\right\},\tag{56}$$

under Variable/Constant-Coefficient Constraints (38), (39), (54) and (55). There exist two branches of Solitonic Solutions (56) because of the "±" signs.

5 Conclusions

The atmosphere has been said to be an evolutionary agent essential to the shaping of a planet. In oceanic science, natural science, engineering, medical science and daily life, liquids have been commonly seen. Plasmas have been believed to be possibly the most abundant form of ordinary matter in the Universe.

In this paper, we have investigated Eq. (1), a generalized variable-coefficient KdV-mKdV equation for the atmosphere, oceanic fluids and plasmas, in which the coefficients $\mu_0(t)$, $\mu_1(t)$ and $\mu_2(t)$ represent the quadratic-nonlinear, cubic-nonlinear and dispersive effects, respectively, while $-\mu_3(t)+x\mu_4(t)$ and $\mu_4(t)A$ correspond to the dissipative and line-damping terms. Special cases of Eq. (1) in fluid mechanics, plasma dynamics and other fields have been listed out. With symbolic computation, beginning with Presumption (5), which is different from those presented in the existing literatures, we have worked out

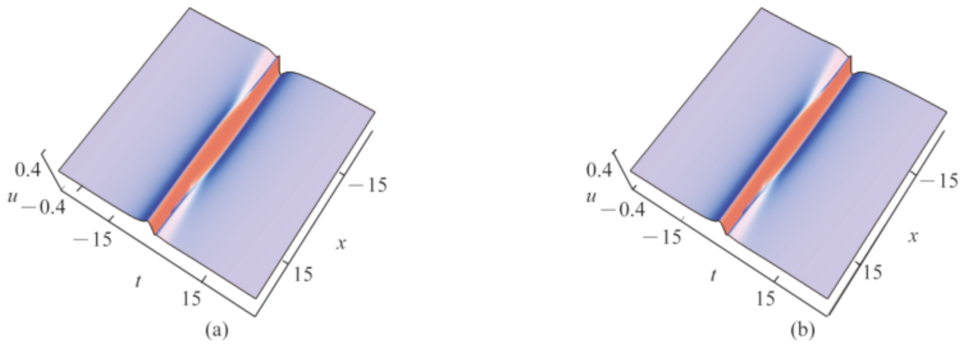

Fig. 2. Waves via Solitonic Solutions (49) with (a) $\mu_0(t)=\frac{t}{2}$, $\mu_1(t)=t^2+1$, $\mu_3(t)=\frac{t}{2}$, $\eta_4=2$, $\eta_5=2$ and $A=1$, with the "_" sign in Expressions (46), (b) the same as (a) except for the "+" sign in Expressions (46).

(1) Scaling Transformations (2);

(2) Bilinear Forms (8) or (9) through the binary Bell polynomials and Scaling Transformations (2), under Variable/Constant-Coefficient Constraints (4), which are dependent on $\mu_0(t)$, $\mu_2(t)$, $\mu_3(t)$ and $\mu_4(t)$;

(3) N-Soliton Solutions (10) and (13) according to Bilinear Forms (9) under Variable/Constant-Coefficient Constraints (4) and (12), which are dependent on $\mu_2(t)$, $\mu_3(t)$ and $\mu_4(t)$, along with One-Soliton Solutions (10) and (14), Two-Soliton Solutions (10) and (15), Three-Soliton Solutions (10) and (16), as well as Fig. 1a to show the interaction between the two dark solitons, and Fig. 1b to display the interaction between the two bright solitons;

(4) Bilinear Auto-Bäcklund Transformations (5), (7a), (22) and (29), with the Hirota method, under Variable/Constant-Coefficient Constraints (4) and (21), which are dependent on $\mu_0(t)$, $\mu_2(t)$, $\mu_3(t)$ and $\mu_4(t)$, along with Solitonic Solutions (5), (7a) and (34), under Variable/Constant-Coefficient Constraints (4), (21) and (32).

In addition, with symbolic computation, Painlevé-Type Auto-Bäcklund Transformations (36), (37) and (40)-(43) have been worked out, under Variable-Coefficient Constraints (38) and (39), which have not been obtained in the existing literatures, either, while have been seen to depend on A, $\mu_0(t)$, $\mu_1(t)$, $\mu_2(t)$, $\mu_3(t)$ and $\mu_4(t)$, along with

(1) Solitonic Solutions (49) under Variable/Constant-Coefficient Constraints (38), (39), (46), (47) and (48), as well as Fig. 2 to display certain waves via Solitonic Solutions (49);

(2) Solitonic Solutions (53) under Variable-Coefficient Constraints (38), (39), (50), (51) and (52);

(3) Solitonic Solutions (56) under Variable/Constant-Coefficient Constraints (38), (39), (54) and (55).

It has been stated that those coefficients correspond to the quadratic-nonlinear, cubic-nonlinear, dispersive, dissipative and line-damping effects in the atmosphere, oceanic fluids and plasmas.

Appendix: Bell-polynomial preliminary

Bell polynomials have been said to provide a relatively-direct way to get the bilinear forms for certain nonlinear evolution equations, instead of the dependent variable transformations (Bell, 1934; Lambert et al., 1994; Wang et al., 2017).

Bell (1934); Lambert et al. (1994) and Wang et al. (2017) have presented the following:

(1) The Bell polynomials:

$$Y_{hx}(v)\equiv Y_h(v_1,\cdots,v_h)=\mathrm{e}^{-v}\partial_x^h\mathrm{e}^{v}, \tag{57}$$

where $h=1,2,\cdots$, v is a C^∞ function of x, $v_h=\partial_x^h v$, and the subscripts in the notation $Y_{hx}(v)$ denote the highest-order derivatives of v with respect to x, e.g.,

$$Y_x=v_x,\ \ Y_{2x}=v_{2x}+v_x^2,\ \ Y_{3x}=v_{3x}+3v_xv_{2x}+v_x^3,\cdots \tag{58}$$

(2) The two-dimensional Bell polynomials:

$$Y_{mx,nt}(\theta)\equiv Y_{m,n}(\theta_{1,1},\cdots,\theta_{1,n},\cdots,\theta_{m,1},\cdots,\theta_{m,n})=\mathrm{e}^{-\theta}\partial_x^m\partial_t^n\mathrm{e}^{\theta}, \tag{59}$$

where θ is a C^∞ function of x and t, $\theta_{k,\varsigma}=\partial_x^k\partial_t^\varsigma\theta$, $k=1,\cdots,m$, $\varsigma=1,\cdots,n$, with m and n being the nonnegative integers, e.g.,

$$Y_{x,t}=\theta_{x,t}+\theta_x\theta_t, Y_{2x,t}=\theta_{2x,t}+\theta_{2x}\theta_t+2\theta_{x,t}\theta_x+\theta_x^2\theta_t,\ \cdots \tag{60}$$

(3) The binary Bell polynomials:

$$\mathcal{Y}_{mx,nt}(p,q)\equiv Y_{mx,nt}(\psi_{1,1},\cdots,\psi_{1,n},\cdots,\psi_{m,1},\cdots,\psi_{m,n})\Big|_{\psi_{k,\varsigma}=\begin{cases}p_{k,\varsigma}, & \text{if } k+\varsigma \text{ is odd}\\ q_{k,\varsigma}, & \text{if } k+\varsigma \text{ is even}\end{cases}}, \tag{61}$$

where $p(x,t)$ and $q(x,t)$ are both the C^∞ functions of x and t, $\psi_{k,\varsigma}$'s are the complex functions of p and q, $p_{k,\varsigma}=\partial_x^k\partial_t^\varsigma p$ and $q_{k,\varsigma}=\partial_x^k\partial_t^\varsigma q$, e.g.,

$$\mathcal{Y}_x(p,q)=p_x,\ \ \mathcal{Y}_{2x}(p,q)=q_{2x}+p_x^2,$$
$$\mathcal{Y}_{x,t}(p,q)=p_xp_t+q_{xt},$$
$$\mathcal{Y}_{3x}(p,q)=p_{3x}+3p_xq_{2x}+p_x^3,\ \cdots \tag{62}$$

Matveev and Salle (1991); Wadati (1975) as well as Cariello and Tabor (1989) have linked the $\mathcal{Y}$ polynomials to the Hirota operators, i.e.,

$$\mathcal{Y}_{mx,nt}\left[p=\ln\left(\frac{f}{g}\right), q=\ln(fg)\right]=(fg)^{-1}D_x^m D_t^n f\cdot g, \qquad (63)$$

where $f(x,t)$ and $g(x,t)$ are the C^∞ functions of x and t, while D_x and D_t are the Hirota operators defined by

$$D_x^m D_t^n f(x,t)\cdot g(x,t)\equiv \left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^n f(x,t)g(x',t')\Big|_{x'=x,t'=t}, \qquad (64)$$

with x' and t' being the formal variables.

References

Plasma (physics), 2021. https://en.wikipedia.org/wiki/Plasma(physics).

Bell, E.T., 1934. Exponential Polynomials, *The Annals of Mathematics*, 35(2), 258–277.

Cariello, F. and Tabor, M., 1989. Painlevé expansions for nonintegrable evolution equations, *Physica D*: *Nonlinear Phenomena*, 39(1), 77–94.

Chen, S.S., Tian, B., Chai, J., Wu, X.Y. and Du, Z., 2020c. Lax pair, binary Darboux transformations and dark-soliton interaction of a fifth-order defocusing nonlinear Schrodinger equation for the attosecond pulses in the optical fiber communication, *Waves in Random and Complex Media*, 30(3), 389–402.

Chen, S.S., Tian, B., Sun, Y. and Zhang, C.R., 2019. Generalized darboux transformations, rogue waves, and modulation instability for the coherently coupled nonlinear schrödinger equations in nonlinear optics, *Annalen der Physik*, 531(8), 1900011.

Chen, Y.Q., Tian, B., Qu, Q.X., Li, H., Zhao, X.H., Tian, H.Y. and Wang, M., 2020a. Ablowitz-Kaup-Newell-Segur system, conservation laws and Backlund transformation of a variable-coefficient Korteweg-de Vries equation in plasma physics, fluid dynamics or atmospheric science, *International Journal of Modern Physics B*, 34(25), 2050226.

Chen, Y.Q., Tian, B., Qu, Q.X., Li, H., Zhao, X.H., Tian, H.Y. and Wang, M., 2020b. Reduction and analytic solutions of a variable-coefficient Korteweg-de Vries equation in a fluid, crystal or plasma, *Modern Physics Letters B*, 34(26), 2050287.

Ding, C.C., Gao, Y.T. and Deng, G.F., 2019. Breather and hybrid solutions for a generalized (3+1)-dimensional B-type Kadomtsev-Petviashvili equation for the water waves, *Nonlinear Dynamics*, 97(4), 2023–2040.

Deng, G.F., Gao, Y.T., Ding, C.C. and Su, J.J., 2020b. Solitons and breather waves for the generalized Konopelchenko-Dubrovsky-Kaup-Kupershmidt system in fluid mechanics, ocean dynamics and plasma physics, *Chaos*, *Solitons & Fractals*, 140, 110085.

Deng, G.F., Gao, Y.T., Su, J.J., Ding, C.C. and Jia, T.T., 2020a. Solitons and periodic waves for the (2+1)-dimensional generalized Caudrey-Dodd-Gibbon-Kotera-Sawada equation in fluid mechanics, *Nonlinear Dynamics*, 99(2), 1039–1052.

Ding, C.C., Gao, Y.T., Deng, G.F. and Wang, D., 2020. Lax pair, conservation laws, Darboux transformation, breathers and rogue waves for the coupled nonautonomous nonlinear Schrodinger system in an inhomogeneous plasma, *Chaos*, *Solitons & Fractals*, 133, 109580.

Djoudi, W. and Zerarka, A., 2016. Exact structures for the KdV-mKdV equation with variable coefficients via the functional variable method, *Optik*, 127, 9621–9626.

Du, X.X., Tian, B., Qu, Q.X., Yuan, Y.Q. and Zhao, X.H., 2020. Lie group analysis, solitons, self-adjointness and conservation laws of the modified Zakharov-Kuznetsov equation in an electron-positron-ion magnetoplasma, *Chaos*, *Solitons & Fractals*, 134, 109709.

Du, X.X., Tian, B., Yuan, Y.Q. and Du, Z., 2019. Symmetry reductions, group-invariant solutions, and conservation laws of a (2+1)-dimensional nonlinear Schrodinger equation in a Heisenberg ferromagnetic spin chain, *Annalen der Physik*, 531(11), 1900198.

Farlex, 2021. Atmosphere, Farlex, Inc., https://encyclopedia.thefreedictionary.com/atmosphere.

Feng, Y.J., Gao, Y.T., Jia, T.T. and Li, L.Q., 2019. Soliton interactions of a variable-coefficient three-component AB system for the geophysical flows, *Modern Physics Letters B*, 33(29), 1950354.

Feng, Y.J., Gao, Y.T., Li, L.Q. and Jia, T.T., 2020. Bilinear form, solitons, breathers and lumps of a (3+1)-dimensional generalized Konopelchenko-Dubrovsky-Kaup-Kupershmidt equation in ocean dynamics, fluid mechanics and plasma physics, *The European Physical Journal Plus*, 135(3), 272.

Gao, X.Y., 2019. Mathematical view with observational/experimental consideration on certain (2+1)-dimensional waves in the cosmic/laboratory dusty plasmas, *Applied Mathematics Letters*, 91, 165–172.

Gao, X.Y., Guo, Y.J. and Shan, W.R., 2020a. Hetero-Bäcklund transformation and similarity reduction of an extended (2+1)-dimensional coupled Burgers system in fluid mechanics, *Physics Letters A*, 384(31), 126788.

Gao, X.Y., Guo, Y.J. and Shan, W.R., 2020b. Water-wave symbolic computation for the Earth, Enceladus and Titan: The higher-order Boussinesq-Burgers system, auto- and non-auto-Backlund transformations, *Applied Mathematics Letters*, 104, 106170.

Gao, X.Y., Guo, Y.J. and Shan, W.R., 2020c. Comment on "Bilinear form, solitons, breathers and lumps of a (3+1)-dimensional generalized Konopelchenko-Dubrovsky-Kaup-Kupershmidt equation in ocean dynamics, fluid mechanics and plasma physics" [Eur. Phys. J. Plus (2020) 135: 272], *The European Physical Journal Plus*, 135, 631.

Gao, X.Y., Guo, Y.J. and Shan, W.R., 2020d. Bilinear forms through the binary Bell polynomials, N solitons and Backlund transformations of the Boussinesq-Burgers system for the shallow water, *Communications in Theoretical Physics*, 72, 095002.

Gao, X.Y., Guo, Y.J. and Shan, W.R., 2020e. Shallow water in an open sea or a wide channel: Auto- and non-auto-Backlund transformations with solitons for a generalized (2+1)-dimensional dispersive long-wave system, *Chaos*, *Solitons & Fractals*, 138, 109950.

Gao, X.Y., Guo, Y.J. and Shan, W.R., 2021a. Oceanic studies via a variable-coefficient nonlinear dispersive-wave system in the Solar System, *Chaos*, *Solitons & Fractals*, 142, 110367.

Gao, X.Y., Guo, Y.J. and Shan, W.R., 2021b. In oceanography, acoustics and hydrodynamics: An extended coupled (2+1)-dimensional Burgers system, *Chinese Journal of Physics*, 70, 264–270.

Gao, X.Y., Guo, Y.J. and Shan, W.R., 2021c. Cosmic dusty plasmas via a (3+1)-dimensional generalized variable-coefficient Kadomtsev-Petviashvili-Burgers-type equation: auto-Backlund transformations, solitons and similarity reductions plus observational/experimental supports, *Waves in Random and Complex Media*, doi:10.1080/17455030.2021.1942308. (in press)

Gao, X.T., Tian, B., Shen, Y. and Feng, C.H., 2021d. Comment on "Shallow water in an open sea or a wide channel: Auto- and non-auto-Backlund transformations with solitons for a generalized (2+1)-dimensional dispersive long-wave system", *Chaos*, *Solitons &*

Fractals, 151, 111222.

Grave, M., Camata, J.J. and Coutinho, A.L.G.A., 2020. A new convected level-set method for gas bubble dynamics, *Computers & Fluids*, 209, 104667.

Hirota, R., 1980. *The Direct Method in Soliton Theory*, Springer, Berlin.

Hu, C.C., Tian, B., Yin, H.M., Zhang, C.R. and Zhang, Z., 2019b. Dark breather waves, dark lump waves and lump wave-soliton interactions for a (3+1)-dimensional generalized Kadomtsev-Petviashvili equation in a fluid, *Computers & Mathematics with Applications*, 78(1), 166–177.

Hu, L., Gao, Y.T., Jia, T.T., Deng, G.F. and Li, L.Q., 2021. Higher-order hybrid waves for the (2+1)-dimensional Boiti-Leon-Manna-Pempinelli equation for an irrotational incompressible fluid via the modified Pfaffian technique, *Zeitschrift für angewandte Mathematik und Physik*, 72, 75.

Hu, L., Gao, Y.T., Jia, S.L., Su, J.J. and Deng, G.F., 2019a. Solitons for the (2+1)-dimensional Boiti-Leon-Manna-Pempinelli equation for an irrotational incompressible fluid via the Pfaffian technique, *Modern Physics Letters B*, 33(30), 1950376.

Jia, T.T., Gao, Y.T., Deng, G.F. and Hu, L., 2019. Quintic time-dependent-coefficient derivative nonlinear Schrodinger equation in hydrodynamics or fiber optics: bilinear forms and dark/anti-dark/gray solitons, *Nonlinear Dynamics*, 98(1), 269–282.

Jia, T.T., Gao, Y.T., Yu, X. and Li, L.Q., 2021. Lax pairs, infinite conservation laws, Darboux transformation, bilinear forms and solitonic interactions for a combined Calogero-Bogoyavlenskii-Schiff-type equation, *Applied Mathematics Letters*, 114, 106702.

Khorram, S., 2020. Bedload sediment rate prediction for the sand transport along coastal waters in ocean management strategy, *China Ocean Engineering*, 34(6), 840–852.

Lambert, F., Loris, I., Springael, J. and Willer, R., 1994. On a direct bilinearization method: Kaup's higher-order water wave equation as a modified nonlocal Boussinesq equation, *Journal of Physics A*: *Mathematical and General*, 27(15), 5325–5334.

Lambert, F. and Springael, J., 2001. On a direct procedure for the disclosure of Lax pairs and Bäcklund transformations, *Chaos*, *Solitons & Fractals*, 12(14-15), 2821–2832.

Lambert, F. and Springael, J., 2008. Soliton equations and simple combinatorics, *Acta Applicandae Mathematicae*, 102(2-3), 147–178.

Li, C.Y., Huang, W., Wu, R.H. and Sheremet, A., 2020a. Weather induced quasi-periodic motions in estuaries and bays: Meteorological tide, *China Ocean Engineering*, 34(3), 299–313.

Li, L.Q., Gao, Y.T., Hu, L., Jia, T.T., Ding, C.C. and Feng, Y.J., 2020b. Bilinear form, soliton, breather, lump and hybrid solutions for a (2+1)-dimensional Sawada-Kotera equation, *Nonlinear Dynamics*, 100(3), 2729–2738.

Liu, F.Y., Gao, Y.T., Yu, X., Ding, C.C., Deng, G.F. and Jia, T.T., 2021a. Painlevé analysis, Lie group analysis and soliton-cnoidal, resonant, hyperbolic function and rational solutions for the modified Korteweg-de Vries-Calogero-Bogoyavlenskii-Schiff equation in fluid mechanics/plasma physics, *Chaos*, *Solitons & Fractals*, 144, 110559.

Liu, F.Y., Gao, Y.T., Yu, X., Li, L.Q., Ding, C.C. and Wang, D., 2021b. Lie group analysis and analytic solutions for a (2+1)-dimensional generalized Bogoyavlensky-Konopelchenko equation in fluid mechanics and plasma physics, *European Physical Journal Plus*, 136, 656.

Liu, F.Y., Gao, Y.T., Yu, X., Hu, L. and Wu, X.H., 2021c. Hybrid solutions for the (2+1)-dimensional variable-coefficient Caudrey-Dodd-Gibbon-Kotera-Sawada equation in fluid mechanics, *Chaos, Solitons & Fractals*, Ms. No. CHAOS-D-21-02144R1. (in press)

Liu, S.H., Tian, B., Qu, Q.X., Li, H., Zhao, X.H., Du, X.X. and Chen, S.S., 2020b. Breather, lump, shock and travelling-wave solutions for a (3+1)-dimensional generalized Kadomtsev-Petviashvili equation in fluid mechanics and plasma physics, *International Journal of Computer Mathematics*, 98, 1130.

Liu, S.H., Tian, B., Qu, Q.X., Zhang, C.R., Hu, C.C. and Wang, M., 2020a. Lump, mixed lump-stripe, mixed rogue wave-stripe and breather wave solutions for a (3+1)-dimensional generalized Calogero-Bogoyavlenskii-Schiff equation, *Modern Physics Letters B*, 34(23), 2050243.

Ma, Y.X., Tian, B., Qu, Q.X., Wei, C.C. and Zhao X., 2021a. Backlund transformations, kink soliton, breather- and travelling-wave solutions for a (3+1)-dimensional B-type Kadomtsev-Petviashvili equation in fluid dynamics, *Chinese Journal of Physics*, doi:10.1016/j.cjph.2021.07.001. (in press)

Ma, Y.X., Tian, B., Qu, Q.X., Tian, H.Y. and Liu, S.H., 2021b. Bilinear Backlund transformation, breather- and travelling-wave solutions for a (2 + 1)-dimensional extended Kadomtsev-Petviashvili II equation in fluid mechanics, *Modern Physics Letters B*, 35, 2150315.

Ma, Y.X., Tian, B., Qu, Q.X., Yang, D.Y. and Chen, Y.Q., 2021c. Painleve analysis, Backlund transformations and traveling-wave solutions for a (3+1)-dimensional generalized Kadomtsev-Petviashvili equation in a fluid, *International Journal of Modern Physics B*, 35, 2150108.

Matsuno, Y., 1984. *Bilinear Transformation Method*, Academic Press, Orlando.

Matveev, V.B. and Salle, M.A., 1991. *Darboux Transformations and Solitons*, Springer, Berlin.

Meng, G.Q., Gao, Y.T., Yu, X., Shen, Y.J. and Qin, Y., 2012. Painlevé analysis, Lax pair, Bäcklund transformation and multi-soliton solutions for a generalized variable-coefficient KdV-mKdV equation in fluids and plasmas, *Physica Scripta*, 85(5), 055010.

Shen, Y. and Tian, B., 2021b. Bilinear auto-Backlund transformations and soliton solutions of a (3+1)-dimensional generalized nonlinear evolution equation for the shallow water waves, *Applied Mathematics Letters*, 122, 107301.

Shen, Y., Tian, B., Liu, S.H. and Yang, D.Y., 2021c. Bilinear Backlund transformation, soliton and breather solutions for a (3+1)-dimensional generalized Kadomtsev-Petviashvili equation in fluid dynamics and plasma physics, *Physica Scripta*, 96, 075212.

Shen, Y., Tian, B., Zhang, C.R., Tian, H.Y. and Liu, S.H., 2021a. Breather-wave, periodic-wave and traveling-wave solutions for a (2+1)-dimensional extended Boiti-Leon-Manna-Pempinelli equation for an incompressible fluid, *Modern Physics Letters B*, 35, 2150261.

Shen, Y., Tian, B. and Zhou, T.Y., 2021d. In nonlinear optics, fluid dynamics and plasma physics: symbolic computation on a (2+1)-dimensional extended Calogero-Bogoyavlenskii-Schiff system, *European Physical Journal Plus*, 136, 572.

Shen, Y., Tian, B. and Liu, S.H., 2021e. Solitonic fusion and fission for a (3+1)-dimensional generalized nonlinear evolution equation arising in the shallow water waves, *Physics Letters A*, 405, 127429.

Shen, Y., Tian, B., Zhao, X., Shan, W.R. and Jiang, Y., 2021f. Bilinear form, bilinear auto-Backlund transformation, breather and lump solutions for a (3+1)-dimensional generalized Yu-Toda-Sasa-Fukuyama equation in a two-layer liquid or a lattice, *Pramana-Journal of Physics*, doi: 10.1007/s12043-021-02163-4. (in press)

Su, J.J., Gao, Y.T., Deng, G.F. and Jia, T.T., 2019a. Solitary waves, breathers, and rogue waves modulated by long waves for a model of a baroclinic shear flow, *Physics Review E*, 100(4), 042210.

Su, J.J., Gao, Y.T. and Ding, C.C., 2019b. Darboux transformations and rogue wave solutions of a generalized *AB* system for the geophysical flows, *Applied Mathematics Letters*, 88, 201–208.

Tang, B., Wang, X.M., Fan Y.Z. and Qu, J.F., 2016. Exact solutions for a generalized KdV-MKdV equation with variable coefficients, *Mathematical Problems in Engineering*, 2016, 5274243.

Tian, H.Y., Tian, B., Yuan, Y.Q. and Zhang, C.R., 2021a. Superregular solutions for a coupled nonlinear Schrodinger system in a two-mode nonlinear fiber, *Physica Scripta*, 96(4), 045213.

Tian, H.Y., Tian, B., Zhang, C.R. and Chen, S.S., 2021b. Darboux dressing transformation and superregular breathers for a coupled nonlinear Schrodinger system with the negative coherent coupling in a weakly birefringent fiber, *International Journal of Computer Mathematics*, doi: 10.1080/00207160.2021.1900568. (in press)

Triki, H., Taha, T.R. and Wazwaz, A.M., 2010. Solitary wave solutions for a generalized KdV-CmKdV equation with variable coefficients, *Mathematics and Computers in Simulation*, 80(9), 1867–1873.

Wadati, M., 1975. Wave propagation in nonlinear lattice. I, *Journal of the Physical Society of Japan*, 38(3), 673–680.

Wang, D., Gao, Y.T., Ding, C.C. and Zhang, C.Y., 2020c. Solitons and periodic waves for a generalized (3+1)-dimensional Kadomtsev-Petviashvili equation in fluid dynamics and plasma physics, *Communications in Theoretical Physics*, 72(11), 115004.

Wang, D., Gao, Y.T., Su, J.J. and Ding, C.C., 2020d. Bilinear forms and soliton solutions for a (2+1)-dimensional variable-coefficient nonlinear Schrodinger equation in an optical fiber, *Modern Physics Letters B*, 34(30), 2050336.

Wang, D., Gao, Y.T., Yu, X., Li, L.Q. and Jia, T.T., 2021c. Bilinear form, solitons, breathers, lumps and hybrid solutions for a (3+1)-dimensional Date-Jimbo-Kashiwara-Miwa equation, *Nonlinear Dynamics*, 104, 1519.

Wang, K., Wu, L., Li, Y.Z. and Sun, X.P., 2020a. Study on the overload and dwell-fatigue property of titanium alloy in manned deep submersible, *China Ocean Engineering*, 34(5), 738–745.

Wang, M., Tian, B., Hu, C.C. and Liu, S.H., 2021a. Generalized Darboux transformation, solitonic interactions and bound states for a coupled fourth-order nonlinear Schrodinger system in a birefringent optical fiber, *Applied Mathematics Letters*, 119, 106936.

Wang, M., Tian, B., Liu, S.H., Shan, W.R. and Jiang, Y., 2021b. Soliton, multiple-lump and hybrid solutions of a (2+1)-dimensional generalized Hirota-Satsuma-Ito equation for the water waves, *European Physical Journal Plus*, 136, 635.

Wang, P., Tian, B., Liu, W.J., Jiang, Y. and Xue, Y.S., 2012. Interactions of breathers and solitons of a generalized variable-coefficient Korteweg-de Vries-modified Korteweg-de Vries equation with symbolic computation, *The European Physical Journal D*, 66, 233.

Wang, M., Tian, B., Qu, Q.X., Du, X.X., Zhang, C.R. and Zhang, Z., 2019a. Lump, lumpoff and rogue waves for a (2+1)-dimensional reduced Yu-Toda-Sasa-Fukuyama equation in a lattice or liquid, *The European Physical Journal Plus*, 134(11), 578.

Wang, M., Tian, B., Sun, Y., Yin, H.M. and Zhang, Z., 2019b. Mixed lump-stripe, bright rogue wave-stripe, dark rogue wave-stripe and dark rogue wave solutions of a generalized Kadomtsev-Petviashvili equation in fluid mechanics, *Chinese Journal of Physics*, 60, 440–449.

Wang, M., Tian, B., Sun, Y. and Zhang, Z., 2020b. Lump, mixed lump-stripe and rogue wave-stripe solutions of a (3+1)-dimensional nonlinear wave equation for a liquid with gas bubbles, *Computers & Mathematics with Applications*, 79(3), 576–587.

Wang, M., Tian, B., Qu, Q.X., Zhao, X.H., Zhang, Z. and Tian, H.Y., 2020e. Lump, lumpoff, rogue wave, breather wave and periodic lump solutions for a (3+1)-dimensional generalized Kadomtsev–Petviashvili equation in fluid mechanics and plasma physics, *International Journal of Computer Mathematics*, 97, 2474–2486.

Wang, Y.F., Tian, B. and Jiang, Y., 2017. Soliton fusion and fission in a generalized variable-coefficient fifth-order Korteweg-de Vries equation in fluids, *Applied Mathematics and Computation*, 292, 448–456.

Yang, D.Y., Tian, B., Qu, Q.X., Yuan, Y.Q., Zhang, C.R. and Tian, H.Y., 2020. Generalized Darboux transformation and the higher-order semirational solutions for a nonlinear Schrodinger system in a birefringent fiber, *Modern Physics Letters B*, 34, 2150013.

Yang, D.Y., Tian, B., Qu, Q.X., Zhang, C.R., Chen, S.S. and Wei, C.C., 2021a. Lax pair, conservation laws, Darboux transformation and localized waves of a variable-coefficient coupled Hirota system in an inhomogeneous optical fiber, *Chaos, Solitons & Fractals*, 150, 110487.

Yang, D.Y., Tian, B., Qu, Q.X., Li, H., Zhao, X.H., Chen, S.S. and Wei, C.C., 2021b. Darboux-dressing transformation, semi-rational solutions, breathers and modulation instability for the cubic-quintic nonlinear Schrödinger system with variable coefficients in a non-Kerr medium, twin-core nonlinear optical fiber or waveguide, *Physica Scripta*, 96(4), 045210.

Yang, D.Y., Tian, B., Tian, H.Y., Chen, Y.Q., Shan, W.R. and Jiang, Y., 2021c. Interaction between the breather and breather-like soliton, and breather-to-soliton conversions of a variable-coefficient coupled Hirota system in an inhomogeneous optical fiber, *Optik*, doi:10.1016/j.ijleo.2021.166815.

Zhang, C.R., Tian, B., Qu, Q.X., Liu, L. and Tian, H.Y., 2020. Vector bright solitons and their interactions of the couple Fokas-Lenells system in a birefringent optical fiber, *Zeitschrift für angewandte Mathematik und Physik*, 71, 18.

Zhang, C.R., Tian, B., Sun, Y. and Yin, H.M., 2019. Binary Darboux transformation and vector-soliton-pair interactions with the negatively coherent coupling in a weakly birefringent fiber, *EPL* (*Europhysics Letters*), 127(4), 40003.

Zhao, X., Tian, B., Qu, Q.X., Yuan, Y.Q., Du, X.X. and Chu, M.X., 2020. Dark-dark solitons for the coupled spatially modulated Gross-Pitaevskii system in the Bose-Einstein condensation, *Modern Physics Letters B*, 34(26), 2050282.

Zhao, X., Tian, B., Tian, H.Y. and Yang, D.Y., 2021. Bilinear Bäcklund transformation, Lax pair and interactions of nonlinear waves for a generalized (2+1)-dimensional nonlinear wave equation in nonlinear optics/fluid mechanics/plasma physics, *Nonlinear Dynamics*, 103(2), 1785–1794.

Zhou, T.Y., Tian, B., Chen, S.S., Wei, C.C. and Chen, Y.Q., 2021. Backlund transformations, Lax pair and solutions of the Sharma-Tasso-Olver-Burgers equation for the nonlinear dispersive waves, *Modern Physics Letters B*, Ms. No. MPLB-D-21-00122R1. (in press)