China Ocean Eng., Vol. 30, No. 3, pp. 359 – 374

DOI 10.1007/s13344-016-0024-8, ISSN 0890-5487

Modified Joint Distribution of Wave Heights and Periods*

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(Received 18 November 2013; received revised form 18 January 2015; accepted 20 March 2015)

ABSTRACT

The modified versions of the linear theoretical model of Longuet-Higgins (1983) are derived in this work and also compared with the laboratory experiments carried out in MARINTEK. The main feature of modifications is to replace the mean frequency in the formulation with the peak frequency of the wave spectrum. These two alternative forms of joint distributions are checked in three typical random sea states characterized by the initial wave steepness. In order to further explore the properties of these models, the associated marginal distributions of wave heights and wave periods are also researched with the observed statistics and some encouraging results are obtained.

Key words: *Longuet-Higgins joint distribution*; *mixed sea state*; *nonlinear wave series*; *wave height*; *wave period*

1. Introduction

In ocean engineering, it is very important to provide an exact description on ocean surface waves. Since wave heights and periods are not statistically independent, their joint probability distribution is normally demanded not only in scientific research but also in industry application.

The first theoretical model of joint distribution of wave heights and periods was derived on the basis of a narrowband approximation (Longuet-Higgins, 1975) and characterized by having the axis of symmetry with respect to the mean wave period which is normally in conflict with the commonly observed results in the analysis of recorded field data (Chakrabarti and Cooley, 1977; Goda, 1978). Almost simultaneously, the asymmetric pattern of the joint distribution of the positive maxima and the time interval between them was developed on the same assumption by a group of researchers at CNEXO (Cavanié *et al*., 1976). However, it involves the use of a spectral width parameter which is determined by the spectral moment up to fourth order and thus depends rather critically on the tail behavior of the spectral density. For arbitrary spectral bandwidth, Lindgren and Rychlik (1982) dealt with the joint distribution of local peak-to-trough excursion and the associated time interval (half period), proposing a lengthy and perhaps accurate approximation. However, possibly because it was not given in a closed form and was elaborated mathematically, this distribution has not been much used in practice.

Later, Longuet-Higgins (1983) revised his theoretical model and proposed a new function which has the same merit as the Cavanié's distribution in being asymmetric for normalized period but only

* The project was financially supported by the European Union (Grant No. 234175) and the Portuguese Foundation for Science and Technology (Grant No. SFRH/BD/98983/2013).

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depends on the lower-order spectral moments. Since then, various degrees of empirical investigation of the joint distribution and associated comparison among different theoretical models had been done frequently in many following studies (e.g., Haver, 1987; Srokosz, 1988).

Besides, improvement of the Longuet-Higgins (1983) theoretical model was still continued. For example, in the work of Stansell *et al*. (2004), they relaxed the restriction that the local wave frequency should be always positive, and derived a new joint distribution for the amplitude of wave envelope and local wave period when the sampling frequency is equal to the local wave frequency. The theoretical results are favorable when compared with data from wave field in extreme storms and from numerical simulations of broadband process with Pierson-Moskowitz spectrum.

Meanwhile, starting from the representation of surface elevation in spatial domain rather than in the traditional temporal domain, a joint distribution of wavelengths and amplitudes was obtained for linear waves (Xu *et al*., 2004), which is essentially another form of Longuet-Higgins joint distribution considering that the fundamental assumptions are still based upon the stationary, homogeneous and Gaussian process.

All these studies related to joint distributions only addressed one single wave system, which is relatively well understood now. However, it is well known that, in some other cases the wave spectrum is presented with two peak frequency such as the combined wind wave and swell wave systems, which can be detected with relatively high frequency up to 20%–26%, both in the open ocean and at coastal sites (Cummings *et al*., 1981; Guedes Soares, 1991; Boukhanovsky and Guedes Soares, 2009; Lucas *et al*., 2011). The probabilistic structure of the bivariate distribution of wave heights and periods in sea states with two-peaked spectra was examined by Rodriguez and Guedes Soares (1999a) by means of numerically simulated wave records, from which the asymmetric and bimodal nature of the distribution was observed in nine representative groups classified in terms of two dimensionless parameters (Guedes Soares, 1984).

Until now, the Longuet-Higgins (1983) joint distribution is still a widely accepted theoretical model in describing the distribution of wave height and associated period in a single wave system (Zhang *et al*., 2013b). However, for mixed sea state, this model is not suitable in theory considering that the carrier frequency is not the mean frequency of the bimodal spectrum any more (Guedes Soares and Nolasco, 1992). Moreover, the calculation of mean frequency of each component is also not easy and quite inaccurate if the inter-modal distance is very small. One alternative method is to use the peak frequency as the carrier wave frequency directly for each wave system.

The relationship between mean frequency and peak frequency is constant for a narrow-band theoretical wave spectrum. However in real sea states there are uncertainties in estimating the spectral parameters from the time records as discussed in Rodriguez *et al*. (1999a). The uncertainty in significant wave height is relatively small as it is a variable related to the integral of the spectrum an operation that tends to smooth the local irregularities in the spectral ordinates. However the mean period results from the first moment of the spectrum and thus it is sensitive to the variability of the high and low frequency components of the spectrum, giving it a higher variability than the peak period that is not sensitive to the changes in the high frequency tail of the spectrum, which still involves some uncertainties (Rodriguez *et al*., 1999b; Rodriguez and Guedes Soares, 1999b).

Furthermore, the choice of the peak period instead of the mean period also makes it more meaningful to deal with the individual component of double peaked spectra, which are better fitted to data using the peak period (Guedes Soares and Henriques, 1998) than the average period in the initial formulation of Guedes Soares (1984).

It is well known that Longuet-Higgins (1952) first described the wave height distribution in the context of linear theory of Gaussian noise (Rice, 1944, 1945). As reviewed by Guedes Soares (2003), numerous empirical and theoretical models of wave heights have been suggested since then (Naess, 1985; Tayfun, 1990; Boccotti, 2000). Later it is found that the second-order nonlinear correction, due to bound wave effects, has no effect on the crest-to-trough wave height (Tayfun, 1980; Tayfun and Fedele, 2007; Petrova and Guedes Soares, 2008). Recently, the third-order nonlinear wave-wave interactions between free wave modes, described quantitatively by means of the coefficient of kurtosis, are suggested for the large amplitude events and the increased probability of occurrence of abnormal waves, as shown in a series of studies on full-scale data (Guedes Soares *et al*., 2003, 2004; Petrova *et al*., 2006, 2007) and laboratory measurements (Mori, 2003; Onorato *et al*., 2006, 2009; Shemer and Sergeeva, 2009; Shemer *et al*., 2010; Cherneva *et al*., 2009, 2011). In addition, the number of waves is also an important factor in determining the extreme wave height in the presence of larger Benjamin-Feir Index (Mori and Janssen, 2006; Cherneva *et al*., 2011; Guedes Soares *et al*., 2011).

As to the statistical properties of the periods, they were first studied by Rice (1944, 1945) in connection with the level crossing problem of a random process. However, Rice's distribution has not been applied to the analysis of ocean wave periods because it seems to be more appropriate for a relatively high frequency process. Later, Bretschneider (1959) postulated that the squares of the wave periods are Rayleigh distributed and some other more advanced empirical models are also proposed on the ground of a large set of field data such as those in the work of Davidan *et al*. (1985) and Nair *et al*. (2003). Besides, it has to be mentioned that many wave period distributions actually are derived from the existing theoretical models of joint distribution of wave amplitudes and periods.

The main purpose of this paper is to propose two alternative forms of the Longuet-Higgins (1983) joint distributions which are based on replacing the carrier wave frequency with the peak frequency of the system. The original and modified models proposed in this paper are studied in different random sea states generated in the offshore basin of Marintek, Norway, strictly satisfying the deepwater and narrow-band conditions. Meanwhile, to further complement the analysis of the joint distributions, the associated marginal distributions of wave heights and periods are compared with the observed statistics as well.

The paper is organized as follows: Section 2 gives a short review and concise derivation on the theoretical models adopted in the paper. Section 3 briefly introduces the laboratory facilities and experimental plans. Section 4 is devoted to the comparison between observed results and those theoretical models in different random sea states. Section 5 summarizes the useful conclusions and casts some light on the future research of joint distribution.

2. Basic Theory

2.1 Longuet-Higgins Distribution

Consider the deep water condition and let $\eta(t)$ represent the surface elevation at a fixed point in time *t*. If $S(\omega)$ is a frequency spectrum of $\eta(t)$ and $m_j = \int_0^{\infty} \omega^j S(\omega)\mathrm{d}\omega$ defines the spectral moment, then the mean frequency $\omega_{01} = m_1/m_0$ can be obtained. For narrow-band ocean waves, the surface elevation can be represented as a carrier wave with fixed frequency ω_{01}, modulated by a complex wave envelope with amplitude ρ and phase ϕ which are real and slowly varying functions of time *t*:

$$\eta(t) = \mathrm{Re}\left\{\rho \exp(\mathrm{i}\phi)\exp\left(\mathrm{i}\omega_{01}t\right)\right\}, \tag{1}$$

Moreover, $T_{01} = 2\pi/\omega_{01}$, $\nu = \left[\left(m_0 m_2/m_1^2\right)-1\right]^{1/2}$ and $s = H_\mathrm{s}\omega_\mathrm{p}^2/(2g)$ are defined as mean wave period, wave spectral width parameter, and wave steepness, respectively, where H_s is significant wave height and ω_p means peak frequency. In accordance with the above assumption and definition, it can be concluded that the spectral energy is concentrated in a small frequency interval and the individual wave height *h* is twice the wave amplitude *a* which is equal to the magnitude ρ of the envelope function, i.e., $h = 2a = 2\rho$. The wave period can be expressed by the inverse of the rate of change of the total phase, that is, $\tau = 2\pi/\dot{\varphi}$ where $\dot{\varphi} = \omega_{01} + \dot{\phi}$.

Furthermore, the heights and periods can be normalized as follows:

$$R = h/\sqrt{8m_0}, \tag{2}$$

$$T = \tau/T_{01}. \tag{3}$$

After changing the variables of the joint probability density function $p\left(\rho,\dot{\phi}\right)$ (presented in Longuet-Higgins, 1975), a dimensionless joint distribution of wave heights and periods is given (Longuet-Higgins, 1983):

$$p\left(R,T\right) = \frac{2L\left(\nu\right)R^2}{\nu\sqrt{\pi}T^2}\exp\left\{-R^2\left[1+\frac{1}{\nu^2}\left(1-\frac{1}{T}\right)^2\right]\right\}, \tag{4}$$

where $L\left(\nu\right)$ is a normalizing factor introduced to correct the total probability to unity. After the integration of Eq. (4) over its range, the normalizing factor is presented by

$$L\left(\nu\right) = 2\sqrt{1+\nu^2}\Big/\left(1+\sqrt{1+\nu^2}\right). \tag{5}$$

The position of the mode in Eq. (4) is found from the condition that both $\partial p/\partial R$ and $\partial p/\partial T$ vanish simultaneously. This leads to

$$\begin{cases} R = 1/\sqrt{1+\nu^2} \\ T = 1/1+\nu^2 \end{cases} \tag{6}$$

The value of $p\left(R,T\right)$ at this point is therefore

$$p_{\max}=\left[2L\Big/\left(\mathrm{e}\sqrt{\pi}\right)\right]\left(1+\nu^{2}\right)\Big/\nu\ . \tag{7}$$

The marginal distributions of wave heights and wave periods are derived from the joint distribution by integrating over all values of periods and heights in the interval $\left(0,\infty\right)$, respectively.

$$p\left(R\right)=R\exp\left(-R^{2}\right)L\left(\nu\right)\left[1+\operatorname{erf}\left(R/\nu\right)\right], \tag{8}$$

$$p\left(T\right)=\frac{L\left(\nu\right)}{2\nu T^{2}}\left[1+\frac{1}{\nu^{2}}\left(1-\frac{1}{T}\right)^{2}\right]^{-3/2}, \tag{9}$$

where the error function is

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_{0}^{x}\exp\left(-t^{2}\right)\mathrm{d}t\ . \tag{10}$$

2.2 Modified Longuet-Higgins Distribution I

It is widely accepted that the ocean waves are characterized by the JONSWAP spectrum and that the narrow-band condition is strictly satisfied for the unidirectional waves mechanically generated in the laboratory. Consequently, the peak frequency ω_{p} is very close to the mean frequency ω_{01} and most energy is concentrated around the peak frequency in the spectrum. In this sense, it is reasonable to use ω_{p} work as the carrier frequency as what normally is done in the numerical wave modeling. Moreover, the mean frequency ω_{01} calculated directly from bimodal wave spectrum is not the carrier frequency any more in the mixed sea state. To derive the alternative Longuet-Higgins joint distribution, the surface elevation is now represented in a new form as shown in:

$$\eta\left(t\right)=\mathrm{Re}\left\{\rho_{1}\exp\left(\mathrm{i}\phi_{1}\right)\exp\left(\mathrm{i}\omega_{\mathrm{p}}t\right)\right\}. \tag{11}$$

Since the major derivation processes are almost the same as those of the Longuet-Higgins joint distribution, only the critical procedures are presented below. Since the carrier frequency has been changed, it is necessary to define a new kind of spectral moment to replace the original central moment.

$$\tilde{\mu}_{n}=\int_{0}^{\infty}\left(\omega-\omega_{\mathrm{p}}\right)^{n}S\left(\omega\right)\mathrm{d}\omega\ . \tag{12}$$

Thus the new first-order spectral moment is given by $\tilde{\mu}_{1}=m_{1}-\omega_{\mathrm{p}}m_{0}$. For a narrow-band spectrum, the mean frequency will approach the peak frequency with the increasing steepness and thus $\tilde{\mu}_{1}\to 0$. Consequently, the four normally distributed random variables utilized in the derivation process can still be treated as independent variables for the reason that the correlation matrix is approximately diagonal. Furthermore, it has been empirically found that the representative period parameters are interrelated, thus it is feasible to set up a relationship between the mean period and peak period for the purpose of retaining the wave spectral width parameter ν in the modified joint distribution.

$$T_{01} = \beta T_{\mathrm{p}} . \tag{13}$$

Now the wave periods will be normalized by the peak period rather than the mean period while the normalized parameter of wave heights is still the same as before. After some algebraic operations, the modified joint distribution, named as Longuet-Higgins joint distribution I, is:

$$p(R,T_1) = \frac{2\alpha C(\alpha) R^2}{\sqrt{\pi} T_1^2} \exp\left\{ -R^2 \left[1 + \alpha^2 \left(1 - \frac{1}{T_1} \right)^2 \right] \right\}, \tag{14}$$

where

$$C(\alpha) = 2\sqrt{1+\alpha^2} \Big/ \left(\sqrt{1+\alpha^2} + \alpha \right), \tag{15}$$

$$\alpha = \beta \Big/ \sqrt{(\beta-1)^2 + \nu^2} . \tag{16}$$

The modified joint distribution also depends on one parameter α which is determined by wave spectral width parameter ν and the ratio of mean period to peak period β. The new position of the mode can be found in the same way as before:

$$\begin{cases} R = \alpha \big/ \sqrt{1+\alpha^2} \\ T_1 = \alpha^2 \big/ (1+\alpha^2) \end{cases}. \tag{17}$$

And the probability of the density mode is

$$p_{\max} = \left[2C \big/ \left(\mathrm{e}\sqrt{\pi} \right) \right] (1+\alpha^2) \big/ \alpha . \tag{18}$$

Furthermore, the modified marginal distributions of wave heights and wave periods are given by

$$p(R) = R\exp(-R^2) C(\alpha) \left[1 + \mathrm{erf}(\alpha R) \right], \tag{19}$$

$$p(T_1) = \frac{\alpha C(\alpha)}{2T_1^2} \left[1 + \alpha^2 \left(1 - \frac{1}{T_1} \right)^2 \right]^{-3/2} . \tag{20}$$

2.3 Modified Longuet-Higgins Distribution II

It is obvious that some terms have been ignored in the derivation of Longuet-Higgins joint distribution I due to the assumption of $\tilde{\mu}_1 \to 0$. The merits of taking this limit will be seen in the later comparison. Now it is possible to propose a more exact joint distribution without any hypothesis. After much more tedious algebraic operations, the modified distribution, named as Longuet-Higgins joint distribution II, is obtained.

$$p(R,T_1) = \frac{2\beta L(\nu) R^2}{\nu \sqrt{\pi} T_1^2} \exp\left\{ -R^2 \left[1 + \frac{1}{\nu^2} \left(1 - \frac{\beta}{T_1} \right)^2 \right] \right\}. \tag{21}$$

The position of the mode is also found with the following expressions

$$\begin{cases} R = 1 \big/ \sqrt{1+\nu^2} \\ T_1 = \beta / 1+\nu^2 \end{cases} \tag{22}$$

And the value of $p(R,T_1)$ at this point is

$$p_{\max} = \left[2L\Big/\left(\beta e\sqrt{\pi}\right)\right]\left(1+\nu^2\right)\Big/\nu \ . \tag{23}$$

As anticipated, the marginal distribution of wave heights is still the same as that expressed in Eq. (8) and will not be repeated here. The modified marginal function of wave periods is derived in the same manner as before, leading to:

$$p\left(T_1\right) = \frac{\beta L\left(\nu\right)}{2\nu T_1^2}\left[1+\frac{1}{\nu^2}\left(1-\frac{\beta}{T_1}\right)^2\right]^{-3/2} . \tag{24}$$

It is noted that the original and modified marginal distributions of wave heights expressed in Eqs. (8) and (19) are both nearly Rayleigh distributed although they have a correction factor that serves to reduce the probability of small amplitude waves and to increase the probability of waves near the mode, shifting the mode slightly to the higher values. The correction has an exponentially small effect on the tail of the Rayleigh distribution which is of greatest interest in engineering. The higher-order correction due to third-order nonlinear interactions in relatively narrow-band and long-crested waves is indicated by the modified Edgeworth-Rayleigh (MER) distribution (Mori and Janssen, 2006):

$$p\left(R\right) = 2R\exp\left(-R^2\right)\left[1+\frac{\lambda_{40}}{6}\left(R^4-4R^2+2\right)\right], \tag{25}$$

where λ_{40} is the coefficient of kurtosis that is normally associated with the occurrence of abnormal waves. A more general third-order theoretical distribution (GC model) is proposed by Tayfun and Fedele (2007). However, the discrepancy between them is so small that it can be neglected in most cases (Cherneva *et al*., 2009).

3. Facilities and Experimental Data

The measured wave time series of this study come from a set of laboratory experiments run in the wave basin of MARINTEK with dimensions 80 m long and 50 m wide as sketched in Fig. 1. The wave surface elevations are recorded by 10 capacitance wave gauges deployed along the centerline of the basin where the water depth is 2 m. The gauge closest to the wave-maker is 10 m away. The double-flap wavemaker is installed at one of the short walls of the basin and the sloping beach at the opposite side serves to absorb the incident wave energy. The length scale of the experiments is 1:50. Thus the duration of each time series is more than 3 hours in full-scale. All initial parameters of the three typical experiments are listed in Table 1.

The initial condition generated at the wave maker is a typical random sea state described by the JONSWAP wave spectrum. It needs to be stressed that each initial surface elevation in the experiments is synthesized by the linear superposition of a large number of sinusoidal components with different sets of uniformly distributed random phases. The variance of the wavelet amplitudes is determined if the peak period and Philips parameter α which is in dependence of significant wave height, are specified. All the initial wave spectra have the same peak enhancement factor, i.e., $\gamma = 3$. More descriptions about the facility, the laboratory experiments and the associated characteristics of the produced series in the basin can be found in detail in other works (Cherneva *et al*., 2009; Cherneva and

Guedes Soares, 2011; Zhang *et al*., 2013a, 2013b).

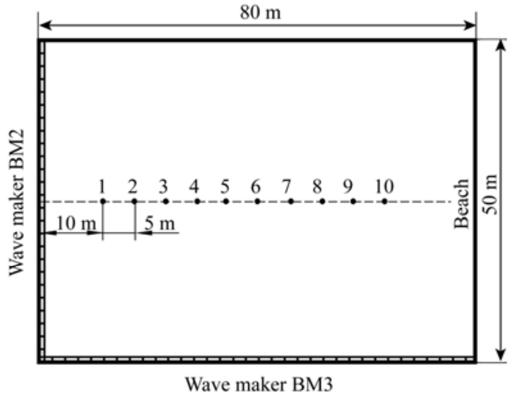

Fig. 1. Layout of the MARINTEK wave basin and gauge locations.

Table 1 Parameters of the three different experiments

No.	H_s (m)	T_p (s)	γ	ε	Symbol
8201	3.5	10.0)	3.0	0.070	Square
8202	7.0	10.0	3.0	0.141	Circle
8219	9.0	10.0	3.0	0.181	Triangle

To have sufficiently good and reliable statistics, the following observed statistics are calculated on the basis of both zero up-crossing and down-crossing waves. For the sake of clarity, different symbols will be used to identify different sea states in the following figures.

4. Comparisons of Experiment and Theory

4.1 Spatial Variations of Basic Wave Parameters

As shown in Fig. 2a, it is apparent that the wave spectral width decreases along the wave basin as a result of energy dissipation, which is mainly in the decreased density of the saturation range in the wave spectrum (Fedele *et al*., 2010). Apparently, the discrepancy in the same location is so small that the spectral width parameter cannot be used to identify the difference among these sea states. Moreover, it is evident that $\nu_{\max} < 0.6$, i.e., all the wave series generated in the laboratory are strictly narrow-band.

Compared with the variation of spectral width parameter, the disparities of steepness in different cases are so pronounced as illustrated in Fig. 2b that these laboratory experiments can be categorized into three typical groups belonging to sea states: low, moderate and severe, respectively. Furthermore, the decreasing tendency of wave steepness along the offshore basin is most strikingly evident in the severe sea state due to the serious energy dissipation in the major form of wave breaking, while in the low sea state it is obscured and thus almost maintains a constant level as the wave propagates downstream the wave basin.

With reference to Fig. 2c, it strongly supports the conclusion that the different representative wave periods are interrelated with each other, in particular for the mean wave period and peak period considering that an analogue approximately linear regression model exists in all sea states. Actually, both variables are typical such that the issue of which one being used to work as a parameter in describing the sea state has already been argued for a long time. Normally, the wave periods will decrease along the wave basin (Zhang *et al*., 2013b). The slight overall increasing tendencies in all sea states mean that the mean period approaches the peak period downstream the wave basin due to the decreased spectral width as discussed before. Furthermore, it is obvious as the sea state becomes more and more severe, mean wave period will approach peak period as well.

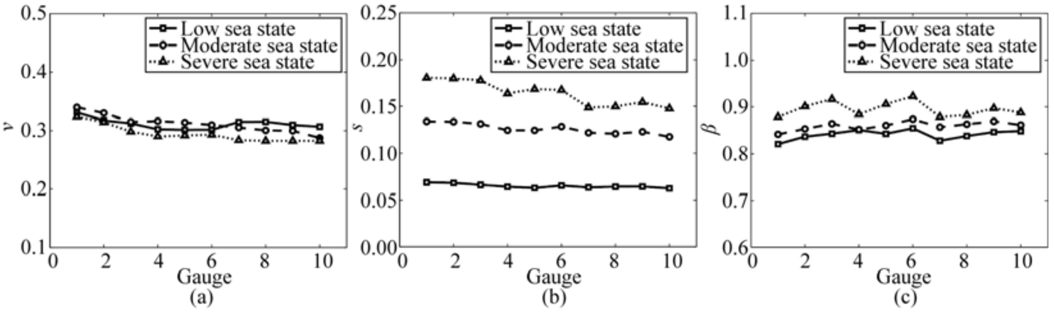

Fig. 2. Spatial variations of basic wave parameters. (a) wave spectral width, (b) wave steepness, (c) the ratio of mean period to peak period. Square, circle and triangle represent low, moderate and severe sea states, respectively.

4.2 Joint Distributions of Wave Heights and Wave Periods

For economy of space, in the following sections the comparison will only focus on the measurements at Gauge 10 where the nonlinear effect has been fully developed in all sea states.

The original Longuet-Higgins (1983) joint distributions, expressed in Eq. (4), and the other two modified theoretical joint distributions represented by Eqs. (14) and (21) are compared with the observed statistics in Fig. 3 and Fig. 4, respectively. Contours take the values (0.99, 0.9, 0.8, 0.6, 0.4, 0.2)×p_t (or p_o) from the centre contour outwards for both theoretical distribution (heavy solid line) and observed result (colored area) where p_t (or p_o) means the theoretical (or observed) value of density mode.

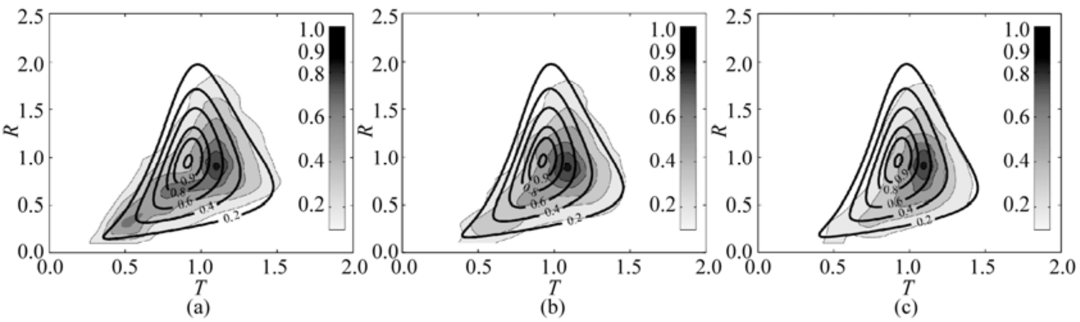

Fig. 3. Original Longuet-Higgins joint distributions versus experimental results at Gauge 10 in three typical sea states. (a), (b) and (c) correspond to three sea states listed in Table 1, respectively.

Fig. 3 clearly reveals that the original Longuet-Higgins joint distributions deviate from the experimental results in all sea states, especially in the area around the density mode. Moreover, the theoretical joint distributions do not display too much difference among different sea states for the reason that they are determined only by the wave spectral width parameters which are almost equal to each other as stated earlier in Fig. 2a. However, the observed statistics of joint distribution present a large discrepancy among these three sea states which implies that the model of Longuet-Higgins joint distribution should be further improved. But until now, only few efforts have been contributed to this subject and little improvement has been achieved due to the rapidly increased complexity when simultaneously involving two correlated random variables in one distribution function. Whatever, the Longuet-Higgins model still provides the basic description for the joint distribution of wave heights and periods, especially in the domain with lower probability such as 0.2–0.4.

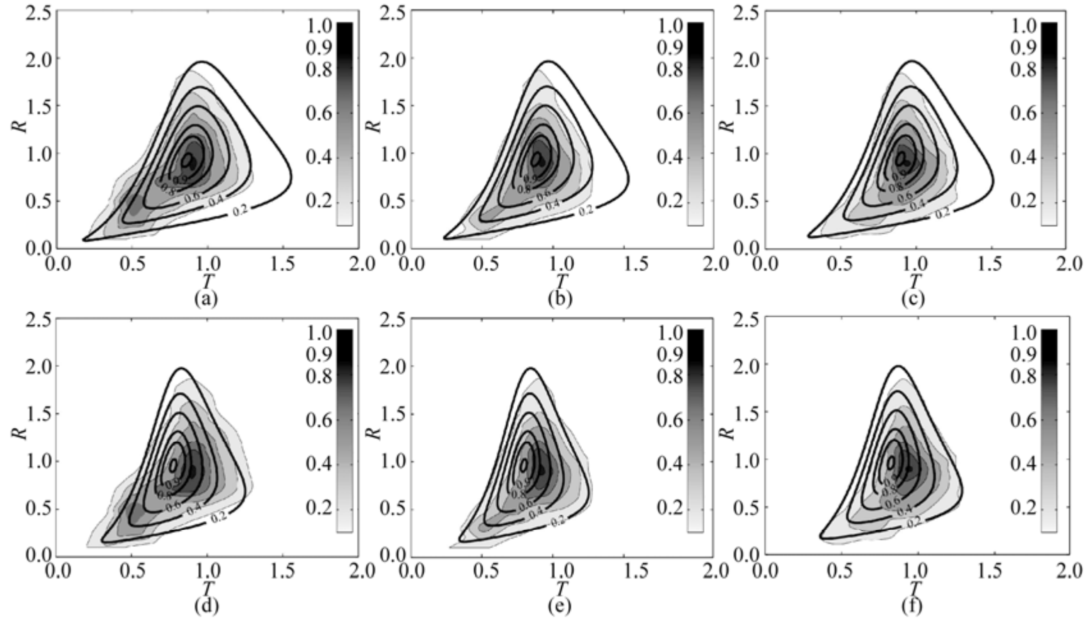

Fig. 4. Modified Longuet-Higgins joint distributions versus experimental results at Gauge 10 in three typical sea states. The three columns correspond to three sea states listed in Table 1 and the theoretical distributions (solid contour line) in the two rows are from Eqs. (14) and (21), respectively.

Apparently, in the upper panel of Fig. 4, i.e., (a)–(c), the modified Longuet-Higgins joint distribution I expressed by Eq. (14) almost catches the location of the density mode under all circumstances even although the area of the contour is further enlarged compared to the observed result. Thus it is reasonable to believe that this simplified theoretical model is able to predict the most frequent combination of wave period and wave height in a time series. Meanwhile, in the lower panel of Fig. 4, i.e., (d)–(f), it can be concluded that the modified Longuet-Higgins joint distribution II represented by Eq. (21) is actually an exact alternative theoretical model of the original Longuet-Higgins joint distribution for the reason that the deviations of the location of density mode are slightly decreased but still significant. Anyway, the alternative form of Longuet-Higgins joint distribution which can be potentially applied in mixed sea states has been formulated correctly herein.

Moreover, in the frame of linear theory it seems impossible to eliminate the difference between theoretical models and empirical distributions. In other words, it is the nonlinearity of the wave series that mainly leads to these disagreements which are clearly indicated on the domain around the density mode.

On the other hand, the conclusions drawn above can be confirmed by all recently derived theoretical models, considering that they are all based on the same linear assumption associated with Longuet-Higgins joint distribution, but there never exists a perfect model applicable to all sea states or in other words, those improvements are confined to some aspects to a certain degree (Stansell *et al.*, 2004; Xu *et al.*, 2004). In order to further explore the defects existing in the linear model of joint distribution, it is better to concentrate on the distribution of one single variable, that is, to analyze the marginal density functions of wave heights and wave periods, respectively.

4.3 Marginal Distribution of Wave Heights

As shown in Fig. 5, various probability density functions of wave heights are exhibited and compared with the observed statistics at Gauge 10, where the theoretical distributions expressed by Eqs. (8), (19) and (25) are represented by light solid line (LH), dot dash line (LH1) and heavy dash line (MER), respectively. Note that the light solid line actually denotes the marginal distribution from modified Longuet-Higgins joint distribution II as well.

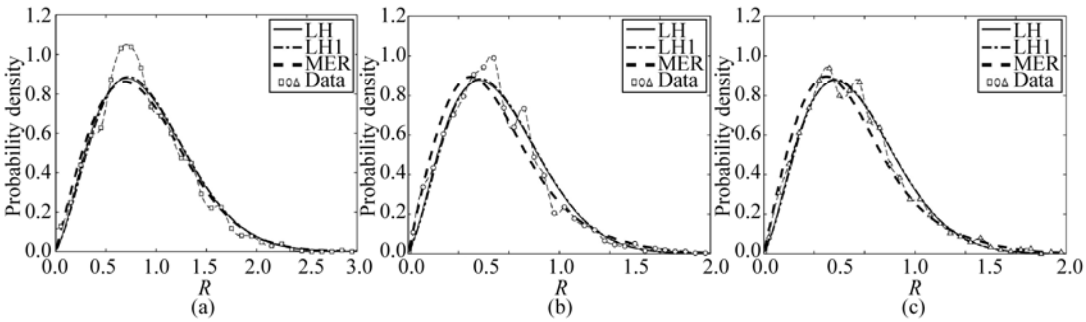

Fig. 5. Theoretical probability distributions of wave heights versus observed statistics at Gauge 10 in different sea states.

In the low sea state, e.g., Fig. 5a, the three theoretical distributions are almost coincident with each other and quasi-Rayleigh distributed due to the negligible nonlinear effect. That is to say, all theoretical models can describe most observed distribution of wave heights perfectly except for the small part around the density mode which can be attributed to the variability and uncertainty in statistics considering that only one short wave series is available analyzed in this work.

In the moderate and severe sea states, e.g., Fig. 5b and Fig. 5c, the original and modified probability density functions of wave heights (LH and LH1) obtained by integrating the joint distributions still keep an agreement with each other while the third-order nonlinear model (MER) exhibits a movement to the left as a result of containing nonlinear effect. It is identified that the modified linear models (LH and LH1) can fit the middle section close to the density mode reasonably well while the nonlinear MER model is more suitable to predict the head and the tail of the wave height distributions. As a matter of fact, part of these observations are strongly supported by those conclusions obtained in the exceedance distributions of wave heights described in many other studies (Tayfun and Fedele, 2007; Cherneva *et al*., 2009). Hence, the errors induced by the distribution of wave heights can be neglected in the theoretical joint distributions since the discrepancies between the modified linear models (LH and LH1) and the experimental results are not so large.

4.4 Marginal Distribution of Wave Periods

The study on wave period distribution is much limited because it is really difficult to find a universal distribution law such as the Rayleigh distribution in the case of wave heights in all sea states. In Fig. 6, the observed statistics of wave periods at Gauge 10 are compared with the original and modified probability density functions, respectively.

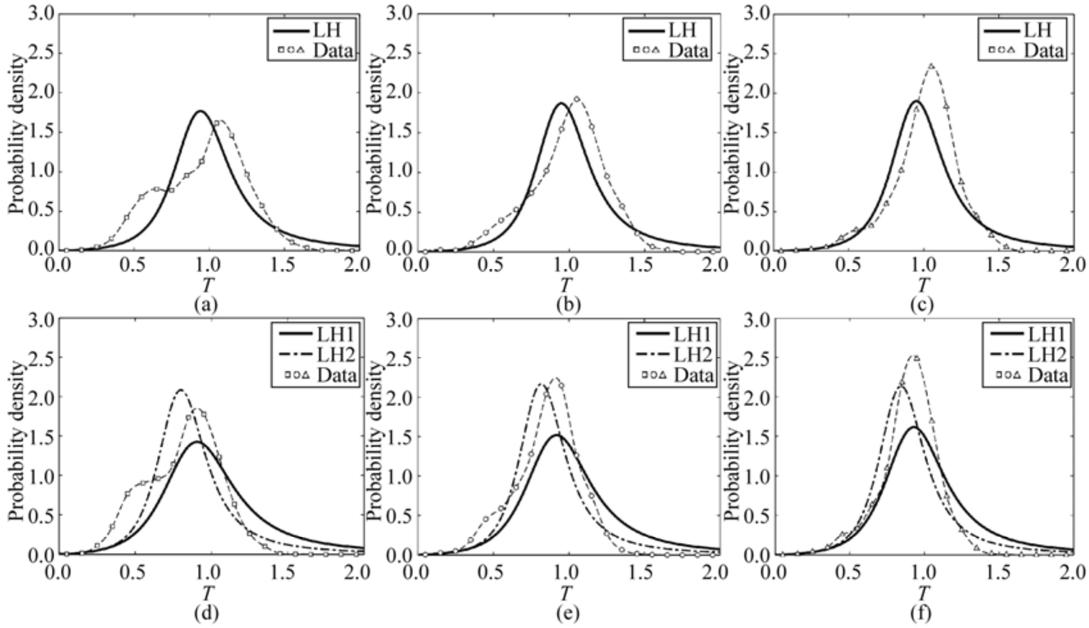

Fig. 6. Theoretical probability density functions of wave periods versus experimental results at Gauge 10 in different sea states. The three columns correspond to three typical sea states listed in Table 1. The theoretical model on the upper panel is the original distribution while the modified ones are plotted on the lower panel.

Firstly, focusing on the experimental results presented on the upper panel, it is visible that the periods of individual waves in a wave train exhibit a distribution narrower than that of wave heights, and the spread lies mainly in the range of 0.5 to 1.5 times the mean wave period which is a little narrower than the scope proposed by Goda (2000). Moreover, as the sea states become more and more severe (from left to right), the observed wave period distribution will become much steeper and narrower. It is also noticeable that the empirical probability distribution generates a hump in the interval (0.5, 1) in the low sea state and the reason is still not clear today. What needs to be pointed out is that the same phenomenon is also observed in the laboratory experiments carried out in the offshore basin of CEHIPAR, Spain (Zhang *et al*., 2013b). Obviously, the original linear theoretical model (LH), represented by Eq. (9), gives a poor description on the experimental results almost in all sea states.

Secondly, moving attention to the lower panel, the first modified theoretical model (LH1), represented by Eq. (20), can capture the locations of the density mode in all sea states although it is not narrow enough compared with the observed statistics. The second modified theoretical model (LH2), expressed by Eq. (24), does not show too much improvement comparing to the original theoretical predictions on the upper panel.

Now it is clear that the main problem in the theoretical joint distribution of wave heights and periods arises from the poor description of wave period distribution although there is a small deviation in the wave height distribution as well. Additionally, the wave period is strongly sensitive to the initial spectral shape (Massel, 1996) and thus displays a totally different distribution in different sea states, especially in the case with small initial wave steepness. It also implies that the reasons for the discrepancy are totally different between the low sea state and the other two more severe sea states. Although the new forms of wave period distributions cannot solve the problem completely, they highlight the right direction to improve the joint distribution.

5. Conclusions

In this paper, modified versions of Longuet-Higgins joint distribution of wave heights and periods have been proposed successfully by replacing the mean frequency with the peak frequency of the wave system. Comparison was made with three typical experiments carried out in the offshore basin of MARINTEK with different initial wave steepness, allowing a discussion of the validity of the original and modified Longuet-Higgins joint distributions and thus casting some light on how to further extend this kind of theoretical model to nonlinear wave time series.

The location of the theoretical density mode in the original Longuet-Higgins joint distribution deviates from the experimental observation to various degrees in all sea states. Inspection of Eq. (4) demonstrates that the original theoretical joint distribution cannot give too many discrepancies among different sea states since it is only determined by the wave spectral width parameter, which is almost invariant in all sea states.

The related marginal distributions of wave heights are approximately Rayleigh distributed and their deviations from the observed statistics can be neglected compared with the errors induced by the marginal distributions of wave periods. The observed wave period distributions exhibit such a large variation in different sea states that they cannot be fitted well by all existed linear theoretical models.

Finally, it has to be admitted that the modified linear theoretical models proposed in this paper cannot significantly improve the joint distribution, but are alternative formulations that can be the object of further improvement. The first modified theoretical joint distribution can catch the location of the mode of the distribution in all sea states although the contour lines are enlarged compared with the observed statistics. The second modified theoretical model is essentially an alternative form of Longuet-Higgins joint distribution which could not fundamentally eliminate the disparities with experiments but is theoretically possible to be used to model the joint distribution of each component in a mixed sea state.

It is evident that the wave period distribution is strongly sensitive to the initial sea state and varies largely in different cases. Thus the critical point is definitely on the distribution of wave periods which needs to be further improved.

Acknowledgements – This work has been developed in the scope of the project "EXTREME SEAS – Design for Ship Safety in Extreme Seas", which is funded by the European Union under the grant agreement No. 234175. The first author also acknowledged the financial support provided by Portuguese Foundation for Science and Technology under the grant SFRH/BD/98983/2013. The data were obtained at MARINTEK, Trondheim in 1999 as a part of the project "Interactions between Waves and Currents", partially funded by the European Union under contract ERBFMGECT980135 for access to Large Scale Facilities.

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