Analysis and Mathematical Physics (2024) 14:107
https://doi.org/10.1007/s13324-024-00967-2

A note on closed quasi-Einstein manifolds

Wagner Oliveira Costa-Filho[1]

Received: 22 June 2024 / Revised: 4 September 2024 / Accepted: 5 September 2024

Abstract

The notion of m-quasi-Einstein manifolds originates from the study of Einstein warped product metrics and they are influential in constructing for many physical models. For example, these manifolds arises for extremal isolated horizons in the theory of black holes. In a recent work by Cochran (arXiv:2404.17090v1, 2024), the author studied Killing vector fields on closed m-quasi-Einstein manifolds. In this short paper, we will give another proof of his main result involving the scalar curvature, which holds for all values of m and is based on the use of known formulae related to quasi-Einstein metrics.

Keywords m-Quasi-Einstein manifolds · Killing vector fields · Scalar curvature · Closed manifolds

Mathematics Subject Classification Primary 53C25; Secondary 53C21 · 53C65

1 Introduction

Because of their relevance in both geometric analysis and general relativity, the study of Riemannian (or semi-Riemannian) manifolds admitting an Einstein-like structure is an attractive subject in modern Mathematical Physics and Differential Geometry. Among the enormous literature on the topic, we indicate Deshmukh and Al-Sodais in [9], Andrade and de Melo in [1], Wylie in [15] and Araújo, Freitas and Santos in [2].

In this sense, an n-dimensional Riemannian manifold (M^n, g), $n \geq 2$, is called an m-quasi-Einstein manifold, for a non-null constant $m \in \mathbb{R}$, if there exist a smooth vector field X on M and a scalar $\lambda \in \mathbb{R}$ satisfying the following equation

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g - \frac{1}{m}X^\flat \otimes X^\flat = \lambda g, \tag{1.1}$$

✉ Wagner Oliveira Costa-Filho
wagner.filho@arapiraca.ufal.br

1 Campus Arapiraca, Federal University of Alagoas, Alagoas CEP 57309-005, Brazil

Published online: 17 September 2024

where Ric, $\mathcal{L}_X g$, $X^\flat$ and $\otimes$ stand for the Ricci tensor, the Lie derivative of the metric g in the direction of X, the dual 1-form to X associated the metric g and the tensorial product, respectively. We notice that, when X vanishes identically M^n is an Einstein manifold.

These manifolds play an essential role in investigating the solutions of the Einstein field equations. Naturally, when m is a positive integer the m-quasi-Einstein manifolds correspond to some warped product Einstein metrics. See, for instance, Barros and Ribeiro in [5].

In this branch, a fundamental application of the called smooth metric measure spaces is as a description of quasi-Einstein manifolds (see Case in [7]). It is worth pointing out that, commonly, the study of m-quasi-Einstein manifolds is considered when X is a gradient of a smooth function on M^n (see Ribeiro and Tenenblat in [14]). Furthermore, non-gradient m-quasi-Einstein metrics has been playing an important role in the theory of near-horizon geometries and can be taken in the formulation of perfect fluid spacetimes. Moreover, when m goes to infinity, this structure reduces to the one associated to a Ricci soliton. For more details we refer to Bahuaud, Gunasekaran, Kunduri and Woolgar in [3] and references therein. See also Poddar, Sharma and Subramanian in [12].

In this setting, we quote to Bahuaud, Gunasekaran, Kunduri and Woolgar in [4], Güler and De in [11] and Ghosh in [10] for some examples of m-quasi-Einstein manifolds and related facts.

In what follows, S denotes the scalar curvature of (M^n, g), defined as the trace of Ric. Recall that a smooth vector field X defined on a Riemannian manifold (M^n, g) is said to be a Killing vector field if $\mathcal{L}_X g = 0$. Killing vector fields have a number of notable features. Besides, on the impact in the geometry as well as in the topology of a spacetime, there is a rich variety of questions where Killing vector fields play a central role. Equally, the assumption of admitting a Killing vector field on a compact manifold has been used in the literature from several points of view.

In this direction, very recently Cochran in [8] established, among other properties, the following interesting result.

Theorem 1.1 *Let (M^n, g) be a closed m-quasi-Einstein manifold. Then,*

a) S is constant if and only if X is a Killing vector field.
b) If X is divergence-free, then X is a Killing vector field.

Our purpose in this note is to give an alternative and unified proof of the above fact, making use of a classical integral formula involving the Lie derivative and well-known identities referring to m-quasi-Einstein metrics. The new approach is valid for all $m \neq 0$, in contrast to this previous proof which also assumed the constant $m \neq -2$.

As a consequence of our discussion, we also reobtain a triviality result for m-quasi-Einstein manifolds due to Bahuaud et al. in [4].

2 Proof of theorem

In this section we present the proof of theorem 1.1. To begin with, we recall some useful auxiliary results. All Riemannian manifolds (M^n, g) are assumed to be connected and oriented. The Lie derivative of the metric g in the direction of Y reads as

$$(\mathcal{L}_Y g)(U, V) = \langle \nabla_U Y, V\rangle + \langle U, \nabla_V Y\rangle,$$

for all smooth vector fields Y, U and V on M^n, where ∇ stands for the Levi-Civita connection of g. Taking trace in above equality, we see that $tr(\mathcal{L}_Y g) = 2\,\mathrm{div}Y$, where div denotes the divergence of a vector field. In general, for an arbitrary smooth vector field Y on M^n one has

$$\mathrm{div}(Y^\flat \otimes Y^\flat) = (\mathrm{div}Y)Y^\flat + (\nabla_Y Y)^\flat. \tag{2.1}$$

We also find that

$$\mathrm{div}(|Y|^2 Y) = |Y|^2\mathrm{div}Y + 2\langle \nabla_Y Y, Y\rangle. \tag{2.2}$$

Similarly, we have

$$\mathrm{div}((\mathrm{div}Y)Y) = (\mathrm{div}Y)^2 + \langle \nabla(\mathrm{div}Y), Y\rangle. \tag{2.3}$$

On the other hand, it can be proved (see Poor in [13], page 170) that for each vector field Y on a closed Riemannian manifold (M^n, g) it holds that

$$\frac{1}{2}\int_M |\mathcal{L}_Y g|^2\, dM = \int_M (|\nabla Y|^2 + (\mathrm{div}Y)^2 - \mathrm{Ric}(Y, Y))\, dM. \tag{2.4}$$

Next, notice that the trace of the Eq. (1.1) provide us

$$S + \mathrm{div}X = \frac{1}{m}|X|^2 + n\lambda. \tag{2.5}$$

At this point, we remember the following key identity obtained by Barros and Ribeiro, which can be found in [5], page 215. Below, Δ denotes the Laplacian operator on M^n.

Lemma 2.1 *Let (M^n, g) be a m-quasi-Einstein manifold. Then,*

$$\frac{1}{2}\Delta|X|^2 = |\nabla X|^2 + \frac{2}{m}|X|^2 div\, X - Ric(X, X).$$

As an application of Lemma 2.1 and for the sake of completeness, we would like to present a proof of Proposition 2.4 in [4], page 8, by following the argument used in that work. It is worth mentioning that essentially the authors in [4] also proved the above Lemma. We state the following result:

Proposition 2.2 *Let (M^n, g) be a closed m-quasi-Einstein manifold with divergence-free vector field X. If $m < 0$ and $\lambda \leq 0$, then X is identically zero.*

Proof According to (1.1),

$$\mathrm{Ric}(X, X) + \langle \nabla_X X, X\rangle = \frac{1}{m}|X|^4 + \lambda |X|^2.$$

Setting $\mathrm{div}X = 0$ in (2.2), by the divergence theorem we have

$$\int_M \langle \nabla_X X, X\rangle \, dM = 0.$$

Hence, Lemma 2.1 implies that

$$\int_M (|\nabla X|^2 - \frac{1}{m}|X|^4 - \lambda |X|^2) \, dM = 0,$$

which gives, under the assumptions, $X = 0$.

Alternatively, we may invoke the second item from Theorem 1.1 and apply the maximum principle for the Laplacian to deduce that $|X|^2$ is constant. Again by Lemma 2.1, we verify that M^n is an Einstein manifold. □

In order to prove the main result, we also need the following nice formula due to Barros and Gomes in [6], page 244. Namely,

Lemma 2.3 *Let (M^n, g) be a m-quasi-Einstein manifold. Then, for all vector field Y on M^n, we have*

$$div\,((\mathcal{L}_X g)Y) = 2div\left(\lambda Y + \frac{1}{m}(X^\flat \otimes X^\flat)Y\right) - g(\nabla S, Y) - 2\langle \nabla Y, Ric\rangle.$$

Now, we are in position to procced with the proof of the theorem.

Proof Initially, suppose that S is constant. We have from (2.5) that

$$\nabla(\mathrm{div}X) = \frac{1}{m}\nabla|X|^2.$$

Thus, taking into account the expression (2.3) we get that

$$\mathrm{div}((\mathrm{div}X)X) = (\mathrm{div}X)^2 + \frac{2}{m}\langle \nabla_X X, X\rangle.$$

From (2.2) and the above equality, applying the divergence theorem we obtain

$$\int_M |X|^2 \mathrm{div}X \, dM = m\int_M (\mathrm{div}X)^2 \, dM. \tag{2.6}$$

Note that, from identity (2.4), Lemma 2.1 and by the divergence theorem, we have

$$\frac{1}{2}\int_M |\mathcal{L}_X g|^2 \, dM = -\frac{2}{m}\int_M |X|^2 \mathrm{div}X \, dM + \int_M (\mathrm{div}X)^2 \, dM.$$

Plugging the relation (2.6) in the above equality we derive that

$$\frac{1}{2}\int_M |\mathcal{L}_X g|^2 \, dM = -\int_M (\mathrm{div}X)^2 \, dM \leq 0.$$

Therefore, we deduce that $\mathcal{L}_X g = 0$.

Reciprocally, let us suppose that X is a Killing vector field on M^n. In this case, the fundamental Eq. (1.1) becomes

$$\mathrm{Ric} = \lambda g - J_m, \tag{2.7}$$

where $J_m = -\frac{1}{m} X^\flat \otimes X^\flat$. Next, choosing $Y = \nabla S$ and using again the divergence theorem, we are able to use Lemma 2.3 and identity (2.7) to obtain that

$$\int_M |\nabla S|^2 \, dM = -2\int_M \langle \nabla^2 S, \mathrm{Ric}\rangle \, dM = 2\int_M \langle \nabla^2 S, J_m\rangle \, dM. \tag{2.8}$$

Here, ∇^2 stands for the Hessian operator on M^n. Now, on the one hand, we have that

$$\mathrm{div}(J_m(\nabla S)) = (\mathrm{div} J_m)(\nabla S) + \langle \nabla^2 S, J_m\rangle. \tag{2.9}$$

On the other hand, according to formula (2.1) and $\mathrm{div}X = 0$, from (2.9) we infer that

$$\mathrm{div}(J_m(\nabla S)) = -\frac{1}{m}\langle \nabla_X X, \nabla S\rangle + \langle \nabla^2 S, J_m\rangle. \tag{2.10}$$

Thus, by integrating in (2.10) and using that X is a Killing field we have

$$\int_M \langle \nabla^2 S, J_m\rangle \, dM = \frac{1}{m}\int_M \langle \nabla_X X, \nabla S\rangle \, dM = \frac{1}{2m}\int_M S\Delta |X|^2 \, dM.$$

Therefore, from (2.8) and (2.5) we get that

$$\int_M |\nabla S|^2 \, dM = \frac{1}{m^2}\int_M |X|^2 \Delta |X|^2 \, dM = -\frac{1}{m^2}\int_M |\nabla |X|^2|^2 \, dM \leq 0.$$

Hence, (M^n, g) has constant scalar curvature (implying that $|X|$ also is constant), and we finish the proof of the first item.

The second item is immediate. Indeed, supposing $\mathrm{div}X = 0$, we make use of Lemma 2.1 to conclude

$$\frac{1}{2}\Delta |X|^2 = |\nabla X|^2 - \mathrm{Ric}(X, X).$$

Finally, on integrating this identity and comparing with the formula (2.4) we can write

$$\int_M |\mathcal{L}_X g|^2 \, dM = 0.$$

So, X is a Killing vector field, completing the proof of the theorem. ◻

Author Contributions I'm the only author.

Funding This manuscript has no funding support.

Data availability Data sharing was not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The author states that there is no conflict of interest.

References

1. Andrade, M., de Melo, A.P.: Some characterizations of compact Einstein-type manifolds. Lett. Math. Phys. **114**(1), 35 (2024)
2. Araújo, M., Freitas, A., Santos, M.: About an integral inequality and rigidity of m-quasi-Einstein manifolds. Lett. Math. Phys. **113**(6), 115 (2023)
3. Bahuaud, E., Gunasekaran, S., Kunduri, H.K., Woolgar, E.: Static near-horizon geometries and rigidity of quasi-Einstein manifolds. Lett. Math. Phys. **112**(6), 116 (2022)
4. Bahuaud, E., Gunasekaran, S., Kunduri, H.K., Woolgar, E.: Rigidity of quasi-Einstein metrics: the incompressible case. Lett. Math. Phys. **114**(1), 8 (2024)
5. Barros, A., Ribeiro, E.: Integral formulae on quasi-Einstein manifolds and applications. Glasg. Math. J. **54**(1), 213–223 (2012)
6. Barros, A.A., Gomes, J.N.V.: Triviality of compact m-quasi-Einstein manifolds. Result. Math. **71**(1–2), 241–250 (2017)
7. Case, J.S.: Smooth metric measure spaces and quasi-Einstein metrics. Int. J. Math. **23**(10), 1250110 (2012)
8. Cochran, E.: *Killing fields on compact m-quasi-Einstein manifolds*, arXiv preprint arXiv:2404.17090 (2024)
9. Deshmukh, S., Al-Sodais, H.: A note on almost Ricci solitons. Anal. Math. Phys. **10**(4), 76 (2020)
10. Ghosh, A.: m-quasi-Einstein metrics satisfying certain conditions on the potential vector field. Mediterr. J. Math. **17**(4), 115 (2020)
11. Güler, S., De, U.C.: Generalized quasi-Einstein metrics and applications on generalized Robertson-Walker spacetimes. J. Math. Phys. **63**(8), 083501 (2022)
12. Poddar, R., Sharma, R., Subramanian, B.: Certain classification results on m-quasi Einstein manifolds. Result. Math. **78**(5), 197 (2023)
13. Poor, W.A.: *Differential geometric structures*. McGraw-Hill Book Company, New York etc. XIII, 338 p. DM 102.13 (1981)
14. Ribeiro, E., Jr., Tenenblat, K.: Noncompact quasi-Einstein manifolds conformal to a Euclidean space. Math. Nachr. **294**(1), 132–144 (2021)
15. Wylie, W.: Rigidity of compact static near-horizon geometries with negative cosmological constant. Lett. Math. Phys. **113**(2), 29 (2023)