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Two-component integrable extension of general heavenly equation

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Abstract
We introduce an integrable two-component extension of the general heavenly equation and prove that the solutions of this extension are in one-to-one correspondence with 4-dimensional hyper-para-Hermitian metrics. Furthermore, we demonstrate that if the metrics in question are hyper-para-Kähler, then our system reduces to the general heavenly equation. We also present an infinite hierarchy of nonlocal symmetries, as well as a recursion operator, for the system under study.

1 Introduction

An integrable partial differential equation in four independent variables discovered by Schief in [40],

$$\begin{aligned}(\lambda_2-\lambda_4)(\lambda_1-\lambda_3)w_{13}w_{24}-(\lambda_3-\lambda_4)(\lambda_1-\lambda_2)w_{12}w_{34}\\ -(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)w_{14}w_{23}=0,\end{aligned} \tag{1}$$

that became known as the general heavenly equation [8], serves as a natural framework for anti-self-dual gravity and has a number of important applications in physics and geometry [8, 18, 28, 29, 39]. Here and below λ_i are arbitrary pairwise distinct real constants; as usual, all functions and other objects are henceforth assumed sufficiently smooth for computations to make sense, and for any $h = h(x^1, \dots, x^4)$ we write $h_i = \partial h/\partial x^i$ and $h_{ij} = \partial^2 h/\partial x^i \partial x^j$.

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In the present paper, we introduce and study a two-component integrable extension of (1),

$$\begin{aligned}
&(\lambda_3-\lambda_4)(\lambda_1-\lambda_2)v_1u_2u_{34}+(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)(v_1u_4u_{23}-v_3u_4u_{12}+v_3u_2u_{14})\\
&\quad-(\lambda_2-\lambda_4)(\lambda_1-\lambda_3)(v_1u_3u_{24}-v_4u_3u_{12}+v_4u_2u_{13})=0,\\
&(\lambda_3-\lambda_4)(\lambda_1-\lambda_2)v_1u_2v_{34}+(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)(v_1u_4v_{23}-v_3u_4v_{12}+v_3u_2v_{14})\\
&\quad-(\lambda_2-\lambda_4)(\lambda_1-\lambda_3)(v_1u_3v_{24}-v_4u_3v_{12}+v_4u_2v_{13})=0.
\end{aligned} \tag{2}$$

In particular, we show that (2) admits a Lax pair $L_i(\lambda)\psi=0$, $i=0,1$ with the Lax operators

$$\begin{aligned}
L_0(\lambda)&=(\lambda_2-\lambda_4)(\lambda_1-\lambda)u_2v_4\partial_{x^1}-(\lambda_1-\lambda_4)(\lambda_2-\lambda)u_4v_1\partial_{x^2}\\
&\quad+(\lambda_1-\lambda_2)(\lambda_4-\lambda)u_2v_1\partial_{x^4}\\
L_1(\lambda)&=(\lambda_2-\lambda_3)(\lambda_1-\lambda)u_2v_3\partial_{x^1}-(\lambda_1-\lambda_3)(\lambda_2-\lambda)u_3v_1\partial_{x^2}\\
&\quad+(\lambda_1-\lambda_2)(\lambda_3-\lambda)u_2v_1\partial_{x^3},
\end{aligned} \tag{3}$$

where λ is the spectral parameter. We also present a recursion operator and an infinite hierarchy of nonlocal symmetries for (2).

System (2) extends (1) in the following sense: if w satisfies the general heavenly Eq. (1) then

$$u=w_1,\quad v=w_2 \tag{4}$$

satisfy (2), and the Lax operators (3) reduce under (4) to those for (1), so (1) can be seen as a reduction of (2).

We show that (2) admits a nice geometric interpretation: solutions of (2) describe hyper-para-Hermitian metrics on 4-manifolds, and locally all such metrics are associated to solutions of (2); see Sect. 2 below for further details. Under the reduction (4) hyper-para-Hermitian metrics associated with (2) become hyper-para-Kähler, as shown in Theorem 2 below.

The system (2) admits also a remarkable reduction beyond (1): if either u or v is proportional to x^1 then (2) reduces to a well-studied integrable PDE in three independent variables, known as the dispersionless Hirota equation [11, 25, 27, 32], the ABC equation [21, 34, 43], or the Veronese web equation [3, 20]. Consequently, to any solution of the said PDE there corresponds a solution of (2) and therefore a hyper-para-Hermitian metric; see Sect. 2 for details.

Note that system (2) is homogeneous which makes it easy to look for algebraic solutions thereof that are zero sets of polynomials in dependent and independent variables, cf. [29] where a similar feature is discussed at length for (1).

Let us also point out that both equations of (2) have the same symbol.

The rest of the paper is organized as follows: Sect. 2 provides background on hyper-para-Hermitian structures and lists our main results which are then proved in Sect. 3. Finally, in Sect. 4, we present an infinite hierarchy of nonlocal symmetries, and a recursion operator for (2).

2 Geometric structures and main results

By definition [16, Section 6.1], see also [6, 7], hyper-para-Hermitian metrics admit a triple of anti-commuting vector bundle endomorphisms $I, J, K : TM \to TM$, where M is a smooth 4-manifold under study and TM its tangent bundle, such that J is a complex structure, I and K are para-complex structures,

$$IK = -KI = J$$

and

$$g(X, Y) = -g(JX, JY) = g(IX, IY) = g(KX, KY),$$

for any $X, Y \in T_x M$, $x \in M$. Recall that hyper-para-Hermitian stuructures are also known as pseudo-hyperhermitian in [5, 12] or hyper-Hermitian of neutral signature [26].

The underlying algebra spanned by (I, J, K) is often referred to as split-quaternions [16] or para-quaternions [5] one, and it can be easily shown that the triple itself uniquely determines the conformal class $[g]$, see [6] and [26] for higher-dimensional generalizations. The hyper-para-Hermitian structures are necessarily anti-self-dual and are characterized in the class of anti-self-dual metrics by the condition that the corresponding twistor space, defined as the space of α-surfaces, fibers over a projective line [12].

In this paper we work in a local coordinate system (x^1, x^2, x^3, x^4) on M. Our key result is the following theorem giving a characterization of hyper-para-Hermitian metrics on 4-manifolds.

Theorem 1 *The conformal class of any hyper-para-Hermitian metric on a 4-dimensional manifold contains a representative that can be locally put in the form*

$$g = \omega^1 \odot \omega^4 - \omega^2 \odot \omega^3$$

where $\odot$ denotes the symmetric tensor product and 1-forms ω^i, $i = 1, \ldots, 4$ are defined as

$$\begin{aligned}
\omega^1 &= dx^4 - \lambda_4 \left(\frac{1}{\lambda_2 - \lambda_4} \frac{v_1}{v_4} dx^1 + \frac{1}{\lambda_1 - \lambda_4} \frac{u_2}{u_4} dx^2 \right) \\
\omega^2 &= \frac{1}{\lambda_4} dx^4 - \frac{1}{\lambda_4} \left(\frac{\lambda_2}{\lambda_2 - \lambda_4} \frac{v_1}{v_4} dx^1 + \frac{\lambda_1}{\lambda_1 - \lambda_4} \frac{u_2}{u_4} dx^2 \right) \\
\omega^3 &= dx^3 - \lambda_3 \left(\frac{1}{\lambda_2 - \lambda_3} \frac{v_1}{v_3} dx^1 + \frac{1}{\lambda_1 - \lambda_3} \frac{u_2}{u_3} dx^2 \right) \\
\omega^4 &= \frac{1}{\lambda_3} dx^3 - \frac{1}{\lambda_3} \left(\frac{\lambda_2}{\lambda_2 - \lambda_3} \frac{v_1}{v_3} dx^1 + \frac{\lambda_1}{\lambda_1 - \lambda_3} \frac{u_2}{u_3} dx^2 \right)
\end{aligned}$$

and functions u and v satisfy system (2) *which is integrable and admits an isospectral Lax pair with the Lax operators* (3).

Remark 1 Notice that for some applications it could be convenient to use a slightly different Lax pair $X_i(\psi) = 0$, $i = 0, 1$ with the Lax operators $X_i(\lambda) = ((\lambda_1 - \lambda_2)u_2v_1)^{-1}L_i(\lambda)$, $i = 0, 1$; the explicit form of X_i can be found in (7), cf. the relevant discussion in the next section at the end of proof of Theorem 1.

Alternative descriptions of hyper-para-Hermitian structures through integrable systems have been studied in [9, 12, 26], cf. also e.g. [3, 4, 13, 15] and references therein for integrable systems in connection with hypercomplex and Einstein–Weyl structures and (anti)self-duality. Let us also mention here that certain descriptions of hyper-Hermitian structures using a system of two-second order PDEs can be found in [14], where the structures under study were referred to as weak heavenly spaces and in [10], where the said structures were treated from the perspective of reduction of a partial differential system governing self-dual conformal structures.

Notice that both equations in (2) share the same symbol, which, in turn, recovers the conformal class by means of the characteristic variety coinciding with the null cone of $[g]$.

The relation of system (2) to the general heavenly Eq. (1) is explained in the following result.

Theorem 2 *If w satisfies the general heavenly Eq.* (1) *then u and v given by* (4)*, i.e., $u = w_1$ and $v = w_2$, satisfy* (2).

Moreover, a hyper-para-Hermitian metric is hyper-para-Kähler if and only if its conformal class contains a representative that can be locally put in the form from Theorem 1 with (u, v) *given by* (4) *for some function w satisfying* (1).

The hyper-para-Kähler structures appearing in Theorem 2 can be introduced as Ricci-flat hyper-para-Hermitian structures (see [12, 33]). According to the Mason–Newman formalism [33], they are described by an integrable system with a Lax pair whose Lax operators are vector fields that are divergence-free with respect to some volume form. One realization of this formalism is provided by the general heavenly Eq. (1) (we refer to [18, 29, 40] for details). Other approaches include the Plebański equations and the Husain–Park equation, among others, cf. [18, 39].

Remark 2 The second assertion of Theorem 2 can be expressed in terms of a curvature. Namely, in order for a hyper-para-Hermitian structure to be hyper-para-Kähler the Obata connection associated with the former has to be Ricci-flat. Recall that the Obata connection is the unique connection ∇ such that $\nabla I = \nabla J = \nabla K = 0$. It was initially introduced within the framework of hyper-Hermitian structures [36], but its definition extends seamlessly to the neutral signature, see [1]. In this context it coincides with the Chern connection of webs [35] (the webs provide a convenient viewpoint on the structures under study as explained in the next section).

Explicitly, the connection in question is defined by the following expression (see also [26, formula (2.1)])

$$\begin{aligned}\nabla_X Y = \pi_H(j[\pi_H(X), j\pi_H(Y)] + [\pi_V(X), \pi_H(Y)])\\ +\pi_V(j[\pi_V(X), j\pi_V(Y)] + [\pi_H(X), \pi_V(Y)])\end{aligned}$$

where $[\cdot, \cdot]$ is the Lie bracket of vector fields, and π_V and π_H are projections to the factors of the decomposition $TM = V \oplus H$, where

$$V = \operatorname{span}\{L_0(\lambda_1), L_1(\lambda_1)\}, \quad H = \operatorname{span}\{L_0(\lambda_2), L_1(\lambda_2)\},$$

and $j : TM \to TM$ is a mapping satisfying $j^2 = 1$ and uniquely determined by properties

$$j : V \to H, \quad j : H \to V, \quad j(\pi_H(X)) - \pi_H(X) \in T, \quad j(\pi_V(X)) - \pi_V(X) \in T,$$

for any $X \in TM$, where $T = \operatorname{span}\{L_0(\lambda_3), L_1(\lambda_3)\}$. In the present context

$$V = \operatorname{span}\{u_4\partial_{x^2} - u_2\partial_{x^4}, u_3\partial_{x^2} - u_2\partial_{x^3}\}, \quad H = \operatorname{span}\{v_4\partial_1 - v_1\partial_{x^4}, v_3\partial_{x^1} - v_1\partial_{x^3}\}$$

and

$$\begin{aligned}
j(u_3\partial_{x^2} - u_2\partial_{x^3}) &= \frac{u_2}{v_1}(v_3\partial_{x^1} - v_1\partial_{x^3}),\\
j(u_4\partial_{x^2} - u_2\partial_{x^4}) &= C\frac{u_2}{v_1}(v_4\partial_{x^1} - v_1\partial_{x^4}),\\
j(v_3\partial_{x^1} - v_1\partial_{x^3}) &= \frac{v_1}{u_2}(u_3\partial_{x^2} - u_2\partial_{x^3}),\\
j(v_4\partial_{x^1} - v_1\partial_{x^4}) &= C^{-1}\frac{v_1}{u_2}(u_4\partial_{x^2} - u_2\partial_{x^4}).
\end{aligned}$$

where $C = \frac{(\lambda_3-\lambda_1)(\lambda_4-\lambda_2)}{(\lambda_4-\lambda_1)(\lambda_3-\lambda_2)}$.

As mentioned in Introduction, it is easily seen that if either u or v is proportional to x^1 then (2) reduces to a PDE in three independent variables x^2, x^3, x^4. The following result is immediate:

Proposition 1 *If $v = cx^1$, where c is an arbitrary nonzero constant, then* (2) *reduces to*

$$\begin{aligned}
&(\lambda_3 - \lambda_4)(\lambda_1 - \lambda_2)u_2u_{34} + (\lambda_2 - \lambda_3)(\lambda_1 - \lambda_4)u_4u_{23}\\
&\quad - (\lambda_2 - \lambda_4)(\lambda_1 - \lambda_3)u_3u_{24} = 0,
\end{aligned} \tag{5}$$

which is nothing but the ABC equation in three independent variables x^2, x^3, x^4, involving x^1 as a mere parameter, where $A = (\lambda_3 - \lambda_4)(\lambda_1 - \lambda_2)$, $B = (\lambda_2 - \lambda_3)(\lambda_1 - \lambda_4)$ and $C = -(\lambda_2 - \lambda_4)(\lambda_1 - \lambda_3)$ satisfy $A + B + C = 0$.

Geometrically, in view of [11], Proposition 1 means that to any 3-dimensional hyper-CR Einstein-Weyl structure (or, equivalently, a Veronese web) associated to a solution of (5), there corresponds a hyper-para-Hermitian metric admitting ∂_{x^1} as a Killing vector. This can be seen as an explicit manifestation of the Jones–Tod reduction, see [17], and also [11].

Extending the above to the case when u or v is a linear function of all coordinates x^i with a nonzero coefficient at x^1 is left as an exercise for the reader.

3 Hyper-para-hermitian structures

The goal of this section is to give proofs of Theorems 1 and 2. To this end we shall employ a description of hyper-para-Hermitian structures via Kronecker webs. Similar concepts have also been recently exploited in [26, 39] and earlier in [11, 27] in the context of 3-dimensional Einstein–Weyl geometry.

Recall that the webs are special families of foliations, originally introduced as reductions of certain bi-Hamiltonian systems [43] (see also [38]), where they were studied in full generality. In the 4-dimensional case needed in this paper, the webs are defined as follows (they are referred to as the isotypic Kronecker webs in [26], or Kronecker webs of the 3-web type in [39]).

Definition A *Kronecker web* on a 4-dimensional manifold is a 1-parameter family of foliations $\{\mathcal{F}_\lambda\}_{\lambda\in\mathbb{R}}$ such that, locally, there exist point-wise independent 1-forms $\alpha^1, \dots, \alpha^4$ such that

$$T\mathcal{F}_\lambda = \ker\left(\alpha^1 + \lambda\alpha^2, \alpha^3 + \lambda\alpha^4\right).$$

Notice that we can equivalently write

$$T\mathcal{F}_\lambda = \operatorname{span}\{X_0(\lambda), X_1(\lambda)\}$$

for $X_0(\lambda) = Y_2 - \lambda Y_1$, $X_1(\lambda) = Y_4 - \lambda Y_3$, where $(Y_i)_{i=1,\dots,4}$ is a frame on M dual to the co-frame $(\alpha^i)_{i=1,\dots,4}$.

The aforementioned correspondence between hyper-para-Hermitian structures and Kronecker webs on 4-dimensional manifolds is established in the following way: the endomorphisms I and K are defined such that the corresponding eigenspaces $\mathcal{D}_I^+, \mathcal{D}_I^-$, $\mathcal{D}_K^+$ and $\mathcal{D}_K^-$, associated with the eigenvalues ± 1, respectively, are given by $T\mathcal{F}_{\lambda_i}$ for certain fixed values of λ_i, $i = 1, 2, 3, 4$ (see Corollaries 2.3, 2.5 and the discussion on page 461 in [26]). Note that the complex structure J is induced from I and K by means of the composition $J = IK$. Additionally, from the viewpoint of the anti-self-dual metrics, the leaves of foliations $\{\mathcal{F}_\lambda\}_{\lambda\in\mathbb{R}}$ are α-submanifolds of $[g]$ and the collection of all leaves is usually referred to as the twistor space, see [12].

With this in mind, assuming the above Kronecker web description of the hyper-para-Hermitian structures on M, we proceed to the proof of Theorem 1.

Proof of Theorem 1 Let $\lambda_1, \dots, \lambda_4 \in \mathbb{R}$ be four distinct values of λ. There are functions f^i, g^i, $i = 1, \dots, 4$, defined locally in a neighborhood of a point $x \in M$, such that the corresponding foliations $\mathcal{F}_{\lambda_i}$ are defined as $f^i = \text{const}$, $g^i = \text{const}$, respectively. Without loss of generality, since the foliations $\mathcal{F}_\lambda$ corresponding to different values of λ are transversal, one can define local coordinates in the neighborhood of x as $x^i = f^i$. Moreover, let $u = g^1$ and $v = g^2$.

We shall look for the λ-dependent vector fields $X_0(\lambda)$ and $X_1(\lambda)$, spanning $T\mathcal{F}_\lambda$ for any $\lambda \in \mathbb{R}$. They can be written in the most general form as

$$X_j(\lambda) = (a_j^1 + \lambda b_j^1)\partial_{x^1} + (a_j^2 + \lambda b_j^2)\partial_{x^2} + (a_j^3 + \lambda b_j^3)\partial_{x^3} + (a_j^4 + \lambda b_j^4)\partial_{x^4},\quad j = 0, 1,$$

for some functions a_j^i and b_j^i, $i = 1, \ldots, 4$, $j = 0, 1$, in a neighborhood of x. Now, since dx^i annihilates both $X_0(\lambda_i)$ and $X_1(\lambda_i)$ we get $a_j^i = -\lambda_i b_j^i$ and consequently

$$X_j(\lambda) = (\lambda - \lambda_1)b_j^1\partial_{x^1} + (\lambda - \lambda_2)b_j^2\partial_{x^2} + (\lambda - \lambda_3)b_j^3\partial_{x^3} + (\lambda - \lambda_4)b_j^4\partial_{x^4},\quad j = 0, 1.$$

Further on, du annihilates $X_0(\lambda_1)$ and $X_1(\lambda_1)$ and dv annihilates $X_0(\lambda_2)$ and $X_1(\lambda_2)$. These conditions give equations

$$\begin{aligned} (\lambda_1 - \lambda_2)u_2 b_j^2 + (\lambda_1 - \lambda_3)u_3 b_j^3 + (\lambda_1 - \lambda_4)u_4 b_j^4 = 0, \qquad j = 0, 1 \\ (\lambda_2 - \lambda_1)v_1 b_j^1 + (\lambda_2 - \lambda_3)v_3 b_j^3 + (\lambda_2 - \lambda_4)v_4 b_j^4 = 0, \qquad j = 0, 1. \end{aligned} \tag{6}$$

Moreover, $X_0(\lambda)$ and $X_1(\lambda)$ are defined up to a $GL(2)$-action. Hence, it can be assumed without loss of generality that $b_0^3 = 0$, $b_0^4 = 1$, $b_1^3 = 1$ and $b_1^4 = 0$. There are four b_j^i left, with $i = 1, 2$ and $j = 0, 1$. However, system (6) fixes them uniquely:

$$b_0^1 = \frac{(\lambda_2 - \lambda_4)}{(\lambda_1 - \lambda_2)}\frac{v_4}{v_1},\quad b_1^1 = \frac{(\lambda_2 - \lambda_3)}{(\lambda_1 - \lambda_2)}\frac{v_3}{v_1},\quad b_0^2 = -\frac{(\lambda_1 - \lambda_4)}{(\lambda_1 - \lambda_2)}\frac{u_4}{u_2},$$
$$b_1^2 = -\frac{(\lambda_1 - \lambda_3)}{(\lambda_1 - \lambda_2)}\frac{u_3}{u_2}.$$

We get

$$\begin{aligned} X_0(\lambda) &= (\lambda - \lambda_1)\frac{(\lambda_2 - \lambda_4)}{(\lambda_1 - \lambda_2)}\frac{v_4}{v_1}\partial_{x^1} - (\lambda - \lambda_2)\frac{(\lambda_1 - \lambda_4)}{(\lambda_1 - \lambda_2)}\frac{u_4}{u_2}\partial_{x^2} + (\lambda - \lambda_4)\partial_{x^4}, \\ X_1(\lambda) &= (\lambda - \lambda_1)\frac{(\lambda_2 - \lambda_3)}{(\lambda_1 - \lambda_2)}\frac{v_3}{v_1}\partial_{x^1} - (\lambda - \lambda_2)\frac{(\lambda_1 - \lambda_3)}{(\lambda_1 - \lambda_2)}\frac{u_3}{u_2}\partial_{x^2} + (\lambda - \lambda_3)\partial_{x^3}. \end{aligned} \tag{7}$$

Note that setting $L_i(\lambda) = (\lambda_1 - \lambda_2)u_2 v_1 X_i(\lambda)$ recovers the Lax operators from (3). By assumption, $X_0(\lambda)$ and $X_1(\lambda)$ span an integrable distribution for any λ. Hence, since the commutator of $X_i(\lambda)$ does not involve ∂_{x^3} and ∂_{x^4}, $X_0(\lambda)$ and $X_1(\lambda)$ necessarily commute. One readily checks that vanishing of the commutator of $X_i(\lambda)$ is equivalent to system (2). Likewise, it is easily seen that vanishing of the commutator of $L_i(\lambda)$ modulo a certain linear combination of $L_i(\lambda)$ is equivalent to system (2). Thus an overdetermined linear system for ψ, $L_i(\lambda)\psi = 0$, $i = 0, 1$, whose compatibility condition is nothing but (2), provides an isospectral Lax pair for (2) involving a parameter λ, which establishes integrability of (2). □

The above proof of Theorem 1, which gives a local correspondence between solutions to (2) and the hyper-para-Hermitian structures on M, can in a sense be seen as a geometric derivation of the system (2) along with its Lax pair. One of the key points here is finding suitable coordinates and writing the Kronecker web arising in the proof in these coordinates, so that the partial differential system locally describing hyper-para-Hermitian structures on M takes a particularly simple form (2). Notice that the said coordinates could be interpreted as eigenfunctions of the associated operators (7) in a fashion reminiscent of the approach presented in [18] for the case of equation (1).

As we realized post factum, in principle one could also have possibly arrived at (2) as an integrable extension of (1) in a different, more algebraic fashion using a procedure from the appendix of [24] for finding multicomponent integrable extensions for various heavenly-type equations through a certain generalization of the Lax pairs of the equations in question. Indeed, applying the said procedure to the Lax pair for (1) with the Lax operators obtained from $X_i(\lambda)$ from Remark 1 using the substitution (4), yields the Lax operators

$$\tilde{X}_0(\lambda) = (\lambda-\lambda_1)\frac{(\lambda_2-\lambda_4)}{(\lambda_1-\lambda_2)}q^{(0,1)}\partial_{x^1} - (\lambda-\lambda_2)\frac{(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_2)}q^{(0,2)}\partial_{x^2} + (\lambda-\lambda_4)\partial_{x^4},$$
$$\tilde{X}_1(\lambda) = (\lambda-\lambda_1)\frac{(\lambda_2-\lambda_3)}{(\lambda_1-\lambda_2)}q^{(1,1)}\partial_{x^1} - (\lambda-\lambda_2)\frac{(\lambda_1-\lambda_3)}{(\lambda_1-\lambda_2)}q^{(1,2)}\partial_{x^2} + (\lambda-\lambda_3)\partial_{x^3}.$$

The compatibility condition for the Lax pair $\tilde{X}_i(\lambda)\psi = 0$, $i = 0, 1$ yields a first-order four-component integrable system for $q^{(i,j)}$ extending (1), and it can be shown that the said system for $q^{(i,j)}$ is, in a fairly non-obvious fashion, essentially equivalent to (2).

Before proceeding to prove Theorem 2 we need the following lemma:

Lemma 1 *Suppose that* (u, v) *is a solution to* (2)*. Then, for any two nowhere vanishing functions* $a, b\colon \mathbb{R}^2 \to \mathbb{R}$*, the pair* $(\tilde{u}, \tilde{v})$ *defined as*

$$\tilde{u}(x^1, x^2, x^3, x^4) = a(x^1, u(x^1, x^2, x^3, x^4)),$$
$$\tilde{v}(x^1, x^2, x^3, x^4) = b(x^2, v(x^1, x^2, x^3, x^4)),$$

where a_u *and* b_v *are nonzero, is a solution to* (2) *descending to the same hyper-para-Hermitian metric.*

Proof Observe that, in view of the proof of Theorem 1, the functions $\tilde{u}$ and $\tilde{v}$ belong to the rings of functions constant on leaves of foliations $\mathcal{F}_{\lambda_1}$ and $\mathcal{F}_{\lambda_2}$, respectively. Hence $\tilde{u}$ and $\tilde{v}$ can replace u and v in the derivation of the Lax pair in the proof of Theorem 1, and therefore in system (2) itself, and the result follows. □

Proof of Theorem 2. The first part of the theorem can be readily verified by direct computations. In order to prove the second part, we shall work using the coordinates from Theorem 1. The Lax distribution span$\{L_0(\lambda), L_1(\lambda)\}$ is annihilated by two λ-

dependent 1-forms that can be taken in the following form

$$
\begin{aligned}
\alpha_0(\lambda) =&(\lambda_1-\lambda_2)(\lambda_4-\lambda)\left(\frac{u_3}{u_2}dx^1+\frac{v_3}{v_1}dx^2+\frac{u_3v_3}{u_2v_1}dx^3\right)\\
&+\left((\lambda_1-\lambda_4)(\lambda_2-\lambda)\frac{u_4v_3}{u_2v_1}-(\lambda_2-\lambda_4)(\lambda_1-\lambda)\frac{u_3v_4}{u_2v_1}\right)dx^4\\
\alpha_1(\lambda) =&(\lambda_1-\lambda_2)(\lambda_3-\lambda)\left(\frac{u_4}{u_2}dx^1+\frac{v_4}{v_1}dx^2+\frac{u_4v_4}{u_2v_1}dx^3\right)\\
&+\left((\lambda_2-\lambda_3)(\lambda_1-\lambda)\frac{u_4v_3}{u_2v_1}-(\lambda_1-\lambda_3)(\lambda_2-\lambda)\frac{u_3v_4}{u_2v_1}\right)dx^4
\end{aligned}
$$

Let $\beta=\alpha_0\wedge\alpha_1$. According to [39, Corollary 2.4] a hyper-para-Hermitian structure is divergence-free (which in the terminology of the present paper means that the structure is hyper-para-Kähler) if and only if there is a nowhere vanishing function $c\colon M\to\mathbb{R}$ such that $c\beta$ is closed. Since L_0 and L_1 commute modulo a linear combination thereof with nonconstant coefficients, see the proof of Theorem 1, we have

$$d\alpha_i=0 \mod \alpha_0,\alpha_1,$$

and consequently

$$d\beta=\varphi\wedge\beta$$

where φ is explicitly written as

$$
\begin{aligned}
\varphi(\lambda)=&\frac{1}{(\lambda_1-\lambda_2)(\lambda-\lambda_3)u_2v_1}\left((\lambda_2-\lambda_3)(\lambda-\lambda_1)(u_{12}(v_3-u_{23}v_1)\right.\\
&\left.+(\lambda_1-\lambda_3)(\lambda-\lambda_2)(u_2v_{13}-u_3v_{12})\right)dx^3\\
&+\frac{1}{(\lambda_1-\lambda_2)(\lambda-\lambda_4)u_2v_1}\left((\lambda_2-\lambda_4)(\lambda-\lambda_1)(u_{12}v_4-u_{24}v_1)\right.\\
&\left.+(\lambda_1-\lambda_4)(\lambda-\lambda_2)(u_2v_{14}-u_4v_{12})\right)dx^4
\end{aligned}
$$

Notice that φ is specified only up to transformations $\varphi\mapsto\varphi_{A^0,A^1}=\varphi+A^0\alpha_0+A^1\alpha_1$ for some functions A^0 and A^1. Since the result is local, we can apply the Poincaré lemma for differential forms, and get that function c exists if and only if φ_{A^0,A^1} is exact for an appropriate choice of A^0 and A^1. Moreover φ_{A^0,A^1} has to be a function on M (i.e. it cannot depend on λ). We now consider the equation

$$\varphi_{A^0,A^1}=\varphi+A^0\alpha_0+A^1\alpha_1=df$$

where A^0, A^1 and f are unknown. Examining the coefficients at dx^1 and dx^2, we obtain an algebraic system for A^i's. This system can be effortlessly solved, resulting in

$$A^0 = \frac{1}{(\lambda_1-\lambda_2)(\lambda-\lambda_4)}\left(\frac{u_4v_1f_2-u_2v_4f_1}{u_3v_4-u_4v_3}\right),$$
$$A^1 = \frac{1}{(\lambda_1-\lambda_2)(\lambda-\lambda_3)}\left(\frac{u_2v_3f_1-u_3v_1f_2}{u_3v_4-u_4v_3}\right).$$

Then, the coefficients at dx^3 and dx^4 reduce to differential equations for f, which, in turn, can be written in the form

$$L_0(f) = (\lambda-\lambda_4)\varphi_4, \qquad L_1(f) = (\lambda-\lambda_3)\varphi_3, \tag{8}$$

where φ_3 and φ_4 stand for the coefficients defined as $\varphi = \varphi_3 dx^3 + \varphi_4 dx^4$. The system always satisfies the compatibility condition, which can be verified directly. Hence, it yields a solution f. However, in general, f depends on λ. Imposing the condition that $\partial_\lambda f = 0$ and substituting $\lambda = \lambda_1$ and $\lambda = \lambda_2$ in (8) gives the following system

$$u_2v_1f_3 - u_3v_1f_2 = u_2v_{13} - u_3v_{12},$$
$$u_2v_1f_3 - u_2v_3f_1 = u_{23}v_1 - u_{12}v_3,$$
$$u_2v_1f_4 - u_4v_1f_2 = u_2v_{14} - u_4v_{12},$$
$$u_2v_1f_4 - u_2v_4f_1 = u_{24}v_1 - u_{12}v_4.$$

It turns out that this system can be rewritten as

$$\left(\partial_{x^3} - \frac{u_3}{u_2}\partial_{x^2}\right)(f - \ln(v_1)) = 0,$$
$$\left(\partial_{x^3} - \frac{v_3}{v_1}\partial_{x^1}\right)(f - \ln(u_2)) = 0,$$
$$\left(\partial_{x^4} - \frac{u_4}{u_2}\partial_{x^2}\right)(f - \ln(v_1)) = 0,$$
$$\left(\partial_{x^4} - \frac{v_4}{v_1}\partial_{x^1}\right)(f - \ln(u_2)) = 0.$$

Denoting $g = f - \ln(v_1)$ and $h = f - \ln(u_2)$ and using the fact that the vector fields $\partial_{x^3} - \frac{u_3}{u_2}\partial_{x^2}$ and $\partial_{x^4} - \frac{u_4}{u_2}\partial_{x^2}$ commute, and so do $\partial_{x^3} - \frac{v_3}{v_1}\partial_{x^1}$ and $\partial_{x^4} - \frac{v_4}{v_1}\partial_{x^1}$, we get that g and h can be written as $g = g(x^1, u)$ and $h = h(x^2, v)$, because x^1 and u are constant along the first pair of vector fields, and x^2 and v are constant along the second pair of vector fields. Further, expressing f in terms of g and h and comparing the expressions gives

$$\ln(u_2) - \ln(v_1) = h - g.$$

By Lemma 1 there is a different solution to (2) such that $\ln(\tilde{u}_2) - \ln(\tilde{v}_1) = 0$. Indeed, one can adjust functions a and b from Lemma 1 in such a way that the counterparts of h and g for $\tilde{u}$ and $\tilde{v}$ cancel out. Consequently,

$$\ln\left(\frac{\tilde{u}_2}{\tilde{v}_1}\right) = 0.$$

Hence, we have $\tilde{u}_2 = \tilde{v}_1$ and it implies that the 1-form

$$\tilde{u}dx^1 + \tilde{v}dx^2$$

is closed with respect to coordinates (x^1, x^2), meaning that locally $\tilde{u} = w_1$ and $\tilde{v} = w_2$, for some function w, i.e. (4) holds. Furthermore, the Lax pair in Theorem 1, under assumption (4), reduces to the Lax pair for (1), as can be found, for instance, in [18, formula (4.11)]. It follows that w is a solution to (1). □

4 Symmetries and Recursion Operator

Now turn to the study of symmetries for (2). To this end we first note that the Lax operators $X_i(\lambda)$ given by (7) from the proof of Theorem 1 are linear in λ and thus can be written as

$$X_i(\lambda) = X_i^{(0)} - \lambda X_i^{(1)}, \quad i = 0, 1 \tag{9}$$

with obvious expressions for $X_i^{(j)}$.

Consider now the following 'adjoint Lax pair' for (2):

$$[Q(\lambda), X_i(\lambda)] = 0, \quad i = 0, 1, \tag{10}$$

where $[\cdot, \cdot]$ is again the usual Lie bracket of vector fields and

$$Q(\lambda) = \sum_{j=1}^{2} \xi^j(\lambda)\partial_{x^j} \tag{11}$$

(in this connection recall that $[X_0(\lambda), X_1(\lambda)] = 0$ modulo (2), so (10) is compatible by virtue of (2)).

The formal expansions

$$\xi^j(\lambda) = \sum_{r=0}^{\infty} \xi^j_{(r)}\lambda^r, \quad j = 1, 2 \tag{12}$$

give rise to an infinite hierarchy of nonlocal variables $\xi^j_{(r)}$ associated with (2) as follows (here and below we put subscripts in the round brackets to indicate that these subscripts do *not* refer to derivatives; also, in the present paper the round brackets around multiple subscripts do *not* indicate symmetrization).

Let $Q_{(r)} = \sum_{j=1}^{2} \xi_{(r)}^{j} \partial_{x^j}$. Then substituting (11) and (12) into (10) yields

$$\left[Q_0, X_i^{(0)}\right] = 0, \quad i = 0, 1 \tag{13}$$

and a set of recursion relations

$$[Q_{(r)}, X_i^{(0)}] = [Q_{(r-1)}, X_i^{(1)}], \quad i = 0, 1, \quad r = 1, 2, \dots \tag{14}$$

relating $Q_{(r)}$ and $Q_{(r-1)}$.

Equations (13) and (14) can be solved with respect to $\partial \xi_{(r)}^{j} / \partial x^m$, $m = 3, 4$, to yield relations of general form

$$\partial \xi_{(r)}^{j} / \partial x^m = A_{(m,r)}^{j}, \quad m = 3, 4, \quad j = 1, 2, \quad r = 0, 1, 2, \dots$$

which recursively define $\xi_{(r)}^{j}$ starting from $r = 0$; here $A_{(m,r)}^{j}$ are certain functions, a bit too cumbersome to spell out here in full, of $\xi_{(s)}^{j}$, $s = 0, \dots, r$ and x- and y-derivatives of those, and of a number of first- and second-order derivatives of u and v.

In other words, we have here an infinite-dimensional differential covering over (2) with the nonlocal variables $\xi_{(r)}^{j}$, $j = 1, 2, r = 0, 1, 2, \dots$; the said covering is defined via (13) and (14). For generalities on nonlocal variables, differential coverings and nonlocal symmetries the reader is referred to [23, 32, 41] and references therein.[1]

With this in mind, we arrive at the following result

Proposition 2 *The flows*

$$u_{\tau_{(r)}} = \xi_{(r)}^{2} u_2, \quad v_{\tau_{(r)}} = -\xi_{(r)}^{1} v_1, \quad r = 0, 1, 2, \dots, \tag{15}$$

with $\xi_{(r)}^{k}$ *defined above, are compatible with* (2)*, and thus define an infinite hierarchy of nonlocal symmetries for* (2) *with the characteristics* $\Xi_{(r)} = (\xi_{(r)}^{2} u_2, -\xi_{(r)}^{1} v_1)^{\mathrm{T}}$, $r = 0, 1, 2, \dots$.

The flow

$$u_\tau = \xi^2 u_2, \quad v_\tau = -\xi^1 v_1, \tag{16}$$

which can be seen as a generating function for (15), is also compatible with (2), and thus (2) also admits a nonlocal symmetry with the characteristic $\Xi = (\xi^2 u_2, -\xi^1 v_1)^{\mathrm{T}}$ *involving the parameter* λ *through* ξ^i *defined above.*

Here and below the superscript T indicates the transposed matrix.

Note that infinite hierarchies of symmetries like the above one are a common occurrence for integrable partial differential systems, and for integrable systems in more

[1] Note, however, that e.g. the authors of [23] refer to what we call nonlocal symmetries as to the shadows of nonlocal symmetries.

than two independent variables the symmetries in question are usually nonlocal, cf. e.g. [23, 31, 32, 34, 37, 41, 42] and references therein.

Proof of Proposition 2. We begin with proving the second part of the theorem. It is straightforward to verify that $U = \xi^2 u_2$ and $V = -\xi^1 v_1$ satisfy the linearized version of (2), that is,

$$\begin{aligned}
&(\lambda_3-\lambda_4)(\lambda_1-\lambda_2)(V_1u_2u_{34}+v_1U_2u_{34}+v_1u_2U_{34})\\
&\quad+(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)(V_1u_4u_{23}+v_1U_4u_{23}+v_1u_4U_{23}\\
&\quad-V_3u_4u_{12}-v_3U_4u_{12}-v_3u_4U_{12}+V_3u_2u_{14}+v_3U_2u_{14}+v_3u_2U_{14})\\
&\quad-(\lambda_2-\lambda_4)(\lambda_1-\lambda_3)(V_1u_3u_{24}+v_1U_3u_{24}+v_1u_3U_{24}\\
&\quad-V_4u_3u_{12}-v_4U_3u_{12}-v_4u_3U_{12}+V_4u_2u_{13}+v_4U_2u_{13}+v_4u_2U_{13})=0,\\
&(\lambda_3-\lambda_4)(\lambda_1-\lambda_2)(V_1u_2v_{34}+v_1U_2v_{34}+v_1u_2V_{34})\\
&\quad+(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)(V_1u_4v_{23}+v_1U_4v_{23}+v_1u_4V_{23}\\
&\quad-V_3u_4v_{12}-v_3U_4v_{12}-v_3u_4V_{12}+V_3u_2v_{14}+v_3U_2v_{14}+v_3u_2V_{14})\\
&\quad-(\lambda_2-\lambda_4)(\lambda_1-\lambda_3)(V_1u_3v_{24}+v_1U_3v_{24}+v_1u_3V_{24}\\
&\quad-V_4u_3v_{12}-v_4U_3u_{12}-v_4u_3V_{12}+V_4u_2v_{13}+v_4U_2v_{13}+v_4u_2V_{13})=0,
\end{aligned}\tag{17}$$

modulo (2), (10) and differential consequences thereof, i.e., (2) admits a nonlocal symmetry with the characteristic $(\xi^2 u_2, -\xi^1 v_1)^{\mathrm{T}}$ (and, as per the well-known general results, the flow (16) is compatible with (2), cf. e.g. [23, 37] and references therein). Upon taking into account linearity of (17) in U and V and substituting the expansions (12) into (17) it is easily seen that $U = \xi^2_{(s)} u_2$ and $V = -\xi^1_{(s)} v_1$ for $s = 0, 1, 2, \ldots$ also satisfy (17) modulo (2), (13), and (14) for $r = 1, \ldots, s$, and differential consequences thereof, so (2) indeed admits an infinite hierarchy of nonlocal symmetries with the characteristics $(\xi^2_{(s)} u_2, -\xi^1_{(s)} v_1)^{\mathrm{T}}$, $s = 0, 1, 2, \ldots$. Compatibility of (15) with (2) immediately follows from this, cf. e.g. [23, 37] and references therein. □

Let us also point out the following two nontrivial local conservation laws for (2):

$$\sum_{j=1}^{4} \partial_{x^j} \sigma^j_{(m)} = 0, \quad m = 1, 2, \tag{18}$$

where

$$\begin{aligned}
&\sigma^1_{(1)}=0,\quad \sigma^2_{(1)}=(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)u_4v_3/v_1-(\lambda_2-\lambda_4)(\lambda_1-\lambda_3)u_3v_4/v_1,\\
&\sigma^3_{(1)}=(\lambda_2-\lambda_4)(\lambda_1-\lambda_3)u_2v_4/v_1,\quad \sigma^4_{(1)}=-(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)u_2v_3/v_1,\\
&\sigma^1_{(2)}=(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)u_4v_3/u_2-(\lambda_2-\lambda_4)(\lambda_1-\lambda_3)u_3v_4/u_2,\quad \sigma^2_{(2)}=0,\\
&\sigma^3_{(2)}=-(\lambda_1-\lambda_4)(\lambda_2-\lambda_3)u_4v_1/u_2,\quad \sigma^4_{(2)}=(\lambda_2-\lambda_4)(\lambda_1-\lambda_3)u_3v_1/u_2
\end{aligned}$$

As usual, we write local conservation laws as differential identities that hold modulo (2) and differential consequences thereof, cf. e.g. [23, 37].

While infinite hierarchies of symmetries like the one presented in Proposition 2 is typical for integrable systems, cf. e.g. [23, 37] and references therein, it is also natural to ask whether (2) admits other structures related to integrability like, say, a recursion operator.

Using the technique from [41], we have arrived at the following result that can be readily verified by straightforward but tedious computation (for background on recursion operators like the one below, being essentially Bäcklund auto-transformations for the linearized version of the system under study, see [30] and cf. e.g. [19, 22, 23, 31, 32, 34, 41, 42]):

Theorem 3 *Let the flow*

$$u_\tau = U, \quad v_\tau = V, \tag{19}$$

where U and V are assumed to be functions of independent variables, jet variables intrinsic to (2)*, and some nonlocal variables for* (2)*, be compatible with* (2) *and thus define a nonlocal symmetry with the characteristic $(U, V)^{\mathrm{T}}$ for* (2).

Consider nonlocal variables ζ^i defined by the relations

$$[X_i^{(1)}, S] = \left([X_i^{(0)}, R]\right)^{TD}, \quad i = 0, 1 \tag{20}$$

where $X_i^{(j)}$ are defined via (9)*, and*

$$R = -(V/v_1)\partial_{x^1} + (U/u_2)\partial_{x^2}, \quad S = \sum_{j=1}^{2} \zeta^j \partial_{x^j} \tag{21}$$

and the superscript TD means that expressions involving the derivatives of U and V like U_{x^i} and V_{x^j} should be replaced by $D_{x^i}U$ and $D_{x^j}V$, where D_{x^i} are total derivatives.

Then the new flow

$$u_\sigma = \tilde{U}, \quad v_\sigma = \tilde{V}, \tag{22}$$

where

$$\tilde{U} = u_2\zeta^2, \quad \tilde{V} = -v_1\zeta^1, \tag{23}$$

is also compatible with (2) *and thus defines another nonlocal symmetry with the characteristic $(\tilde{U}, \tilde{V})^{\mathrm{T}}$ for* (2).

In other words, the relations (20), (21)*, and* (23) *define a recursion operator for* (2) *associating to any nonlocal symmetry with the characteristic $(U, V)^{\mathrm{T}}$ for* (2) *a new (again in general nonlocal) symmetry with the characteristic $(\tilde{U}, \tilde{V})^{\mathrm{T}}$ for* (2).

The above recursion operator is a natural generalization of the recursion operator for general heavenly equation found in [41].

Consider the recursion operator from Theorem 3 in more detail. First of all note that the Eq. (20) defining the nonlocal variables ζ^i can be spelled out as follows:

$$
\begin{aligned}
\partial\zeta^1/\partial x^k = & \frac{(\lambda_1-\lambda_k)u_k\partial\zeta^1/\partial x^2}{u_2(\lambda_1-\lambda_2)} - \frac{(\lambda_2-\lambda_k)v_k\partial\zeta^1/\partial x^1}{v_1(\lambda_1-\lambda_2)} \\
& -\frac{(\lambda_2-\lambda_k)}{v_1^2(\lambda_1-\lambda_2)}\sum_{j=1}^{2}(v_k v_{1j}-v_1 v_{jk})\zeta^j \\
& +\frac{\lambda_k D_{x^k}V}{v_1} - \frac{(\lambda_1-\lambda_k)\lambda_2 u_k D_{x^2}V}{u_2 v_1(\lambda_1-\lambda_2)} + \frac{(\lambda_2-\lambda_k)\lambda_1 v_k D_{x^1}V}{v_1^2(\lambda_1-\lambda_2)} \\
& +\frac{(\lambda_2-\lambda_k)(-v_k v_{12}+v_1 v_{2k})\lambda_1 U}{u_2 v_1^2(\lambda_1-\lambda_2)} \\
& -\frac{(\lambda_1-\lambda_k)(-u_k v_{12}+u_2 v_{1k})\lambda_2 V}{u_2 v_1^2(\lambda_1-\lambda_2)}, \quad k=3,4 \\
\partial\zeta^2/\partial x^k = & \frac{(\lambda_1-\lambda_k)u_k\partial\zeta^2/\partial x^2}{u_2(\lambda_1-\lambda_2)} - \frac{(\lambda_2-\lambda_k)v_k\partial\zeta^2/\partial x^1}{v_1(\lambda_1-\lambda_2)} \\
& +\frac{(\lambda_1-\lambda_k)}{u_2^2(\lambda_1-\lambda_2)}\sum_{j=1}^{2}(u_k u_{2j}-u_2 u_{jk})\zeta^j \\
& -\frac{\lambda_k D_{x^k}U}{u_2} + \frac{(\lambda_1-\lambda_k)\lambda_2 u_k D_{x^2}U}{u_2^2(\lambda_1-\lambda_2)} - \frac{(\lambda_2-\lambda_k)\lambda_1 v_k D_{x^1}U}{u_2 v_1(\lambda_1-\lambda_2)} \\
& -\frac{(\lambda_2-\lambda_k)(-v_k u_{12}+v_1 u_{2k})\lambda_1 U}{u_2^2 v_1(\lambda_1-\lambda_2)} \\
& +\frac{(\lambda_1-\lambda_k)(-u_k u_{12}+u_2 u_{1k})\lambda_2 V}{u_2^2 v_1(\lambda_1-\lambda_2)}, \quad k=3,4 \qquad (24)
\end{aligned}
$$

Denote by $\mathfrak{R}$ the recursion operator defined by (21), (23), and (20) or equivalently by(21), (23), and (24), so that we can write $(\tilde{U},\tilde{V})^{\mathrm{T}}=\mathfrak{R}(U,V)^{\mathrm{T}}$ meaning that a (again in general nonlocal) symmetry with the characteristic $(\tilde{U},\tilde{V})^{\mathrm{T}}$ is related to the (in general nonlocal) symmetry with the characteristic $(U,V)^{\mathrm{T}}$ via (21), (23), and (20) or (24); let us stress that $\mathfrak{R}$ is a correspondence and not really an operator because the relations (24) define ζ^i nonuniquely.

Notice that one now can construct (in general nonlocal) symmetries for system (2) beyond those from Proposition 2 by applying the recursion operator from Theorem 3 e.g. to 'obvious' symmetries like, say, those with the characteristics $(u_i,v_i)^{\mathrm{T}}$, $i=1,\ldots,4$, that result from the translation invariance of (2).

Indeed, let $\eta^i_{(s)}$ denote the nonlocal variables obtained from ζ^i defined via (20) (or equivalently (24)) by substituting into (24) $U=u_s$, $V=v_s$, and accordingly $D_{x^j}U=u_{js}$ and $D_{x^j}V=v_{js}$, where $j,s=1\ldots,4$ and $i=1,2$. Then we have $\mathfrak{R}(u_s,v_s)^{\mathrm{T}}=(u_2\eta^2_{(s)},-v_1\eta^1_{(s)})^{\mathrm{T}}$, $s=1,\ldots,4$, so even the action of $\mathfrak{R}$ on the simplest symmetries with the characteristics $(u_s,v_s)^{\mathrm{T}}$ produces fairly complicated nonlocal symmetries (note that in spite of our best efforts the nonlocal variables $\eta^i_{(s)}$ do not appear to lend themselves to any simplification).

In closing note that comparing the relations (10) and (20), we see that the nonlocal symmetry for (2) with the characteristic $\Xi = (u_2\xi^2, -v_1\xi^1)^{\mathrm{T}}$, see above, is an 'eigenfunction' of $\mathfrak{R}$ with the eigenvalue $1/\lambda$ as we have $\mathfrak{R}\Xi = \Xi/\lambda$; once again, it should be kept in mind that $\mathfrak{R}$ is a correspondence rather than an actual operator.

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Author contributions All authors wrote the main manuscript text and contributed equally to the findings of the paper.

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Declarations

Conflict of interest The authors declare no Conflict of interest.

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