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Spectral properties of Sturm–Liouville operators on infinite metric graphs

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Abstract

This paper mainly deals with the Sturm–Liouville operator

$$\mathbf{H} = \frac{1}{w(x)}\left(-\frac{\mathrm{d}}{\mathrm{d}x}p(x)\frac{\mathrm{d}}{\mathrm{d}x} + q(x)\right), \ x \in \Gamma$$

acting in $L^2_w(\Gamma)$, where Γ is a metric graph. We establish a relationship between the bottom of the spectrum and the positive solutions of quantum graphs, which is a generalization of the classical Allegretto–Piepenbrink theorem. Moreover, we prove the Persson-type theorem, which characterizes the infimum of the essential spectrum.

Keywords Sturm–Liouville operators · Metric graphs · Spectrum

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1 Introduction

The main object of the present paper is the self-adjoint Sturm–Liouville operator in the Hilbert space $L^2_w(\Gamma)$ associated with the differential expression

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$$lf(x) = \frac{1}{w(x)}\left(-\left(p(x)f'(x)\right)' + q(x)f(x)\right), \ x \in \Gamma, \tag{1.1}$$

where Γ is a metric graph and the matching conditions imposed at inner vertices are the *Kirchhoff conditions*. Throughout this paper, we always assume that $1/p$, q, $w \in L^1_{loc}(\Gamma)$; in addition, $p(\cdot)$, $w(\cdot) > 0$ a.e. on Γ, and $q(\cdot)$ is real-valued.

In the last two decades, differential operators on metric graphs have attracted huge attentions due to numerous applications in mathematical physics and engineering ([5, 11, 12, 23, 26] and references therein). Particularly, there has been an increasing interest in the spectral theory of Sturm–Liouville operators on metric graphs (see [5, 6, 8, 11, 12, 15, 16, 25] and references therein). From the mathematical point of view, such a system is interesting because it exhibits a mixed dimensionality being locally one-dimensional but globally multi-dimensional of many different types.

Consider the following form in $L^2_w(\Gamma)$

$$\mathbf{t}_q^0[f] = \int_\Gamma \left[p(x)\left|f'(x)\right|^2 + q(x)\left|f(x)\right|^2\right] \mathrm{d}x$$

defined on the domain

$$\mathrm{dom}(\mathbf{t}_q^0) = \{f \in H_c^1(\Gamma; p, w) : f|_{\partial\Gamma} = 0\}.$$

Here $H_c^1(\Gamma; p, w)$ denotes the subspace of $H^1(\Gamma; p, w)$ with compact support, and

$$H^1(\Gamma; p, w) := \{f \in L^2_w(\Gamma) : \textstyle\int_\Gamma p(x)\left|f'(x)\right|^2 \mathrm{d}x < \infty, \\ f \text{ is continuous and edgewise absolutely continuous}\}.$$

If the form $\mathbf{t}_q^0$ is closable, denote $\mathbf{H}_{\mathbf{t}_q}$ the self-adjoint operator associated with $\mathbf{t}_q = \overline{\mathbf{t}_q^0}$.

The purpose of this paper is to develop Allegretto–Piepenbrink-type theorem (Theorem 3.1) and Persson-type theorem (Theorem 4.1) for the operator $\mathbf{H}_{\mathbf{t}_q}$, which are classical topics and we refer to the papers cited in this paragraph for historical remarks [1–4, 18–20, 22]. More precisely, we establish a relationship between the bottom of the spectrum and the positive solutions of quantum graphs, which is a generalization of the classical Allegretto–Piepenbrink theorem. Moreover, we prove the Persson-type theorem, which characterizes the infimum of the essential spectrum. It should be mentioned that the quantities $\inf\sigma\left(\mathbf{H}_{\mathbf{t}_q}\right)$ and $\inf\sigma_{ess}\left(\mathbf{H}_{\mathbf{t}_q}\right)$ are of fundamental importance for several reasons. For instance, within the framework of parabolic equation theory, the quantity $\inf\sigma\left(\mathbf{H}_{\mathbf{t}_q}\right)$ can serve as an indicator of the rate at which the system converges towards equilibrium. Furthermore, the condition $\inf\sigma_{ess}\left(\mathbf{H}_{\mathbf{t}_q}\right) = +\infty$ is satisfied precisely when $\mathbf{H}_{\mathbf{t}_q}$ possesses a purely discrete spectrum.

In the last decades, Allegretto–Piepenbrink-type theorem has been investigated for strongly local Dirichlet forms [19], positive Schrödinger operators on general weighted graphs [17] and Schrödinger operators on $\mathbb{R}^d$ with singular potentials [22]. Here, we generalize this theorem to our context and provide a simple proof along the lines of ([9, 21]). Recently, a Persson-type theorem for the Schrödinger operators ($p = w = 1$) on infinite metric graphs has been given in [1] by Akduman and

Pankov. Moreover, Lenz and Stollmann present a Persson-type theorem valid for all regular Dirichlet forms satisfying a spatial local compactness condition [18]; they also discuss a generalization to certain Schrödinger type operators, where the negative part of measure perturbations has to fulfill some Kato condition.In this direction, we present concrete conditions on the coefficients $1/p$, q, w and the lengths of the graph edges, which guarantee the validity of the Persson-type theorem for the Sturm–Liouville operator on infinite metric graphs. It is worth noting that, in the case when $q \geq 0$, our results (Theorems 3.1 and 4.1) seem to be covered by the literature on Dirichlet forms, particularly [19, Corollary 2.4 and Theorem 3.3] and [18, Theorem 5.3]. However, this paper strives to present a more comprehensible and approachable exposition, making it more accessible to those who may not be intimately familiar with Dirichlet forms.

Let us now finish the introduction by describing the content of the article. In Sect. 2, we review necessary notions and facts on infinite metric graphs. Sections 3 and 4 are devoted to investigating the Allegretto–Piepenbrink-type theorem and the Persson-type theorem respectively for the Sturm–Liouville operator $\mathbf{H}_{\mathbf{t}_q}$ on infinite metric graphs.

2 Preliminaries on metric graphs

In what follows, $\Gamma = (\mathcal{E}, \mathcal{V})$ will be a graph with countably infinite sets of vertices $\mathcal{V}$ and edges $\mathcal{E}$. A graph is called connected if for any two vertices there is a path connecting them. For every vertex $v \in \mathcal{V}$, we denote the set of edges incident to the vertex v by $\mathcal{E}_v$ and

$$\deg_\Gamma(v) := \#\{e : e \in \mathcal{E}_v\}$$

is called *the degree* of a vertex $v \in \mathcal{V}$. Moreover, the boundary of Γ is defined as

$$\partial\Gamma = \left\{v \in \mathcal{V}(\Gamma) : \deg_\Gamma(v) = 1\right\}.$$

The graph Γ is said to be a *metric graph* if each edge e is assigned a positive length $|e| \in (0, \infty)$. This enables us to equip Γ with a topology and metric. By assigning each edge a direction and calling one of its vertices the initial vertex $o(e)$ and the other one the terminal vertex $t(e)$, every edge $e \in \mathcal{E}(\Gamma)$ can be identified with a copy of the interval $I_e = [0, |e|]$. The distance $\rho(x, y)$ between two points x and y in Γ is defined as the length of the shortest path that connects these points. Since the graph is connected, the distance is well defined. In addition, there is a natural measure, dx, on Γ which coincides with the Lebesgue measure on each edge. In particular, integration over Γ makes sense. For further details we refer to, e.g., [5, Chapter 1.3].

Throughout this paper, we shall always make the following assumptions.

Hypothesis 2.1 *The graph Γ is connected and locally finite* ($\deg_\Gamma(v) < \infty$ *for every* $v \in \mathcal{V}$).

Hypothesis 2.2 *There is a finite upper bound for lengths of graph edges:*

$$\sup_{e \in \mathcal{E}(\Gamma)} |e| = d^* < \infty.$$

In fact, Hypothesis 2.2 is imposed for a convenience only. We denote by $L_w^2(\Gamma)$ the space of all complex-valued functions which are weighted square-integrable on Γ with respect to the measure $\mathrm{d}x$. More explicitly, this space consists of all measurable functions f such that $f_e \in L_w^2(e)$ for all $e \in \mathcal{E}(\Gamma)$ and

$$\|f\|^2_{L_w^2(\Gamma)} = \sum_{e\in\mathcal{E}(\Gamma)} \|f\|^2_{L_w^2(e)} < \infty.$$

We also need the standard space $L^1_{loc}(\Gamma)$ with respect to the measure $\mathrm{d}x$. It consists of all functions which are absolutely integrable on every edge.

In this paper, we impose at each inner vertex v the following condition:

$$\begin{cases} f \text{ is continuous at } v, \\ \sum\limits_{e\in\mathcal{E}_v} \left(pf'\right)_e(v) = 0, \end{cases} \tag{2.1}$$

which is the so-called Kirchhoff vertex condition. Here $\left(pf'\right)_e(v)$ is the quasi-derivative in the outgoing direction at the vertex v, f_e denotes the restriction of a function f onto the edge e.

Notation 2.1 *In this paper,* $\mathbb{N}$ *denotes the set of positive integers and* $\mathbb{N}_0$ *denotes the set of nonnegative integers.*

3 Allegretto–Piepenbrink-type theorem on metric graphs

In this section, we formulate the Allegretto–Piepenbrink-type theorem for the operator $\mathbf{H}_{\mathbf{t}_q}$, presupposing that the form $\mathbf{t}_q^0$ is closable. It should be mentioned that we allow multigraphs, that is, we allow multiple edges and loops. We shall define a function as a solution of the equation $ly = \lambda y$, $\lambda \in \mathbb{C}$ if it is continuous and edgewise absolutely continuous on Γ, satisfies the equation on each edge $e \in \mathcal{E}(\Gamma)$, and fulfills the Kirchhoff conditions at inner vertices.

Theorem 3.1 (Allegretto–Piepenbrink-type theorem) *For any* $\lambda \in \mathbb{R}$,
(1) *if there exists a positive solution* $y > 0$ *on* Γ *for* $ly = \lambda y$, *then* $\inf\sigma(\mathbf{H}_{\mathbf{t}_q}) \geq \lambda$;
(2) *if* $\inf\sigma(\mathbf{H}_{\mathbf{t}_q}) > \lambda$, *then there exists a positive solution* $y > 0$ *on* Γ *for* $ly = \lambda y$.

Proof (1) Let y be a positive solution of $ly = \lambda y$. Then for any $\eta(\cdot) \in \mathrm{dom}(\mathbf{t}_{\mathrm{q}}^0)$ denote $g(x) = \frac{\eta(x)}{y(x)}$, and thus we have $g \in \mathrm{dom}(\mathbf{t}_{\mathrm{q}}^0)$. Note that

$$\begin{aligned} \mathbf{t}_q^0[\eta] &= \int_\Gamma [p(x)\left|\eta'(x)\right|^2 + q(x)\,|\eta(x)|^2]\mathrm{d}x, \\ &= \int_\Gamma [p|g'y|^2 + p|gy'|^2 + pg'y\overline{gy}' + p\overline{g}'\overline{y}gy' + q\,|\eta|^2]\mathrm{d}x, \end{aligned}$$

and

$$\int_{\Gamma} p|g|^2|y'|^2\mathrm{d}x = p|g|^2y'\overline{y}\Big|_{\partial\Gamma} - \int_{\Gamma}(p|g|^2y')'\overline{y}\mathrm{d}x$$
$$= -\int_{\Gamma}(py')'|g|^2\overline{y}\mathrm{d}x - \int_{\Gamma} py'\overline{y}(g'\overline{g}+\overline{g}'g)\mathrm{d}x.$$

Therefore, taking into account that $y > 0$ hence $y = \bar{y}$, we get:

$$\mathbf{t}_q^0[\eta] = \int_{\Gamma}\left(p|g'y|^2 + \lambda w|g|^2|y|^2\right)\mathrm{d}x \geq \lambda\int_{\Gamma} w|\eta|^2\mathrm{d}x$$

for every $\eta(\cdot) \in \mathrm{dom}(\mathbf{t}_{\mathrm{q}}^0)$. This shows that the lower bound of the form $\mathbf{t}_q$ and thus the operator $\mathbf{H}_{\mathbf{t}_q}$ is not less than λ.

(2) Assume that $\inf\sigma(\mathbf{H}_{\mathbf{t}_q}) > \lambda$. Let $\Gamma' \subset \Gamma$ be any finite compact subgraph obtained by cutting through the interior of edges. Denote

$$\mathbf{t}_{q,\Gamma'}^0[f] := \int_{\Gamma'}[p(x)\left|f'(x)\right|^2 + q(x)\left|f(x)\right|^2]\mathrm{d}x, \tag{3.1}$$

and

$$\mathrm{dom}(\mathbf{t}_{q,\Gamma'}^0) := \{f \in H^1(\Gamma'; p, w) : f|_{\partial\Gamma'} = 0,\ \mathbf{t}_{q,\Gamma'}^0[f] < \infty\}.$$

Note that $\mathrm{dom}(\mathbf{t}_{q,\Gamma'}^0) \subset \mathrm{dom}(\mathbf{t}_q^0)$ in the sense that every function in $\mathrm{dom}(\mathbf{t}_{q,\Gamma'}^0)$ can be extended to be in $\mathrm{dom}(\mathbf{t}_q^0)$ by setting it zero on remaining edges. Thus the form $\mathbf{t}_{q,\Gamma'}^0$ is lower semibounded and closable. Now we define the Dirichlet operator $\mathbf{H}_{\mathrm{D}}^{\Gamma'}$ as follows:

$$\mathbf{H}_{\mathrm{D}}^{\Gamma'}f = lf,$$
$$\mathrm{dom}(\mathbf{H}_{\mathrm{D}}^{\Gamma'}) = \{f \in L_w^2(\Gamma') : f|_{\partial\Gamma'} = 0,\ f \text{ is edgewise absolutely continuous and satisfies (2.1) at inner vertices}, lf \in L_w^2(\Gamma')\}.$$

Then according to the representation theorem, it is standard to show that the self-adjoint operator associated with the closure $\mathbf{t}_{q,\Gamma'} = \overline{\mathbf{t}_{q,\Gamma'}^0}$ is the Dirichlet operator $\mathbf{H}_{\mathrm{D}}^{\Gamma'}$ in $L_w^2(\Gamma')$,which directly yields that

$$\inf\sigma\left(\mathbf{H}_{\mathrm{D}}^{\Gamma'}\right) > \lambda. \tag{3.2}$$

By cutting through the interior of edges of Γ, we can establish a sequence of finite subgraphs Γ_n, with $n \in \mathbb{N}$, that collectively constitute an increasing exhaustion of Γ. Then in view of [24, Theorem 4.1 and Theorem 5.2.2] and (3.2), it is easy to see that there exists a solution $u_n(x)$ of $ly = \lambda y$ that is positive on Γ_n. Define $y_n(x) = \frac{u_n(x)}{u_n(o)}$. Then it can be seen that

$$ly_n = \lambda y_n,\ \inf_{x\in\Gamma_n} y_n(x) > 0 \text{ and } y_n(o) = 1.$$

Next, we shall prove that $\{y_n\}_{n=m}^{\infty}$ and $\left\{py_n'\right\}_{n=m}^{\infty}$ are uniformly bounded and equicontinuous on each $e \in \mathcal{E}(\Gamma_m)$. By using the Harnack inequality given by F. Gesteszy and Z. Zhou in their work [7] and the Kirchhoff conditions at inner vertices, it is easy to obtain that for each positive integer n, there exist positive constants C_{m1} and C_{m2} (depending only on m) such that

$$C_{m2} \leq \inf_{\substack{x\in\Gamma_m\\ m<n}} y_n(x) \leq \sup_{\substack{x\in\Gamma_m\\ m<n}} y_n(x) \leq C_{m1}. \tag{3.3}$$

This means that $\{y_n(x)\}_{n=m}^{\infty}$ is uniformly bounded on Γ_m. For each $e \in \mathcal{E}(\Gamma_m)$ and $x_1, x_2 \in e$,

$$|y_n(x_1) - y_n(x_2)| = \left|\int_{x_1}^{x_2} \frac{1}{p} py_n' \mathrm{d}t\right| \leq \sup_{x\in e} \left|py_n'\right| \int_{x_1}^{x_2} \frac{1}{p} \mathrm{d}t,$$

thus we can show that $\{y_n(x)\}_{n=m}^{\infty}$ is equicontinuous on each $e \in \mathcal{E}(\Gamma_m)$ if $\left\{py_n'(x)\right\}_{n=m}^{\infty}$ is uniformly bounded on $e \in \mathcal{E}(\Gamma_m)$. Note that $y_n(x)$ is a solution of the equation $ly = \lambda y$ on Γ_n, which yields that $py_n'(\cdot)$ is absolutely continuous on each $e \in \mathcal{E}(\Gamma_n)$. Therefore, following the mean value theorem for integrals, one has for each $e \in \mathcal{E}(\Gamma_m)$, there exists a point $x_e \in e$ such that

$$py_n'(x_e) = \frac{\int_e \frac{1}{p} \cdot py_n' \mathrm{d}t}{\int_e \frac{1}{p} \mathrm{d}t} \leq \frac{2 \max_{x\in e} |y_n(x)|}{\int_e \frac{1}{p} \mathrm{d}t} \leq \frac{2C_{m1}}{\int_e \frac{1}{p} \mathrm{d}t}. \tag{3.4}$$

Since $ly_n = \lambda y_n$, for $x \in e$, one has

$$|py_n'(x) - py_n'(x_e)| \leq \int_e (|q| + |\lambda| w) |y_n(t)| \mathrm{d}t$$

and thus

$$|py_n'(x)| \leq |py_n'(x_e)| + \int_e (|q| + |\lambda| w) |y_n(t)| \mathrm{d}t.$$

Then the uniform boundedness of $\left\{py_n'(x)\right\}_{n=m}^{\infty}$ on each $e \in \mathcal{E}(\Gamma_m)$ follows from (3.4) and the uniform boundedness of $\{y_n(x)\}_{n=m}^{\infty}$ on Γ_m. Moreover, for each $e \in \mathcal{E}(\Gamma_m)$ and $x_1, x_2 \in e$,

$$\left|py_n'(x_1) - py_n'(x_2)\right| \leq \int_{x_1}^{x_2} (|q| + |\lambda| w) |y_n(t)| \mathrm{d}t \leq C \cdot |x_1 - x_2|,$$

which yields the equicontinuity of $\left\{py_n'(x)\right\}_{n=m}^{\infty}$ on each $e \in \mathcal{E}(\Gamma_m)$.

Then it follows from Arzela-Ascoli Theorem that there exists a subsequence $\left\{y_{n_j}\right\} \subset \{y_n\}_{n=m}^{\infty}$ such that $\left\{y_{n_j}\right\}$ and $\left\{py_{n_j}'\right\}$ are uniformly convergent on each edge of Γ_m.

By diagonalization, we can extract a subsequence $\left\{y_{n_{j,j}}(x)\right\}$ and edgewise continuous functions f and g defined on Γ such that for all $\Gamma_m \subset \Gamma$,

$$\sup_{x\in\Gamma_m} \left|y_{n_{j,j}} - f\right| + \sup_{x\in\Gamma_m} \left|py'_{n_{j,j}} - g\right| \to 0, \text{ as } j \to \infty.$$

Based the above considerations, now we aim to prove that the function f is a positive solution of the equation $ly = \lambda y$ on Γ. For each $e \in \mathcal{E}(\Gamma)$,

$$y_{n_j}(x) - y_{n_j}(o(e)) = \int_{o(e)}^{x} \frac{1}{p} \cdot py'_{n_j} \mathrm{d}t.$$

Then it follows from the dominated convergence theorem that

$$f(x) - f(o(e)) = \int_{o(e)}^{x} \frac{1}{p} g \mathrm{d}t,$$

which yields that $pf' = g$. Also,

$$py'_{n_{j,j}}(x) - py'_{n_{j,j}}(o(e)) = \int_{o(e)}^{x} (q - \lambda w)\, y_{n_{j,j}}(t) \mathrm{d}t,$$

which yields that $g' = (q - \lambda w) f$. Clearly, f satisfies the Kirchhoff conditions at inner vertices. Moreover, it is immediately seen from (3.3) that $f > 0$ on Γ. The proof is completed. □

Remark 3.1 Observe that following the proof of Theorem 3.1, one can also easily derive the inequality $\inf \sigma(\mathbf{H}_{\mathbf{t}_q}) \geq \lambda$ if there exists a positive function y that satisfies $ly - \lambda y \geq 0$ on every edge $e \in \mathcal{E}(\Gamma)$, and if this function fulfills the Kirchhoff conditions at inner vertices, along with the requirement that y and its derivative py' are edgewise absolutely continuous.

4 Persson-type theorem on metric graphs

In this section, we illustrate that under the following Hypothesis 4.1, the Persson-type theorem for $\mathbf{H}_{\mathbf{t}_q}$ can be given, see Theorem 4.1. Here and thereafter we use the following notation $a_+ = \max\{a, 0\}$ and $a_- = -\min\{a, 0\}$.

Hypothesis 4.1 (1) $1/p \in L^{\eta}(\Gamma)$, $\eta \in [1, +\infty]$, $q \in L^1_{loc}(\Gamma)$, $w \in L^1_{loc}(\Gamma)$;
(2) *there exists a compact subgraph* $\Gamma_c \subset \Gamma$ *such that*

$$C_w := \operatorname*{ess\,inf}_{x\in\Gamma\backslash\Gamma'} w > 0;$$

(3) $\inf_{e\in\mathcal{E}(\Gamma)} |e| = d_* > 0$; (4) $C_q := \sup_{e\in\mathcal{E}(\Gamma)} \int_e q_- \mathrm{d}t < +\infty$.

Remark 4.1 Taking into account Lemma 4.3, it is apparent that the lower semibounded form $\mathbf{t}_q^0$ is closable whenever Hypothesis 4.1 is satisfied.

Remark 4.2 Due to [6, Theorem 3.5 (viii)] and Hypothesis 4.1 (3), the essential spectrum of $\mathbf{H}_{\mathbf{t}_q}$ is always non-empty, i.e., $\sigma_{\text{ess}}\left(\mathbf{H}_{\mathbf{t}_q}\right) \neq \emptyset$.

Notation 4.1 *Fix a vertex $o \in \Gamma$. Throughout this section, for any integer $n > 0$, let $\Gamma^{(n)} \subset \Gamma$ be the union of all edges e such that both endpoints of e are at a distance at most n from o.*

To give the Persson-type theorem, let us introduce the form $\mathbf{t}_{q,n}^0$ as follows:

$$\mathbf{t}_{q,n}^0[f] = \int_{\Gamma\setminus\Gamma^{(n)}} p(x)\left|f'(x)\right|^2 + q(x)\left|f(x)\right|^2 \mathrm{d}x,$$

and

$$\mathrm{dom}(\mathbf{t}_{q,n}^0) = \{f \in H_c^1(\overline{\Gamma\setminus\Gamma^{(n)}}; p, w) : f|_{\partial\left(\overline{\Gamma\setminus\Gamma^{(n)}}\right)} = 0,\ f|_{\overline{\Gamma\setminus\Gamma^{(n)}}\cap\Gamma^{(n)}} = 0\}.$$

In view of Remark 4.5, it follows that $\mathbf{t}_{q,n}^0$ is closable whenever Hypothesis 4.1 is satisfied. Then we denote $\mathbf{H}_{\mathbf{t}_{q,n}}$ as the self-adjoint operator associated with $\mathbf{t}_{q,n} := \overline{\mathbf{t}_{q,n}^0}$.

Theorem 4.1 *(Persson-type theorem) Suppose Hypothesis 4.1 holds. Then*

$$\inf \sigma_{ess}\left(\mathbf{H}_{\mathbf{t}_q}\right) = \lim_{n\to\infty} \inf \sigma\left(\mathbf{H}_{\mathbf{t}_{q,n}}\right).$$

Remark 4.3 Based on the definitions of $\mathbf{H}_{\mathbf{t}_{q,n}}$, one observes that the sequence $\left\{\inf \sigma\left(\mathbf{H}_{\mathbf{t}_{q,n}}\right)\right\}$ is nondecreasing. Consequently, the limit $\lim_{n\to\infty} \inf \sigma\left(\mathbf{H}_{\mathbf{t}_{q,n}}\right)$ either exists as a finite value or diverges to $+\infty$.

Before proving the Persson-type theorem, we need some preliminary lemmas and notations.

Lemma 4.1 *Suppose the conditions $(1)-(3)$ in Hypothesis 4.1 are satisfied. For every $\epsilon > 0$, there exists a constant C_ϵ such that for all $e \in \mathcal{E}(\Gamma)$,*

$$\sup_{x\in e}|f(x)|^2 \leq \epsilon \int_e p(x)\left|f'(x)\right|^2 \mathrm{d}x + C_\epsilon \int_e w(x)\left|f(x)\right|^2 \mathrm{d}x \tag{4.1}$$

for every $f \in H^1(\Gamma; p, w)$.

Proof Since $1/p \in L^\eta(\Gamma)$ for some $\eta \in [1, +\infty]$, for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $e \in \mathcal{E}(\Gamma)$ and $x \in e$, one has

$$\int_{e\cap\Gamma(x;\delta)} \frac{1}{p(t)}\mathrm{d}t < \frac{\epsilon}{2}, \tag{4.2}$$

where $\Gamma(x;\delta)=\{y\in\Gamma\,|\rho(x,y)\le\delta\}$. We can assume that $\delta<\frac{d_*}{2}$. In fact, it is easy to prove the case when $\eta=1$ or $\eta=+\infty$; for $\eta\in(1,+\infty)$, this can be seen with the help of the Hölder inequality. For instance, if $1/p\in L^2(\Gamma)$, for every $\epsilon>0$, there exists $\delta(\epsilon)>0$ such that for all $x\in\Gamma$,

$$\int_{e\cap\Gamma(x;\delta)}\frac{1}{p(t)}\mathrm{d}t\le\left(\int_{e\cap\Gamma(x;\delta)}1^2\mathrm{d}t\int_{e\cap\Gamma(x;\delta)}\frac{1}{p^2(t)}\mathrm{d}t\right)^{\frac{1}{2}}$$
$$<d_*\cdot\left(\int_{e\cap\Gamma(x;\delta)}\frac{1}{p^2(t)}\mathrm{d}t\right)^{\frac{1}{2}}<\frac{\epsilon}{2}.$$

Under the condition on w, there exists $c>0$ such that for all $x\in\Gamma$,

$$\int_{\Gamma(x;\frac{\delta}{2})}w(t)\mathrm{d}t>c. \tag{4.3}$$

For $f\in H^1(\Gamma;p,w)$ and $x,y\in e\in\mathcal{E}(\Gamma)$, one has

$$|f(x)|^2\le2|f(y)|^2+2|f(x)-f(y)|^2=2|f(y)|^2+2\left|\int_x^y f'(t)\,\mathrm{d}t\right|^2$$
$$\le2|f(y)|^2+2\int_x^y p(t)\left|f'(t)\right|^2\mathrm{d}t\int_x^y\frac{1}{p(t)}\mathrm{d}t. \tag{4.4}$$

We multiply (4.4) by $w(y)$ and integrate over $I(x,e)=e\cap\Gamma(x;\delta)$ for arbitrary $e\in\mathcal{E}(\Gamma)$ and $x\in e$. Note that the volume of $I(x,e)$ is no less than δ. Then

$$\begin{aligned}&|f(x)|^2\int_{I(x,e)}w(y)dy\\&\le2\int_{I(x,e)}|f(y)|^2w(y)dy+2\int_{I(x,e)}w(y)\left(\int_x^y p(t)\left|f'(t)\right|^2\mathrm{d}t\int_x^y\frac{1}{p(t)}\mathrm{d}t\right)dy\\&\le2\int_{I(x,e)}|f(y)|^2w(y)dy+2\int_{I(x,e)}w(y)dy\int_e p(y)\left|f'(y)\right|^2dy\int_{I(x,e)}\frac{1}{p(y)}dy.\end{aligned} \tag{4.5}$$

Dividing the both sides of the last inequality from (4.5) by $\int_{I(x,e)}w(y)dy$ and letting $C_\epsilon=\frac{2}{c}$, one proves the statement. □

Lemma 4.2 *Suppose Hypothesis 4.1 holds. Then for every $\epsilon>0$, there exists a constant C_ϵ such that*

$$\int_\Gamma q_-(x)\,|f(x)|^2\,\mathrm{d}x\le\epsilon\int_\Gamma p(x)\left|f'(x)\right|^2\mathrm{d}x+C_\epsilon\int_\Gamma w(x)\,|f(x)|^2\,\mathrm{d}x \tag{4.6}$$

for all $f\in H^1(\Gamma;p,w)$. Moreover, the form $\mathbf{t}_q^0$ is lower semibounded.

Proof The claim (4.6) is a direct consequence of Lemma 4.1 in virtue of

$$\begin{aligned}\int_\Gamma q_-(x)\,|f(x)|^2\,\mathrm{d}x &= \sum_{e\in\mathcal{E}(\Gamma)}\int_e q_-(x)\,|f(x)|^2\,\mathrm{d}x \\ &\le \left(\sup_{e\in\mathcal{E}(\Gamma)}\int_e q_-(x)\mathrm{d}x\right)\sum_{e\in\mathcal{E}(\Gamma)}\left\|f^2\right\|_{L^\infty(e)}.\end{aligned}$$

Taking into account inequality (4.6) and letting $\epsilon = \frac{1}{2}$, we obtain

$$\begin{aligned}\mathbf{t}_q^0[f] &= \int_\Gamma [p(x)\left|f'(x)\right|^2 + q_+(x)\,|f(x)|^2]\mathrm{d}x - \int_\Gamma q_-(x)\,|f(x)|^2\,\mathrm{d}x \\ &\ge -C_{\frac{1}{2}}\int_\Gamma w(x)\,|f(x)|^2\,\mathrm{d}x,\end{aligned}$$

which implies that the form $\mathbf{t}_q^0$ is lower semibounded. □

Remark 4.4 Suppose Hypothesis 4.1 holds. Then Lemma 4.2 also holds for the graphs $\overline{\Gamma\backslash\Gamma^{(n)}}$ according to the definition of $\Gamma^{(n)}$.

Lemma 4.3 *Suppose Hypothesis 4.1 holds. The following form*

$$\begin{aligned}\mathbf{s}_q^0[f] &= \int_\Gamma [p(x)\left|f'(x)\right|^2 + q(x)\,|f(x)|^2]\mathrm{d}x, \\ \mathrm{dom}(\mathbf{s}_\mathrm{q}^0) &= \left\{f\in H^1(\Gamma;p,w) : \mathbf{s}_q^0[f] < \infty,\ f|_{\partial\Gamma} = 0\right\}\end{aligned}$$

is lower semibounded and closed. Moreover, $\mathbf{t}_q \subset \mathbf{s}_q^0$.

Proof Define

$$\begin{aligned}\mathbf{q}_-[f] &:= -\int_\Gamma q_-(x)\,|f(x)|^2\,\mathrm{d}x, \\ \mathrm{dom}(\mathbf{q}_-) &:= \{f\in L_w^2(\Gamma) : |\mathbf{q}_-[f]| < \infty\}.\end{aligned}$$

It follows from Lemma 4.2 that the form $\mathbf{q}_-$ is infinitesimally $\mathbf{s}_{q+}^0$ bounded. It is obvious that $\mathbf{s}_{q+}^0$ is lower semibounded and closed. Applying KLMN theorem [13], we complete the proof. □

Remark 4.5 Suppose Hypothesis 4.1 holds. The forms

$$\begin{aligned}\mathbf{s}_{q,n}^0[f] &= \int_{\Gamma\backslash\Gamma^{(n)}} [p(x)\left|f'(x)\right|^2 + q(x)\,|f(x)|^2]\mathrm{d}x, \\ \mathrm{dom}(\mathbf{s}_{q,n}^0) &= \left\{f\in H^1(\Gamma\backslash\Gamma^{(n)};p,w) : \mathbf{s}_{q,n}^0[f] < \infty,\ f|_{\partial\left(\overline{\Gamma\backslash\Gamma^{(n)}}\right)} = 0,\ f|_{\overline{\Gamma\backslash\Gamma^{(n)}}\cap\Gamma^{(n)}} = 0\right\}\end{aligned}$$

are lower semibounded and closed. Moreover, $\mathbf{t}_{q,n} \subset \mathbf{s}_{q,n}^0$.

Lemma 4.4 *Let* $\mathbf{s}$ *be a closed quadratic form on* $L^2_w(\Gamma)$ *that is bounded from below and let* $\mathbf{H}$ *be the corresponding self-adjoint operator. Assume that there is a normalized sequence* f_n *in* $\operatorname{dom}(\mathbf{s})$ *that converges weakly to zero. Then*

$$\inf \sigma_{ess}(\mathbf{H}) \leq \liminf_{n\to\infty} \mathbf{s}[f_n].$$

Proof see [10]. □

Now we are in a position to prove the Persson-type Theorem for the operator $\mathbf{H}_{\mathbf{t}_q}$.

Proof of Theorem 4.1 Firstly, we prove that

$$\inf \sigma_{ess}\left(\mathbf{H}_{\mathbf{t}_q}\right) \geq \lim_{n\to\infty} \inf \sigma\left(\mathbf{H}_{\mathbf{t}_{q,n}}\right) =: r. \tag{4.7}$$

For any $\lambda \in \sigma_{ess}\left(\mathbf{H}_{\mathbf{t}_q}\right)$, we shall prove that $\lambda \geq r$. From Weyl theorem, one can choose a sequence $\{u_m\}_{m=1}^{\infty} \subset \operatorname{dom}\left(\mathbf{H}_{\mathbf{t}_q}\right)$ such that

$$\|u_m\|_{L^2_w(\Gamma)} = 1 \text{ for all } m, \tag{4.8}$$

$$\begin{aligned} &u_m \xrightarrow{w} 0 \text{ as } m \to \infty, \\ &\left\|\mathbf{H}_{\mathbf{t}_q} u_m - \lambda u_m\right\|_{L^2_w(\Gamma)} \to 0 \text{ as } m \to \infty. \end{aligned}$$

Denote

$$\widetilde{\Gamma^{(n)}} := \Gamma^{(n)} \cup \left\{e \in \mathcal{E}(\Gamma) : e \text{ is an edge with only one vertex } v \text{ in } \Gamma^{(n)}\right\}.$$

Then $\Gamma \backslash \widetilde{\Gamma^{(n)}}$ is the union of all edges which do not have vertices in $\Gamma^{(n)}$. Now define functions φ_n on Γ such that

$$\varphi_n = 0 \text{ for } x \in \Gamma^{(n)}, \tag{4.9}$$

$$\varphi_n = 1 \text{ for } x \in \Gamma \backslash \widetilde{\Gamma^{(n)}}, \tag{4.10}$$

$$0 \leq \varphi_n \leq 1, \tag{4.11}$$

$$\frac{\sqrt{p}}{\sqrt{w}} \varphi_n' \text{ bounded in } \Gamma.$$

Let e be an edge with only one vertex v_e in $\Gamma^{(n)}$. Without loss of generality, assume that v_e is the initial vertex of e. Then for $x \in e$, we can define $\varphi_n(x) = 1 - \frac{\int_x^{t(e)} \frac{\sqrt{w}}{\sqrt{p}} \mathrm{d}t}{\int_e \frac{\sqrt{w}}{\sqrt{p}} \mathrm{d}t}$.

Now denote $f_{m,n} := \varphi_n u_m$. We are going to prove that $\left\{f_{m,n}\right\}_{m=1}^{\infty} \subset \operatorname{dom}(\mathbf{t}_{q,n})$ and as $m \to \infty$,

$$\mathbf{t}_{q,n}\left[f_{m,n}\right] \leq \lambda + o(1) \tag{4.12}$$

and

$$\left\| f_{m,n} \right\|^2_{L^2_w(\Gamma\setminus\Gamma^{(n)})} = 1 + o(1). \tag{4.13}$$

We observe that for any $u_m \in \mathrm{dom}\big(\mathbf{H}_{\mathbf{t}_q}\big)$, there exists a sequence $\{g_{k,m}\}_{k=1}^{\infty} \subset \mathrm{dom}\big(\mathbf{t}_q^0\big)$ such that

$$\left\| g_{k,m} - u_m \right\|_{\mathbf{t}_q} \to 0 \text{ as } k \to \infty. \tag{4.14}$$

In order to prove $\{f_{m,n}\}_{m=1}^{\infty} \in \mathrm{dom}(\mathbf{t}_{q,n})$, it is enough to prove that $\varphi_n g_{k,m} \in \mathrm{dom}\big(\mathbf{t}_{q,n}^0\big)$ and $\left\|\varphi_n g_{k,m} - \varphi_n u_m\right\|_{\mathbf{t}_{q,n}} \to 0$ as $k \to \infty$. In fact, by use of the properties of φ_n and the fact $\{g_{k,m}\}_{k=1}^{\infty} \subset \mathrm{dom}\big(\mathbf{t}_q^0\big)$, we get $\{\varphi_n g_{k,m}\}_{k=1}^{\infty} \subset \mathrm{dom}(\mathbf{q}_n)$ and

$$\begin{aligned}
\int_{\Gamma\setminus\Gamma^{(n)}} p \left|(\varphi_n g_{k,m})'\right|^2 \mathrm{d}x &\le 2\left(\int_{\Gamma\setminus\Gamma^{(n)}} p\left|\varphi_n g'_{k,m}\right|^2 \mathrm{d}x + \int_{\Gamma\setminus\Gamma^{(n)}} p \left|\varphi'_n g_{k,m}\right|^2 \mathrm{d}x\right)\\
&\le C\left(\int_{\Gamma\setminus\Gamma^{(n)}} p\left|g'_{k,m}\right|^2 \mathrm{d}x + \int_{\Gamma\setminus\Gamma^{(n)}} \left|g_{k,m}\right|^2 w \mathrm{d}x\right),
\end{aligned}$$

where C is some constant, which implies that $\varphi_n g_{k,m} \in \mathrm{dom}\big(\mathbf{t}_{q,n}^0\big)$. From Lemma 4.2, it follows that

$$\begin{aligned}
\left\|g_{k,m} - u_m\right\|_{\mathbf{t}_q} &= \mathbf{t}_q\left[g_{k,m} - u_m\right] + (c+1)\int_\Gamma w\left|g_{k,m} - u_m\right|^2 \mathrm{d}x\\
&\ge \frac{1}{2}\int_\Gamma p\left|(g_{k,m} - u_m)'\right|^2 \mathrm{d}x + \int_\Gamma q_+ \left|g_{k,m} - u_m\right|^2 \mathrm{d}x\\
&\quad - C_{\frac{1}{2}}\int_\Gamma w\left|g_{k,m} - u_m\right|^2 \mathrm{d}x + \int_\Gamma (c+1)\, w\left|g_{k,m} - u_m\right|^2 \mathrm{d}x,
\end{aligned}$$

where $c > 0$ is some positive constant such that $\mathbf{t}_q \ge -c$. Therefore, we see from (4.14) and Lemma 4.2 that the expressions

$$\int_\Gamma p\left|(g_{k,m} - u_m)'\right|^2 \mathrm{d}x, \quad \int_\Gamma w\left|g_{k,m} - u_m\right|^2 \mathrm{d}x \text{ and } \int_\Gamma |q|\left|g_{k,m} - u_m\right|^2 \mathrm{d}x$$

all tend to zero as $k \to \infty$. Hence

$$\begin{aligned}
&\left\|\varphi_n g_{k,m} - \varphi_n u_m\right\|_{\mathbf{t}_{q,n}}\\
&= \int_{\Gamma\setminus\Gamma^{(n)}} p\left|\left(\varphi_n\left(g_{k,m} - u_m\right)\right)'\right|^2 + \left(q + (c_n+1)\, w\right)\left|\varphi_n\left(g_{k,m} - u_m\right)\right|^2 \mathrm{d}x\\
&\le C\left(\int_\Gamma p\left|(g_{k,m} - u_m)'\right|^2 \mathrm{d}x + \int_\Gamma w\left|g_{k,m} - u_m\right|^2 \mathrm{d}x\right)\\
&\quad + \int_\Gamma \left(|q| + (c_n+1)\, w\right)\left|g_{k,m} - u_m\right|^2 \mathrm{d}x \to 0, \text{ as } k \to \infty,
\end{aligned}$$

where $c_n > 0$ are positive constants such that $\mathbf{t}_{q,n} \geq -c_n$. This proves $\left\{f_{m,n}\right\}_{m=1}^{\infty} \subset \operatorname{dom}(\mathbf{t}_{q,n})$.

Next, we aim to prove (4.12). In virtue of the properties of $\{u_m\}_{m=1}^{\infty}$,

$$\begin{aligned}\mathbf{t}_q\left[u_m\right]-\lambda=(\mathbf{H}_{\mathbf{t}_q} u_m, u_m)-\lambda &\leq \left\|\mathbf{H}_{\mathbf{t}_q} u_m\right\|_{L_w^2(\Gamma)}\|u_m\|_{L_w^2(\Gamma)}-\lambda\|u_m\|_{L_w^2(\Gamma)} \\ &\leq \left\|\mathbf{H}_{\mathbf{t}_q} u_m-\lambda u_m\right\|_{L_w^2(\Gamma)},\end{aligned}$$

and thus

$$\mathbf{t}_q\left[u_m\right] \leq \lambda+o(1). \tag{4.15}$$

Therefore, it follows from Lemma 4.2 that as $m \to \infty$,

$$\int_{\Gamma} p\left|u_m'\right|^2 \mathrm{d}x \leq \lambda+\int_{\Gamma} q_-\left|u_m\right|^2 \mathrm{d}x+o(1) \tag{4.16}$$

$$\leq \lambda+\frac{1}{2} \int_{\Gamma} p\left|u_m'\right|^2 \mathrm{d}x+C_{\frac{1}{2}} \int_{\Gamma} w\left|u_m\right|^2 \mathrm{d}x+o(1) \tag{4.17}$$

which implies that

$$\int_{\Gamma} p\left|u_m'\right|^2 \mathrm{d}x \leq 2\left(\lambda+C_{\frac{1}{2}}+o(1)\right). \tag{4.18}$$

We also observe that for fixed n,

$$\int_{\Gamma^{(n)}}\left|u_m\right|^2 w \mathrm{d}x \to 0 \text{ as } m \to \infty. \tag{4.19}$$

Let e_o be an edge incident to o. Then $u_m(o)=u_m(t)-\int_o^t u_m'(s) \mathrm{d}s$ for $t \in e_o$. Multiplying the both sides of the last equality by $w(t)$ and integrating over e_o, we get:

$$u_m(o) \int_{e_o} w(t) \mathrm{d}t=\int_{e_o} w(t) u_m(t) \mathrm{d}t-\int_{e_o} w(t) \int_0^t u_m'(s) \mathrm{d}s \mathrm{d}t,$$

then

$$\begin{aligned}|u_m(o)| \int_{e_o} w(t) \mathrm{d}t &\leq\left|\int_{e_o} w(t) u_m(t) \mathrm{d}t\right|+\left|\int_{e_o} w(t) \int_0^t u_m'(s) \mathrm{d}s \mathrm{d}t\right| \\ &\leq\left(\int_{e_o} w(t) \mathrm{d}t \int_{e_o} w(t)\left|u_m(t)\right|^2 \mathrm{d}t\right)^{\frac{1}{2}} \\ &\quad+\left(\int_{e_o} w(t) \mathrm{d}t \int_{e_o} \frac{1}{p(t)} \mathrm{d}t \int_{e_o} p(t)\left|u_m'(t)\right|^2 \mathrm{d}t\right)^{\frac{1}{2}}.\end{aligned}$$

Therefore, it follows from (4.8) and (4.18) that $\{u_m(o)\}$ is bounded. Moreover, for any $x_1, x_2 \in \Gamma^{(n)}$

$$\left|u_m\left(x_1\right)-u_m\left(x_2\right)\right|=\left|\int_{x_1}^{x_2} u_m'(t) \mathrm{d}t\right| \leq\left(\int_{x_1}^{x_2} \frac{1}{p} \mathrm{d}t \cdot \int_{x_1}^{x_2} p\left|u_m'\right|^2 \mathrm{d}x\right)^{\frac{1}{2}}. \tag{4.20}$$

Relation (4.20) together with (4.18) and the boundedness of $\{u_m(o)\}$ yields that $\{u_m\}$ are uniformly bounded and uniformly equicontinuous on $\Gamma^{(n)}$. Then it follows from Arzela-Ascoli theorem that there is a subsequence $\{u_{m_k}\}$, which is convergent in $L^2_w(\Gamma^{(n)})$. Since $u_{m_k} \xrightarrow{w} 0$ as $k \to \infty$, the limit function must be zero, that is $\|u_{m_k}\|_{L^2_w(\Gamma^{(n)})} \to 0$ as $k \to \infty$. But then the original sequence itself must have this property, since otherwise we could get a contradiction by applying the arguments above to a suitable subsequence. Hence (4.19) is proved.

By use of the definition of φ_n, (4.18) and (4.19), one has

$$\begin{aligned}
&\int_{\Gamma\setminus\Gamma^{(n)}} p\left|(\varphi_n u_m)'\right|^2 \mathrm{d}x \\
&\leq \int_{\Gamma\setminus\Gamma^{(n)}} p\left|\varphi_n u_m'\right|^2 \mathrm{d}x + 2\int_{\Gamma\setminus\Gamma^{(n)}} p\left|\varphi_n u_m'\right|\left|\varphi_n' u_m\right| \mathrm{d}x + \int_{\Gamma\setminus\Gamma^{(n)}} p\left|\varphi_n' u_m\right|^2 \mathrm{d}x \\
&\leq \int_{\Gamma\setminus\Gamma^{(n)}} p\left|u_m'\right|^2 \mathrm{d}x + C\left(\int_{\widetilde{\Gamma^{(n)}}} p\left|u_m'\right|^2 \mathrm{d}x \int_{\widetilde{\Gamma^{(n)}}} |u_m|^2\, w\mathrm{d}x\right)^{\frac{1}{2}} + C\int_{\widetilde{\Gamma^{(n)}}} |u_m|^2\, w\mathrm{d}x \\
&\leq \int_{\Gamma} p\left|u_m'\right|^2 \mathrm{d}x + o(1) \text{ as } m \to \infty. \qquad (4.21)
\end{aligned}$$

From Lemma 4.1, it follows that for every $\epsilon > 0$, there exists a constant C_ϵ such that

$$\begin{aligned}
\int_{\widetilde{\Gamma^{(n)}}} |q|\,|u_m|^2 \mathrm{d}x &= \sum_{e\in\mathcal{E}\left(\widetilde{\Gamma^{(n)}}\right)} \int_e |q|\,|u_m|^2 \mathrm{d}x \leq \left(\sup_{e\in\mathcal{E}\left(\widetilde{\Gamma^{(n)}}\right)} \int_e |q|\,\mathrm{d}t\right) \sum_{e\in\mathcal{E}\left(\widetilde{\Gamma^{(n)}}\right)} \left\|u_m^2\right\|_{L^\infty(e)} \\
&\leq \epsilon C_n \int_{\widetilde{\Gamma^{(n)}}} p\left|u_m'\right|^2 \mathrm{d}x + C_\epsilon C_n \int_{\widetilde{\Gamma^{(n)}}} w\,|u_m|^2 \mathrm{d}x,
\end{aligned}$$

where $C_n := \sup_{e\in\mathcal{E}\left(\widetilde{\Gamma^{(n)}}\right)} \int_e |q|\,\mathrm{d}t$. This together with (4.18) and (4.19) yields that

$$\int_{\widetilde{\Gamma^{(n)}}} |q|\,|u_m|^2 \mathrm{d}x = o(1) \text{ as } m \to \infty.$$

Therefore,

$$\begin{aligned}
\int_{\Gamma\setminus\Gamma^{(n)}} q\,|\varphi_n u_m|^2 \mathrm{d}x &= \int_{\widetilde{\Gamma^{(n)}}\setminus\Gamma^{(n)}} q\,|\varphi_n u_m|^2 \mathrm{d}x + \int_{\Gamma\setminus\widetilde{\Gamma^{(n)}}} q\,|u_m|^2 \mathrm{d}x \\
&= \int_{\widetilde{\Gamma^{(n)}}} q\,|\varphi_n u_m|^2 \mathrm{d}x + \int_{\Gamma} q\,|u_m|^2 \mathrm{d}x - \int_{\widetilde{\Gamma^{(n)}}} q\,|u_m|^2 \mathrm{d}x \\
&= \int_{\Gamma} q\,|u_m|^2 \mathrm{d}x + o(1) \text{ as } m \to \infty. \qquad (4.22)
\end{aligned}$$

Relation (4.22) together with (4.15) and (4.21) implies (4.12).

To prove (4.13), we note from (4.9), (4.10), (4.11) and (4.19) that

$$
\begin{aligned}
0 &\le \int_{\Gamma} |u_m|^2 \, w\mathrm{d}x - \int_{\Gamma\setminus\Gamma^{(n)}} |\varphi_n u_m|^2 \, w\mathrm{d}x = \int_{\Gamma} |u_m|^2 \, w\mathrm{d}x - \int_{\Gamma} |\varphi_n u_m|^2 \, w\mathrm{d}x \\
&= \int_{\Gamma} |u_m|^2 \, w\mathrm{d}x - \left[\int_{\Gamma\setminus\widetilde{\Gamma^{(n)}}} |\varphi_n u_m|^2 \, w\mathrm{d}x + \int_{\widetilde{\Gamma^{(n)}}} |\varphi_n u_m|^2 \, w\mathrm{d}x \right] \\
&= \int_{\widetilde{\Gamma^{(n)}}} |u_m|^2 \, w\mathrm{d}x - \int_{\widetilde{\Gamma^{(n)}}} |\varphi_n u_m|^2 \, w\mathrm{d}x \to 0 \text{ as } m \to \infty.
\end{aligned}
$$

Hence (4.13) follows from (4.8).

It follows from the definition of r that for any given number $\epsilon > 0$, there exists a positive number N such that for all $n > N$,

$$
\inf \sigma \left(\mathbf{H}_{\mathbf{t}_{q,n}} \right) \ge r - \epsilon, \text{ i.e.,} \inf_{f \in \operatorname{dom}(\mathbf{t}_{q,n})} \frac{\mathbf{t}_{q,n}[f]}{\|f\|^2_{L^2_w(\Gamma\setminus\Gamma^{(n)})}} \ge r - \epsilon.
$$

This yields that for every $f \in \operatorname{dom}(\mathbf{t}_{q,n})$,

$$
\mathbf{t}_{q,n}[f] \ge (r - \epsilon) \|f\|^2_{L^2_w(\Gamma\setminus\Gamma^{(n)})}.
$$

Combining this with (4.12) and (4.13) we immediately get

$$
(r - \epsilon)[1 + o(1)] \le \mathbf{t}_{q,n}\left[f_{m,n}\right] \le \lambda + o(1) \text{ as } m \to \infty.
$$

Since ϵ is arbitrary, (4.7) is proved.

The reverse inequality follows from Lemma 4.4. In fact, we can pick a sequence of functions $f_n \in \operatorname{dom}(\mathbf{t}_{q,n})$ vanishing on $\Gamma^{(n)}$ and satisfying $\|f_n\|^2_{L^2_w(\Gamma)} = 1$ such that

$$
\left| \inf \sigma \left(\mathbf{H}_{\mathbf{t}_{q,n}} \right) - \mathbf{t}_{q,n}[f_n] \right| \le \frac{1}{n}
$$

for all $n \in \mathbb{N}$. Then $\{f_n\}$ converges weakly to zero. Moreover, by construction

$$
\lim_{n\to\infty} \mathbf{t}_q[f_n] = \lim_{n\to\infty} \mathbf{t}_{q,n}[f_n] = \lim_{n\to\infty} \inf \sigma \left(\mathbf{H}_{\mathbf{t}_{q,n}} \right).
$$

Now Lemma 4.4 gives the desired inequality. □

Remark 4.6 With slightly modifications, Theorem 4.1 can be extended to the case when we only assume $1/p,\ w \in L^1_{loc}(\Gamma)$ and $q \ge 0$ without the restriction $\inf_{e\in\mathcal{E}(\Gamma)} |e| = d_* > 0$.

Remark 4.7 Theorem 4.1 admits an obvious extension to the following general case. Suppose Hypothesis 4.1 holds. Define

$$\mathbf{s}_q\left[f\right]=\int_{\Gamma}[p(x)\left|f'(x)\right|^2+q(x)\left|f(x)\right|^2]\mathrm{d}x, \tag{4.23}$$

$$\mathbf{s}_{q,n}\left[f\right]=\int_{\Gamma\backslash\Gamma^{(n)}}[p(x)\left|f'(x)\right|^2+q(x)\left|f(x)\right|^2]\mathrm{d}x \tag{4.24}$$

on the respective domains

$$dom(\mathbf{s}_q)=\{f\in H^1(\Gamma;p,w):|\mathbf{q}\left[f\right]|<\infty\},$$
$$dom(\mathbf{s}_{q,n})=\{f\in H^1(\Gamma\backslash\Gamma^{(n)};p,w):|\mathbf{q}_n\left[f\right]|<\infty\}.$$

By a similar proof to that of Lemma 4.3, one has $\mathbf{s}_q$ and $\mathbf{s}_{q,n}$ are all semibounded and closed; respectively, denote $\mathbf{H}_{\mathbf{s}_q}$ and $\mathbf{H}_{\mathbf{s}_{q,n}}$ the corresponding self-adjoint operators. Then

$$\inf\sigma_{ess}\left(\mathbf{H}_{\mathbf{s}_q}\right)=\lim_{n\to\infty}\inf\sigma\left(\mathbf{H}_{\mathbf{s}_{q,n}}\right).$$

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Declarations

Conflict of interest The authors declare no competing interests.

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