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ORIGINAL ARTICLE

Logarithmic coefficient bounds for the class of Bazilevič functions

Navneet Lal Sharma[1] **· Teodor Bulboacă**[2]

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Abstract

If $\mathcal{S}$ denotes the class of all univalent functions in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then the logarithmic coefficients γ_n of $f \in \mathcal{S}$ are defined by

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n(f) z^n,\ z \in \mathbb{D}.$$

The logarithmic coefficients were brought to the forefront by I.M. Milin in the 1960's as a method of calculating the coefficients a_n for $f \in \mathcal{S}$. He concerned himself with logarithmic coefficients and their role in the theory of univalent functions, while in 1965 Bazilevič also pointed out that the logarithmic coefficients are crucial in problems concerning the coefficients of univalent functions. In this paper we estimate the bounds for the logarithmic coefficients $|\gamma_n(f)|$ when f belongs to the class $\mathcal{B}(\alpha, \beta)$ of Bazilevič function of type (α, β).

Keywords Univalent function · Starlike · Convex functions · Bazilevič functions · Logarithmic coefficient

Mathematics Subject Classification 30C45 · 30C50 · 30C55

Dedicated to the memory of Professor Yaşar Polatoğlu (1950–2021).

✉ Navneet Lal Sharma
nlsharma@gsv.ac.in, sharma.navneet23@gmail.com

Teodor Bulboacă
bulboaca@math.ubbcluj.ro, teodor.bulboaca@ubbcluj.ro

1 Department of Mathematics, Gati Shakti Vishwavidyalaya (A Central University Under the Ministry of Railways, Government of India) Vadodara, Lalbaug, Vadodara, Gujarat 390004, India

2 Faculty of Mathematics and Computer Science, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania

1 Preliminaries

Let $\mathcal{A}$ be the class of analytic functions f defined in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. If $f \in \mathcal{A}$, then f has the following Taylor series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all univalent (i.e., one-to-one) functions in $\mathbb{D}$. A function $f \in \mathcal{A}$ is said to be starlike function if f satisfies the condition $\operatorname{Re}\left(zf'(z)/f(z)\right) > 0$ for all $z \in \mathbb{D}$. Similarly, a function $f \in \mathcal{A}$ is said to be convex function if f satisfies the condition $\operatorname{Re}\left(1 + zf''(z)/f'(z)\right) > 0$ for $z \in \mathbb{D}$. A function $f \in \mathcal{A}$ is said to be close-to-convex if there exists a real number θ and a function $g \in \mathcal{S}^*$ such that $\operatorname{Re}\left(\mathrm{e}^{i\theta} zf'(z)/g(z)\right) > 0$ in $\mathbb{D}$. We denote by $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{K}$, the classes of starlike functions, convex functions and close-to-convex functions, respectively. With the class $\mathcal{S}$ being of the first priority, its subclasses such as $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{K}$ have been extensively studied in the literature and they appear in different contexts (see the books [8, 11, 13].)

A function $f \in \mathcal{A}$ is called Bazilevič functions of type (α, β) for $\alpha > 0$ and $\beta \in \mathbb{R}$ if f is given by

$$f(z) = \left[(\alpha + i\beta) \int_0^z g^{\alpha}(t)\, p(t)\, t^{i\beta - 1} dt\right]^{1/(\alpha + i\beta)}, \ z \in \mathbb{D},$$

where $g \in \mathcal{S}^*$ and p is an analytic function in $\mathbb{D}$ with $p(0) = 1$, and $\operatorname{Re} p(z) > 0$ for all $z \in \mathbb{D}$. It is well-know that this definition is equivalent to

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)} \left(\frac{f(z)}{g(z)}\right)^{\alpha} \left(\frac{f(z)}{z}\right)^{i\beta}\right] > 0, \ z \in \mathbb{D},$$

for $g \in \mathcal{S}^*$. We denote by $\mathcal{B}(\alpha, \beta)$ the class of Bazilevič function of type (α, β). In [3], Bazilevič proved that $\mathcal{B}(\alpha, \beta) \subset \mathcal{S}$ for $\alpha > 0$ and $\beta \in \mathbb{R}$. For an appropriate choice of the parameters α and β one can obtain the subfamilies of $\mathcal{B}(\alpha, \beta)$. For example:

(i) the class $\mathcal{B}(\alpha, 0) =: \mathcal{B}(\alpha)$ is called the Bazilevič function of type α, see [3];
(ii) the class $\mathcal{B}(1) =: \mathcal{K}$;
(iii) if we choose $g(z) = z$ and $\beta = 0$, then the class $\mathcal{B}(\alpha, \beta)$ reduces to the class $\mathcal{B}_1(\alpha)$ of Bazilevič functions with logarithmic growth;
(iv) the class $\mathcal{B}_1(0) = \mathcal{S}^*$ and the class $\mathcal{B}_1(1) =: \mathcal{R}$, i.e. the well-known class of functions whose derivative has positive real part in $\mathbb{D}$.

A Bazilevič function form belongs to the largest known subclass of $\mathcal{S}$ which has specific expressions. Indeed, the following relations are well-known (see [3, 8, 11, 13]):

$$\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{K} \subset \mathcal{B}(\alpha) \subset \mathcal{B}(\alpha, \beta) \subset \mathcal{S}.$$

2 Introduction to the logarithmic coefficients problem

The logarithmic coefficients γ_n of $f \in \mathcal{S}$ are defined by the formula

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n(f) z^n \; z \in \mathbb{D}, \tag{2.1}$$

where $\log 1 = 0$, and we will simply write $\gamma_n(f) = \gamma_n$ when there is no confusion. These coefficients play an important role for various estimates in the theory of univalent functions, and some authors use γ_n in place of $2\gamma_n$. Next note that differentiating (2.1), and equating the corresponding coefficients we get

$$\gamma_1 = \frac{a_2}{2}, \quad \gamma_2 = \frac{1}{2}\left(a_3 - \frac{a_2^2}{2}\right), \quad \gamma_3 = \frac{1}{2}\left(a_4 - a_2 a_3 + \frac{a_2^3}{3}\right).$$

For example, the logarithmic coefficients for the *Koebe function* $\kappa(z) := z/(1 - e^{i\theta}z)^2 \in \mathcal{S}$ are $\gamma_n(\kappa) = e^{(in\theta)}/n$, and it is well-known that the Koebe function is the extremal for many basic functionals defined on $\mathcal{S}$. Consequently, it could expected that if $f \in \mathcal{S}$, then $|\gamma_n| \le n$ holds for $n \ge 1$. However, this is not true.

If $f \in \mathcal{S}$, then $|\gamma_1| \le 1$ and $|\gamma_2| \le \dfrac{1}{2} + e^{-2} = 0.635\ldots$ (using the Fekete-Szegő inequality), see [8, Theorem 3.8].

The upper bound of the logarithmic coefficients γ_n for $f \in \mathcal{S}$ appears to be more harder and non-significant for $n \ge 3$. But for $f \in \mathcal{S}^*$, the inequality $|\gamma_n| \le 1/n$ and for $f \in \mathcal{C}$, the inequality $|\gamma_n| \le 1/2n$ holds (see [1, 8]). Attempting to extend this inequality for the class $\mathcal{K}$ is also more difficult because the inequality $|\gamma_n| \le 1/n$ fails when $n = 2$ for $f \in \mathcal{K}$, see [10].

Recently, Ponnusamy et al. [14] studied on the problem related to the logarithmic coefficients bounds for certain subfamilies of univalent functions, while in [15] they also determined the sharp bounds for the inverse logarithmic coefficients for the class $\mathcal{S}$ and some of its important geometrically characterized subclasses.

In a series of papers in the 1960's I.M. Milin drew attention to the logarithmic coefficients as a means of estimating the coefficients a_n for $f \in \mathcal{S}$, and he concerned himself with logarithmic coefficients and their role in the theory of univalent functions. According to Milin, $|\gamma_n|$ for $f \in \mathcal{S}$ cannot be much bigger than $1/n$ in an average sense. This result is known as Milin's lemma and holds a prominent place in the history:

Milin Lemma. [8, p. 151] For each $f \in \mathcal{S}$,

$$\sum_{k=1}^{n} k|\gamma_k|^2 \le \sum_{k=1}^{n} \frac{1}{k} + \delta, \; \text{for } \delta < 0.312,$$

where δ is called the Milin constant and it cannot be reduced to zero. This lemma and the third Lebedev-Milin inequality leads to the remarkable bound $|c_n| < e^{\delta/2} < 1.17$ for the coefficients of odd univalent function $h(z) = c_1 z + c_2 z^2 + \ldots$ (by more carefully estimation I.M. Milin improved the estimate to 1.14). However, Milin conjectured that:

Milin's Conjecture. For each $f \in \mathcal{S}$

$$\sum_{k=1}^{n} k(n-k+1)|\gamma_k|^2 \leq \sum_{k=1}^{n}(n-k+1)\frac{1}{k}, \text{ for } n \in \mathbb{N}. \tag{2.2}$$

The equality holds in (2.2) only for the Koebe function.

In 1984 Louis de Branges [7] proved this conjecture asserts that $\delta = 0$ in an average sense (see also [9], and the inequality (2.2)) is known as a de Branges's inequality. The de Brange's inequality is a source of many interesting inequalities involving logarithmic coefficients of univalent function. In [4, 5] de Branges explored the problem of logarithmic coefficients and came up with the following inequality for the class $\mathcal{S}$:

Theorem A *If $f \in \mathcal{S}$, then*

$$\sum_{k=1}^{\infty} k\left|\gamma_k - \frac{1}{k}\right|^2 \leq \frac{1}{2}\log\frac{1}{\alpha}, \tag{2.3}$$

where $\alpha = \lim_{r\to 1}(1-r^2)M(r,f)$ is the Hayman index of f and the positive real axis is the direction of maximal growth.

The proof of the inequality (2.3) is based on Milin's reformulation of the Grunsky inequalities (see [8]). Bazilevič also estimated the value $\sum_{k=1}^{\infty} k|\gamma_k|^2 r^k$ which after multiplication by π is equal to the area of the image of the disk $|z| < r < 1$ mapped by the function $\log(f(z)/z)$, $f \in \mathcal{S}$. He suggested in his review of A.Z. Grinshpan's thesis a conjecture that this value does not exceed $\log\left[1/(1-r^2)\right]$ for all $0 < r < 1$.

After two summations by parts, Milin's inequality (2.2) leads to the following result as one of its consequences (see [2, 12]):

$$\sum_{k=1}^{\infty} k|\gamma_k|^2 r^k \leq \sum_{k=1}^{\infty}\frac{r^k}{k} = \log\frac{1}{1-r}, \quad 0 \leq r < 1. \tag{2.4}$$

Using the relation (2.4), Ye [16] proved the following logarithmic coefficient bounds for $f \in \mathcal{B}(\alpha)$:

Theorem B [16, Theorem 1] *If $f \in \mathcal{B}(\alpha)$, then we have*

$$|\gamma_n| \leq A_1 n^{-1}(1+\alpha)\log(1+n), \ n \in \mathbb{N},$$

where A_1 is a constant. The exponent -1 is the best possible.

Motivated by the above investigations, in this article we study the bound of logarithmic coefficients γ_n for the class of Bazilevič function of type (α, β). We will prove our main result using the relation (2.4) and two previous results presented the next section. We begin by stating our main result and useful lemmas, while the proof of the main result will appear in Sect. 4.

3 Main results

Theorem 1 *Let $f \in \mathcal{B}(\alpha, \beta)$, with $\alpha > 0$ and $\beta \in \mathbb{R}$. Then the logarithmic coefficients γ_n of f satisfy the inequality*

$$|\gamma_n| < C(\alpha, \beta)\, n^{-1} \log(1+n), \ \ for\ n \in \mathbb{N},$$

where $C(\alpha, \beta)$ is an absolute constant given by

$$C(\alpha,\beta) = \frac{\mathrm{e}}{2}\left[\frac{6}{\log 2} + 4\big(\alpha + |\beta| + |\alpha + i\beta|\big)\left(\frac{1}{\log 2} + 2\right) + 2\sqrt{2}\sqrt{\frac{1}{1+\log 2} + 2}\ \cdot \right.$$
$$\left.\sqrt{\Big[\big(\alpha + |\alpha + i\beta|\big)^2 + 2\,|\beta|\,\big(\alpha + |\alpha + i\beta|\big)\Big]\left(\frac{1}{2\log 2} + 4\right) + \frac{|\beta|^2}{2\log 2}}\ \right]. \tag{3.1}$$

The exponent -1 is the best possible.

Next we present three useful lemmas which are the main tools to prove our theorem.

Lemma 1 [16, Lemma 1] *Let $f \in \mathcal{S}$. Then, for $z = r\mathrm{e}^{i\theta}$, $1/2 \le r < 1$, we have*

$$(i) \quad L_1 = \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{zf'(z)}{f(z)}\right|^2 d\theta < 1 + \frac{4}{1-r}\log\frac{1}{1-r},$$

and

$$(ii) \quad L_2 = \frac{1}{2\pi}\int_{1/2}^{r}\int_0^{2\pi} \left|\frac{zf'(z)}{f(z)}\right|^2 d\theta dr < 1 + 2\log\frac{1}{1-r^2}.$$

Lemma 2 *Let $f \in \mathcal{B}(\alpha, \beta)$ with $\alpha > 0$, $\beta \in \mathbb{R}$ and $g \in \mathcal{S}^*$. Then, for $z = r\mathrm{e}^{i\theta}$, $1/2 \le r < 1$, we have*

$$N_1 = \frac{1}{2\pi}\left|\int_0^{2\pi} \frac{zf'(z)}{f(z)}\ \exp\left(i\arg\left\{\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}\right\}\right) d\theta\right|$$
$$\le 3 + 2\big(\alpha + |\beta| + |\alpha + i\beta|\big)\left(1 + 2\log\frac{1}{1-r^2}\right). \tag{3.2}$$

Proof Set $w(z) := \dfrac{zf'(z)}{f(z)} = u(z) + iv(z)$, that is $\operatorname{Re} w = u$, $\operatorname{Im} w = v$. Since $f \in \mathcal{B}(\alpha, \beta) \subset \mathcal{S}$, the function w is analytic in $\mathbb{D}$, hence from the Cauchy-Riemann relations we easily get the well-known formula

$$\frac{\partial w(z)}{\partial r} = \frac{\partial u(z)}{\partial r} + i\frac{\partial v(z)}{\partial r} = \frac{1}{r}\left(\frac{\partial v(z)}{\partial \theta} - i\frac{\partial u(z)}{\partial \theta}\right), \; z = re^{i\theta}. \tag{3.3}$$

According to (3.3), for any $1/2 \leq r_0 \leq r < 1$ and $\theta \in [0, 2\pi]$ we have

$$w\left(re^{i\theta}\right) - w\left(r_0e^{i\theta}\right) = \int_{r_0}^{r} \frac{\partial w(z)}{\partial r} dr = \int_{r_0}^{r} \frac{1}{r}\left(\frac{\partial v(z)}{\partial \theta} - i\frac{\partial u(z)}{\partial \theta}\right) dr.$$

Since $z = re^{i\theta}$, using the triangle inequality in the above relation, one can obtain

$$\begin{aligned}
N_1 &= \frac{1}{2\pi}\left|\int_0^{2\pi}\left(w\left(r_0e^{i\theta}\right) + w\left(re^{i\theta}\right) - w\left(r_0e^{i\theta}\right)\right)\exp\left(i\arg\left\{\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}\right\}\right)d\theta\right| \\
&\leq \frac{1}{2\pi}\left|\int_0^{2\pi} w\left(r_0e^{i\theta}\right)\exp\left(i\arg\left\{\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}\right\}\right)d\theta\right| \\
&+ \frac{1}{2\pi}\left|\int_{r_0}^{r}\frac{1}{r}\int_0^{2\pi}\left(\frac{\partial v(z)}{\partial \theta} - i\frac{\partial u(z)}{\partial \theta}\right)\exp\left(i\arg\left\{\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}\right\}\right)d\theta dr\right| \\
&= N_{11} + N_{12},
\end{aligned} \tag{3.4}$$

where

$$N_{11} := \frac{1}{2\pi}\left|\int_0^{2\pi} w\left(r_0e^{i\theta}\right)\exp\left(i\arg\left\{\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}\right\}\right)d\theta\right|$$

and

$$N_{12} := \frac{1}{2\pi}\left|\int_{r_0}^{r}\int_0^{2\pi}\frac{1}{r}\left(\frac{\partial v(z)}{\partial \theta} - i\frac{\partial u(z)}{\partial \theta}\right)\exp\left(i\arg\left\{\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}\right\}\right)d\theta dr\right|.$$

Next, we find convenient upper bounds for the above two integrals N_{11} and N_{12}.

First, setting $r_0 = 1/2$, from the distortion theorem for the class $\mathcal{S}$ [6] it follows that

$$N_{11} \leq \frac{1}{2\pi}\int_0^{2\pi}\left|w\left(r_0e^{i\theta}\right)\right|d\theta \leq \max_{\theta\in[0,2\pi]}\left|w\left(r_0e^{i\theta}\right)\right| \leq \frac{1+r_0}{1-r_0} = 3. \tag{3.5}$$

Secondly, if we denote

$$I_1(z) := \exp\left(i \arg\left\{\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}\right\}\right) = \exp\left(i \arg\left\{\left(\frac{f(z)}{z}\right)^{\alpha+i\beta}\left(\frac{z}{g(z)}\right)^{\alpha}\right\}\right)$$
$$= \exp\left(i \arg\left(\frac{f(z)}{z}\right)^{\alpha+i\beta} - i \arg\left(\frac{g(z)}{z}\right)^{\alpha}\right),$$

a simple computation shows that

$$\left|\frac{\partial I_1(z)}{\partial\theta}\right| = \left|I_1 \frac{\partial}{\partial\theta}\left(i \arg\left(\frac{f(z)}{z}\right)^{\alpha+i\beta} - i \arg\left(\frac{g(z)}{z}\right)^{\alpha}\right)\right|$$
$$= \left|i(\alpha+i\beta)\frac{\partial}{\partial\theta}\left(\arg\frac{f(z)}{z}\right) - i\alpha\frac{\partial}{\partial\theta}\left(\arg\frac{g(z)}{z}\right)\right|, \tag{3.6}$$

hence

$$\left|\frac{\partial I_1(z)}{\partial\theta}\right| = \left|\frac{\partial}{\partial\theta}\arg\left\{\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}\right\}\right|. \tag{3.7}$$

Since

$$\operatorname{Re}\frac{zf'(z)}{f(z)} = 1 + \frac{\partial}{\partial\theta}\left(\arg\frac{f(z)}{z}\right),$$

and similarly for the function g, from (3.6) and the triangle's inequality, we deduce that

$$\left|\frac{\partial I_1(z)}{\partial\theta}\right| = \left|(\alpha+i\beta)\left(\operatorname{Re}\frac{zf'(z)}{f(z)} - 1\right) - \alpha\left(\operatorname{Re}\frac{zg'(z)}{g(z)} - 1\right)\right|$$
$$= \left|(\alpha+i\beta)\operatorname{Re}\frac{zf'(z)}{f(z)} - \alpha\operatorname{Re}\frac{zg'(z)}{g(z)} - i\beta\right|$$
$$\le |\alpha+i\beta|\left|\frac{zf'(z)}{f(z)}\right| + \alpha\left|\frac{zg'(z)}{g(z)}\right| + |\beta|. \tag{3.8}$$

Using the above notation for I_1, we can rewrite the integral N_{12} in the form

$$N_{12} = \frac{1}{2\pi}\left|\int_{r_0}^{r}\int_0^{2\pi}\frac{1}{r}\left(\frac{\partial v(z)}{\partial\theta} - i\frac{\partial u(z)}{\partial\theta}\right)I_1(z)d\theta dr\right|.$$

If we use an integration by parts in the formula of N_{12}, it follows that

$$\begin{aligned} N_{12} &= \frac{1}{2\pi}\left|\int_{r_0}^{r}\int_0^{2\pi}\frac{1}{r}\left(\frac{\partial v(z)}{\partial\theta}-i\frac{\partial u(z)}{\partial\theta}\right)I_1(z)d\theta dr\right| \\ &= \frac{1}{2\pi}\left|\int_{r_0}^{r}\frac{1}{r}\left[\Big(I_1(z)\big(v(z)-iu(z)\big)\Big)\Big|_{\theta=0}^{\theta=2\pi}-\int_0^{2\pi}\big(v(z)-iu(z)\big)\frac{\partial I_1(z)}{\partial\theta}d\theta\right]dr\right| \\ &= \frac{1}{2\pi}\left|0+\int_{r_0}^{r}\frac{1}{r}\int_0^{2\pi}iw(z)\frac{\partial I_1(z)}{\partial\theta}d\theta dr\right| \le \frac{1}{2\pi}\int_{r_0}^{r}\frac{1}{r}\int_0^{2\pi}|w(z)|\left|\frac{\partial I_1(z)}{\partial\theta}\right|d\theta dr, \end{aligned}$$

and using the inequality (3.8) together with the fact that $r \ge r_0 = 1/2$, we have

$$\begin{aligned} N_{12} &\le \frac{1}{2\pi}\int_{r_0}^{r}\frac{1}{r}\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|\left[|\alpha+i\beta|\left|\frac{zf'(z)}{f(z)}\right|+\alpha\left|\frac{zg'(z)}{g(z)}\right|+|\beta|\right]d\theta dr \\ &\le \frac{2}{2\pi}\left[|\alpha+i\beta|\int_{r_0}^{r}\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|^2 d\theta dr+\alpha\int_{r_0}^{r}\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|\left|\frac{zg'(z)}{g(z)}\right|d\theta dr\right. \\ &\quad \left.+|\beta|\int_{r_0}^{r}\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|d\theta dr\right]. \end{aligned}$$

Since $f, g \in \mathcal{S}$, by using the Schwarz's inequality for double integrals for the last two integrals and applying the inequality (ii) of Lemma 1 for the first integral, the above inequality leads to

$$\begin{aligned} N_{12} &< 2|\alpha+i\beta|\left(1+2\log\frac{1}{1-r^2}\right) \\ &\quad +\frac{2\alpha}{2\pi}\left[\int_{r_0}^{r}\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|^2 d\theta dr\cdot\int_{r_0}^{r}\int_0^{2\pi}\left|\frac{zg'(z)}{g(z)}\right|^2 d\theta dr\right]^{1/2} \\ &\quad +\frac{2|\beta|}{2\pi}\left[\int_{r_0}^{r}\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|^2 d\theta dr\cdot\int_{r_0}^{r}\int_0^{2\pi}d\theta dr\right]^{1/2}. \end{aligned}$$

Using again the inequality (ii) of Lemma 1, we find that

$$\begin{aligned} N_{12} &< 2|\alpha+i\beta|\left(1+2\log\frac{1}{1-r^2}\right)+2\alpha\left[\left(1+2\log\frac{1}{1-r^2}\right)^2\right]^{1/2}+L \\ &= 2\left(|\alpha+i\beta|+\alpha\right)\left(1+2\log\frac{1}{1-r^2}\right)+L, \end{aligned}$$

where

$$L := \frac{2|\beta|}{2\pi}\left[\int_{r_0}^{r}\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|^2 d\theta dr \cdot \int_{r_0}^{r}\int_0^{2\pi} d\theta dr\right]^{1/2}$$
$$= 2|\beta|\left[\frac{1}{2\pi}\int_{r_0}^{r}\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|^2 d\theta dr \cdot \frac{1}{2\pi}\int_{r_0}^{r}\int_0^{2\pi} d\theta dr\right]^{1/2}.$$

For the above first integral, according to the inequality (ii) of Lemma 1, we have

$$\frac{1}{2\pi}\int_{r_0}^{r}\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|^2 d\theta dr < 1 + 2\log\frac{1}{1-r^2},$$

while straight computation shows that the value of the second integral is

$$\frac{1}{2\pi}\int_{r_0}^{r}\int_0^{2\pi} d\theta dr = r - r_0 = r - \frac{1}{2} < 1, \text{ for } \frac{1}{2} \le r < 1.$$

Since the right-hand sides of the above two inequalities are positive numbers, we finally get

$$L < 2|\beta|\left(1 + 2\log\frac{1}{1-r^2}\right)^{1/2} < 2|\beta|\left(1 + 2\log\frac{1}{1-r^2}\right),$$

therefore

$$N_{12} \le 2\left(|\alpha + i\beta| + \alpha + |\beta|\right)\left(1 + 2\log\frac{1}{1-r^2}\right). \tag{3.9}$$

From the relation (3.4) combined with the inequalities (3.5) and (3.9), we get the desired result (3.2). □

Lemma 3 *Let $f \in \mathcal{B}(\alpha,\beta)$ with $\alpha > 0$, $\beta \in \mathbb{R}$ and $g \in \mathcal{S}^*$. Then, for $z = re^{i\theta}$, $1/2 \le r < 1$, we have*

$$N_2 = \frac{1}{2\pi}\left|\int_0^{2\pi}\frac{zf'(z)}{f(z)}\exp\left(2i\arg\left\{\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}\right\}\right)e^{in\theta}d\theta\right|$$
$$< 2\left(\frac{1}{n^2} + \frac{2}{n}\log\frac{1}{1-r^2}\right)^{1/2}\left(\left[\left(\alpha + |\alpha + i\beta|\right)^2 + 2|\beta|\left(\alpha + |\alpha + i\beta|\right)\right]\cdot\right.$$
$$\left.\left(1 + \frac{4}{1-r}\log\frac{1}{1-r}\right) + |\beta|^2\right)^{1/2}.$$

Proof From the formula (2.1) and using the fact that $\mathrm{e}^{im\theta} = \dfrac{1}{im}\dfrac{\partial \mathrm{e}^{im\theta}}{\partial\theta}$, $m \in \mathbb{C}$, since $z = r\mathrm{e}^{i\theta}$ we get

$$\begin{aligned}
\mathrm{e}^{in\theta}\,\frac{zf'(z)}{f(z)} &= \mathrm{e}^{in\theta}\left(1 + z\frac{d}{dz}\log\frac{f(z)}{z}\right) = \mathrm{e}^{in\theta} + 2\sum_{k=1}^{\infty} k\gamma_k r^k \mathrm{e}^{i(n+k)\theta} \\
&= \frac{1}{i}\,\frac{\partial}{\partial\theta}\left(\frac{\mathrm{e}^{in\theta}}{n} + 2\sum_{k=1}^{\infty} k\gamma_k r^k \frac{\mathrm{e}^{i(n+k)\theta}}{n+k}\right) \\
&= \frac{1}{i}\,\frac{\partial}{\partial\theta} G(r,\theta),
\end{aligned} \tag{3.10}$$

where

$$G(r,\theta) := \frac{\mathrm{e}^{in\theta}}{n} + 2\sum_{k=1}^{\infty} k\gamma_k r^k \frac{\mathrm{e}^{i(n+k)\theta}}{n+k}. \tag{3.11}$$

Setting

$$I_2(z) := \exp\left(2i \arg\left\{\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}\right\}\right),$$

from (3.10), we obtain

$$N_2 = \frac{1}{2\pi}\left|\int_0^{2\pi} \frac{zf'(z)}{f(z)}\,\mathrm{e}^{in\theta} I_2(z)d\theta\right| = \frac{1}{2\pi}\left|\int_0^{2\pi} \frac{1}{i}\,\frac{\partial}{\partial\theta} G(r,\theta) I_2(z)d\theta\right|,$$

integrating by parts the integral N_2 and we get

$$\begin{aligned}
N_2 &= \frac{1}{2\pi}\left|\Big(G(r,\theta)I_2(z)\Big)\Big|_{\theta=0}^{\theta=2\pi} - \int_0^{2\pi} G(r,\theta)\frac{\partial}{\partial\theta}\left(2i\arg\left\{\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}\right\}\right)d\theta\right| \\
&= \frac{1}{2\pi}\left|0 - 2i\int_0^{2\pi} G(r,\theta)\frac{\partial}{\partial\theta}\left(\arg\left\{\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}\right\}\right)d\theta\right| \\
&\leq \frac{2}{2\pi}\int_0^{2\pi} |G(r,\theta)|\left|\frac{\partial}{\partial\theta}\left(\arg\left\{\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}\right\}\right)\right|d\theta.
\end{aligned}$$

Using the Schwarz's inequality it follows that

$$\begin{aligned}
N_2 &\leq 2\left[\frac{1}{2\pi}\int_0^{2\pi} |G(r,\theta)|^2\,d\theta\right]^{1/2}\cdot\left[\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\partial}{\partial\theta}\left(\arg\left\{\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}\right\}\right)\right|^2 d\theta\right]^{1/2} \\
&= 2\,N_{21}^{1/2}\,N_{22}^{1/2},
\end{aligned} \tag{3.12}$$

where

$$N_{21} := \frac{1}{2\pi}\int_0^{2\pi} |G(r,\theta)|^2\, d\theta, \tag{3.13}$$

and

$$N_{22} := \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{\partial}{\partial\theta}\left(\arg\left\{\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}\right\}\right)\right|^2 d\theta. \tag{3.14}$$

Next, we find the required upper bounds for the both integrals N_{21} and N_{22} defined by (3.13) and (3.14).

First, we evaluate the integral N_{21} by using the value of $G(r,\theta)$ given by (3.11), that is

$$N_{21} = \frac{1}{2\pi}\int_0^{2\pi} |G(r,\theta)|^2\, d\theta = \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{e^{in\theta}}{n} + 2\sum_{k=1}^{\infty} k\gamma_k r^k \frac{e^{i(n+k)\theta}}{n+k}\right|^2 d\theta,$$

and using the *Parseval-Gutzmer* formula, since $2kn \leq (k+n)^2$, we get

$$N_{21} = \frac{1}{n^2} + 4\sum_{k=1}^{\infty} k^2|\gamma_k|^2 r^{2k}\frac{1}{(n+k)^2} \leq \frac{1}{n^2} + \frac{2}{n}\sum_{k=1}^{\infty} k|\gamma_k|^2 r^{2k}.$$

Consequently, the inequality (2.4) leads to

$$N_{21} \leq \frac{1}{n^2} + \frac{2}{n}\log\frac{1}{1-r^2}. \tag{3.15}$$

Secondly, from the relation (3.14), that is

$$N_{22} = \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{\partial}{\partial\theta}\left(\arg\left\{\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}\right\}\right)\right|^2 d\theta,$$

using (3.7) and (3.8), we have

$$\begin{aligned}
N_{22} &= \frac{1}{2\pi}\int_0^{2\pi}\left[|\alpha+i\beta|\left|\frac{zf'(z)}{f(z)}\right| + \alpha\left|\frac{zg'(z)}{g(z)}\right| + |\beta|\right]^2 d\theta \\
&\leq \frac{|\alpha+i\beta|^2}{2\pi}\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|^2 d\theta + \frac{\alpha^2}{2\pi}\int_0^{2\pi}\left|\frac{zg'(z)}{g(z)}\right|^2 d\theta + \frac{|\beta|^2}{2\pi}\int_0^{2\pi} d\theta \\
&\quad + \frac{2\alpha|\alpha+i\beta|}{2\pi}\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|\left|\frac{zg'(z)}{g(z)}\right| d\theta + \frac{2\alpha|\beta|}{2\pi}\int_0^{2\pi}\left|\frac{zg'(z)}{g(z)}\right| d\theta \\
&\quad + \frac{2|\alpha+i\beta||\beta|}{2\pi}\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right| d\theta.
\end{aligned}$$

Applying Schwarz's inequality in the 4th, 5th and 6th terms of the above inequality and using the inequality (i) of Lemma 1, we get

$$N_{22} < \left(|\alpha+i\beta|^2+\alpha^2+2\alpha|\alpha+i\beta|\right)\left(1+\frac{4}{1-r}\log\frac{1}{1-r}\right)+|\beta|^2$$
$$+2\left(\alpha|\beta|+|\beta||\alpha+i\beta|\right)\left(1+\frac{4}{1-r}\log\frac{1}{1-r}\right)^{1/2}.$$

Using the fact that

$$1+\frac{4}{1-r}\log\frac{1}{1-r}>1,$$

it follows

$$N_{22} < \left[\left(\alpha+|\alpha+i\beta|\right)^2+2|\beta|\left(\alpha+|\alpha+i\beta|\right)\right]\left(1+\frac{4}{1-r}\log\frac{1}{1-r}\right)+|\beta|^2. \tag{3.16}$$

Finally, from (3.12), using the inequalities (3.15) and (3.16), we conclude that

$$N_2 \le 2N_{21}^{1/2}N_{22}^{1/2} < 2\left(\frac{1}{n^2}+\frac{2}{n}\log\frac{1}{1-r^2}\right)^{1/2}\left(\left[\left(\alpha+|\alpha+i\beta|\right)^2+2|\beta|\left(\alpha+|\alpha+i\beta|\right)\right]\cdot\right.$$
$$\left.\left(1+\frac{4}{1-r}\log\frac{1}{1-r}\right)+|\beta|^2\right)^{1/2},$$

and the proof of the lemma is complete. □

4 Proof of main result

Proof of Theorem 1 If we suppose that $f\in\mathcal{B}(\alpha,\beta)$, then by the definition of the class $\mathcal{B}(\alpha,\beta)$ we have

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}\right]>0,\ z\in\mathbb{D},$$

for $\alpha>0$, $\beta\in\mathbb{R}$, and $g\in\mathcal{S}^*$. Setting

$$h(z):=\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta},\ z\in\mathbb{D}, \tag{4.1}$$

it follows that $\operatorname{Re}h(z)>0$ for all $z\in\mathbb{D}$, and we also have

$$h(z)=2\operatorname{Re}h(z)-\overline{h(z)},\ z\in\mathbb{D}. \tag{4.2}$$

From (2.1), we get

$$\frac{zf'(z)}{f(z)} = 1 + z\left(\log\frac{f(z)}{z}\right)' = 1 + 2\sum_{n=1}^{\infty} n\gamma_n z^n,$$

then

$$2n\gamma_n = \frac{1}{2\pi i}\int_{|z|=r} \frac{zf'(z)}{f(z)} z^{-(n+1)} dz,\ n = 1, 2, \ldots,$$

where the sense of the curve $|z| = r$ is the direct (trigonometric) one, and for $z = re^{i\theta}$ the above relation becomes

$$2n|\gamma_n|r^n = \frac{1}{2\pi}\left|\int_0^{2\pi} \frac{zf'(z)}{f(z)} e^{-in\theta} d\theta\right|.$$

Using the relations (4.1) and (4.2), it follows that

$$\begin{aligned}
2n|\gamma_n|r^n &= \frac{1}{2\pi}\left|\int_0^{2\pi} h(z)\left(\frac{g(z)}{f(z)}\right)^{\alpha}\left(\frac{z}{f(z)}\right)^{i\beta} e^{-in\theta} d\theta\right| \\
&\leq \frac{1}{2\pi}\left|\int_0^{2\pi} 2\,\mathrm{Re}\, h(z)\left(\frac{g(z)}{f(z)}\right)^{\alpha}\left(\frac{z}{f(z)}\right)^{i\beta} e^{-in\theta} d\theta\right| \\
&+ \frac{1}{2\pi}\left|\int_0^{2\pi} \overline{h(z)}\left(\frac{g(z)}{f(z)}\right)^{\alpha}\left(\frac{z}{f(z)}\right)^{i\beta} e^{-in\theta} d\theta\right| \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} 2\,\mathrm{Re}\, h(z)\left|\left(\frac{g(z)}{f(z)}\right)^{\alpha}\left(\frac{z}{f(z)}\right)^{i\beta}\right| d\theta \\
&+ \frac{1}{2\pi}\left|\int_0^{2\pi} h(z)\overline{\left(\frac{g(z)}{f(z)}\right)^{\alpha}}\,\overline{\left(\frac{z}{f(z)}\right)^{i\beta}} e^{in\theta} d\theta\right| \\
&= J_1 + J_2,
\end{aligned} \tag{4.3}$$

where

$$J_1 := \frac{1}{\pi}\int_0^{2\pi} \mathrm{Re}\, h(z)\left|\left(\frac{g(z)}{f(z)}\right)^{\alpha}\left(\frac{z}{f(z)}\right)^{i\beta}\right| d\theta \tag{4.4}$$

and

$$J_2 := \frac{1}{2\pi}\left|\int_0^{2\pi} h(z)\overline{\left(\frac{g(z)}{f(z)}\right)^{\alpha}}\,\overline{\left(\frac{z}{f(z)}\right)^{i\beta}} e^{in\theta} d\theta\right|. \tag{4.5}$$

Now, we determine the upper bounds for the both integrals J_1 and J_2 defined in (4.4) and (4.5).

Since $\operatorname{Re} h(z) > 0$, $z \in \mathbb{D}$, we have $J_1 > 0$. Therefore, from (4.4) and (4.1), it follows that

$$\begin{aligned}
J_1 &= \frac{1}{\pi}\left|\int_0^{2\pi} \operatorname{Re} h(z) \left|\left(\frac{g(z)}{f(z)}\right)^{\alpha}\left(\frac{z}{f(z)}\right)^{i\beta}\right| d\theta\right| \\
&= \frac{1}{\pi}\left|\operatorname{Re}\left(\int_0^{2\pi} h(z) \left|\left(\frac{g(z)}{f(z)}\right)^{\alpha}\left(\frac{z}{f(z)}\right)^{i\beta}\right| d\theta\right)\right| \\
&\leq \frac{1}{\pi}\left|\int_0^{2\pi} h(z) \left|\left(\frac{g(z)}{f(z)}\right)^{\alpha}\left(\frac{z}{f(z)}\right)^{i\beta}\right| d\theta\right| \\
&= \frac{1}{\pi}\left|\int_0^{2\pi} \frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta} \left|\left(\frac{g(z)}{f(z)}\right)^{\alpha}\left(\frac{z}{f(z)}\right)^{i\beta}\right| d\theta\right| \\
&= \frac{1}{\pi}\left|\int_0^{2\pi} \frac{zf'(z)}{f(z)} \exp\left[i \arg\left\{\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}\right\}\right] d\theta\right|,
\end{aligned}$$

and using the inequality (3.2) of Lemma 2, we find

$$J_1 \leq 6 + 4\big(\alpha + |\beta| + |\alpha + i\beta|\big)\left(1 + 2\log\frac{1}{1-r^2}\right), \quad \frac{1}{2} \leq r = |z| < 1. \tag{4.6}$$

Also, from (4.5) and using (4.1), we get

$$\begin{aligned}
J_2 &= \frac{1}{2\pi}\left|\int_0^{2\pi} \frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}\left\{\overline{\left(\frac{g(z)}{f(z)}\right)^{\alpha}}\;\overline{\left(\frac{z}{f(z)}\right)^{i\beta}}\right\} \mathrm{e}^{in\theta} d\theta\right| \\
&= \frac{1}{2\pi}\left|\int_0^{2\pi} \frac{zf'(z)}{f(z)} \exp\left[2i \arg\left\{\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}\right\}\right] \mathrm{e}^{in\theta} d\theta\right|,
\end{aligned}$$

and from Lemma 3, we obtain

$$\begin{aligned}
J_2 < 2\left(\frac{1}{n^2} + \frac{2}{n}\log\frac{1}{1-r^2}\right)^{1/2}&\left(\left[\big(\alpha + |\alpha + i\beta|\big)^2 + 2|\beta|\big(\alpha + |\alpha + i\beta|\big)\right]\cdot\right. \\
&\left.\left(1 + \frac{4}{1-r}\log\frac{1}{1-r}\right) + |\beta|^2\right)^{1/2}, \quad \frac{1}{2} \leq r = |z| < 1.
\end{aligned} \tag{4.7}$$

The relation (4.3) combined with the inequalities (4.6) and (4.7) leads to

$$\begin{aligned}
2n|\gamma_n|r^n &\leq J_1 + J_2 \\
&< 6 + 4\big(\alpha + |\beta| + |\alpha + i\beta|\big)\left(1 + 2\log\frac{1}{1-r^2}\right) + 2\left(\frac{1}{n^2} + \frac{2}{n}\log\frac{1}{1-r^2}\right)^{1/2}\cdot
\end{aligned}$$

$$\left(\left[\left(\alpha+|\alpha+i\beta|\right)^2+2|\beta|\left(\alpha+|\alpha+i\beta|\right)\right]\left(1+\frac{4}{1-r}\log\frac{1}{1-r}\right)+|\beta|^2\right)^{1/2},$$
$$\frac{1}{2}\le r=|z|<1.$$

Taking $\frac{1}{2}\le r:=r_n=1-\frac{1}{n+1}<1, n=1,2,\ldots$, and using the fact that $r^2<r$ for $1/2<r<1$, we deduce that

$$2n|\gamma_n|\left(1-\frac{1}{n+1}\right)^n<6+4\left(\alpha+|\beta|+|\alpha+i\beta|\right)\left(1+2\log(1+n)\right)+2\left(\frac{1}{n^2}+\frac{2}{n}\log(1+n)\right)^{1/2}\cdot$$
$$\left(\left[\left(\alpha+|\alpha+i\beta|\right)^2+2|\beta|\left(\alpha+|\alpha+i\beta|\right)\right]\left(1+4(1+n)\log(1+n)\right)+|\beta|^2\right)^{1/2},\ n\in\mathbb{N},$$

which is equivalent to

$$2n|\gamma_n|\left(1-\frac{1}{n+1}\right)^n<\log(1+n)\left[\frac{6}{\log(1+n)}+4\left(\alpha+|\beta|+|\alpha+i\beta|\right)\left(\frac{1}{\log(1+n)}+2\right)\right.$$
$$+\frac{2}{n^{1/2}}\left(\frac{1}{n\log(1+n)}+2\right)^{1/2}(1+n)^{1/2}\left(\left[\left(\alpha+|\alpha+i\beta|\right)^2+2|\beta|\left(\alpha+|\alpha+i\beta|\right)\right]\cdot\right.$$
$$\left.\left.\left(\frac{1}{(1+n)\log(1+n)}+4\right)+\frac{|\beta|^2}{(1+n)\log(1+n)}\right)^{1/2}\right],\ n\in\mathbb{N}.$$

The above inequality could be written as

$$|\gamma_n|<n^{-1}\log(1+n)\,G(n),\ n\in\mathbb{N}, \tag{4.8}$$

where

$$G(n)\ :=\frac{1}{2}\left(1+\frac{1}{n}\right)^n\left[\frac{6}{\log(1+n)}+4\left(\alpha+|\beta|+|\alpha+i\beta|\right)\left(\frac{1}{\log(1+n)}+2\right)\right.$$
$$+2\left(1+\frac{1}{n}\right)^{1/2}\left(\frac{1}{n\log(1+n)}+2\right)^{1/2}\left(\left[\left(\alpha+|\alpha+i\beta|\right)^2+2|\beta|\left(\alpha+|\alpha+i\beta|\right)\right]\cdot\right.$$
$$\left.\left.\left(\frac{1}{(1+n)\log(1+n)}+4\right)+\frac{|\beta|^2}{(1+n)\log(1+n)}\right)^{1/2}\right].$$

It is easy to see that

$$G(n)<\frac{\mathrm{e}}{2}\,H(n),\ n\in\mathbb{N}, \tag{4.9}$$

where

$$H(n) := \frac{6}{\log(1+n)} + 4\big(\alpha + |\beta| + |\alpha + i\beta|\big)\left(\frac{1}{\log(1+n)} + 2\right)$$
$$+2\left(1+\frac{1}{n}\right)^{1/2}\left(\frac{1}{n\log(1+n)}+2\right)^{1/2}\left(\Big[\big(\alpha+|\alpha+i\beta|\big)^2+2|\beta|\big(\alpha+|\alpha+i\beta|\big)\Big]\cdot\right.$$
$$\left.\left(\frac{1}{(1+n)\log(1+n)}+4\right)+\frac{|\beta|^2}{(1+n)\log(1+n)}\right)^{1/2}$$

is a strictly decreasing function on $\mathbb{N}$. Therefore, since

$$H(1) = \frac{6}{\log 2} + 4\big(\alpha + |\beta| + |\alpha + i\beta|\big)\left(\frac{1}{\log 2} + 2\right) + 2\sqrt{2}\sqrt{\frac{1}{1+\log 2}+2}\cdot$$
$$\sqrt{\Big[\big(\alpha+|\alpha+i\beta|\big)^2+2\,|\beta|\,\big(\alpha+|\alpha+i\beta|\big)\Big]\left(\frac{1}{2\log 2}+4\right)+\frac{|\beta|^2}{2\log 2}},$$

from (4.9), we obtain

$$G(n) < \frac{\mathrm{e}}{2}H(n) \le \frac{\mathrm{e}}{2}H(1),\ \ n \in \mathbb{N},$$

hence the inequality (4.8) leads to

$$|\gamma_n| < C(\alpha,\beta)\,n^{-1}\log(1+n),\ \text{ for } n \ge 1,$$

where $C(\alpha,\beta) = \mathrm{e}/2 \cdot H(1)$ is given by (3.1).

Since the Koebe function $k \in \mathcal{B}(\alpha,\beta)$ and the logarithmic coefficients satisfies the relation $|\gamma_n| = 1/n$, $n \in \mathbb{N}$, it follows the exponent -1 is the best possible. □

Remark 1 In particular, if we take $\beta = 0$ in Theorem 3.1 then we get the result of [16, Theorem 1].

Concluding remarks

In this paper, we extend we extend Theorem 1 of [16] for the classes of Bazilevič functions $\mathcal{B}(\alpha,\beta)$ by using a new technique based on a consequence of Millin's and of the other two previous results.

Thus, we obtained an estimate of an integral given by Lemma 3.3 which we used to prove our main theorem. The proof of this lemma uses the Parseval-Gutzmer formula which allows us to obtain an estimate for the upper bounds of the modules of the logarithmic coefficients $\gamma_n(f)$ if $f \in \mathcal{B}(\alpha,\beta)$, $\alpha > 0$ and $\beta \in \mathbb{R}$. For the case $\beta = 0$, this theorem reduces to the result of [16].

Our result could be used for some further studies connected with logarithmic coefficient estimations for some subclasses or for the classes of Bazilevič functions $\mathcal{B}(\alpha,\beta)$.

An interesting open problem is to find the smallest constant $C(\alpha, \beta)$ such that the inequality of Theorem 1 holds for all $f \in \mathcal{B}(\alpha, \beta)$ and $n \in \mathbb{N}$.

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