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Eigenvalue for a problem involving the fractional (p, q)-Laplacian operator and nonlinearity with a singular and a supercritical Sobolev growth

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Abstract

In this paper, we are interested in studying the multiplicity, uniqueness, and nonexistence of solutions for a class of singular elliptic eigenvalue problems for the Dirichlet fractional (p, q)-Laplacian. The nonlinearity considered involves supercritical Sobolev growth. Our approach is variational together with the sub- and supersolution methods, and in this way we can address a wide range of problems not yet contained in the literature. Even when $W_0^{s_1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ failing, we establish $\|u\|_{L^\infty(\Omega)} \leq C[u]_{s_1,p}$ (for some $C > 0$), when u is a solution.

Keywords Eigenvalue problem · Fractional p-Laplacian · Sobolev spaces · Supercritical Sobolev growth

Mathematics Subject Classification 35J75 · 35R11 · 35J67 · 35A15

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. In this paper, we study the following singular eigenvalue problem for the Dirichlet fractional (p, q)-Laplacian

$$\begin{cases} (-\Delta_p)^{s_1} u + (-\Delta_q)^{s_2} u = \lambda \left[u(x)^{-\eta} + f(x, u) \right] & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ u > 0 & \text{in } \Omega \end{cases} \qquad (P_\lambda)$$

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with $\lambda > 0$, $0 < s_2 < s_1 < 1$, $0 < \eta < 1$ and $1 < q < p$.

The fractional p-laplacian operator $(-\Delta_p)^s$ is defined as

$$(-\Delta_p)^s u(x) = C(N, s, p) \lim_{\varepsilon \searrow 0} \int\limits_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} \, dy \,,$$

for all $x \in \mathbb{R}^n$, where $C(N, s, p)$ is a normalization factor. The fractional p-Laplacian is a nonlocal version of the p-Laplacian and is an extension of the fractional Laplacian $(p = 2)$.

In (P_λ), we have the sum of two such operators. So, in problem (P_λ), the differential operator is nonhomogeneous, and this is a source of difficulties in the study of (P_λ). Boundary value problems, driven by a combination of two or more operators of different natures, arise in many mathematical models of physical processes. One of the first such models was introduced by Cahn-Hilliard [5] describing the process of separation of binary alloys. Other applications can be found in Bahrouni-Radulescu-Repovs [1] (on transonic flow problems). Problems with or without singularity involving fractional operators have been considered in different directions, as we can see in [6, 7, 20]. In [8, 19], the authors study singular systems, considering operators of the types (p, q)-Laplacian and fractional (p, q)-Laplacian, respectively. However, none of the works addressed operators of distinct fractional powers or nonlinearities involving supercritical powers.

In the reaction of (P_λ) , $\lambda > 0$ is a parameter, $u \mapsto u^{-\eta}$ with $0 < \eta < 1$ is a singular term and $f(z, x)$ is a Carathéodory perturbation (that is, for all $x \in \mathbb{R}$, $z \mapsto f(z, x)$ is measurable on Ω and for a.e. $z \in \Omega$, $x \mapsto f(z, x)$ is continuous). Unlike many authors, we will not assume that for a.e. $z \in \Omega$, $f(z, \cdot)$ is $(p - 1)$-superlinear near $+\infty$. However, this superlinearity of the perturbation $f(z, \cdot)$ is not formulated using the very common in the literature Ambrosetti-Rabinowitz condition (the AR-condition, for short), see Ref. [2]. The main goal of the paper is to explore the existence of a positive solution to (P_λ). Using variational tools from the critical point theory together with truncations and comparison techniques, we show that (P_λ) has a positive solution.

Throughout this paper, to simplify notation, we omit the constant $C(N, s, p)$. From now on, given a subset Ω of R^N we set $\Omega^c = R^N \setminus \Omega$ and $\Omega^2 = \Omega \times \Omega$. The fractional Sobolev spaces $W^{s,p}(\Omega)$ are defined to be the set of functions $u \in L^p(\Omega)$ such that

$$[u]_{s,p} = \left(\int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dxdy \right)^{\frac{1}{p}} < \infty.$$

and we defined the space $W_0^{s,p}(\Omega)$ by

$$W_0^{s,p}(\Omega) = \left\{ u \in W^{s,p}(\Omega); \;\; u = 0 \;\text{ in }\; \Omega^c \right\}.$$

In [3] the authors showed that,

$$W_0^{s_1,p}(\Omega) \hookrightarrow W_0^{s_2,q}(\Omega).$$

Thus, the ideal space to study the problem (P_λ) is $W_0^{s_1,p}(\Omega)$.

The main spaces that will be used in the analysis of problem (P_λ) are the Sobolev space $W_0^{s_1,p}(\Omega)$ and the Banach space

$$C^0_{s_1}(\overline{\Omega}) = \Big\{u \in C^0(\overline{\Omega});\ \frac{u}{d_\Omega^{s_1}} \text{ has a continuous extension to } \overline{\Omega}\Big\}.$$

where d_Ω is the distance function, $d_\Omega = \operatorname{dist}(x, \partial\Omega)$.

On account of the Poincaré inequality, we have that $[.]_{s,p}$ is a norm of the Sobolev space $W_0^{s_1,p}(\Omega)$. Moreover, in [3] the authors show that

$$[u]_{s_2,p} \leq \frac{C}{s_2(s_1 - s_2)}\,[u]_{s_1,p}\,, \quad \text{for all } \ u \in W_0^{s_1,p}(\Omega),$$

for $0 < s_2 < s_1 < 1$ and $1 < p < q < \infty$, in other words, we have $W_0^{s_1,p}(\Omega) \hookrightarrow W_0^{s_2,q}(\Omega)$.

The Banach space $C^0_{s_1}(\overline{\Omega})$ is ordered with positive (order) cone

$$(C^0_{s_1}(\overline{\Omega}))_+ = \left\{ f \in C^0_{s_1}(\overline{\Omega});\quad f \geq 0 \ \text{ in } \ \Omega \right\}$$

which is nonempty and has topological interior

$$\operatorname{int}\left(C^0_{s_1}(\overline{\Omega})_+\right) = \left\{ v \in C^0_{s_1}(\overline{\Omega});\quad v > 0 \ \text{ in } \ \Omega \ \text{ and } \ \inf \frac{v}{d_\Omega^{s_1}} > 0 \right\}.$$

Given $u, v \in W_0^{s_1,p}(\Omega)$ with $u \leq v$ we denote

$$[u, v] = \{h \in W_0^{s_1,p}(\Omega);\ u(x) \leq h(x) \leq v(x) \ \text{ for a. a. } \Omega\}$$
$$[u) = \{h \in W_0^{s_1,p}(\Omega);\ \ u(x) \leq h(x) \text{ for a. a. } \Omega\}.$$

2 The hypotheses

The hypotheses on the perturbation $f(x,t)$ are following:

H: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(x, 0) = 0$ for a. a. $x \in \Omega$ and for each $t > 0$ fixed $f(\cdot, t)$, $\frac{1}{f(\cdot,t)} \in L^\infty(\Omega)$, moreover

(i) $\displaystyle\lim_{n\to\infty} \frac{F(x,t)}{t^p} = \infty$ uniformly for a. a. $x \in \Omega$, where $F(x,t) = \int_0^t f(x,s)\mathrm{d}s$;

(ii) If $e(x,t) = \left[1 - \dfrac{p}{1-\eta}\right] t^{1-\eta} + f(x,t).t - pF(x,t)$, then there exists $\beta \in (L^1(\Omega))_+$ such that

$$e(x,t) \le e(x,s) + \beta(x) \ \text{ for a.e. } \ x \in \Omega \ \text{ all } \ 0 \le t \le s.$$

(iii) There exist $\delta > 0$ and $\tau \in (1,q)$ and $c_0 > 0$ such that,

$$c_0 t^{\tau-1} \le f(x,t) \ \text{ for a.e. } \ x \in \Omega \ \text{ all } \ t \in [0,\delta]$$

and for $s > 0$, we have

$$0 < m_s \le f(x,t) \ \text{ for a.e. } \ x \in \Omega \ \text{ all } \ t \ge s.$$

(iv) For every $\rho > 0$, there exists $\widehat{E}_\rho > 0$ such that for a.e. $x \in \Omega$, the function

$$t \mapsto f(x,t) + \widehat{E}_\rho t^{p-1}$$

is nondecreasing on $[0,\rho]$.

(v) We assume that there exists a number $\theta > 0$ such that

$$\limsup_{t\to\infty} \frac{f(x,t)}{t^{p^*_{s_1}-1+\theta}} < +\infty \text{ uniformly in } x.$$

(vi) At last, we assume that there exists a sequence (M_k) with $M_k \to \infty$ and such that, for each $r \in (p, p^*_{s_1})$,

$$t \in [0, M_k] \Longrightarrow \frac{f(x,t)}{t^{r-1}} \le \frac{f(x,M_k)}{(M_k)^{r-1}} \text{ uniformly in } x.$$

The classical AR-condition restricts $f(x,.)$ to have at least $(\mu-1)$-polynomial growth near ∞. In contrast, the quasimonotonicity condition that we use in this work (see hypothesis $\mathbf{H}$ (ii)), does not impose such a restriction on the growth of $f(x,.)$ and permits also the consideration of superlinear nonlinearities with slower growth near ∞ (see the examples below). Besides, hypothesis ($\mathbf{H}$ (ii)) is a slight extension of a condition used by Li-Yang [14, condition (f_4)].

There are convenient ways to verify ($\mathbf{H}$ (ii)). So, the hypothesis ($\mathbf{H}$ (ii)) holds, if we can find $M > 0$ such that for a.e. $x \in \Omega$

- $t \mapsto \dfrac{t^{-\eta} + f(x,t)}{t^{p-1}}$ is nondecreasing on $[M,\infty)$.
- or $t \mapsto e(x,t)$ is nondecreasing on $[M,\infty)$.

Hypothesis ($\mathbf{H}$ (iii)) implies the presence of a concave term near zero, while hypothesis ($\mathbf{H}$ (iv)) is a one-sided local Hölder condition. It is satisfied if, for a.e. $x \in \Omega$, $f(x,.)$ is differentiable, and for every $\rho > 0$, we can find $\widehat{c}_\rho$ such that

$$-\widehat{c}_\rho t^{p-1} \le f'_t(x,t)t \ \text{ for a.e. } \ x \in \Omega, \ \text{ all } \ 0 \le t \le \rho.$$

Below we list two examples of functions that satisfy the conditions (**H**)

- The function $f_1(x,t) = \begin{cases} t^{\tau-1} & \text{if } 0 \le t \le 1, \\ t^{p^*_{s_1}-1+\theta} & \text{if } t > 1, \end{cases}$ with $1 < \tau < q < p < \theta < p^*_{s_1}$ satisfies the hipotheses (**H**) and also the AR-condition.
- The function $f_2(x,t) = \begin{cases} t^{\tau-1} & \text{if } 0 \le t \le 1, \\ t^{p^*_{s_1}-1+\theta} \ln t + t^{s-1} & \text{if } t > 1, \end{cases}$ with $1 < \tau < q < p, \ 1 < s < p$ satisfies the hipotheses (**H**) but does not satisfy the AR-condition.

3 Preliminary

For any $r > 1$ consider the function $J_r : \mathbb{R} \to \mathbb{R}$ given by $J_r(t) = |t|^{r-2}.t$. Thus, using the arguments of [21], there exists $c_r > 0$ and $\tilde{c_r} > 0$ such that

$$\langle J_r(z) - J_r(w), z - w\rangle \ge \begin{cases} c_r|z-w|^r, & \text{if } r \ge 2, \\ c_r\dfrac{|z-w|^2}{(|z|+|w|)^{2-r}}, & \text{if } r \le 2. \end{cases} \tag{1}$$

$$|J_r(t_1) - J_r(t_2)| \le \begin{cases} \tilde{c_r}|t_1-t_2|^{r-1}, & \text{if } r \le 2, \\ \tilde{c_r}|t_1-t_2|^2.\,(|t_1|+|t_2|)^{r-2}, & \text{if } r \ge 2. \end{cases} \tag{2}$$

Lemma 1 *Let $u, v \in W^{s,r}_0(\Omega)$ and denote $w = u - v$. Then,*

$$\int_{\mathbb{R}^{2N}} \frac{\big(J_r(u(x)-u(y)) - J_r(v(x)-v(y))\big)\big(w(x)-w(y)\big)}{|x-y|^{N+sr}}\mathrm{d}x\mathrm{d}y$$
$$\ge \begin{cases} c_r\,[u-v]^r_{s,r}\,, & \text{if } r \ge 2, \\ c_r\dfrac{[u-v]^2_{s,r}}{\big([u]_{s,r}+[v]_{s,r}\big)^{2-r}}, & \text{if } r \le 2. \end{cases}$$

Proof The case $r \ge 2$, the result is an immediate application of the above inequality.
Case $r \le 2$. Note that, using the Holder inequality we have

$$\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^r}{|x-y|^{N+sr}}\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^r}{|x-y|^{N+sr}}.\frac{\big(|u(x)-u(y)|+|v(x)-v(y)|\big)^{\frac{r(2-r)}{2}}}{\big(|u(x)-u(y)|+|v(x)-v(y)|\big)^{\frac{r(2-r)}{2}}}\mathrm{d}x\mathrm{d}y$$
$$= \int_{\mathbb{R}^{2N}} \left[\frac{|u(x)-u(y)|}{(|u(x)-u(y)|+|v(x)-v(y)|)^{\frac{(2-r)}{2}}\,|x-y|^{\frac{N+sr}{2}}}\right]^r$$
$$\frac{\big(|u(x)-u(y)|+|v(x)-v(y)|\big)^{\frac{r(2-r)}{2}}}{|x-y|^{\frac{2-r}{2}}}\mathrm{d}x\mathrm{d}y$$
$$\le \left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{\big(|u(x)-u(y)|+|v(x)-v(y)|\big)^{2-r}|x-y|^{N+sr}}\mathrm{d}x\mathrm{d}y\right)^{\frac{r}{2}}\big([u]_{s,r}+[v]_{s,r}\big)^{\frac{r(2-r)}{2}}$$

Thus, using the inequality (1) we have

$$\left(\frac{[u-v]_{s,r}^{r}}{\left([u]_{s,r}+[v]_{s,r}\right)^{\frac{r(2-r)}{2}}}\right)^{\frac{2}{r}} \leq \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{(|u(x)-u(y)|+|v(x)-v(y)|)^{2-r}\,|x-y|^{N+sr}}\mathrm{d}x\mathrm{d}y$$
$$\leq \frac{1}{c_r}\int_{\mathbb{R}^{2N}} \frac{\big(J_r(u(x)-u(y))-J_r(v(x)-v(y))\big)\big((u-v)(x)-(u-v)(y)\big)}{|x-y|^{N+sr}}\mathrm{d}x\mathrm{d}y.$$

□

For every $1<r<\infty$, denote by $A_{s,r}: W_0^{s,r}(\Omega)\to \left(W_0^{s,r}(\Omega)\right)^*$ the nonlinear map defined by

$$\langle A_{s,r}(u),\varphi\rangle = \int_{\mathbb{R}^{2N}} \frac{J_r(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sr}}\mathrm{d}x\mathrm{d}y, \quad \text{for all } u,\varphi\in W_0^{s,r}(\Omega).$$

An immediate consequence of Lemma 1 is the following proposition

Proposition 1 *The map $A_{s,r}: W_0^{s,r}(\Omega)\to \left(W_0^{s,r}(\Omega)\right)^*$ maps bounded sets to bounded sets, is continuous, strictly monotone and satisfies,*

$$u_n \rightharpoonup u \;\; in \;\; W_0^{s,r}(\Omega) \;\; and \;\; \limsup_{n\to\infty}\langle A_{s,r}(u_n),(u_n-u)\rangle \leq 0 \Rightarrow u_n\to u \;\; in \;\; W_0^{s,r}(\Omega).$$

Proof Indeed, using the inequality (2) we have

$$\|A_{s,r}(u)-A_{s,r}(w)\|_* \leq \begin{cases} \tilde{c_r}\,[u-w]_{s,r}^{r-1}, & \text{if } r\leq 2,\\ \tilde{c_r}\,[u-w]_{s,r}^{2}\,.\left([u]_{s,r}+[w]_{s,r}\right)^{r-2}, & \text{if } r\geq 2.\end{cases}$$

and thus $A_{s,r}$ maps bounded sets to bounded sets, is continuous.

Moreover, if $p\geq 2$ then using also the Lemma 1 results,

$$\lim_{n\to\infty} c_r\,[u_n-u]_{s,r}^2$$
$$\leq \lim_{n\to\infty}\int_{\mathbb{R}^{2N}} \frac{\big(J_r(u_n(x)-u_n(y))-J_r(u(x)-u(y))\big)\big((u_n-u)(x)-(u_n-u)(y)\big)}{|x-y|^{N+sr}}\mathrm{d}x\mathrm{d}y$$
$$=\limsup_{n\to\infty}\Big\langle A_{s,r}(u_n)-A_{s,r}(u),u_n-u\Big\rangle\leq 0,$$

and if $p\leq 2$ let's use again the Lemma 1 and obtain

$$c_r\frac{[u_n-u]_{s,r}^2}{\left([u_n]_{s,r}+[u]_{s,r}\right)^{2-r}}$$
$$\leq \int_{\mathbb{R}^{2N}} \frac{\big(J_r(u_n(x)-u_n(y))-J_r(u(x)-u(y))\big)\big((u_n-u)(x)-(u_n-u)(y)\big)}{|x-y|^{N+sr}}\mathrm{d}x\mathrm{d}y$$
$$=\Big\langle A_{s,r}(u_n)-A_{s,r}(u),u_n-u\Big\rangle$$

thus, if $u_n \rightharpoonup u$ in $W_0^{s,r}(\Omega)$ and $\limsup_{n\to\infty} A_{s,r}(u_n).(u_n - u) \leq 0$ then, there exists $M > 0$ such that $\|u_n\|_{s,r} \leq M$ and thus

$$\lim_{n\to\infty} c_r \frac{[u_n - u]_{s,r}^2}{\left(M + [u]_{s,r}\right)^{2-r}} \leq \lim_{n\to\infty} c_r \frac{[u_n - u]_{s,r}^2}{\left([u_n]_{s,r} + [u]_{s,r}\right)^{2-r}}$$
$$\leq \limsup_{n\to\infty} \left\langle A_{s,r}(u_n) - A_{s,r}(u), u_n - u \right\rangle \leq 0.$$

Consequently, for all $1 < p < \infty$, we have $u_n \to u$ in $W_0^{s,r}(\Omega)$. □

The following result is a natural improvement of [15, Lemma 9] to the Dirichlet fractional (p, q)-Laplacian.

Proposition 2 *(Weak comparison principle) Let* $0 < s_1 < s_2 < 1$, $1 < q < p$, Ω *be bounded in* $\mathbb{R}^N$ *and* $u, v \in W_0^{s_1,p}(\Omega) \cap C_{s_1}^0(\overline{\Omega})$. *Suppose that,*

$$\left\langle A_{s_1,p}(u) + A_{s_2,q}(u), (u-v)^+ \right\rangle \leq \left\langle A_{s_1,p}(v) + A_{s_2,q}(v), (u-v)^+ \right\rangle$$

then $u \leq v$.

Proof The proof is a straightforward calculation, but for convenience of the reader we present a sketch of it. By considering the equations for both p and q, and subtracting them and adjusting the terms, we obtain

$$\left\langle A_{s_1,p}(u) + A_{s_2,q}(u), (u-v)^+ \right\rangle - \left\langle A_{s_1,p}(v) + A_{s_2,q}(v), (u-v)^+ \right\rangle \leq 0. \tag{3}$$

Using the identity

$$J_m(b) - J_m(a) = (m-1)(b-a) \int_0^1 |a + t(b-a)|^{m-2} \mathrm{d}t$$

for $a = v(x) - v(y)$ and $b = u(x) - u(y)$, we have

$$J_m(u(x) - u(y)) - J_m(v(x) - v(y)) = (m-1)\,[(u-v)(x) - (u-v)(y)]\, Q_m(x, y),$$

where $Q_m(x, y) = \displaystyle\int_0^1 |(v(x) - v(y)) + t[(u-v)(x) - (u-v)(y)]|^{m-2}\, \mathrm{d}t$.

We have $Q_m(x, y) \geq 0$ and $Q_m(x, y) = 0$ only if $v(x) = v(y)$ and $u(x) = u(y)$. Rewriting the integrands in (3) we obtain

$$\int_{\mathbb{R}^{2N}} \left(\frac{(p-1)\,[(u-v)(x) - (u-v)(y)]\, Q_p(x, y)}{|x-y|^{N+sp}} \right) ((u-v)^+(x) - (u-v)^+(y))\mathrm{d}x\mathrm{d}y$$
$$+ \int_{\mathbb{R}^{2N}} \left(\frac{(q-1)\,[(u-v)(x) - (u-v)(y)]\, Q_q(x, y)}{|x-y|^{N+sq}} \right) ((u-v)^+(x)$$

$$-(u-v)^+(y))\mathrm{d}x\mathrm{d}y \leq 0.$$

We now consider

$$\psi = u - v = (u-v)^+ - (u-v)^-, \quad \varphi = (u-v)^+ = \psi^+.$$

It follows from the last inequality that

$$\int_{\mathbb{R}^{2N}} \left(\frac{(p-1)(\psi(x)-\psi(y))(\psi^+(x)-\psi^+(y))Q_p(x,y)}{|x-y|^{N+sp}} \right) \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^{2N}} \left(\frac{(q-1)(\psi(x)-\psi(y))(\psi^+(x)-\psi^+(y))Q_q(x,y)}{|x-y|^{N+sq}} \right) \mathrm{d}x\mathrm{d}y \leq 0.$$

Applying the inequality $(\xi-\eta)(\xi^+-\eta^+) \geq |\xi^+-\eta^+|^2$ we obtain

$$\int_{\mathbb{R}^{2N}} \frac{(p-1)|\psi^+(x)-\psi^+(y)|^2 Q_p(x,y)}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^{2N}} \frac{(q-1)|\psi^+(x)-\psi^+(y)|^2 Q_q(x,y)}{|x-y|^{N+sq}} \mathrm{d}x\mathrm{d}y \leq 0.$$

Thus, at almost every point (x, y) we have $\psi^+(x) = \psi^+(y)$ or

$$Q_p(x,y) = Q_q(x,y) = 0.$$

Since $Q_p(x,y) = Q_q(x,y) = 0$ also imply $\psi^+(x) = \psi^+(y)$, we conclude that

$$(u-v)^+(x) = C \geq 0, \quad \forall x \in \mathbb{R}^N$$

and since, $u, v \in W_0^{s_1,p}(\Omega)$, results that $C = 0$ and consequently $u \leq v$. □

Proposition 3 *(Strong comparison principle) Let* $0 < s_1 < s_2 < 1$, $1 < q < p$, Ω *be bounded in* $\mathbb{R}^N$, $g \in C^0(\mathbb{R}) \cap BV_{loc}(\mathbb{R})$, $u, v \in W_0^{s_1,p}(\Omega) \cap C^0_{s_1}(\overline{\Omega})$ *such that* $u \neq v$ *and* $K > 0$ *satisfy,*

$$\begin{cases} (-\Delta_p)^{s_1} u + (-\Delta_p)^{s_1} u + g(u) \leq (-\Delta_p)^{s_1} v + (-\Delta_q)^{s_2} v + g(v) \leq K & \text{weakly in } \Omega, \\ 0 < u \leq v & \text{in } \Omega. \end{cases}$$

then $u \leq v$ *in* Ω. *In particular, if* $u, v \in int[(C^0_{s_1}(\overline{\Omega})^+)]$ *then* $v - u \in int[(C^0_{s_1}(\overline{\Omega})^+)]$.

Proof Without loss of generality, we may assume that g is nondecreasing and $g(0) = 0$. In fact, by Jordan's decomposition we can find $g_1, g_2 \in C^0(\mathbb{R})$ nondecreasing such that $g(t) = g_1(t) - g_2(t)$ and $g_1(0) = 0$.

Since, $u \neq v$ by continuity, we can find $x_0 \in \Omega$, $\rho, \varepsilon > 0$ such that $\overline{B_\rho(x_0)} \subset \Omega$ and

$$\sup_{\overline{B_\rho(x_0)}} u < \inf_{\overline{B_\rho(x_0)}} v - \varepsilon.$$

Hence, for all $\eta > 1$ close enough to 1 we have

$$\sup_{B_\rho(x_0)} \eta u < \inf_{B_\rho(x_0)} v - \frac{\varepsilon}{2}.$$

Define $w_\eta \in W_0^{s_1,p}(\Omega \backslash \overline{B_\rho(x_0)})$ by

$$w_\eta(x) = \begin{cases} \eta u(x), \text{ if } \ x \in \overline{B_\rho(x_0)}^c, \\ v(x), \quad \text{if } \ x \in \overline{B_\rho(x_0)}, \end{cases}$$

so $w_\eta \le v(x)$ in $\overline{B_\rho(x_0)}$ and by the nonlocal superposition principle ([11], Proposition 2.6) we have weakly in $\Omega \backslash \overline{B_\rho(x_0)}$

$$(-\Delta_p)^{s_1} w_\eta \le \eta^{p-1}(-\Delta_p)^{s_1} u - C_\rho \varepsilon^{p-1} \ \text{ and } \ (-\Delta_q)^{s_2} w_\eta \le \eta^{q-1}(-\Delta_q)^{s_2} u - C_\rho \varepsilon^{q-1}$$

for some $C_\rho > 0$ and all $\eta > 1$ close enough to 1. Further, we have weakly in $\Omega \backslash \overline{B_\rho(x_0)}$

$$\begin{aligned}
&(-\Delta_p)^{s_1} w_\eta + (-\Delta_q)^{s_2} w_\eta + g(w_\eta) \le \eta^{p-1}(-\Delta_p)^{s_1} u \\
&+ \eta^{q-1}(-\Delta_q)^{s_2} u + g(w_\eta) - C_\rho \varepsilon^{q-1} - C_\rho \varepsilon^{p-1} \\
&\le \eta^{p-1}\Big((-\Delta_p)^{s_1} u + (-\Delta_q)^{s_2} u + g(u)\Big) \\
&+ \Big(g(w_\eta) - \eta^{p-1} g(u)\Big) - C_\rho \varepsilon^{q-1} - C_\rho \varepsilon^{p-1} \\
&\le \Big((-\Delta_p)^{s_1} u + (-\Delta_q)^{s_2} u + g(u)\Big) + \Big(g(w_\eta) - \eta^{p-1} g(u)\Big) \\
&+ K\big(\eta^{p-1} - 1\big) - C_\rho \varepsilon^{q-1} - C_\rho \varepsilon^{p-1} \\
&\le \Big((-\Delta_p)^{s_1} v + (-\Delta_q)^{s_2} v + g(v)\Big) + \Big(g(w_\eta) - \eta^{p-1} g(u)\Big) + K\big(\eta^{p-1} - 1\big) \\
&- C_\rho \varepsilon^{q-1} - C_\rho \varepsilon^{p-1}.
\end{aligned}$$

Since

$$\Big(g(w_\eta) - \eta^{p-1} g(u)\Big) + K\big(\eta^{p-1} - 1\big) \to 0$$

uniformly in $\Omega \backslash \overline{B_\rho(x_0)}$ as $\eta \to 1^+$, we have, for all $\eta > 1$ close enough to 1,

$$\begin{cases} (-\Delta_p)^{s_1} w_\eta + (-\Delta_p)^{s_1} w_\eta + g(w_\eta) \le (-\Delta_p)^{s_1} v + (-\Delta_q)^{s_2} v + g(v) \le K \\ \qquad\qquad \text{weakly in } \ \Omega \backslash \overline{B_\rho(x_0)}, \\ 0 < w_\eta \le v \ \text{ in } \ \Big(\Omega \backslash \overline{B_\rho(x_0)}\Big). \end{cases}$$

Testing with $\varphi = (w_\eta - v)^+ \in W_0^{s_1}(\Omega)\backslash\overline{B_\rho(x_0)}$, recalling the monotonicity of g, and applying Proposition 2 we get $v > w_\eta$ in $\Omega)\backslash\overline{B_\rho(x_0)}$. So we have

$$v \geq \eta u \geq u.$$

In particular, if $u, v \in \operatorname{int}\left[(C_{s_1}^0(\overline{\Omega}))_+\right]$ then

$$\inf_\Omega \frac{v-u}{d_\Omega^{s_1}} \leq \inf_\Omega \frac{(\eta-1)u}{d_\Omega^{s_1}} > 0$$

and so $v - u \in \operatorname{int}\left[(C_{s_1}^0(\overline{\Omega}))_+\right]$. □

4 An auxiliary problem

Firstly, we will need to define, with the help of the real sequence defined in **H**(vii), a sequence of auxiliary equations that will be important for our purpose. More specifically, for each $k \in \mathbb{N}$, we define the auxiliary truncation functions by choosing $r \in \left(p, p_{s_1}^*\right)$ such that $p_{s_1}^* - r < \theta$ and we set

$$f_k(x,t) = \begin{cases} 0, & \text{if } t \leq 0 \\ f(x,t), & \text{if } 0 \leq t \leq M_k \\ \dfrac{f(x, M_k)}{(M_k)^{r-1}} t^{r-1}, & \text{if } t \geq M_k. \end{cases} \tag{4}$$

Notice that we define f_k to be such that r in its definition is independent of k. We see that we are really truncating our original function, making it subcritical for large arguments. Furthermore, in view of conditions **H**(vi), **H**(vii) and the choice of θ, we can prove that, for k big enough, f_k satisfies, for a constant $C > 0$,

$$|f_k(x,t)| \leq C\,(M_k)^{2\theta}\,|t|^{r-1}. \tag{5}$$

Indeed, for all $t > 0$, condition **H**(vii) and (4) gives

$$f_k(x,t) \leq \frac{f(x, M_k)}{(M_k)^{r-1}} t^{r-1}$$

and, by **H**(vi), if k is sufficiently large,

$$\frac{f(x, M_k)}{(M_k)^{r-1}} \leq C\,(M_k)^{p_{s_1}^* - r + \theta} \leq C\,(M_k)^{2\theta}.$$

For each $k \in \mathbb{N}$, let us consider the following auxiliary problem

$$\begin{cases} (-\Delta_p)^{s_1} u + (-\Delta_q)^{s_2} u = \lambda\left[u(x)^{-\eta} + f_k(x,u)\right] & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ u > 0 & \text{in } \Omega \end{cases} \tag{$P_{k,\lambda}$}$$

with $\lambda > 0$, $0 < s_1 < s_2 < 1$, $0 < \eta < 1$ and $1 < q < p$.

By the hypotheses (**H**), the hypotheses on the truncation $f_k(x,t)$ are following:

H$_k$: $f_k : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f_k(x,0) = 0$ for a. a. $x \in \Omega$ and

(i) $f_k(x,t) \leq \alpha_k(x)[1 + t^{r-1}]$ for a. a. $x \in \Omega$ all $t \geq 0$ with $\alpha_k \in L^\infty(\Omega)$ and $p < r < p^*_{s_1} = \dfrac{NP}{N - s_1 p}$;

(ii) $\lim_{t\to\infty} \dfrac{F_k(x,t)}{t^p} = \infty$ uniformly for a. a. $x \in \Omega$, where $F_k(x,t) = \int_0^t f_k(x,s)\mathrm{d}s$;

(iii) If $e_k(x,t) = \left[1 - \dfrac{p}{1-\eta}\right] t^{1-\eta} + f_k(x,t).t - pF_k(x,t)$, then there exists $\beta_k \in (L^1(\Omega))_+$ such that

$$e_k(x,t) \leq e_k(x,s) + \beta_k(x) \ \text{ for a.e. } \ x \in \Omega \ \text{ all } \ 0 \leq t \leq s.$$

(iv) There exist $\delta > 0$ and $\tau \in (1,q)$ and $c_0 > 0$ such that,

$$c_0 t^{\tau-1} \leq f_k(x,t) \ \text{ for a.e. } \ x \in \Omega \ \text{ all } \ t \in [0,\delta]$$

and for all $s > 0$, we have

$$0 < m_{k,s} \leq f_k(x,t) \ \text{ for a.e. } \ x \in \Omega \ \text{ all } \ t \geq s.$$

(v) For every $\rho > 0$, there exists $\widehat{E}_{k,\rho} > 0$ such that for a.e. $x \in \Omega$, the function

$$t \mapsto f_k(x,t) + \widehat{E}_{k,\rho} t^{p-1}$$

is nondecreasing on $[0,\rho]$.

The hypothesis (**H**$_k$ (i)) holds by (5), (**H**$_k$ (ii)) holds by (4) and $p < r$. We will prove first that (**H**$_k$ (iv)) holds. Since $\delta > 0$, $\tau \in (1,q)$ and $c_0 > 0$, if $\delta < M_k$, we have

$$c_0 t^{\tau-1} \leq f(x,t) = f_k(x,t) \ \text{ for a.e. } \ x \in \Omega \ \text{ all } \ t \in [0,\delta].$$

For $s > 0$, we have

- $0 < s \leq t \leq M_k$,

$$f_k(x,t) = f(x,t) \geq m_s > 0,$$

by (**H** (iii)).

- $0 < s \leq M_k < t$,

$$f_k(x,t) = \frac{f\,(x,M_k)}{(M_k)^{r-1}} t^{r-1} \geq \frac{f\,(x,M_k)}{(M_k)^{r-1}} M_k^{r-1} = f\,(x,M_k) > 0.$$

• $0 < M_k < s \le t$,

$$f_k(x,t) = \frac{f(x,M_k)}{(M_k)^{r-1}} t^{r-1} \ge \frac{f(x,M_k)}{(M_k)^{r-1}} M_k^{r-1} = f(x,M_k) > 0.$$

So, for all $s > 0$ we have

$$f_k(x,t) \ge m_{k,s} > 0 \ \text{ for a.e. } \ x \in \Omega \ \text{ all } \ t \ge s,$$

with $m_{k,s} = \max\left\{ m_s, \inf_{x\in\Omega} f(x,M_k) \right\} > 0$.

To prove that ($\mathbf{H}_k$ (iii)) holds it is sufficiently verify that there is a constant $C_k > 0$ such that $t \mapsto e_k(x,t)$ is nondecreasing on $[C_k,\infty)$. Since for $t \ge M_k$ we have

$$\begin{aligned} e_k(x,t) &= \left[1 - \frac{p}{1-\eta}\right] t^{1-\eta} + f_k(x,t).t - pF_k(x,t) \\ &= \left[1 - \frac{p}{1-\eta}\right] t^{1-\eta} + \frac{f(x,M_k)}{(M_k)^{r-1}} t^r - p\int_0^{M_k} f(x,s)ds - \int_{M_k}^t \frac{f(x,M_k)}{(M_k)^{r-1}} s^{r-1} ds \\ &= \left[1 - \frac{p}{1-\eta}\right] t^{1-\eta} + \frac{f(x,M_k)}{(M_k)^{r-1}} t^r - p\int_0^{M_k} f(x,s)ds - \frac{f(x,M_k)}{(M_k)^{r-1}} \frac{1}{r}[t^r - M_k^r]. \end{aligned}$$

Hence

$$\frac{\partial}{\partial t} e_k(x,t) = [1-\eta-p]\, t^{-\eta} + (r-1)\frac{f(x,M_k)}{(M_k)^{r-1}} t^{r-1}.$$

Notice that $\frac{\partial}{\partial t} e_k(x,t) \ge 0$ if

$$[1-\eta-p]\, t^{-\eta} + (r-1)\frac{f(x,M_k)}{(M_k)^{r-1}} t^{r-1} \ge 0,$$

or equivalently, if

$$t \ge \left(-[1-\eta-p] \frac{(M_k)^{r-1}}{(r-1)f(x,M_k)} \right)^{\frac{1}{r+\eta}}.$$

We can consider

$$C_k = \left(-[1-\eta-p] \frac{(M_k)^{r-1}}{(r-1)m_{k,s}} \right)^{\frac{1}{r+\eta}},$$

where $m_{k,s}$ is as in ($\mathbf{H}_k$ (iv)). Hence, $t \mapsto e_k(x,t)$ is nondecreasing on $[C_k,\infty)$. The proof of ($\mathbf{H}_k$ (v)) follows from (4) and ($\mathbf{H}$ (iv)).

Definition 1 A function $u \in W_0^{s_1,p}(\Omega)$ is a weak solution of the problem ($P_{k,\lambda}$) if, $u^{-\eta}\varphi \in W_0^{s_1,p}(\Omega)$ for all $\varphi \in W_0^{s_1,p}(\Omega)$ and

$$\left\langle A_{s_1,p}(u) + A_{s_2,q}(u), \varphi \right\rangle = \int_\Omega \lambda \left[u^{-\eta} + f_k(x,u)\right]\varphi \mathrm{d}x, \quad \text{for all } \varphi \in W_0^{s_1,p}(\Omega).$$

The difficulty that we encounter in the analysis of problem ($P_{k,\lambda}$) is that the energy (Euler) function of the problem $I_\lambda : W_0^{s_1,p}(\Omega) \to \mathbb{R}$ defined by

$$I_\lambda(u) = \frac{1}{p}[u]_{s_1,p}^p + \frac{1}{q}[u]_{s_2,p}^q - \lambda \int_\Omega \left[\frac{1}{1-\eta}(u^+)^{1-\eta} + F_k(x,u^+)\right]\mathrm{d}x. \tag{6}$$

for all $u \in W_0^{s_1,p}(\Omega)$, is not C^1 (due to the singular term). So, we can not use the minimax methods of critical point theory directly on $I_\lambda(.)$. We have to find ways to bypass the singularity and deal with C^1-functionals.

The hypotheses $\mathbf{H}$ (i) and $\mathbf{H}$ (iv) assure us that, there are $c_0 > 0$ and $c_2 > 0$ such that,

$$f_k(x,z) \geq c_0 z^{\tau-1} - c_2 z^{\theta-1}, \quad \text{for a. a. } x \in \Omega \text{ and } z \geq 0. \tag{7}$$

We consider the following auxiliary Dirichilet fractional (p,q)-equation

$$\begin{cases} (-\Delta_p)^{s_1}u + (-\Delta_q)^{s_2}u = \lambda\left[c_0 u(x)^{\tau-1} - c_2 u^{\theta-1}\right] & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ u > 0 & \text{in } \Omega \end{cases} \tag{8}$$

with $0 < s_2 < s_1$, $\lambda > 0$ and $1 < \tau < q < p < \theta < p_s^* = \dfrac{Np}{N-sp}$.

Lemma 2 *If $\underline{u}_\lambda \in W_0^{s_1,p}(\Omega)$ be a weak solution of problem (8). Then $\underline{u}_\lambda \in L^\infty(\Omega)$.*

Proof We denote by $h_\lambda(t) = \lambda c_0 t^{\tau-1} - \lambda c_2 t^{\theta-1}$. Thus,

$$\begin{aligned} &\langle A_{s_1,p}(\underline{u}_\lambda) + A_{s_2,q}(\underline{u}_\lambda), \phi\rangle \\ &= \int_{\mathbb{R}^{2N}} \left(\frac{J_p(\underline{u}_\lambda(x) - \underline{u}_\lambda(y))}{|x-y|^{N+s_1p}} + \frac{J_q(\underline{u}_\lambda(x) - \underline{u}_\lambda(y))}{|x-y|^{N+s_2q}}\right)(\phi(x)-\phi(y))\mathrm{d}x\mathrm{d}y \\ &= \int_\Omega h_\lambda(\underline{u}_\lambda)\phi \mathrm{d}x \end{aligned} \tag{9}$$

for any $\phi \in W_0^{s_1,p}(\Omega)$.

For each $k \in \mathbb{N}$, set

$$\Omega_k := \{x \in \Omega \ : \ u(x) > k\}.$$

Since $\underline{u}_\lambda \in W_0^{s_1,p}(\Omega)$ and $\underline{u}_\lambda \geq 0$ in Ω, we have that $(\underline{u}_\lambda - k)^+ \in W_0^{s_1,p}(\Omega)$. Taking $\phi = (\underline{u}_\lambda - k)^+$ in (9), we obtain

$$\left\langle A_{s_1,p}(\underline{u}_\lambda) + A_{s_2,q}(\underline{u}_\lambda), \phi \right\rangle = \int_\Omega h_\lambda(\underline{u}_\lambda)(\underline{u}_\lambda - k)^+ \mathrm{d}x. \tag{10}$$

Applying the algebraic inequality $|a-b|^{p-2}(a-b)(a^+ - b^+) \geq |a^+ - b^+|^p$ to estimate the left-hand side of (10), we obtain

$$\begin{aligned}
\left(\int_{\Omega_k} (\underline{u}_\lambda - k)^{p_s^*} \mathrm{d}x\right)^{\frac{p}{p_s^*}} &\leq C \int_{\mathbb{R}^{2N}} \frac{|\underline{u}_\lambda(x) - \underline{u}_\lambda(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y \\
&\leq C \left\langle A_{s_1,p}(\underline{u}_\lambda) + A_{s_2,q}(\underline{u}_\lambda), \phi \right\rangle \\
&= C \int_{\Omega_k} h_\lambda(\underline{u}_\lambda)(\underline{u}_\lambda - k)\mathrm{d}x \\
&= C \int_{\Omega_k} \left[\lambda c_0 \underline{u}_\lambda^{\tau-1} - \lambda c_2 \underline{u}_\lambda^{\theta-1}\right](\underline{u}_\lambda - k)\mathrm{d}x \\
&\leq C \int_{\Omega_k} \lambda c_0 \underline{u}_\lambda^{\tau-1}(\underline{u}_\lambda - k)\mathrm{d}x.
\end{aligned} \tag{11}$$

Since $1 < \tau < p$, for $k > 1$ in Ω_k we have

$$\underline{u}_\lambda^{\tau-1}(\underline{u}_\lambda - k) \leq \underline{u}_\lambda^{p-1}(\underline{u}_\lambda - k) \leq 2^{p-1}(\underline{u}_\lambda - k)^p + 2^{p-1}k^{p-1}(\underline{u}_\lambda - k)$$

and thus,

$$\int_\Omega \underline{u}_\lambda^{\tau-1}(\underline{u}_\lambda - k)\mathrm{d}x \leq 2^{p-1}\int_\Omega (\underline{u}_\lambda - k)^p \mathrm{d}x + 2^{p-1}k^{p-1}\int_{\Omega_k}(\underline{u}_\lambda - k)\mathrm{d}x. \tag{12}$$

Applying Hölder's inequality, we obtain

$$\int_{\Omega_k} (\underline{u}_\lambda - k)^p \mathrm{d}x \leq |\Omega_k|^{\frac{p_s^*-p}{p_s^*}} \left(\int_{\Omega_k} (\underline{u}_\lambda - k)^{p_s^*} \mathrm{d}x\right)^{\frac{p}{p_s^*}}. \tag{13}$$

So, using the inequalities (12) and (13) in (11), we have

$$\int_{\Omega_k} (\underline{u}_\lambda - k)^p \mathrm{d}x \leq C_0 \, |\Omega_k|^{\frac{p_s^*-p}{p_s^*}} \left[2^{p-1}\int_{\Omega_k} (\underline{u}_\lambda - k)^p \mathrm{d}x + 2^{p-1}k^{p-1}\int_{\Omega_k}(\underline{u}_\lambda - k)\mathrm{d}x\right].$$

Thus, we obtain

$$\left[1 - 2^{p-1}C_0|\Omega_k|^{\frac{p_s^*-p}{p_s^*}}\right]\int_{\Omega_k} (\underline{u}_\lambda - k)^p \mathrm{d}x \leq 2^{p-1}k^{p-1}\,|\Omega_k|^{\frac{(p_s^*-p)}{p_s^*}}\int_{\Omega_k}(\underline{u}_\lambda - k)\mathrm{d}x.$$

If $k \to \infty$, then $|\Omega_k| \to 0$. Therefore, there exists $k_0 > 1$ such that

$$1 - 2^{p-1} C_0 |\Omega_k|^{\frac{p_s^* - p}{p_s^*}} \geq \frac{1}{2} \quad \text{if } k \geq k_0 > 1.$$

Thus, for such k, we conclude that

$$\frac{1}{2} \int_{\Omega_k} (\underline{u}_\lambda - k)^p \mathrm{d}x \leq 2^{p-1} k^{p-1} C_0 |\Omega_k|^{\frac{p_s^* - p}{p_s^*}} \int_{A_k} (\underline{u}_\lambda - k) \mathrm{d}x. \tag{14}$$

Hölder's inequality and (14) yield

$$\left(\int_{\Omega_k} (\underline{u}_\lambda - k) \mathrm{d}x \right)^p \leq$$
$$|\Omega_k|^{p-1} \int_{\Omega_k} (\underline{u}_\lambda - k)^p \mathrm{d}x \leq |\Omega_k|^{p-1} 2^{p-1} k^{p-1} C_0 |\Omega_k|^{\frac{p_s^* - p}{p_s^*}} \int_{A_k} (\underline{u}_\lambda - k) \mathrm{d}x.$$

Thus,

$$\int_{\Omega_k} (u - k) \mathrm{d}x \leq 2 \tilde{C} k |\Omega_k|^{1+\epsilon}, \quad \forall k \geq k_0, \tag{15}$$

where $\epsilon = \dfrac{p_s^* - p}{p_s^*(p-1)} > 0$ and $\tilde{C} > 0$.

The same arguments used in [16] assures us that $\underline{u}_\lambda \in L^\infty(\Omega)$. Then the nonlinear regularity theory, see [9] says that $\underline{u}_\lambda \in \mathrm{int}(C^0_{s_1}(\Omega))_+$. □

Proposition 4 *For every* $\lambda > 0$, *the problem* (8) *admits a unique positive solution* $\underline{u}_\lambda \in$ *int*$(C^0_{s_1}(\Omega)_+)$ *and* $\underline{u}_\lambda \to 0$ *in* $C^0_{s_1}(\overline{\Omega})$ *as* $\lambda \to 0^+$.

Proof *Existence* Note that, the solutions of the problem (8) are critical points of the functional $\tilde{I}_\lambda : W_0^{s_1,p}(\Omega) \to W_0^{s_2,q}(\Omega)$ given by

$$\tilde{I}_\lambda(u) = \frac{1}{p}[u]^p_{s_1,p} + \frac{1}{q}[u]^q_{s_2,p} - \frac{\lambda c_0}{\tau} \|u^+\|^\tau_\tau + \frac{\lambda c_2}{\theta} \|u^+\|^\theta_\theta, \text{ for all } u \in W_0^{s_1,p}(\Omega) \tag{16}$$

where $\|.\|_t$ denote the norm in space $L^t(\Omega)$.

Since $1 < \tau < q < p < \theta$, then $\tilde{I}_\lambda(tu) \to \infty$ as $t \to \infty$, is that, J_λ is coercive. Also using the Sobolev embedding theorem, we see that $\tilde{I}_\lambda$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $\underline{u}_\lambda \in W_0^{s_1,p}(\Omega)$ such that

$$\tilde{I}_\lambda(\underline{u}_\lambda) = \min \left\{ J_\lambda(u); \quad u \in W_0^{s_1,p}(\Omega) \right\}.$$

Now notice that $1 < \tau < q < p < \theta$ and $u \in \text{int}(C^0_{s_1}(\Omega)_+)$ results

$$\tilde{I}_\lambda(tu) < 0 \ \text{ for } \ t \in (0,1) \ \text{ small enough} \tag{17}$$

thus $\tilde{I}_\lambda(\underline{u}_\lambda) < 0 = \tilde{I}_\lambda(0)$ and therefore $\underline{u}_\lambda \neq 0$.

Using the (17) we have,

$$\tilde{I}'_\lambda(\underline{u}_\lambda) = 0$$

and consequently

$$\begin{aligned}\left\langle A_{s_1,p}(\underline{u}_\lambda) + A_{s_2,q}(\underline{u}_\lambda), \varphi\right\rangle &= \lambda \int_\Omega c_0(\underline{u}_\lambda^+)^{\tau-1}\varphi \mathrm{d}x \\ &-\lambda \int_\Omega c_2(\underline{u}_\lambda^+)^{\theta-1}\varphi \mathrm{d}x, \quad \text{for all } \ \varphi \in W^{s_1,p}_0(\Omega).\end{aligned} \tag{18}$$

Choosing $\varphi = \underline{u}_\lambda^- \in W^{s_1,p}_0(\Omega)$ results

$$\begin{aligned}&\left[\underline{u}_\lambda^-\right]^p_{s_1,p} + \left[\underline{u}_\lambda^-\right]_{s_2,q} \leq \left\langle A_{s_1,p}(\underline{u}_\lambda) + A_{s_2,q}(\underline{u}_\lambda), \underline{u}_\lambda^-\right\rangle \\ &= \lambda \int_\Omega c_0(\underline{u}_\lambda^+)^{\tau-1}\underline{u}_\lambda^- \mathrm{d}x - \lambda \int_\Omega c_2(\underline{u}_\lambda^+)^{\theta-1}\underline{u}_\lambda^- \mathrm{d}x = 0\end{aligned}$$

and therefore $\left[\underline{u}_\lambda^-\right]^p_{s_1,p} = 0$, is that, $\underline{u}_\lambda \geq 0$ and $\underline{u}_\lambda \neq 0$.

Uniqueness To show the uniqueness of the solution, we will use arguments similar to those used in [12]. Let's use the following discrete Picone's inequality from [4]

$$J_r(a-b)\left(\frac{c^r}{a^{r-1}} - \frac{d^r}{b^{r-1}}\right) \leq |c-d|^r, \quad \text{for all } a, b \in \mathbb{R}^*_+, c, d \in \mathbb{R}^+. \tag{19}$$

Let $\underline{u}_\lambda, \underline{v}_\lambda \in W^{s_1,p}_0(\Omega)$ positive solutions of the problem (8). As above, we show that $\underline{u}_\lambda, \underline{v}_\lambda \in \text{int}(C^0_{s_1}(\Omega)_+)$. Thus, using the same arguments as Lemma 2.4 of [12] we have,

$$\frac{\underline{u}_\lambda^p}{\underline{v}_\lambda^{p-1}} \in W^{s_1,p}_0(\Omega).$$

Consider $w_\lambda = (\underline{u}_\lambda^p - \underline{v}_\lambda^p)^+$, thus,

$$\frac{w_\lambda}{\underline{v}_\lambda^{p-1}} = \left(\frac{\underline{u}_\lambda^p}{\underline{v}_\lambda^{p-1}} - \underline{v}_\lambda\right)^+ \in W^{s_1,p}_0(\Omega) \text{ and } \frac{w_\lambda}{\underline{u}_\lambda^{p-1}} = \left(\underline{u}_\lambda - \frac{\underline{v}_\lambda^p}{\underline{u}_\lambda^{p-1}}\right)^+ \in W^{s_1,p}_0(\Omega).$$

We denote by $g_\lambda(t) = \lambda c_0 t^{\tau-p} - \lambda c_2 t^{\theta-p}$. Thus, g is strictly decreasing in $\mathbb{R}^+_0$.

Testing (18) with $\frac{w_\lambda}{\underline{u}_\lambda^{p-1}}$ we have

$$\left\langle A_{s_1,p}(\underline{u}_\lambda) + A_{s_2,q}(\underline{u}_\lambda), \frac{w_\lambda}{\underline{u}_\lambda^{p-1}} \right\rangle = \lambda \int_\Omega c_0 \underline{u}_\lambda^{\tau-1} \frac{w_\lambda}{\underline{u}_\lambda^{p-1}} \mathrm{d}x - \lambda \int_\Omega c_2 \underline{u}_\lambda^{\theta-1} \frac{w_\lambda}{\underline{u}_\lambda^{p-1}} \mathrm{d}x$$
$$= \lambda \int_\Omega c_0 \underline{u}_\lambda^{\tau-p} w_\lambda \mathrm{d}x - \lambda \int_\Omega c_2 \underline{u}_\lambda^{\theta-p} w_\lambda \mathrm{d}x$$
$$= \int_{\{\underline{u}_\lambda > \underline{v}_\lambda\}} g_\lambda(\underline{u}_\lambda)(\underline{u}_\lambda^p - \underline{v}_\lambda^p) \mathrm{d}x$$

and testing (18) with $\frac{w_\lambda}{\underline{v}_\lambda^{p-1}}$ we have

$$\left\langle A_{s_1,p}(\underline{v}_\lambda) + A_{s_2,q}(\underline{v}_\lambda), \frac{w_\lambda}{\underline{v}_\lambda^{p-1}} \right\rangle = \lambda \int_\Omega c_0 \underline{v}_\lambda^{\tau-1} \frac{w_\lambda}{\underline{v}_\lambda^{p-1}} \mathrm{d}x - \lambda \int_\Omega c_2 \underline{v}_\lambda^{\theta-1} \frac{w_\lambda}{\underline{v}_\lambda^{p-1}} \mathrm{d}x$$
$$= \lambda \int_\Omega c_0 \underline{v}^{\tau-p} w_\lambda \mathrm{d}x - \lambda \int_\Omega c_2 \underline{v}_\lambda^{\theta-\tau} w_\lambda \mathrm{d}x$$
$$= \int_{\{\underline{u}_\lambda > \underline{v}_\lambda\}} g_\lambda(\underline{v}_\lambda)(\underline{u}_\lambda^p - \underline{v}_\lambda^p) \mathrm{d}x$$

Thus,

$$\left\langle A_{s_1,p}(\underline{u}_\lambda) + A_{s_2,q}(\underline{u}_\lambda), \frac{w_\lambda}{\underline{u}_\lambda^{p-1}} \right\rangle - \left\langle A_{s_1,p}(\underline{v}_\lambda) + A_{s_2,q}(\underline{v}_\lambda), \frac{w_\lambda}{\underline{v}_\lambda^{p-1}} \right\rangle$$
$$= \int_{\{\underline{u}_\lambda > \underline{v}_\lambda\}} \left[g_\lambda(\underline{u_\lambda}) - g_\lambda(\underline{v}_\lambda) \right] (\underline{u}_\lambda^p - \underline{v}_\lambda^p) \mathrm{d}x.$$

Note that, using the discrete Picone's inequality (19), see (Proposition 3.1, [12]) we have

$$j_p(u(x)-u(y)) \left(\frac{w_\lambda(x)}{\underline{u}_\lambda(x)^{p-1}} - \frac{w_\lambda(y)}{\underline{u}_\lambda(y)^{p-1}} \right) \geq j_p(v(x)-v(y)) \left(\frac{w_\lambda(x)}{\underline{v}_\lambda(x)^{p-1}} - \frac{w_\lambda(y)}{\underline{v}_\lambda(y)^{p-1}} \right)$$

and thus,

$$\left\langle A_{s_1,p}(\underline{u}_\lambda) + A_{s_2,q}(\underline{u}_\lambda), \frac{w_\lambda}{\underline{u}_\lambda^{p-1}} \right\rangle \geq \left\langle A_{s_1,p}(\underline{v}_\lambda) + A_{s_2,q}(\underline{v}_\lambda), \frac{w_\lambda}{\underline{v}_\lambda^{p-1}} \right\rangle.$$

Therefore, since g_λ is strictly decreasing in $\mathbb{R}_0^+$ results

$$0 \leq \int_{\{\underline{u}_\lambda > \underline{v}_\lambda\}} \left[g_\lambda(\underline{u_\lambda}) - g_\lambda(\underline{v}_\lambda) \right] (\underline{u}_\lambda^p - \underline{v}_\lambda^p) \mathrm{d}x \leq 0$$

so we deduce that $\{\underline{u}_\lambda > \underline{v}_\lambda\}$ has null measure, is that, $\underline{u}_\lambda \leq \underline{v}_\lambda$ in Ω. Similarly, using the function test $w_\lambda = (\underline{v}_\lambda^p - \underline{u}_\lambda^p)^+$ we see that $\underline{u}_\lambda \geq \underline{v}_\lambda$ in Ω, and thus $\underline{u}_\lambda = \underline{v}_\lambda$.

Moreover, we have

$$\begin{aligned}\left[\underline{u}_\lambda\right]_{s_1,p}^p &\leq \left[\underline{u}_\lambda\right]_{s_1,p}^p + \left[\underline{u}_\lambda\right]_{s_2,q}^q \\ &= \lambda c_0\|\underline{u}_\lambda\|_\tau^\tau - \lambda c_2\|\underline{u}_\lambda\|_\theta^\theta \\ &\leq \lambda c_0\|\underline{u}_\lambda\|_\tau^\tau \\ &\leq \lambda \hat{c}_0\left[\underline{u}_\lambda\right]_{s_1,p}^\tau,\end{aligned}$$

for some $\hat{c}_0 > 0$. Thus,

$$\left[\underline{u}_\lambda\right]_{s_1,p}^{p-\tau} \leq \lambda\hat{c}_0$$

and therefore, $\underline{u}_\lambda \to 0$ in $W_0^{s_1,p}(\Omega)$ as $\lambda \to 0^+$. Using the nonlinear regularity theorem, see [9], results that

$$\underline{u}_\lambda \to 0 \text{ in } C_{s_1}^0(\overline{\Omega}) \text{ as } \lambda \to 0^+.$$

□

We consider another auxiliary problem,

$$\begin{cases}(-\Delta_p)^{s_1}u + (-\Delta_q)^{s_2}u = \lambda\underline{u}_\lambda^{-\eta} + 1 & \text{in } \Omega,\\ u = 0 & \text{in } \mathbb{R}^N\setminus\Omega,\\ u > 0 & \text{in } \Omega\end{cases} \tag{20}$$

with $\lambda > 0$, $0 < \eta < 1$ and $1 < q < p$.

Proposition 5 *For every $\lambda > 0$, there exists a unique solution $\overline{u}_\lambda \in int\left[(C_{s_1}^0(\overline{\Omega}))_+\right]$ of the problem (20) and a $\lambda_0 > 0$ such that, for all $0 < \lambda \leq \lambda_0$ it holds*

$$\underline{u}_\lambda \leq \overline{u}_\lambda.$$

Proof Note that, the Lemma 14.16 of Gilbarg-Trundiger [10] says that $d_\Omega^{s_1} \in C^2(\Omega_{\delta_0})$, where $\Omega_{\delta_0} = \{x \in \Omega; d_\Omega^{s_1}(x) < \delta_0\}$. Thus, $d_\Omega^{s_1} \in \text{int}\left[(C_{s_1}^0(\Omega))_+\right]$ and so by Proposition 4.1.22 of [17], there exists $c_3 = c_3(\underline{u}_\lambda) > 0$ and $c_4 = c_4(\underline{u}_\lambda) > 0$ such that,

$$c_3 d_\Omega^{s_1} \leq \underline{u}_\lambda \leq c_4 d_\Omega^{s_1}. \tag{21}$$

Since due to (21), $\lambda\underline{u}_\lambda^{-\eta} + 1 \in L^1(\Omega)$. The existence of a weak solution of (20) follows from direct minimization in $W_0^{s_1,p}(\Omega)$ of the functional

$$\frac{1}{p}\,[u]_{s_1,p}^p + \frac{1}{q}\,[u]_{s_2,p}^q - \int_\Omega(\lambda\underline{u}_\lambda^{-\eta} + 1)u\,dx,$$

whereas the uniqueness comes from, for instance, the comparison principle for the Dirichlet fractional (p, q)-Laplacian, Propossition 2. Using the maximum principle, [9], the solution $\overline{u}_\lambda \in \text{int}\left[(C^0_{s_1}(\overline{\Omega})_+)\right]$.

For show the existence of $\lambda_0 > 0$ such that $\underline{u}_\lambda \leq \overline{u}_\lambda$ for all $0 < \lambda \leq \lambda_0$, acting on (20) with $\overline{u}_\lambda$ and obtain

$$
\begin{aligned}
[\overline{u}_\lambda]^p_{s_1,p} &\leq [\overline{u}_\lambda]^p_{s_1,p} + [\overline{u}_\lambda]^q_{s_2,q} \\
&= \lambda \int_\Omega \underline{u}_\lambda^{-\eta}.\overline{u}_\lambda \mathrm{d}x + \int_\Omega \overline{u}_\lambda \mathrm{d}x \\
&= \lambda \int_\Omega \underline{u}_\lambda^{1-\eta}.\frac{\overline{u}_\lambda}{\underline{u}_\lambda}\mathrm{d}x + \int_\Omega \overline{u}_\lambda \mathrm{d}x \\
&\leq \lambda c_5 \int_\Omega \frac{\overline{u}_\lambda}{d_\Omega^{s_1}}\mathrm{d}x + |\Omega|^{\frac{p-1}{p}} \left(\int_\Omega \overline{u}_\lambda^p \mathrm{d}x\right)^{\frac{1}{p}} \quad \text{(Holder inequality)} \\
&\leq \left(\lambda c_5 + \frac{1}{\lambda_1(p)}\right) |\Omega|^{\frac{p-1}{p}} [\overline{u}_\lambda]_{s_1,p} \text{ (Hardy's inequality and first eigenvalue).}
\end{aligned}
$$

So, we have $\{\overline{u}_\lambda\}_{\lambda\in(0,1]}$ is uniformly bounded in $W_0^{s_1,p}(\Omega)$. Using arguments similar to the Lemma 1, (see also Ladyzhenskaya-Ural'tseva [13] Theorem 7.1) results

$$\{\overline{u}_\lambda\}_{\lambda\in(0,1]} \subset L^\infty(\Omega) \text{ is uniformly bounded in } \lambda.$$

The condition **H** (i) implies that there exists $\lambda_0 > 0$ such that,

$$\lambda f_k(x, \overline{u}_\lambda) \leq \lambda \|a\|(1 + \|\overline{u}_\lambda\|^{\theta-1}) \leq 1 \text{ for all } \lambda \in (0, \lambda_0] \text{ and } x \text{ a. a. in } \Omega.$$

For each $\lambda \in (0, \lambda_0]$ consider the Carathéodory function

$$\kappa_\lambda(x,t) = \begin{cases} \lambda[c_0(t^+)^{\tau-1} - c_2(t^+)^{\theta-1}] & \text{if } t \leq \overline{u}_\lambda(x), \\ \lambda[c_0\overline{u}_\lambda(x)^{\tau-1} - c_2\overline{u}_\lambda(x)^{\theta-1}] & \text{if } \overline{u}_\lambda(x) < t. \end{cases}$$

Let $\Psi_\lambda : W_0^{s_1,p} \to \mathbb{R}$ the C^1-functional defined by

$$\Psi_\lambda(u) = \frac{1}{p}[u]^p_{s_1,p} + \frac{1}{q}[u]^q_{s_2,p} - \int_\Omega K_\lambda(x,u)\mathrm{d}x, \quad \text{for all } u \in W_0^{s_1,p}(\Omega)$$

where $K_\lambda(x,t) = \displaystyle\int_0^t \kappa_\lambda(x,s)\mathrm{d}s$.

Note that, Ψ_λ is coercive and sequentially wekly lower semicontinuous. So, there exists $\tilde{u}_\lambda \in W_0^{s_1,p}(\Omega)$ such that

$$\Psi_\lambda(\tilde{u}_\lambda) = \min\left[\Psi_\lambda(u);\ u \in W_0^{s_1,p}(\Omega)\right].$$

Since $1 < \tau < q < p < \theta$ results

$$\Psi_\lambda(tu) < 0 \text{ for } t \in (0,1) \text{ small enough} \tag{22}$$

thus $\Psi_\lambda(\underline{u}_\lambda) < 0 = \Psi_\lambda(0)$ and therefore $\underline{u}_\lambda \neq 0$.

Using the (22) we have,

$$\Psi'_\lambda(\tilde{u}_\lambda) = 0$$

and consequently

$$\left\langle A_{s_1,p}(\tilde{u}_\lambda) + A_{s_2,q}(\tilde{u}_\lambda), \varphi \right\rangle = \int_\Omega \kappa_\lambda(x, \tilde{u}_\lambda)\varphi \mathrm{d}x, \quad \text{for all } \varphi \in W_0^{s_1,p}(\Omega).$$

Choosing $\varphi = -\tilde{u}_\lambda \in W_0^{s_1,p}(\Omega)$, we see that $\tilde{u}_\lambda \geq 0$ and $\tilde{u}_\lambda \neq 0$. Taking $\varphi = (\tilde{u}_\lambda - \overline{u}_\lambda)^+ \in W_0^{s_1,p}(\Omega)$ we find,

From (7), we have that there exits $c_0 > 0$ and $c_2 > 0$ such that $f_k(x,t) \geq c_0 t^{\tau-1} - c_2 t^{\theta-1}$ and so

$$\begin{aligned}
&\left\langle A_{s_1,p}(\tilde{u}_\lambda) + A_{s_2,q}(\tilde{u}_\lambda), (\tilde{u}_\lambda - \overline{u}_\lambda)^+ \right\rangle \\
&= \int_\Omega \kappa_\lambda(x, \tilde{u}_\lambda)(\tilde{u}_\lambda - \overline{u}_\lambda)^+ \mathrm{d}x \\
&= \int_\Omega \lambda[c_0 \overline{u}_\lambda^{\tau-1} - c_2 \overline{u}_\lambda^{\theta-1}](\tilde{u}_\lambda - \overline{u}_\lambda)^+ \mathrm{d}x \\
&\leq \int_\Omega \lambda f_k(x, \overline{u}_\lambda)(\tilde{u}_\lambda - \overline{u}_\lambda)^+ \mathrm{d}x \\
&\leq \int_\Omega [\lambda \underline{u}_\lambda^{-\eta} + 1](\tilde{u}_\lambda - \overline{u}_\lambda)^+ \mathrm{d}x \quad (\text{for all } 0 < \lambda \leq \lambda_0) \\
&= \left\langle A_{s_1,p}(\overline{u}_\lambda) + A_{s_2,q}(\overline{u}_\lambda), (\tilde{u}_\lambda - \overline{u}_\lambda)^+ \right\rangle
\end{aligned}$$

and so, by Proposition 2 $\tilde{u}_\lambda \leq \overline{u}_\lambda$. Moreover, note that,

$$\Psi_\lambda(u) = \tilde{I}_\lambda(u), \quad \text{for all } u \in [0, \overline{u}_\lambda],$$

thus

$$\begin{aligned}
\tilde{I}_\lambda(\tilde{u}_\lambda) = \Psi_\lambda(\tilde{u}_\lambda) &= \min\left[\Psi_\lambda(u);\ u \in W_0^{s_1,p}(\Omega)\right] \\
&= \min\left\{\Psi_\lambda(u);\ u \in [0, \overline{u}_\lambda]\right\} \\
&= \min\left\{\tilde{I}_\lambda(u);\ u \in [0, \overline{u}_\lambda]\right\} \\
&= \tilde{I}_\lambda(\underline{u}_\lambda).
\end{aligned}$$

By Proposition 4 we have $\tilde{u}_\lambda = \underline{u}_\lambda$ and therefore $\underline{u}_\lambda \leq \overline{u}_\lambda$ for all $0 < \lambda \leq \lambda_0$. □

5 Existence of positive solution for $P_{k,\lambda}$

We consider the set

$$\mathcal{L} = \left\{ \lambda > 0; \quad \text{problem } P_{k,\lambda} \text{admits a positive solution} \right\}$$

and the set S_λ of the positive solutions to the problem $P_{k,\lambda}$.

Proposition 6 *Assume the hypotheses* ($\boldsymbol{H}_k$) *hold, then*

i) $\mathcal{L} \neq \emptyset$;
ii) If $\lambda \in \mathcal{L}$, *then* $\underline{u}_\lambda \leq u$ *for all* $u \in S_\lambda$ *and* $S_\lambda \subseteq int[(C^0_{s_1}(\Omega))_+]$.

Proof Let $\lambda_0 > 0$ given in the Proposition 4, so for $\lambda \in (0, \lambda_0]$ we have

$$\underline{u}_\lambda \leq \overline{u}_\lambda \;\text{ and }\; \lambda f(x, \overline{u}_\lambda) \leq 1 \;\text{ for a. a. }\; x \in \Omega. \tag{23}$$

We consider the function

$$g_\lambda(x,t) = \begin{cases} \lambda[\underline{u}_\lambda^{-\eta} + f_k(x, \underline{u}_\lambda)] \text{ if } \; t < \underline{u}_\lambda(x), \\ \lambda[t^{-\eta} + f_k(x,t)] \quad \text{if } \; \underline{u}_\lambda(x) \leq t \leq \overline{u}_\lambda(x), \\ \lambda[\overline{u}_\lambda^{-\eta} + f_k(x, \overline{u}_\lambda)] \text{ if } \; \overline{u}_\lambda(x) < t, \end{cases}$$

and the functional $\Phi_\lambda : W^{s_1,p}_0(\Omega) \to \mathbb{R}$ defined by

$$\Phi_\lambda(u) = \frac{1}{p}\,[u]^p_{s_1,p} + \frac{1}{q}\,[u]^q_{s_2,p} - \int_\Omega G_\lambda(x,u)\mathrm{d}x, \quad \text{for all } \; u \in W^{s_1,p}_0(\Omega)$$

where $G(x,t) = \displaystyle\int_0^t g_\lambda(x,s)\mathrm{d}s$.

By Proposition 3 of [18] we have $\Phi_\lambda \in C^1(W^{s_1,p}_0(\Omega), \mathbb{R})$. Moreover, using the hypotheses ($\mathbf{H}$) we have, Φ_λ is coercive and sequently weakly lower semicontinuous. Thus, there exists $u_\lambda := u_{k,\lambda} \in W^{s_1,p}_0(\Omega)$ such that,

$$\Phi_\lambda(u_\lambda) = \min\left[\Phi_\lambda(u); \; u \in W^{s_1,p}_0(\Omega)\right].$$

Thus, $\Phi'_\lambda(u_\lambda) = 0$, that is,

$$\left\langle A_{s_1,p}(u_\lambda) + A_{s_2,q}(u_\lambda), \varphi \right\rangle = \int_\Omega g_\lambda(x, u_\lambda)\varphi \mathrm{d}x, \quad \text{for all } \; \varphi \in W^{s_1,p}_0(\Omega). \tag{24}$$

Testing the Eq. (24) with $\varphi = (u_\lambda - \overline{u}_\lambda)^+ \in W^{s_1,p}_0(\Omega)$ and using the inequality (23), we find

$$\left\langle A_{s_1,p}(u_\lambda) + A_{s_2,q}(u_\lambda), (u_\lambda - \overline{u}_\lambda)^+ \right\rangle$$

$$
\begin{aligned}
&= \int_\Omega g_\lambda(x, u_\lambda)(u_\lambda - \overline{u}_\lambda)^+ \mathrm{d}x \\
&= \int_\Omega \lambda[\overline{u}_\lambda^{-\eta} + f_k(x, \overline{u}_\lambda)](u_\lambda - \overline{u}_\lambda)^+ \mathrm{d}x \\
&\leq \int_\Omega [\lambda \underline{u}_\lambda^{-\eta} + 1](u_\lambda - \overline{u}_\lambda)^+ \mathrm{d}x \quad (\text{for all } \ 0 < \lambda \leq \lambda_0) \\
&= \left\langle A_{s_1,p}(\overline{u}_\lambda) + A_{s_2,q}(\overline{u}_\lambda), (u_\lambda - \overline{u}_\lambda)^+ \right\rangle
\end{aligned}
$$

and so, by Proposition 2 $u_\lambda \leq \overline{u}_\lambda$.

Analogously, testing (24) with the function $\varphi = (\underline{u}_\lambda - u_\lambda)^+ \in W_0^{s_1,p}(\Omega)$ and using (7), we have,

$$
\begin{aligned}
&\left\langle A_{s_1,p}(u_\lambda) + A_{s_2,q}(u_\lambda), (\underline{u}_\lambda - u_\lambda)^+ \right\rangle = \int_\Omega g_\lambda(x, u_\lambda)(\underline{u}_\lambda - u_\lambda)^+ \mathrm{d}x \\
&= \int_\Omega \lambda[\underline{u}_\lambda^{-\eta} + f_k(x, \underline{u}_\lambda)](\underline{u}_\lambda - u_\lambda)^+ \mathrm{d}x \\
&\geq \int_\Omega \lambda[c_0 \underline{u}_\lambda^{\tau-1} - c_2 \underline{u}^{\theta-1}](\underline{u}_\lambda - u_\lambda)^+ \mathrm{d}x \quad (\text{for all } \ 0 < \lambda \leq \lambda_0) \\
&= \left\langle A_{s_1,p}(\underline{u}_\lambda) + A_{s_2,q}(\underline{u}_\lambda), (\underline{u}_\lambda - u_\lambda)^+ \right\rangle
\end{aligned}
$$

and so, by Proposition 2 we have $u_\lambda \leq \overline{u}_\lambda$.

Therefore,

$$
u_\lambda \in [\underline{u}_\lambda, \overline{u}_\lambda] \Rightarrow u_\lambda \in S_\lambda \Rightarrow (0, \lambda_0] \subseteq \mathcal{L}.
$$

For item (ii), it is sufficient to argue as in the Proposition 4, replacing $\overline{u}_\lambda$ with $u \in S_\lambda$, we show that $\underline{u}_\lambda \leq u$ for all $u \in S_\lambda$. For show that $S_\lambda \subseteq \operatorname{int}[(C^0_{s_1}(\Omega))_+]$ we use the maximum principle, see [9]. □

Proposition 7 *If hypotheses* $(\boldsymbol{H}_k)$ *hold,* $\lambda \in \mathcal{L}$ *and* $\mu \in (0, \lambda)$*, then* $\mu \in \mathcal{L}$*.*

Proof Let $\lambda \in \mathcal{L}$, so we can find $u_\lambda \in S_\lambda \subseteq \operatorname{int}[(C^0_{s_1}(\overline{\Omega}))_+]$. Consider the Dirichlet problem,

$$
\begin{cases}
(-\Delta_p)^{s_1} u + (-\Delta_q)^{s_2} u = \vartheta c_0 u(x)^{\tau-1} - \lambda c_2 u^{\theta-1} & \text{in } \ \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\
u > 0 & \text{in } \Omega
\end{cases}
\tag{25}
$$

with $0 < \vartheta < \lambda$ and $1 < \tau < \underline{q} < p < \theta$. As we did in the proposition, we can find a unique solution $\tilde{u}_\vartheta \in \operatorname{int}[(C^0_{s_1}(\overline{\Omega}))_+]$ to the problem (25) and, in addition, we can show that $\tilde{u}_\vartheta^{-\eta} \in L^1(\Omega)$. Since, for all $0 < \vartheta_1 < \vartheta_2 \leq \lambda$, we have $\vartheta_1 c_0 u(x)^{\tau-1} - \lambda c_2 u^{\theta-1} \leq \vartheta_2 c_0 u(x)^{\tau-1} - \lambda c_2 u^{\theta-1}$, by comparison principle results that $\tilde{u}_{\vartheta_1} \leq \tilde{u}_{\vartheta_2}$. Note that, by Proposition 5 $\tilde{u}_\lambda = \underline{u}_\lambda$, so

$$\tilde{u}_\mu \leq \underline{u}_\lambda \leq u_\lambda.$$

Define the Caracthéodory function,

$$\gamma_\lambda(x,t) = \begin{cases} \mu[\tilde{u}_\mu^{-\eta} + f_k(x,\tilde{u}_\mu)] \text{ if } t < \tilde{u}_\mu(x), \\ \mu[t^{-\eta} + f_k(x,t)] \quad \text{ if } \tilde{u}_\mu(x) \leq t \leq \tilde{u}_\mu(x), \\ \mu[\tilde{u}_\mu^{-\eta} + f_k(x,\tilde{u}_\mu)] \text{ if } \tilde{u}_\mu(x) < t, \end{cases}$$

Let $\Upsilon_\lambda : W_0^{s_1,p}(\Omega) \to \mathbb{R}$ the C^1-functional defined by

$$\Upsilon_\lambda(u) = \frac{1}{p}[u]_{s_1,p}^p + \frac{1}{q}[u]_{s_2,p}^q - \int_\Omega \Gamma_\mu(x,u)\mathrm{d}x, \quad \text{for all } u \in W_0^{s_1,p}(\Omega)$$

where $\Gamma_\lambda(x,t) = \int_0^t \gamma(x,s)\mathrm{d}s$.

Note that, Υ_λ is coercive and sequentially wekly lower semicontinuous. So,

$$\Upsilon_\mu(u_\mu) = \min\left[\Upsilon_\mu(u);\ u \in W_0^{s_1,p}(\Omega)\right].$$

is attained by a function $u_\mu := u_{k,\mu} \in W_0^{s_1,p}(\Omega)$.

Thus, $\Upsilon'_\mu(u_\mu) = 0$, that is,

$$\left\langle A_{s_1,p}(u_\mu) + A_{s_2,q}(u_\mu), \varphi \right\rangle = \int_\Omega \gamma_\mu(x,u_\mu)\varphi\mathrm{d}x, \quad \text{for all } \varphi \in W_0^{s_1,p}(\Omega). \tag{26}$$

Testing the Eq. (26) with $\varphi = (u_\mu - u_\lambda)^+ \in W_0^{s_1,p}(\Omega)$, using the Proposition 2 and $0 < \mu < \lambda$ we show that $u_\mu \leq u_\lambda$. In addition, testing the Eq. (26) with the function $\varphi = (\tilde{u}_\mu - u_\mu)^+ \in W_0^{s_1,p}(\Omega)$, using the Proposition 2 and the fact $\tilde{u}_\mu$ is unique solution of the problem (25), we show $\tilde{u}_\mu \leq u_\mu$.

So we have proved that,

$$u_\mu \in [\tilde{u}_\mu, u_\lambda] \Rightarrow u_\mu \in S_\mu \subseteq \mathrm{int}[(C_{s_1}^0(\overline{\Omega}))_+] \text{ and so } \mu \in \mathcal{L}.$$

□

Proposition 8 *If hypotheses $(\boldsymbol{H}_k)$ hold, $\lambda \in \mathcal{L}$, $u_\lambda \in S_\lambda \subseteq int\left[(C_{s_1}^0(\overline{\Omega}))_+\right]$ and $\mu < \lambda$, then $\mu \in \mathcal{L}$ and there exists $u_\mu \in S_\mu$ such that*

$$u_\lambda - u_\mu \in int\left[(C_{s_1}^0(\overline{\Omega}))_+\right].$$

Proof By Proposition 6 we know that $\mu \in \mathcal{L}$ and we can find $u_\mu := u_{k,\mu} \in S_\mu \subseteq \text{int}\left[(C^0_{s_1}(\overline{\Omega})_+)\right]$ such that $u_\mu \le u_\lambda$. Let $\rho = \|u_\lambda\|_\infty$ and $\widehat{E}_{k,\rho} > 0$ be as postulated by hypothesis $(\mathbf{H}_k)$ (v). We have

$$\begin{aligned}(-\Delta_p)^{s_1} u_\mu(x) &+ (-\Delta_q)^{s_2} u_\mu(x) + \lambda \widehat{E}_{k,\rho} u_\mu(x)^{p-1} - \lambda u_\mu(x)^{-\eta} \\ &\le \mu f_k(x, u_\mu(x)) + \lambda \widehat{E}_{k,\rho} u_\mu(x)^{p-1} \\ &= \lambda \left[f_k(x, u_\mu(x)) + \widehat{E}_{k,\rho} u_\mu(x)^{p-1} \right] - (\lambda - \mu) f_k(x, u_\mu(x)) \\ &\le \lambda \left[f_k(x, u_\mu(x)) + \widehat{E}_{k,\rho} u_\mu(x)^{p-1} \right] \\ &= (-\Delta_p)^{s_1} u_\lambda(x) + (-\Delta_q)^{s_2} u_\lambda(x) + \lambda \widehat{E}_{k,\rho} u_\lambda(x)^{p-1} - \lambda u_\lambda(x)^{-\eta}.\end{aligned}$$

Note that, the function $g(t) = \lambda \widehat{E}_{k,\rho} t^{p-1} - \lambda t^{-\eta}$ is nondecreasing in $\mathbb{R}^+_0$, thus, by Proposition 3 we have $u_\lambda - u_\mu \in \text{int}\left[(C^0_{s_1}(\overline{\Omega}))_+\right]$. ◻

Proposition 9 *Assume that the hypotheses* $(\boldsymbol{H}_k)$ *hold. Then* $\lambda^* = \sup \mathcal{L} < +\infty$, *for each* $k \in \mathbb{N}$.

Proof By hypotheses $H(i)$, (ii) and (iii) we can find $\widehat{\lambda} > 0$ such that

$$t^{p-1} \le \widehat{\lambda} f_k(x, t) \ \text{ for all } \ x \in \Omega, \ \ \text{all } \ t \ge 0. \tag{27}$$

Let $\lambda > \lambda^*$ and suppose that $\lambda \in \mathcal{L}$. Then, there exists $u_\lambda := u_{k,\lambda} \in S_\lambda \subseteq \text{int}[(C^0_{s_1}(\overline{\Omega}))_+]$, that is, u_λ is a solution of the problem $(P_{k,\lambda})$. Consider $\Omega_0 \subset\subset \Omega$ and $m_0 = \min\limits_{\overline{\Omega}} u_\lambda > 0$. For $\delta \in (0, 1)$ small we set $m_0^\delta = m_0 + \delta$. Let $\rho = \|u_\lambda\|_\infty$ and $\widehat{E}_{k,\rho} > 0$ be as postulated by $H(v)$. We have,

$$\begin{aligned}(-\Delta_p)^{s_1} m_0^\delta &+ (-\Delta_q)^{s_2} m_0^\delta + \lambda \widehat{E}_{k,\rho} (m_0^\delta)^{p-1} - \lambda (m_0^\delta)^{-\eta} \\ &\le \lambda \widehat{E}_{k,\rho} (m_0^\delta)^{p-1} + \chi(\delta) \ \ (\text{with } \ \chi(\delta) \to 0^+ \ \text{ as } \ \delta \to 0^+) \\ &= \left[\lambda \widehat{E}_{k,\rho} + 1\right] m_0^{p-1} + \chi(\delta) \\ &\le \widehat{\lambda} f_k(x, m_0) + \lambda \widehat{E}_{k,\rho} (m_0^\delta)^{p-1} + \chi(\delta) \ \ (\text{see } (27)) \\ &= \lambda \left[f_k(x, m_0) + \widehat{E}_{k,\rho} (m_0^\delta)^{p-1} \right] - (\lambda - \widehat{\lambda}) f_k(x, m_0) + \chi(\delta) \\ &\le \lambda \left[f_k(x, u_\lambda(x)) + \widehat{E}_{k,\rho} u_\lambda^{p-1} \right] \ \text{ for } \delta(0, 1) \ \text{ small enough.} \\ &= (-\Delta_p)^{s_1} u_\lambda(x) + (-\Delta_q)^{s_2} u_\lambda(x) + \lambda \widehat{E}_{k,\rho} u_\lambda(x)^{p-1} - \lambda u_\lambda(x)^{-\eta}.\end{aligned}$$

where we have used the hypotheses $H(iv)$, (v) and the fact $\chi(\delta) \to 0^+$ as $\delta \to 0^+$. By strong comparison principle we have

$$u_\lambda - m_0^\delta \in \text{int}[(C^0_{s_1}(\Omega))_+] \ \text{ for } \ \delta \in (0, 1) \ \text{ small enough}$$

which contradicts with the definition of m_0. Consequently, it holds $0 < \lambda^* \le \widehat{\lambda} < \infty$. ◻

Proposition 10 *If hypotheses* $(\boldsymbol{H}_k)$ *hold and* $\lambda \in (0, \lambda^*)$, *then problem* ($P_{k,\lambda}$) *has least two positive solutions*

$$u_0, \hat{u} \in int[(C^0_{s_1}(\Omega))_+] \ \ with \ \ u_0 \le \hat{u} \ \ and \ \ u_0 \ne \hat{u}.$$

Proof Let $0 < \lambda < \vartheta < \lambda^*$. By Proposition 9 $\lambda, \vartheta \in \mathcal{L}$. Thus, by Proposition 8 we can find $u_0 \in S_\lambda \subseteq \text{int}[(C^0_{s_1}(\Omega))_+]$ and $u_\vartheta \in S_\vartheta \subseteq \text{int}[(C^0_{s_1}(\Omega))_+]$ such that

$$u_\vartheta - u_0 \in S_\lambda \subseteq \text{int}[(C^0_{s_1}(\Omega))_+].$$

From Proposition 8, we know that $u_\lambda \le u_0$, hence $u_0^{-\eta} \in L^1(\Omega)$. Consider the Carathéodory function

$$\widehat{\omega}_\lambda(x,t) = \begin{cases} \lambda[u_0^{-\eta} + f_k(x,u_0)] \text{ if } \ t < u_0(x), \\ \lambda[t^{-\eta} + f_k(x,t)] \quad \text{ if } \ u_0(x) \le t \le u_\vartheta(x), \\ \lambda[u_\vartheta^{-\eta} + f_k(x,u_\vartheta)] \text{ if } \ u_\vartheta(x) < t \end{cases}$$

and define the C^1-functional $\widehat{\mu}_\lambda : W_0^{s_1,p}(\Omega) \to \mathbb{R}$ by

$$\widehat{\mu}_\lambda(u) = \frac{1}{p}[u]^p_{s_1,p} + [u]^q_{s_2,p} - \int_\Omega \widehat{W}_\lambda(x,u)dx \ \text{ for all } \ u \in W_0^{s_1,p}(\Omega).$$

where $\widehat{W}_\lambda(t,x) = \int_0^t \widehat{\omega}_\lambda(x,s)\text{d}s.$

Consider also another Carathéodory function

$$\omega_\lambda(x,t) = \begin{cases} \lambda[u_0^{-\eta}(x) + f_k(x,u_0)] \text{ if } \ t \le u_0(x), \\ \lambda[t^{-\eta} + f_k(x,t)] \qquad \text{ if } \ u_0(x) < t \end{cases}$$

and define the C^1-functional $\mu_\lambda : W_0^{s_1,p}(\Omega) \to \mathbb{R}$ by

$$\mu_\lambda(u) = \frac{1}{p}[u]^p_{s_1,p} + [u]^q_{s_2,p} - \int_\Omega W_\lambda(x,u)dx \ \text{ for all } \ u \in W_0^{s_1,p}(\Omega)$$

where $W_\lambda(t,x) = \int_0^t \omega_\lambda(x,s)\text{d}s.$

It is clear that,

$$\widehat{\mu}_\lambda(u)\Big|_{[0,u_\theta]} = \mu_\lambda(u)\Big|_{[0,u_\theta]} \ \text{and} \widehat{\mu}'_\lambda(u)\Big|_{[0,u_\theta]} = \mu'_\lambda(u)\Big|_{[0,u_\theta]} \tag{28}$$

Let $K_\mu = \{u \in W_0^{s_1,p}(\Omega); \mu'(u) = 0\}$. Using the same arguments used in ([18], Proposition 8) we can show that

$$K_{\widehat{\mu}_\lambda} \subseteq [u_0, u_\theta] \cap \text{int}[(C^0_{s_1}(\overline{\Omega}))_+] \tag{29}$$

$$K_{\mu_\lambda} \subseteq [u_0) \cap \mathrm{int}[(C^0_{s_1}(\overline{\Omega}))_+] \tag{30}$$

From (30), we can assume that K_{μ_λ} is finite. Otherwise, we already have an infinity of positive smooth solutions of $(P_{k,\lambda})$ bigger than u_0 and so we are done. In addition, we can assume that

$$K_{\mu_\lambda} \cap [u_0, u_\theta] = \{u_0\}. \tag{31}$$

Moreover, it is clear that $\widehat{\mu}_\lambda$ is coercive and sequentially weakly lower semicontinuous. So there exists $\tilde{u}_0 \in W^{s_1,p}_0(\Omega)$ such that,

$$\widehat{\mu}_{\tilde{u}_0} = \min\left[\widehat{\mu}_\lambda(u); \quad u \in W^{s_1,p}_0(\Omega)\right]$$

from (29) we have

$$\tilde{u}_0 \in K_{\widehat{\mu}_\lambda} \subseteq [u_0, u_\theta] \cap \mathrm{int}[(C^0_{s_1}(\overline{\Omega}))_+]$$

and so, from (28) and (31) results $\tilde{u}_0 = u_0$. Therefore,

$$u_0 \in \mathrm{int}[(C^0_{s_1}(\overline{\Omega}))_+] \text{ is a local } \; W^{s_1,p}_0(\Omega) - \text{minimizer of } \; \mu_\lambda.$$

Consequently, there exists $\rho \in (0, 1)$ such that,

$$\mu_\lambda(u_0) < \inf\left[\mu_\lambda(u); \quad [u - u_0]_{s_1,p} = \rho\right] = m_\lambda.$$

Note that, if $u \in \mathrm{int}[(C^0_{s_1}(\overline{\Omega}))_+]$, then on account of hypothesis $(\mathbf{H}_k\ (ii))$ we have,

$$\mu_\lambda(tu) \to -\infty \;\text{ as }\; t \to \infty$$

and moreover, classical arguments, which can be found in ([18], [2]), along with conditions $(\mathbf{H}_k)$ show that the function μ_λ satisfies the Cerami condition. By mountain pass theorem, there exists $\widehat{u} \in W^{s_1,p}_0(\Omega)$ such that,

$$\widehat{u} \in K_{\mu_\lambda} \subseteq [u_0) \cap \mathrm{int}[(C^0_{s_1}(\overline{\Omega}))_+]$$

and $m_\lambda \leq \mu_\lambda(\widehat{u})$. So, we have $\widehat{u} \in S_\lambda,\ u_0 \leq \widehat{u}$ and $\widehat{u} \neq u_0$. □

Proposition 11 *If hypotheses* $(\boldsymbol{H}_k)$ *hold, then* $\lambda^* \in \mathcal{L}$.

Proof Let $\{\lambda_n\} \subset (0, \lambda^*)$ be such that $\lambda_n \to \lambda^*$. We have $\{\lambda_n\}_{n\geq 1} \subseteq \mathcal{L}$ and of the proof of Proposition 10 we find $u_n \in S_{\lambda_n} \subseteq \mathrm{int}[(C^0_{s_1}(\overline{\Omega}))_+]$ such that,

$$\mu_{\lambda_n}(u_n) = \frac{1}{p}[u_n]^p_{s_1,p} + [u_n]^q_{s_2,q} - \lambda_n \int_\Omega [u_n^{1-\eta} + f_k(x, u_n).u_n]\mathrm{d}x$$

$$= \frac{1}{p}[u_n]^p_{s_1,p} + \frac{1}{q}[u_n]^q_{s_2,q} - [u_n]^p_{s_1,p} - [u_n]^p_{s_2,p} \quad (\text{Since } \ u_n \in S_{\lambda_n})$$
$$= \left(\frac{1}{p} - 1\right)[u_n]^p_{s_1,p} + \left(\frac{1}{q} - 1\right)[u_n]^q_{s_2,q} < 0 \ \text{ for all } \ n \in \mathbb{N}.$$

Moreover, we have

$$\left\langle A_{s_1,p}(u_n) + A_{s_2,q}(u_n), \varphi \right\rangle = \int_\Omega [\lambda_n u_n^{-\eta} + f_k(x, u_n)]\varphi \mathrm{d}x, \quad \text{for all } \ \varphi \in W_0^{s_1,p}(\Omega). \tag{32}$$

Arguing as in the proof of Proposition 13 in [2], we obtain that at least for a subsequence,

$$u_n \to u_* \ \text{ in } \ W_0^{s_1,p}(\Omega) \ \text{ as } n \to \infty.$$

By Proposition 8, $\tilde{u}_{\lambda_1} \leq u_n$ for all $n \in \mathbb{N}$. Therefore, we see $u_* \neq 0$ and $u_*^{-\eta}\varphi \leq \tilde{u}_{\lambda_1}^{-\eta}\varphi \in L^1(\Omega)$ for all $\varphi \in W_0^{s_1,p}(\Omega)$. In (32), we pass to the limit as $n \to \infty$ and we obtain

$$\left\langle A_{s_1,p}(u_*) + A_{s_2,q}(u_*), \varphi \right\rangle = \int_\Omega [\lambda^* u_*^{-\eta} + f_k(x, u_*)]\varphi \mathrm{d}x, \quad \text{for all } \ \varphi \in W_0^{s_1,p}(\Omega).$$

that is,

$$u_* \in S_{\lambda^*} \subseteq \operatorname{int}[(C^0_{s_1}(\overline{\Omega}))_+] \ \text{ and so } \ \lambda^* \in \mathcal{L}.$$

□

So, summarizing the situation for problem ($P_{k,\lambda}$), we can state the following bifurcation-type theorem.

Theorem 1 *If hypotheses ($\boldsymbol{H}_k$) hold, then we can find $\lambda^* > 0$ such that*

1. *For every $\lambda \in (0, \lambda^*)$ problem ($P_{k,\lambda}$) has at least two nontrivial positive solutions*

$$u_0, \hat{u} \in int[(C^0_{s_1}(\Omega))_+] \ \text{ with } \ u_0 \leq \hat{u} \ \text{ and } \ u_0 \neq \hat{u}.$$

2. *For $\lambda = \lambda^*$ problem ($P_{k,\lambda}$) has one nontrivial positive solution*

$$u_* \in int[(C^0_{s_1}(\overline{\Omega}))_+] \ \text{ and so } \ \lambda^* \in \mathcal{L}.$$

3. *For $\lambda > \lambda^*$ problem ($P_{k,\lambda}$) has no nontrivial positive solution.*

6 Existence of positive solution for P_λ

We denote by $u := u_{k,\lambda}$ the solution of the problem ($P_{k,\lambda}$) given by Theorem 1. Thus, we obtain

Proposition 12 *Let $u := u_{k,\lambda} \in W_0^{s_1,p}(\Omega)$ be a positive weak solution to the problem in ($P_{k,\lambda}$), then $u \in L^\infty(\bar{\Omega})$. Moreover, there exists $k > 1$ sufficiently large such that,*

$$\|u\|_\infty \leq M_k.$$

Proof The arguments of the proof is taken from the celebrated article of [22] with appropriate modifications. We will proceed with the smooth, convex and Lipschitz function $g_\epsilon(t) = \left(\epsilon^2 + t^2\right)^{\frac{1}{2}}$ for every $\epsilon > 0$. Moreover, $g_\epsilon(t) \to |t|$ as $t \to 0$ and $\left|g_\epsilon'(t)\right| \leq 1$. Let $0 < \psi \in C_c^\infty(\Omega)$ and choose $\varphi = \psi \left|g_\epsilon'(u)\right|^{p-2} g_\epsilon'(u)$ as the test function.

By Lemma 5.3 of [22] for all $\psi \in C_c^\infty(\Omega) \cap \mathbb{R}^+$, we obtain

$$\langle A_{s_1,p}(g_\epsilon(u)), \psi\rangle + \langle A_{s_2,q}(g_\epsilon(u)), \psi\rangle \leq \lambda \int_\Omega \left(\frac{1}{|u|^\eta} + |f_k(x,u)|\right) \left|g_\epsilon'(u)\right|^{p-1} \psi \mathrm{d}x$$

By Fatou's Lemma as $\varepsilon \to 0$ we have

$$\langle A_{s_1,p}(u), \psi\rangle + \langle A_{s_2,q}(u), \psi\rangle \leq \lambda \int_\Omega \left(\frac{1}{|u|^\eta} + |f_k(x,u)|\right) \psi \mathrm{d}x \tag{33}$$

Define $u_n = \min\{(u - M_k^\gamma)^+, n\}$ for each $n \in \mathbb{N}$ and $\gamma > 0$. Let $\beta > 1$, $\delta > 0$ and consider $\psi_\delta = (u_n + \delta)^\beta - \delta^\beta$. Thus, $\psi_\delta = 0$ in $\{u \leq M_k^\gamma\}$ and using ψ_δ in (33) we obtain

$$\langle A_{s_1,p}(u), \psi_\delta\rangle + \langle A_{s_2,q}(u), \psi_\delta\rangle \leq \lambda \int_\Omega \left(\frac{1}{|u|^\eta} + |f_k(x,u)|\right) ((u_n + \delta)^\beta - \delta^\beta)\mathrm{d}x$$

By Lemma 5.4 in [22] to follow the estimates,

$$\begin{aligned}
&\langle A_{s_1,p}(u), \psi_\delta\rangle + \langle A_{s_2,q}(u), \psi_\delta\rangle \\
&\geq \beta \left(\frac{p}{\beta + p - 1}\right)^p \left[(u_n + \delta)^{\frac{\beta+p-1}{p}}\right]_{s_1,p}^p + \beta \left(\frac{q}{\beta + q - 1}\right)^q \left[(u_n + \delta)^{\frac{\beta+q-1}{q}}\right]_{s_2,q}^q \\
&\geq \beta \left(\frac{p}{\beta + p - 1}\right)^p \left[(u_n + \delta)^{\frac{\beta+p-1}{p}}\right]_{s_1,p}^p
\end{aligned}$$

consequently,

$$\beta \left(\frac{p}{\beta + p - 1}\right)^p \left[(u_n + \delta)^{\frac{\beta+p-1}{p}}\right]_{s_1,p}^p \leq \lambda \int_\Omega \left(\frac{1}{|u|^\eta} + |f_k(x,u)|\right) ((u_n + \delta)^\beta - \delta^\beta)\mathrm{d}x$$

and thus,

$$\left[(u_n+\delta)^{\frac{\beta+p-1}{p}}\right]_{s_1,p}^{p} \le \lambda\frac{1}{\beta}\left(\frac{\beta+p-1}{p}\right)^{p}\int_{\Omega}\left(\frac{1}{|u|^{\eta}}+|f_k(x,u)|\right) ((u_n+\delta)^{\beta}-\delta^{\beta})\mathrm{d}x \tag{34}$$

Using the estimates (5), for $M_k > 1$ we have,

$$\begin{aligned}
&\int_{\Omega}\left(\frac{1}{|u|^{\eta}}+|f_k(x,u)|\right)((u_n+\delta)^{\beta}-\delta^{\beta})\mathrm{d}x\\
&\le \int_{\Omega}\left(\frac{1}{|u|^{\eta}}+C.M_k^{2\theta}|u|^{r-1}\right)((u_n+\delta)^{\beta}-\delta^{\beta})\mathrm{d}x\\
&= \int_{\{u\ge M_k^{\gamma}\}}\left(\frac{1}{|u|^{\eta}}+C.M_k^{2\theta}|u|^{r-1}\right)((u_n+\delta)^{\beta}-\delta^{\beta})\mathrm{d}x\\
&= \int_{\{u\ge M_k^{\gamma}\}}((u_n+\delta)^{\beta}-\delta^{\beta})\mathrm{d}x+\int_{\{u\ge M_n\}}C.M_k^{2\theta}|u|^{r-1}((u_n+\delta)^{\beta}-\delta^{\beta})\mathrm{d}x\\
&\le \int_{\{u\ge M_k^{\gamma}\}}M_k^{2\theta}((u_n+\delta)^{\beta}-\delta^{\beta})\mathrm{d}x+\int_{\{u\ge M_k^{\gamma}\}}C.M_k^{2\theta}|u|^{r-1}((u_n+\delta)^{\beta}-\delta^{\beta})\mathrm{d}x\\
&\le \int_{\Omega}M_k^{2\theta}((u_n+\delta)^{\beta}-\delta^{\beta})\mathrm{d}x+\int_{\Omega}C.M_k^{2\theta}|u|^{r-1}((u_k+\delta)^{\beta}-\delta^{\beta})\mathrm{d}x\\
&\le C.M_k^{2\theta}\left(|\Omega|^{\frac{\sigma-1}{\sigma}}+\|u\|_{L^{p_{s_1}^*}(\Omega)}^{r-1}\right)\|(u_n+\delta)^{\beta}\|_{L^{\sigma}(\Omega)}
\end{aligned}$$

where C is a constant independent of k and $\sigma = \dfrac{p_{s_1}^*}{p_{s_1}^*-r+1}$. Moreover, observe that the function $u := u_k$ satisfies $u \le \overline{u}$ where $\overline{u}$ is a supersolution of the problem (20) does not depend on k, we have $\|u\|_{L^{p_{s_1}^*}(\Omega)}^{r-1} \le C_0\|\overline{u}\|_{\infty}^{r-1}$ independent of k. Thus,

$$\begin{aligned}
&\int_{\Omega}\left(\frac{1}{|u|^{\eta}}+|f_k(x,u)|\right)((u_n+\delta)^{\beta}-\delta^{\beta})\mathrm{d}x\\
&\le KM_k^{2\theta}\left(|\Omega|^{\frac{\sigma-1}{\sigma}}+\|\overline{u}\|_{\infty}^{r}\right)\|(u_n+\delta)^{\beta}\|_{L^{\sigma}(\Omega)}\\
&= K_0M_k^{2\theta}\|(u_n+\delta)^{\beta}\|_{L^{\sigma}(\Omega)}
\end{aligned} \tag{35}$$

with K_0 independent of k.

By Sobolev inequality, triangle inequality and $(u_n+\delta)^{\beta+p-1} \geq \delta^{p-1}(u_n+\delta)^{\beta}$

$$\begin{aligned}\left[(u_n+\delta)^{\frac{\beta+p-1}{p}}\right]_{s_1,p}^p &\geq S\|(u_n+\delta)^{\beta}-\delta^{\beta}\|^p_{L^{p^*_{s_1}}(\Omega)}\\ &\geq \left(\frac{\delta}{2}\right)^{p-1}\left[\int_\Omega |(u_n+\delta)^{\frac{p^*_{s_1}\beta}{p}}\mathrm{d}x\right]^{\frac{p}{p^*_{s_1}}} - \delta^{\beta+p-1}|\Omega|^{\frac{p}{p^*_{s_1}}} \quad (36)\\ &\geq \left(\frac{\delta}{2}\right)^{p-1}\|(u_n+\delta)^{\frac{\beta}{p}}\|^p_{L^{p^*_{s_1}}(\Omega)} - M_k^{2\theta}\delta^{\beta+p-1}|\Omega|^{\frac{p}{p^*_{s_1}}},\end{aligned}$$

in the estimate above we using that $M_k > 1$.

Using the estimates (36) and (35) in (34), we obtain

$$\begin{aligned}&\left\|(u_n+\delta)^{\frac{\beta}{p}}\right\|^p_{L^{p^*_{s_1}}(\Omega)} \leq \left(\frac{2}{\delta}\right)^{p-1}\Bigg[\left(\frac{(\beta+p-1)^p}{\beta p^p}\right)\\ &\quad K_0 M_k^{2\theta}\|(u_n+\delta)^{\beta}\|_{L^{\sigma}(\Omega)} + \delta^{\beta+p-1}|\Omega|^{\frac{p}{p^*_s}}\Bigg]\\ &= \left(\frac{2}{\delta}\right)^{p-1}\left(\frac{(\beta+p-1)^p}{\beta p^p}\right)K_0 M_k^{2\theta}\|(u_n+\delta)^{\beta}\|_{L^{\sigma}(\Omega)} + \delta^{\beta}|\Omega|^{\frac{p}{p^*_s}}\\ &\leq \left(\frac{2}{\delta}\right)^{p-1}\left(\frac{(\beta+p-1)^p}{\beta p^p}\right)K_0 M_k^{2\theta}\|(u_n+\delta)^{\beta}\|_{L^{\sigma}(\Omega)} + |\Omega|^{\frac{p}{p^*_s}-1}\int_\Omega (u_n+\delta)^{\beta}\mathrm{d}x\end{aligned}$$

By Holder's inequality, we have

$$\delta^{\beta} = |\Omega|^{-1}\int_\Omega \delta^{\beta}\mathrm{d}x \leq |\Omega|^{-1}\int_\Omega (u_n+\delta)^{\beta}\mathrm{d}x \leq |\Omega|^{-\frac{1}{\sigma}}\|(u_n+\delta)^{\beta}\|_{L^{\sigma}(\Omega)}.$$

Consequently,

$$\begin{aligned}&\left\|(u_n+\delta)^{\frac{\beta}{p}}\right\|^p_{L^{p^*_{s_1}}(\Omega)} \leq \left(\frac{2}{\delta}\right)^{p-1}\left(\frac{(\beta+p-1)^p}{\beta p^p}\right)K_0 M_k^{2\theta}\|(u_n+\delta)^{\beta}\|_{L^{\sigma}(\Omega)}\\ &+|\Omega|^{\frac{p}{p^*_s}-\frac{1}{\sigma}}\|(u_n+\delta)^{\beta}\|_{L^{\sigma}(\Omega)}.\end{aligned}$$

Since, $\frac{1}{\beta}\left(\frac{\beta+p-1}{p}\right)^p \geq 1$ we can deduce that

$$\left\|(u_n+\delta)^{\frac{\beta}{p}}\right\|^p_{L^{p^*_{s_1}}(\Omega)} \leq \frac{1}{\beta}\left(\frac{\beta+p-1}{p}\right)^p M_k^{2\theta}\left\|(u_n+\delta)^{\beta}\right\|_q\left(\frac{K_0}{\delta^{p-1}}+|\Omega|^{\frac{p}{p^*_s}-\frac{1}{\sigma}}\right)$$

Now choose, $\delta > 0$ such that $\delta^{p-1} = K_0|\Omega|^{\frac{1}{\sigma}-\frac{p}{p^*_{s_1}}}$ and $\beta > 1$ such that, $\left(\frac{\beta+p-1}{p}\right)^p \geq \beta^p$. Thus,

$$\left\|(u_n+\delta)^{\frac{\beta}{p}}\right\|^p_{L^{p^*_{s_1}}(\Omega)} \leq CM_k^{2\theta}\beta^{p-1}\left\|(u_n+\delta)^\beta\right\|_{L^\sigma(\Omega)}$$

For $\tau = \sigma\beta$ and $\alpha = \frac{p^*_{s_1}}{\sigma p}$ we obtain,

$$\|u_n+\delta\|^\beta_{L^{\alpha\tau}(\Omega)} \leq CM_k^{2\theta}\beta^{p-1}\|u_n+\delta\|^\beta_{L^\tau(\Omega)}$$

and therefore,

$$\|u_n+\delta\|_{L^{\alpha\tau}(\Omega)} \leq \left(CM_k^{2\theta}\right)^{\frac{\sigma}{\tau}}\left(\frac{\tau}{\sigma}\right)^{(p-1)\frac{\sigma}{\tau}}\|u_n+\delta\|_{L^\tau(\Omega)}.$$

Taking, $\tau_0 = \sigma$, $\tau_{m+1} = \alpha\tau_m = \alpha^{m+1}\sigma$, then after performing m iterations we obtain the inequality

$$\begin{aligned}\|u_n+\delta\|_{L^{\tau_{m+1}}(\Omega)} &\leq \left(CM_k^{2\theta}\right)^{\sum_{i=0}^m \frac{\sigma}{\tau_i}}\left(\prod_{i=1}^m\left(\frac{\tau_i}{\sigma}\right)^{\frac{\sigma}{\tau_i}}\right)^{(p-1)}\|u_n+\delta\|_{L^\tau(\Omega)}\\ &= \left(CM_k^{2\theta}\right)^{\sum_{i=1}^m \frac{1}{\alpha^i}}\left(\prod_{i=1}^m \alpha^{\frac{i}{\alpha^i}}\right)^{(p-1)}\|u_n+\delta\|_{L^\tau(\Omega)}\end{aligned}$$

Therefore, on passing the limit as $m \to \infty$, we get

$$\|u_n\|_{L^\infty(\Omega)} \leq \|u_n+\delta\|_{L^\infty(\Omega)} \leq C^{\frac{\alpha}{\alpha-1}}M_k^{\frac{2\theta\alpha}{\alpha-1}}\alpha^{\frac{(p-1)\alpha}{(\alpha-1)^2}}\|u_n+\delta\|_{L^\sigma(\Omega)} \leq C_1 M_k^{\frac{2\theta\alpha}{\alpha-1}}. \tag{37}$$

In the last inequality we use the fact, $u \leq \overline{u}$, where $\overline{u} \in L^\infty(\Omega)$ is a supersolution of the problem (20) and thus, $u_n = \min\{(u-M_k^\gamma)^+, n\} \leq (u-M_k^\gamma)^+ \leq u^+ \leq \overline{u}$, for each $n \in \mathbb{N}$ and k large enough (such that $\|\overline{u}\| \leq M_k^\gamma$).

Therefore, as $n \to \infty$ we obtain

$$\|(u-M_k^\gamma)^+\|_\infty \leq M_k$$

for M_k sufficiently large and $\frac{2\theta\alpha}{\alpha-1} < 1$. Consequently, since $M_k \to \infty$ as $k \to \infty$ we have, for $\gamma < 1$, there exists $k > 1$ large enough such that,

$$\|u\|_\infty \leq M_k.$$

Also, by (37), the embedding $W_0^{s_1,p}(\Omega) \hookrightarrow L^\sigma(\Omega)$ and since $u_n = \min\{(u - M_k^\gamma)^+, n\} \leq (u - M_k^\gamma)^+ \leq u^+ \leq |u|$ we can establish

$$\|u_n\|_{L^\infty(\Omega)} \leq C M_k^{\frac{2\theta\alpha}{\alpha-1}} [u]_{s_1,p}.$$

Therefore, as $n \to \infty$ we obtain

$$\|u\|_{L^\infty(\Omega)} \leq C M_k^{\frac{2\theta\alpha}{\alpha-1}} [u]_{s_1,p},$$

for $k > 1$ large enough fixed. □

Theorem 2 *If hypotheses* (***H***) *hold, then we can find* $\lambda^* = \lambda^*(k) > 0$ *(k as in Proposition* 12*) such that*

1. *For every* $\lambda \in (0, \lambda^*)$ *problem* (P_λ) *has at least two nontrivial positive solutions*

$$u_0, \hat{u} \in int[(C^0_{s_1}(\Omega))_+] \ \text{ with } \ u_0 \leq \hat{u} \ \text{ and } \ u_0 \neq \hat{u}.$$

2. *For* $\lambda = \lambda^*$ *problem* (P_λ) *has one nontrivial positive solution*

$$u_* \in int[(C^0_{s_1}(\overline{\Omega}))_+] \ \text{ and so } \ \lambda^* \in \mathcal{L}.$$

3. *For* $\lambda > \lambda^*$ *problem* (P_λ) *has no nontrivial positive solution.*

Proof By Theorem 1, for each $\lambda \in (0, \lambda^*]$ and $k \in \mathbb{N}$ there exists $u_{k,\lambda}$ such that,

$$\begin{cases} (-\Delta_p)^{s_1} u + (-\Delta_q)^{s_2} u = \lambda \left[u(x)^{-\eta} + f_k(x,u) \right] & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ u > 0 & \text{in } \Omega. \end{cases} \qquad (P_{k,\lambda})$$

Moreover, 1, 2 and 3 holds to the problem ($P_{k,\lambda}$), by Theorem 1.

Using the Proposition 12, we have $\|u_{k,\lambda}\|_\infty < M_k$ for some $k > 1$ large enough. Thus, $u_\lambda := u_{k,\lambda}(x) \leq M_k$ and therefore $f_k(x, u_\lambda) = f(x, u_\lambda)$, in other words u_λ satisfies the problem (P_λ). □

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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