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Boundedness of fractional heat semigroups generated by degenerate Schrödinger operators

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Abstract

Let $L = -\frac{1}{\omega}\mathrm{div}(A(x)\cdot\nabla) + V$ be a degenerate Schrödinger operator in $\mathbb{R}^n$, where ω is a weight of the Muckenhoupt class A_2, $A(x)$ is a real and symmetric matrix depending on x and satisfies

$$C^{-1}\omega(x)|\xi|^2 \leq A(x)\xi_i\overline{\xi_j} \leq C\omega(x)|\xi|^2$$

for some positive constant C and all x, ξ in $\mathbb{R}^n$, and V is a nonnegative potential belonging to a certain reverse Hölder class with respect to the measure $\omega(x)dx$. By the subordinative formula, various regularity estimates about the fractional heat semigroup $\{e^{-tL^\alpha}\}_{t>0}$ are investigated, where L^α denotes the fractional powers of L for $\alpha \in (0, 1)$. As an application, we obtain the boundedness on the weighted Morrey spaces and BMO type spaces for some operator related to L^α.

Keywords Degenerate Schrödinger operators · Fractional heat semigroup · Fractional Laplacian · Weight Morrey spaces

Mathematics Subject Classification 42B20 · 42B25

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1 Introduction

Let

$$Lf(x) = -\frac{1}{\omega}\mathrm{div}(A(x)\cdot\nabla f(x)) + Vf(x)$$

be a degenerate Schrödinger operator on $\mathbb{R}^n$, where ω is a weight from the Muckenhoupt class A_2, which satisfies

$$\Big(\frac{1}{\omega(B)}\int_B \omega(x)dx\Big)\Big(\frac{1}{\omega(B)}\int_B \omega^{-1}(x)dx\Big) \le C$$

for a fixed constant C and any ball B, $A(x) = (a_{i,j}(x))_{1\le i,j\le n}$ is a real symmetric matrix such that for all n-dimensional vectors $\xi \in (\xi_1, \xi_2, \ldots, \xi_n)$,

$$C^{-1}\omega(x)|\xi|^2 \le \sum_{1\le i,j\le n} a_{i,j}(x)\xi_i\overline{\xi_j} \le C\omega(x)|\xi|^2,$$

and the nonnegative potential V belongs to the reverse Hölder class with respect to the measure $d\mu = \omega(x)dx$.

Throughout this paper, we always assume that ω satisfies both a doubling condition and a reverse doubling condition, i.e., there exist two numbers ν and γ, $0 < \nu \le n \le \gamma$, such that for any ball $B(x,r)$ and $t > 1$, the following inequalities hold:

$$ct^\nu \le \frac{\omega(B(x,tr))}{\omega(B(x,r))} \le Ct^\gamma$$

for some constants c and C independent of the point x. When the inequality on the left is satisfied we say $\omega \in RD_\nu$, while, if the other holds, we write $\omega \in D_\gamma$. For $\omega \in D_\gamma$, a nonnegative potential V, which is locally integrable, belongs to the reverse Hölder class $RH_q(\omega)$ for some $q > \gamma/2$ if there is a positive constant C such that

$$\Big(\frac{1}{\omega(B)}\int_B V^q(x)\omega(x)dx\Big)^{1/q} \le \frac{C}{\omega(B)}\int_B V(x)\omega(x)dx$$

for all balls B in $\mathbb{R}^n$. Let $\delta_0 = 2-\gamma/q$. In the whole paper, we maintain the assumption and a definition of δ_0.

Due to the background of the quantum mechanics, in the fields of partial differential equations and mathematical physics, the Schrödinger type operator L, which is a class of typical second-order differential operators, plays an important role. In recent years, due to the rapid development of fields such as nanotechnology and condensed matter physics, the study of the Schrödinger type operator L has also gained more and more applications and attention. For example, the degenerate Schrödinger operator L is often used in the study of new semiconductor devices and nanostructures such as quantum dots. In addition, with the development of quantum computing and quantum communication, the degenerate Schrödinger operator L has been studied more and more deeply. Therefore, the study of degenerate Schrödinger operators L has important theoretical and application values and has attracted the attention of some mathematicians and physicists.

Specially, when $V = 0$, in order to study the behavior of nonnegative solutions of the degenerate elliptic equation, Fabes et al. in [15] investigated the following second order degenerate elliptic differential operator in divergence form

$$L_0 u = -\frac{1}{\omega}\operatorname{div}(A \cdot \nabla u).$$

Furthermore, Fabes et al. in [14] obtained the fundamental solution Γ_0 of L_0 in a ball. It should be mentioned that if $\omega \in RD_\nu$ with $\nu > 2$,

$$\Gamma_0(x, y) \simeq \frac{|x-y|^2}{\omega(B(x, |x-y|))}.$$

For more information about L_0, we refer to [6–9] and the references therein.

For arbitrary degenerate Schrödinger operators L, $\{T_t^L\}_{t>0} := \{e^{-tL}\}_{t>0}$ denotes the heat semigroup generated by L with the integral kernels denoted by $K_t^L(\cdot, \cdot)$. Since the potential V is non-negative, the following upper bound estimate holds:

$$0 < K_t^L(x, y) \le h_t(x, y),$$

where $h_t(\cdot, \cdot)$ denotes the kernels of the semigroup $\{S_t\}_{t>0} := \{e^{-tL_0}\}_{t>0}$ generated by L_0 on $L^2(d\mu)$. In [10], by a perturbation argument, Dziubański further investigated the regularity properties of $K_t^L(\cdot, \cdot)$ and, as an application, the author obtained the atomic characterization of the Hardy space related to L denoted by $H_L^1(\mathbb{R}^n)$. As a continuation of the previous result, Huang et al. [22] used the square functions generated by $\{e^{-tL}\}_{t>0}$ to characterize $H_L^1(\mathbb{R}^n)$. Harboure et al. [19] studied the behavior of operators associated with L on weighted Lebesgue spaces, weighted Morrey spaces and BMO type spaces, respectively. For more results about degenerate Schrödinger operators, we refer the reader to [4, 23–26] and the references therein.

In this paper, we will investigate the boundedness of several singular integrals related to the fractional degenerate Schrödinger operator L^α, $\alpha \in (0, 1)$, on some weighted function spaces. Under the assumption $\alpha \in (0, 1)$, in Sect. 3, we first analyze the pointwise estimates and regularity properties of the fractional heat semigroups generated by L_0^α and L^α:

$$\begin{cases} \{S_{\alpha,t}\}_{t>0} := \{e^{-tL_0^\alpha}\}_{t>0}; \\ \{T_{\alpha,t}^L\}_{t>0} := \{e^{-tL^\alpha}\}_{t>0}. \end{cases}$$

Denote by $h_{\alpha,t}(\cdot,\cdot)$ and $K_{\alpha,t}^L(\cdot,\cdot)$ the integral kernels of $S_{\alpha,t}$ and $T_{\alpha,t}^L$, respectively. Precisely, for each $t>0$,

$$\begin{cases} S_{\alpha,t}f(x) := \int_{\mathbb{R}^n} h_{\alpha,t}(x,y)f(y)\omega(y)dy; \\ T_{\alpha,t}^L f(x) := \int_{\mathbb{R}^n} K_{\alpha,t}^L(x,y)f(y)\omega(y)dy. \end{cases}$$

For the Laplace operator $-\Delta$, the fractional heat kernel related with $-\Delta$, denoted by $h_{\alpha,t}^{-\Delta}$, is a convolution kernel and can be defined via the Fourier multiplier: $\widehat{h_{\alpha,t}^{-\Delta}}(\xi) = e^{-t|\xi|^{2\alpha}}$. We can use the classical methods to estimate the regularity of $h_{\alpha,t}^{-\Delta}$. See Lemma 2.1, Lemma 2.2 and Remark 2.1 of [29]. For the case of degenerate Schrödinger operators L, we can not define the fractional heat kernels related with L via the Fourier transform. Hence, the methods of [29] are invalid.

Let $\mathscr{L}$ be a second-order differential operator. Unlike the case of the Laplace operator, the fractional heat semigroup related with $\mathscr{L}$ is introduced via the following formulation. For $\alpha \in (0,1)$, the fractional power of $\mathscr{L}$, denoted by $\mathscr{L}^\alpha$, is defined as

$$\mathscr{L}^\alpha(f)(x) := \frac{1}{\Gamma(-\alpha)} \int_0^\infty \left(e^{-t\sqrt{\mathscr{L}}}f(x) - f(x)\right)\frac{dt}{t^{1+2\alpha}}, \quad f \in L^2(\mathbb{R}^n).$$

Then via the subordinative formula, the integral kernel $K_{\alpha,t}^{\mathscr{L}}$ of $e^{-t\mathscr{L}^\alpha}$ (cf. [17]) can be expressed as

$$K_{\alpha,t}^{\mathscr{L}}(x,y) := \int_0^\infty \eta_t^\alpha(s)K_s^{\mathscr{L}}(x,y)ds \ \ \forall\, x,y \in \mathbb{R}^n, \tag{1.1}$$

where $\eta_t^\alpha(\cdot)$ is a non-negative continuous function on $(0,\infty)$ satisfying

$$\begin{cases} \int_0^\infty \eta_t^\alpha(s)ds = 1; \\ \eta_t^\alpha(s) = \frac{1}{t^{1/\alpha}}\eta_1^\alpha(\frac{s}{t^{1/\alpha}}); \\ \eta_t^\alpha(s) \lesssim \frac{t}{s^{1+\alpha}} \ \forall s,t>0; \\ \int_0^\infty s^{-r}\eta_1^\alpha(s)ds < \infty, \quad r>0; \\ \eta_t^\alpha(s) \simeq \frac{t}{s^{1+\alpha}} \ \ \forall s \geq t^{1/\alpha} > 0. \end{cases} \tag{1.2}$$

In this paper, we use the subordinate formula (1.1) to introduce the fractional heat kernels related with L. Using the identity (1.1) and the estimates about $h_t(\cdot,\cdot)$ and $K_t^L(\cdot,\cdot)$, we obtain the size and regularity estimates of $h_{\alpha,t}(\cdot,\cdot)$ and $K_{\alpha,t}^L(\cdot,\cdot)$. See Propositions 3.4 and 3.5. By the regularity estimate obtained in Sect. 3, we can study the boundedness of some operators related to L^α on the weighted Lebesgue spaces $L^p(\mathbb{R}^n,\omega)$ and the weighted Morrey spaces $M_p^\lambda(\mathbb{R}^n,\omega)$.

The classical Morrey spaces $M_p^\lambda(\mathbb{R}^n)$ have been studied extensively and used widely in analysis, geometry, mathematical physics and other related fields, which were originally introduced by Morrey in [30] to investigate the local behavior of solutions of second order elliptic partial differential equations. The advantage of using this functional space lies in the fact that ones can obtain better regularity properties for solutions of the boundary elliptic and parabolic equations in Morrey spaces. However, the regularity results for many partial differential equations can be provided as applications of the boundedness properties of several singular integral operators. By these interesting applications, many mathematicians considered the boundedness properties of singular integral operators in different kinds of functional spaces so called Morrey type spaces. For more information about Morrey spaces and their applications, we refer the reader to [1, 2, 27, 32] and the references therein.

In Sect. 4, motivated by [18, 36], we investigate the boundedness of some operators related to L^α on the weighted Lebesgue spaces $L^p(\mathbb{R}^n,\omega)$ and the weighted Morrey spaces $M_p^\lambda(\mathbb{R}^n,\omega)$. Using the boundedness of the maximal function M_ω, we obtain the $L^p(\mathbb{R}^n,\omega)$-boundedness and the $M_p^\lambda(\mathbb{R}^n,\omega)$-boundedness for the semigroup maximal operators $S^*=\sup_{t>0}S_{\alpha,t}$ and $T^*=\sup_{t>0}T^L_{\alpha,t}$, respectively. See Theorem 4.2 and Corollary 4.5. Denote by $I_{\alpha,\beta}$ and $I^L_{\alpha,\beta}$ the negative powers of L_0^α and L^α, respectively. Precisely,

$$\begin{cases} I_{\alpha,\beta}f(x):=\displaystyle\int_0^\infty S_{\alpha,t}f(x)t^{\beta/2}\frac{dt}{t};\\ I^L_{\alpha,\beta}f(x):=\displaystyle\int_0^\infty T^L_{\alpha,t}f(x)t^{\beta/2}\frac{dt}{t}.\end{cases}$$

We show that $I_{\alpha,\beta}$ and $I^L_{\alpha,\beta}$ are both bounded from $M_p^\lambda(\mathbb{R}^n,\omega)$ to $M_s^\lambda(\mathbb{R}^n,\omega)$ with $1<p<\lambda/\alpha\beta$ and $1/s=1/p-\alpha\beta/\lambda$. See Theorem 4.7. Moreover, the operator $I^L_{\alpha,\beta}$ deserves special attention. In Theorem 4.8, we investigate some behaviours about the mixed operators $I^L_{\alpha,\beta}V^{\sigma/2}$ and $V^{\sigma/2}I^L_{\alpha,\beta}$ for $0<\sigma\le\alpha\beta<\nu$. We point out that, in Theorem 4.8, the boundedness of $I^L_{\alpha,\beta}V^{\sigma/2}$ from $M_p^\lambda(\mathbb{R}^n,\omega)$ to $M_s^\lambda(\mathbb{R}^n,\omega)$ is based on a very stringent assumption that $V\in RH_\infty(\omega)$. Due to $RH_\infty(\omega)\subset RH_q(\omega)$ for $1<q<\infty$, an interesting question is to investigate the boundedness of $I^L_{\alpha,\beta}V^{\sigma/2}$ on weighted Morrey spaces when the potential V belongs to $RH_q(\omega)$. For this purpose, we introduce another class of weighted Morrey type spaces $M_p^{\lambda_1,\lambda_2}(\mathbb{R}^n,\omega)$ defined in Definition 4.13. Under the assumption that $V\in RH_q(\omega)$ and the weight ω satisfies the lower-Ahlfors condition (4.5), we prove that the operators $V^{\sigma/2}I^L_{\alpha,\beta}$ and $I^L_{\alpha,\beta}V^{\sigma/2}$ are bounded from $M_s^{\lambda s/p,s/p}(\mathbb{R}^n,\omega)$ to $M_p^\lambda(\mathbb{R}^n,\omega)$ when s and p satisfy suitable conditions. See Theorems 4.14 and 4.15, respectively.

It is well-known that the $T1$ theorem plays a crucial role in the analysis of BMO-boundedness of Calderón-Zygmund singular integral operators. Recently, for the Hermite operator $H=-\Delta+|x|^2$, in [3], Betancor et al. introduced a $T1$ criterion for Calderón-Zygmund operators related to H on the BMO type space $BMO_H(\mathbb{R}^n)$.

Later, Ma et al. in [28] generalized the $T1$ criterion to the case of Campanato type spaces $BMO^d_{\mathcal{L}}(\mathbb{R}^n)$ related with $\mathcal{L} = -\Delta + V$. Then in [5], Bui et al. extented the $T1$ theorem to make it applicable to a large class of generalized Calderón-Zygmund type operators.

Therefore, at last, we obtain the boundedness of the maximal operator S^* and $I_{\alpha,\beta}$ on BMO type spaces. Besides, we also prove that $I_{\alpha,\beta}$ can be extended to a bounded operator from $M_1^{\alpha\beta-d}(\mathbb{R}^n, \omega)$ to $BMO^d(\mathbb{R}^n, \omega)$. See Theorem 5.3. For the operators related to L^α, we use the $T1$ theorem corresponding to $BMO^d_\rho(\mathbb{R}^n, \omega)$ to obtain the $BMO^d_\rho(\mathbb{R}^n, \omega)$-boundedness for T^*, $I^L_{\alpha,\beta}$ and $I^L_{\alpha,\beta} V^{\sigma/2}$ with $\alpha\beta \geq \sigma$. See Theorems 6.4, 6.6 and 6.7.

Remark 1.1 (i) When the matrix $A(x)$ is the identity matrix and $\omega \equiv 1$, the degenerate Schrödinger operator L^α, $\alpha \in (0, 1)$, comes back to the Schrödinger operator $\mathcal{L}^\alpha = (-\Delta + V)^\alpha$. As far as we know, the result of this paper for boundedness on Morrey spaces $M^\lambda_p(\mathbb{R}^n, \omega)$ (cf. Theorems 4.7, 4.8, 4.11, 4.14 and 4.15) is also new for $\mathcal{L}^\alpha$. In particular, for the degenerate Schrödinger operator L, the results of Theorems 4.11, 4.14 and 4.15 are also new when $\alpha = 1$. For the result on $BMO^d_\rho(\mathbb{R}^n, \omega)$, when L^α comes back to $\mathcal{L}^\alpha$, we can see that Theorem 6.4 comes back to [36, Theorem 3]. Moreover, Theorems 6.6 and 6.7 are also new for $\mathcal{L}^\alpha$.

(ii) When L^α_0 comes back to $\mathcal{L}^\alpha_0 = (-\Delta)^\alpha$, Theorems 5.3 and 5.5 are also new. Moreover, in the non-degenerate case of $\omega \equiv 1$, we know that weak-$L^s(\mathbb{R}^n) \subset M_1^{n/s}(\mathbb{R}^n)$ for any $s > 1$ (see [18, Lemma 4.1] with $\omega \equiv 1$). Therefore, Theorem 5.3 recovers the boundedness of the modified classical fractional integral of order $\alpha\beta$ from weak-$L^{n/\alpha\beta}(\mathbb{R}^n)$ into BMO or, more generally, from weak-L^p into $BMO^d(\mathbb{R}^n)$ for $p \geq n/(\alpha\beta)$ and $d = \alpha\beta - p/n < 1$ (see [18, Theorem2.5] and [16, Theorems 1.1 & 1.2]).

Throughout this article, we will use c and C to denote the positive constants, which are independent of main parameters and may be different at each occurrence. $\mathsf{U} \simeq \mathsf{V}$ indicates that there is a constant $C > 0$ such that $C^{-1}\mathsf{V} \leq \mathsf{U} \leq C\mathsf{V}$, whose right inequality is also written as $\mathsf{U} \lesssim \mathsf{V}$. Similarly, one writes $\mathsf{V} \gtrsim \mathsf{U}$ for $\mathsf{V} \geq C\mathsf{U}$.

2 Preliminaries

To state our main results, the following auxiliary function plays a fundamental role. It was introduced by Shen in [33] (see also [26]) and is defined as

$$\rho(x) := \sup\left\{r > 0 : \frac{r^2}{\omega(B(x,r))} \int_{B(x,r)} V(x)\omega(x)dx \leq 1\right\}, \quad x \in \mathbb{R}^n, \quad (2.1)$$

where $B(x, r)$ denotes a ball in $\mathbb{R}^n$ centered at x and with radius r. It follows from [33] that the auxiliary function $\rho(\cdot)$ determined by $V \in RH_q(\omega)$ satisfies $0 < \rho(x) < \infty$ for any given $x \in \mathbb{R}^n$. The following results concerning the critical radius function (2.1) are well known.

Lemma 2.1 ([26, Lemma 4]) *There exist constants $C_0 \geq 1$ and $N_0 > 0$ such that for any x and y in $\mathbb{R}^n$,*

$$\frac{1}{C_0}\Big(1 + |x-y|/\rho(x)\Big)^{-N_0} \leq \frac{\rho(y)}{\rho(x)} \leq C_0\Big(1 + |x-y|/\rho(x)\Big)^{N_0/(1+N_0)}. \quad (2.2)$$

It should be mentioned that if $V \in RH_q(\omega)$, it satisfies the doubling condition: for every $x \in \mathbb{R}^n$ and $r > 0$,

$$\frac{1}{\omega(B(x,2r))}\int_{B(x,2r)} V(y)\omega(y)dy \lesssim \frac{1}{\omega(B(x,r))}\int_{B(x,r)} V(y)\omega(y)dy.$$

Lemma 2.2 ([26, Lemma 2]) *Assume that $w \in D_\gamma$, $V \in RH_q(\omega)$ with $q > \gamma/2$. Then for every $0 < r < R < \infty$, $y \in \mathbb{R}^n$ we have*

$$\frac{r^2}{w(B(y,r))}\int_{B(y,r)} V(x)\omega(x)dx \lesssim \Big(\frac{r}{R}\Big)^{2-\gamma/q}\frac{R^2}{\omega(B(y,R))}\int_{B(y,R)} V(x)\omega(x)dx.$$

Lemma 2.3 ([10, Lemma 4.4]) *Assume that $V \in RH_q(\omega)$ with $q > \gamma/2$. Then, for any $N > \log_2 C_0 + 1$, there exists a constant C_N such that for any $x \in \mathbb{R}^n$ and $r > 0$,*

$$\frac{1}{(1 + r/\rho(x))^N}\int_{B(x,r)} V(y)\omega(y)dy \leq C_N\frac{\omega(B(x,r))}{r^2}.$$

Definition 2.4 Let $1 \leq p \leq \infty$, $\omega \in A_2$ and $0 \leq \lambda$.

(i) For $1 \leq p < \infty$, the weighted Morrey space $M_p^\lambda(\mathbb{R}^n, \omega)$ is defined as the set of all L^p-locally integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{M_p^\lambda(\mathbb{R}^n,\omega)} := \sup_{B=B(x,r)}\Big(\frac{r^\lambda}{\omega(B)}\int_B |f(x)|^p\omega(x)dx\Big)^{1/p} < \infty.$$

(ii) For $p = \infty$, define $M_\infty^\lambda(\mathbb{R}^n, \omega) := L^\infty(\mathbb{R}^n, \omega)$.

It is easy to see that $\|f\|_{M_p^\lambda(\mathbb{R}^n,\omega)}$ coincides with the standard L^p-norm when $\omega \equiv 1$ and $\lambda = n$. More generally, if the weight ω is Ahlfors of order λ, i.e., $r^\lambda \simeq \omega(B(x,r))$, then $M_p^\lambda(\mathbb{R}^n, \omega) = L^p(\mathbb{R}^n, \omega)$.

The weighted Lipschitz type spaces are defined as follows.

Definition 2.5 For $0 \leq d < 1$, the space $BMO^d(\mathbb{R}^n, \omega)$ is defined as the set of all functions $f \in L^1_{loc}(\mathbb{R}^n, \omega)$ satisfying the following inequality: there exists a constant C such that for any ball $B = B(x,r)$,

$$\frac{1}{r^d\omega(B)}\int_B |f(x) - f_B|\omega(x)dx \leq C, \quad (2.3)$$

where f_B stands for the average of f on B with respect to $\omega(x)dx$, that is, $f_B = \frac{1}{\omega(x)}\int_B f(y)\omega(y)dy$. The norm $\|f\|_{BMO^d}$ is defined as the infimum of the constants C such that (2.3) holds.

Recently, the function spaces related to Schrödinger operators are investigated extensively. For $\mathcal{L} = -\Delta + V$, the Hardy type spaces related to $\mathcal{L}$, denoted by $H^1_{\mathcal{L}}(\mathbb{R}^n)$, were introduced by Dziubański and Zienkiewicz [11, 12]. Using the theory of local Hardy spaces, the authors also established the atomic characterization and the Riesz transform characterization of $H^1_{\mathcal{L}}(\mathbb{R}^n)$. Dziubański et al. in [13] introduced the BMO type space associated with $\mathcal{L}$. Similarly to [11], Dziubański [10] investigated the Hardy space related to degenerate Schrödinger operators. For further information on this topic, we refer the reader to [4, 22–24, 37, 38] and the references therein. Following the idea of [13], the weighted Lipschitz type spaces related to degenerate Schrödinger operators can be defined as follows.

Definition 2.6 For $0 \le d < 1$, the space $BMO^d_L(\mathbb{R}^n, \omega)$ is defined as the set of all locally integrable functions f with respect to $\omega(x)dx$ satisfying there exists a constant C such that for any ball $B = B(x, r)$,

$$\frac{1}{r^d\omega(B)}\int_B |f(x) - f(B, V)|\omega(x)dx \le C,$$

where

$$f(B, V) := \begin{cases} f_B, & r < \rho(x); \\ 0, & r \ge \rho(x), \end{cases}$$

The infimum of the above constants C actually gives a norm.

Remark 2.7 (i) It is well known that in (2.3) above the mean value f_B can be equivalently replaced by arbitrary constant c. For $d = 0$, $BMO^0(\mathbb{R}^n, \omega)$ coincides with the classical weighted BMO space for $\omega \in A_\infty$ (see [31]). Furthermore, if $d > 0$ and ω is a doubling weight, all functions $f \in BMO^d(\mathbb{R}^n, \omega)$ satisfy

$$|f(x) - f(y)| \le C|x - y|^d,$$

which indicates, for this case, the space $BMO^d(\mathbb{R}^n, \omega)$ coincides with integral Lipschitz spaces with respect to the Lebesgue measure, but now just for a doubling weight.

(ii) When $d > 0$ and ω is doubling, the functions in $BMO^d_L(\mathbb{R}^n, \omega)$ can be described by the following pointwise inequalities

$$|f(x) - f(z)| \le C|x - y|^d, \quad |x - y| < \rho(x)$$

and

$$|f(x)| \le C\rho^d(x).$$

As a consequence, for $d > 0$ and a doubling weight ω, the integral Lipschitz space $BMO_L^d(\mathbb{R}^n, \omega)$ defined above also coincides with the integral version corresponding to $\omega \equiv 1$.

3 Regularity estimates of the fractional heat kernel

We first state several known estimates about h_t and K_t^L, which can be seen in [20].

Lemma 3.1 ([20, Theorem 2.4]) *Assume that* $\omega \in RD_\nu \bigcap D_\gamma \bigcap A_2$, $2 < \nu \le \gamma$.

(i) *There exists a positive constant c such that*

$$0 \le h_t(x, y) \lesssim \frac{1}{\omega(B(x, \sqrt{t}))} e^{-|x-y|^2/ct}.$$

(ii) *If* $|x - z| \le |x - y|/4$, *for some* $0 < \eta \le 1$,

$$|h_t(x, y) - h_t(z, y)| \lesssim \min\left\{1, \left(\frac{|x-z|}{\sqrt{t}}\right)^{\eta}\right\} \frac{e^{-|x-y|^2/ct}}{\omega(B(x, \sqrt{t}))}.$$

The following estimates about $K_t^L(\cdot, \cdot)$ and $h_t(\cdot, \cdot) - K_t^L(\cdot, \cdot)$ can be seen in [10, 23].

Lemma 3.2 *Assume that* $\omega \in RD_\nu \bigcap D_\gamma \bigcap A_2$, $2 < \nu \le \gamma$ *and V satisfies a* $RH_q(\omega)$ *condition with* $q > \gamma/2$. *Let* $\delta_0 = 2 - \gamma/q$.

(i) *([10, Theorem 2.2]) For each* $N \ge 0$ *there is a positive constant* C_N *such that*

$$0 \le K_t^L(x, y) \le C_N \frac{e^{-|x-y|^2/ct}}{\omega(B(x, \sqrt{t}))} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$

(ii) *([23, Proposition 3.2]) For any given* $0 < \delta < \min\{\eta, \delta_0\}$, *there exist two positive constants* C_N *and c such that for every* $N > 0$ *and* $|x - z| < |x - y|/4$,

$$|K_t^L(x, y) - K_t^L(z, y)| \le C_N \left(\frac{|x-z|}{\sqrt{t}}\right)^{\delta} \frac{e^{-|x-y|^2/ct}}{\omega(B(x, \sqrt{t}))} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$

Lemma 3.3 ([10, Proposition 5.1]) *Assume that* $\omega \in RD_\nu \bigcap D_\gamma \bigcap A_2$, $2 < \nu \le \gamma$ *and V satisfies a* $RH_q(\omega)$ *condition with* $q > \gamma/2$. *Let* $\delta_0 = 2 - \gamma/q$.

(i) *There exists a positive constant c such that*

$$|h_t(x, y) - K_t^L(x, y)| \lesssim \min\left\{1, \left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta_0}\right\} \frac{e^{-|x-y|^2/ct}}{\omega(B(x, \sqrt{t}))}.$$

(ii) *For any given* $0<\delta<\min\{\eta,\delta_0\}$, $|x-z|<|x-y|/4$ *and* $|x-z|\le\rho(x)$, *there exists a positive constant* c *such that*

$$|(h_t(x,y)-K_t^L(x,y))-(h_t(z,y)-K_t^L(z,y))|\lesssim\Big(\frac{|x-z|}{\rho(x)}\Big)^{\delta}\frac{e^{-|x-y|^2/ct}}{\omega(B(x,\sqrt{t}))}.$$

Below we will give the estimates for $h_{\alpha,t}(\cdot,\cdot)$ and $K^L_{\alpha,t}(\cdot,\cdot)$, respectively.

Proposition 3.4 *Let* $\alpha\in(0,1)$. *Assume that* $\omega\in RD_\nu\bigcap D_\gamma\bigcap A_2$, $2<\nu\le\gamma$.

(i) *For* $x,y\in\mathbb{R}^n$ *and* $t>0$,

$$0\le h_{\alpha,t}(x,y)\lesssim\min\Big\{\frac{1}{\omega(B(x,|x-y|))}\frac{t}{|x-y|^{2\alpha}},\frac{1}{\omega(B(x,t^{1/2\alpha}))}\Big\},$$

which gives

$$0\le h_{\alpha,t}(x,y)\lesssim\frac{1}{\omega(B(x,\sqrt{t^{1/\alpha}+|x-y|^2}))}\frac{t}{(t^{1/\alpha}+|x-y|^2)^{\alpha}}.$$

(ii) *There exists some* $0<\eta_1\le\min\{2\alpha,\eta\}$ *such that for* $|x-z|\le|x-y|/4$,

$$\begin{aligned}&|h_{\alpha,t}(x,y)-h_{\alpha,t}(z,y)|\\&\lesssim\Big(\frac{|x-z|}{t^{1/2\alpha}}\Big)^{\eta_1}\min\Big\{\frac{t^{1+\eta_1/2\alpha}}{\omega(B(x,|x-y|))|x-y|^{2\alpha+\eta_1}},\frac{1}{\omega(B(x,t^{1/2\alpha}))}\Big\},\end{aligned}$$

which gives

$$|h_{\alpha,t}(x,y)-h_{\alpha,t}(z,y)|\lesssim\frac{1}{\omega(B(x,\sqrt{t^{1/\alpha}+|x-y|^2}))}\Big(\frac{|x-z|}{t^{1/2\alpha}}\Big)^{\eta_1}\frac{t}{(t^{1/\alpha}+|x-y|^2)^{\alpha}}.$$

Proof For (i), it can be deduced from (1.1), (1.2) and Lemma 3.1 that

$$\begin{aligned}h_{\alpha,t}(x,y)&\lesssim\int_0^\infty\frac{t}{s^{1+\alpha}}h_s(x,y)ds\\&\lesssim\int_0^\infty\frac{t}{s^{1+\alpha}}\frac{1}{\omega(B(x,\sqrt{s}))}e^{-|x-y|^2/cs}ds.\end{aligned}$$

By the change of variable $s=t^{1/\alpha}u$, we get

$$\begin{aligned}h_{\alpha,t}(x,y)&\lesssim\int_0^\infty\frac{t}{(t^{1/\alpha}u)^{1+\alpha}}\frac{e^{-c|x-y|^2/(t^{1/\alpha}u)}t^{1/\alpha}}{\omega(B(x,t^{1/2\alpha}u))}du\\&=\int_0^\infty\frac{e^{-c|x-y|^2/(t^{1/\alpha}u)}}{\omega(B(x,t^{1/2\alpha}u))}u^{-(1+\alpha)}du.\end{aligned}$$

Letting $|x-y|^2/(t^{1/\alpha}u) = r^2$, we have

$$\begin{aligned} h_{\alpha,t}(x,y) &\lesssim \int_0^\infty \frac{e^{-cr^2}}{\omega(B(x,|x-y|/r))}\frac{tr^{2\alpha-1}}{|x-y|^{2\alpha}}dr \\ &= \frac{t}{|x-y|^{2\alpha}}\Big(\int_0^1+\int_1^\infty\Big)\frac{r^{2\alpha-1}e^{-cr^2}}{\omega(B(x,|x-y|/r))}dr =: I_1+I_2. \end{aligned}$$

For I_2, since $r \ge 1$, the doubling condition of ω can be utilized to derive

$$\begin{aligned} \frac{1}{\omega(B(x,|x-y|/r))} &= \frac{1}{\omega(B(x,|x-y|))}\frac{\omega(B(x,r|x-y|/r))}{\omega(B(x,|x-y|/r))} \\ &\lesssim \frac{r^\gamma}{\omega(B(x,|x-y|))}, \end{aligned}$$

which further yields

$$I_2 \lesssim \frac{t|x-y|^{-2\alpha}}{\omega(B(x,|x-y|))}\int_1^\infty r^{2\alpha-1+\gamma}e^{-cr^2}dr \lesssim \frac{t|x-y|^{-2\alpha}}{\omega(B(x,|x-y|))}.$$

It remains to prove I_1. Since $0<r<1$ and $\omega \in RD_\nu$, we have

$$\omega(B(x,|x-y|/r)) \gtrsim r^{-\nu}\omega(B(x,|x-y|)),$$

which implies

$$I_1 \lesssim \frac{t|x-y|^{-2\alpha}}{\omega(B(x,|x-y|))}\int_0^1 r^{2\alpha-1+\nu}e^{-r^2}dr \lesssim \frac{t|x-y|^{-2\alpha}}{\omega(B(x,|x-y|))}.$$

Therefore, we obtain

$$h_{\alpha,t}(x,y) \lesssim \frac{1}{\omega(B(x,|x-y|))}\frac{t}{|x-y|^{2\alpha}}. \tag{3.1}$$

On the other hand, noting that

$$h_{\alpha,t}(x,y) \lesssim \int_0^\infty \frac{1}{\omega(B(x,\sqrt{s}))}\frac{1}{t^{1/\alpha}}\eta_1^\alpha(s/t^{1/\alpha})ds,$$

we can apply the change of variables $\tau = s/t^{1/\alpha}$ to get

$$\begin{aligned} h_{\alpha,t}(x,y) &= \int_0^\infty \frac{1}{\omega(B(x,\sqrt{\tau t^{1/\alpha}}))}\frac{1}{t^{1/\alpha}}\eta_1^\alpha(\tau)t^{1/\alpha}d\tau \\ &= \int_0^\infty \frac{1}{\omega(B(x,\sqrt{\tau t^{1/\alpha}}))}\eta_1^\alpha(\tau)d\tau \end{aligned}$$

$$= \left(\int_0^1 + \int_1^\infty \right) \frac{1}{\omega(B(x, \sqrt{\tau t^{1/\alpha}}))} \eta_1^\alpha(\tau) d\tau$$
$$=: I_3 + I_4.$$

Now we are in a position to show I_3. By the fact that $0 < \tau < 1$ and $\omega \in D_\gamma$, we obtain

$$I_3 \lesssim \frac{1}{\omega(B(x, t^{1/2\alpha}))} \int_0^1 \tau^{-\gamma/2} \eta_1^\alpha(\tau) d\tau \lesssim \frac{1}{\omega(B(x, t^{1/2\alpha}))}.$$

For I_4, since $\tau \geq 1$ and $\omega \in RD_\nu$, we conclude that

$$I_4 \lesssim \frac{1}{\omega(B(x, t^{1/2\alpha}))} \int_1^\infty \tau^{-\nu/2} \eta_1^\alpha(\tau) d\tau \lesssim \frac{1}{\omega(B(x, t^{1/2\alpha}))}.$$

The above estimates imply that

$$h_{\alpha,t}(x, y) \lesssim \frac{1}{\omega(B(x, t^{1/2\alpha}))}. \tag{3.2}$$

Below we divide the range of $t^{1/2\alpha}$ into two cases.

Case 1: $t^{1/2\alpha} \leq |x - y|$. It follows from (3.1) that

$$\begin{aligned} h_{\alpha,t}(x, y) &\lesssim \frac{1}{\omega(B(x, |x - y|))} \frac{t}{|x - y|^{2\alpha}} \\ &\lesssim \frac{1}{\omega(B(x, \sqrt{t^{1/\alpha} + |x - y|^2}))} \frac{t}{(t^{1/\alpha} + |x - y|^2)^\alpha}. \end{aligned}$$

Case 2: $t^{1/2\alpha} > |x - y|$. By (3.2), we get

$$\begin{aligned} h_{\alpha,t}(x, y) &\lesssim \frac{1}{\omega(B(x, |x - y|))} \frac{t}{t} \\ &\lesssim \frac{1}{\omega(B(x, \sqrt{t^{1/\alpha} + |x - y|^2}))} \frac{t}{(t^{1/\alpha} + |x - y|^2)^\alpha}. \end{aligned}$$

Hence, in any case, we have

$$h_{\alpha,t}(x, y) \lesssim \frac{1}{\omega(B(x, \sqrt{t^{1/\alpha} + |x - y|^2}))} \frac{t}{(t^{1/\alpha} + |x - y|^2)^\alpha}.$$

For (ii), by (1.1) and Lemma 3.1, we can get

$$|h_{\alpha,t}(x, y) - h_{\alpha,t}(z, y)| = \left| \int_0^\infty \eta_t^\alpha(s) \Big(h_s(x, y) - h_s(z, y) \Big) ds \right|$$

$$\lesssim \int_0^\infty \frac{t}{s^{1+\alpha}} \frac{e^{-c|x-y|^2/s}}{\omega(B(x,\sqrt{s}))}\Big(\frac{|x-z|}{\sqrt{s}}\Big)^{\eta_1} ds.$$

By changing variables, we have

$$|h_{\alpha,t}(x,y)-h_{\alpha,t}(z,y)| \lesssim \Big(\frac{|x-z|}{t^{1/2\alpha}}\Big)^{\eta_1} \int_0^\infty \frac{e^{-c|x-y|^2/(t^{1/\alpha}u)}}{\omega(B(x,t^{1/2\alpha}\sqrt{u}))} u^{-(1+\alpha+\eta_1/2)} du.$$

Let $|x-y|^2/(t^{1/\alpha}u)=r^2$. Then

$$\begin{aligned}
&|h_{\alpha,t}(x,y)-h_{\alpha,t}(z,y)| \\
&= \Big(\frac{|x-z|}{t^{1/2\alpha}}\Big)^{\eta_1} \frac{t^{1+\eta_1/2\alpha}}{|x-y|^{2\alpha+\eta_1}} \Big(\int_0^1 + \int_1^\infty\Big) \frac{r^{2\alpha-1+\eta_1} e^{-cr^2}}{\omega(B(x,|x-y|/r))} dr \\
&=: I_1 + I_2.
\end{aligned}$$

For I_2, we know that

$$\frac{1}{\omega(B(x,|x-y|/r))} \lesssim \frac{r^\gamma}{\omega(B(x,|x-y|))}.$$

Therefore,

$$I_2 \lesssim \Big(\frac{|x-z|}{t^{1/2\alpha}}\Big)^{\eta_1} \frac{t^{1+\eta_1/2\alpha}}{|x-y|^{2\alpha+\eta_1}} \frac{1}{\omega(B(x,|x-y|))}.$$

It remains to prove I_1. Since

$$\omega(B(x,|x-y|/r)) \gtrsim r^{-\nu}\omega(B(x,|x-y|)),$$

we can deduce that

$$I_1 \lesssim \Big(\frac{|x-z|}{t^{1/2\alpha}}\Big)^{\eta_1} \frac{t^{1+\eta_1/2\alpha}}{|x-y|^{2\alpha+\eta_1}} \frac{1}{\omega(B(x,|x-y|))}.$$

Therefore, we obtain

$$|h_{\alpha,t}(x,y)-h_{\alpha,t}(z,y)| \lesssim \Big(\frac{|x-z|}{t^{1/2\alpha}}\Big)^{\eta_1} \frac{t^{1+\eta_1/2\alpha}}{|x-y|^{2\alpha+\eta_1}} \frac{1}{\omega(B(x,|x-y|))}.$$

On the other hand, similarly to the proof of (i), we can also get

$$|h_{\alpha,t}(x,y)-h_{\alpha,t}(z,y)| \lesssim \Big(\frac{|x-z|}{t^{1/2\alpha}}\Big)^{\eta_1} \frac{1}{\omega(B(x,t^{1/2\alpha}))}.$$

Hence, in any case, we have

$$|h_{\alpha,t}(x,y)-h_{\alpha,t}(z,y)|\lesssim\frac{(|x-z|/t^{1/2\alpha})^{\eta_1}}{\omega(B(x,\sqrt{t^{1/\alpha}+|x-y|^2}))}\frac{t}{(t^{1/\alpha}+|x-y|^2)^{\alpha}}.$$

□

Similarly to Proposition 3.4, the following estimates of $K^L_{\alpha,t}(\cdot,\cdot)$ can be deduced from Lemma 3.2 and (1.1). So we omit the details.

Proposition 3.5 *Assume that $\omega\in RD_\nu\bigcap D_\gamma\bigcap A_2$, $2<\nu\le\gamma$ and V satisfies a $RH_q(\omega)$ condition with $q>\gamma/2$. Let $\delta_0=2-\gamma/q$.*

(i) *For each $N\ge 0$, there is a positive constant C_N such that*

$$\begin{aligned}&0\le K^L_{\alpha,t}(x,y)\\&\le C_N\min\left\{\frac{t^{1+N/\alpha}}{|x-y|^{2\alpha+2N}}\frac{1}{\omega(B(x,|x-y|))},\frac{1}{\omega(B(x,t^{1/2\alpha}))}\right\}\left(1+\frac{t^{1/2\alpha}}{\rho(x)}+\frac{t^{1/2\alpha}}{\rho(y)}\right)^{-N},\end{aligned}$$

which gives

$$0\le K^L_{\alpha,t}(x,y)\le\frac{C_N}{\omega(B(x,\sqrt{t^{1/\alpha}+|x-y|^2}))}\frac{t}{(t^{1/\alpha}+|x-y|^2)^{\alpha}}\left(1+\frac{t^{1/2\alpha}}{\rho(x)}\right)^{-N}\left(1+\frac{t^{1/2\alpha}}{\rho(y)}\right)^{-N}.$$

(ii) *For any given $0<\delta'<\min\{2\alpha,\eta,\delta_0\}$, there exist two positive constants C_N and c such that for every $N>0$ and $|x-z|<|x-y|/4$,*

$$\begin{aligned}&\left|K^L_{\alpha,t}(x,y)-K^L_{\alpha,t}(z,y)\right|\\&\le C_N\left(\frac{|x-z|}{t^{1/2\alpha}}\right)^{\delta'}\min\left\{\frac{t^{1+N/\alpha+\delta'/2\alpha}}{|x-y|^{2\alpha+2N+\delta'}}\frac{1}{\omega(B(x,|x-y|))},\frac{1}{\omega(B(x,t^{1/2\alpha}))}\right\}\\&\quad\times\left(1+\frac{t^{1/2\alpha}}{\rho(x)}+\frac{t^{1/2\alpha}}{\rho(y)}\right)^{-N},\end{aligned}$$

which gives

$$\begin{aligned}|K^L_{\alpha,t}(x,y)-K^L_{\alpha,t}(z,y)|\le&\frac{C_N}{\omega(B(x,\sqrt{t^{1/\alpha}+|x-y|^2}))}\frac{t}{(t^{1/\alpha}+|x-y|^2)^{\alpha}}\\&\times\left(\frac{|x-z|}{t^{1/2\alpha}}\right)^{\delta'}\left(1+\frac{t^{1/2\alpha}}{\rho(x)}\right)^{-N}\left(1+\frac{t^{1/2\alpha}}{\rho(y)}\right)^{-N}.\end{aligned}$$

Similarly to the proofs of [36, Propositions 5 & 8], we obtain the following proposition.

Proposition 3.6 *Assume that $\omega\in RD_\nu\bigcap D_\gamma\bigcap A_2$, $2<\nu\le\gamma$ and V satisfying a $RH_q(\omega)$ condition with $q>\gamma/2$. Let $\delta_0=2-\gamma/q$.*

(i)
$$|h_{\alpha,t}(x,y)-K^L_{\alpha,t}(x,y)| \lesssim \begin{cases} \Big(\frac{|x-y|}{\rho(x)}\Big)^{\delta_0} \frac{1}{\omega(B(x,\sqrt{t^{1/\alpha}+|x-y|^2}))}\frac{t}{(t^{1/\alpha}+|x-y|^2)^\alpha}, & t^{1/\alpha}\le |x-y|^2;\\ \Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{\delta_0} \frac{1}{\omega(B(x,\sqrt{t^{1/\alpha}+|x-y|^2}))}\frac{t}{(t^{1/\alpha}+|x-y|^2)^\alpha}, & t^{1/\alpha}\ge |x-y|^2.\end{cases}$$

(ii) *For any given* $0<\delta<\min\{\eta,\delta_0\}$, $|x-z|<|x-y|/4$ *and* $|x-z|\le\rho(x)$,

$$|(h_{\alpha,t}(x,y)-K^L_{\alpha,t}(x,y))-(h_{\alpha,t}(z,y)-K^L_{\alpha,t}(z,y))| \lesssim \Big(\frac{|x-z|}{\rho(x)}\Big)^{\delta}\frac{1}{\omega(B(x,\sqrt{t^{1/\alpha}+|x-y|^2}))}\frac{t}{(t^{1/\alpha}+|x-y|^2)^\alpha}.$$

4 Boundedness of the maximal operators and fractional integral operators

In this section, motivated by [19], we apply Propositions 3.4 and 3.5 to prove the boundedness of the maximal operator S^* and T^* which are dominated by

$$G^*f(x):=\sup_{t>0}\int_{\mathbb{R}^n}\frac{1}{\omega(B(x,\sqrt{t^{1/\alpha}+|x-y|^2}))}\frac{t}{(t^{1/\alpha}+|x-y|^2)^\alpha}f(y)\omega(y)dy.$$

The maximal function with respect to the measure $\omega(x)dx$ is defined as

$$M_\omega f(x):=\sup_{x\in B}\frac{1}{\omega(B)}\int_B|f(y)|\omega(y)dy.$$

When ω is a doubling weight, M_ω is bounded on $L^p(\mathbb{R}^n,\omega)$ for $1<p\le\infty$ and is of weak type (1,1) with respect to $\omega(x)dx$. Then we also need the following fractional maximal function $M^{\sigma_1,\gamma_1}_\omega$ which is defined by

$$M^{\sigma_1,\gamma_1}_\omega f(x):=\sup_{x\in B}\Big(\frac{1}{\omega(B)^{1-\sigma_1\gamma_1/\lambda}}\int_B|f(y)|^{\gamma_1}\omega(y)dy\Big)^{1/\gamma_1}.$$

Lemma 4.1 ([35, Lemma 2.5]) *Suppose that* $1<\gamma_1<p<\lambda/\sigma_1$, $1/s=1/p-\sigma_1/\lambda$ *and* ω *is doubling. Then*

$$\|M^{\sigma_1,\gamma_1}_\omega f\|_{L^s(\mathbb{R}^n,\omega)}\lesssim\|f\|_{L^p(\mathbb{R}^n,\omega)}.$$

Theorem 4.2 *Let* $\omega\in A_2\bigcap RD_\nu\bigcap D_\gamma$ *with* $2<\nu\le\gamma$.

(i) S^* *is bounded on* $L^p(\mathbb{R}^n,\omega)$ *for* $1<p\le\infty$ *and is of weak type (1,1) with respect to* $\omega(x)dx$.
(ii) *If* $V\in RH_q(\omega)$ *for* $q>\gamma/2$, T^* *also has the above properties.*

Proof It is sufficient to prove that G^* is dominated by M_ω. Split

$$G^*f(x) = \sup_{t>0}\Big(I_0 + \sum_{k=1}^{\infty} I_k\Big),$$

where

$$I_0 := \int_{B(x,t^{1/2\alpha})} \frac{t}{(t^{1/\alpha}+|x-y|^2)^\alpha} \frac{f(y)\omega(y)dy}{\omega(B(x,\sqrt{t^{1/\alpha}+|x-y|^2}))}$$

and

$$I_k := \int_{B(x,2^k t^{1/2\alpha})\backslash B(x,2^{k-1}t^{1/2\alpha})} \frac{t}{(t^{1/\alpha}+|x-y|^2)^\alpha} \frac{f(y)\omega(y)dy}{\omega(B(x,\sqrt{t^{1/\alpha}+|x-y|^2}))}.$$

It is obvious that $I_0 \lesssim M_\omega f(x)$ and $I_k \lesssim M_\omega f(x)$. The boundedness of M_ω indicates the desire results. □

Next we investigate the behaviors of the fractional integral operators $I_{\alpha,\beta}$ and $I^L_{\alpha,\beta}$, respectively. The kernels of $I_{\alpha,\beta}$ and $I^L_{\alpha,\beta}$ are given by

$$\begin{cases} H_{\alpha,\beta}(x,y) := \int_0^\infty h_{\alpha,t}(x,y)t^{\beta/2}\frac{dt}{t}; \\ K^L_{\alpha,\beta}(x,y) := \int_0^\infty K^L_{\alpha,t}(x,y)t^{\beta/2}\frac{dt}{t}, \end{cases}$$

respectively.

Lemma 4.3 *Let $\omega \in A_2 \bigcap RD_\nu \bigcap D_\gamma$ with $2 < \nu \le \gamma$.*

(i) *For $0 < \alpha\beta < \nu$,*

$$0 \le H_{\alpha,\beta}(x,y) \lesssim \frac{|x-y|^{\alpha\beta}}{\omega(B(x,|x-y|))}. \tag{4.1}$$

(ii) *If $V \in RH_q(\omega)$ with $q > \gamma/2$, then for $0 < \alpha\beta < \nu$ and any $N > 0$,*

$$0 \le K^L_{\alpha,\beta}(x,y) \le C_N \frac{|x-y|^{\alpha\beta}}{\omega(B(x,|x-y|))}\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{-N}.$$

Proof For (i), taking $t^{1/\alpha} = |x-y|^2 s$, we apply Proposition 3.4 (i) to deduce that

$$\begin{aligned} H_{\alpha,\beta}(x,y) &\lesssim \int_0^\infty \frac{1}{\omega(B(x,\sqrt{t^{1/\alpha}+|x-y|^2}))}\frac{t^{\beta/2}}{(t^{1/\alpha}+|x-y|^2)^\alpha}dt \\ &= \int_0^\infty \frac{|x-y|^{\alpha\beta+2\alpha}}{\omega(B(x,\sqrt{|x-y|^2 s+|x-y|^2}))}\frac{s^{\alpha\beta/2+\alpha-1}ds}{(|x-y|^2(s+1))^\alpha} \end{aligned}$$

$$= I_1 + I_2,$$

where

$$\begin{cases} I_1 := |x-y|^{\alpha\beta} \displaystyle\int_0^1 \frac{1}{\omega(B(x,|x-y|\sqrt{1+s}))} \frac{s^{\alpha\beta/2+\alpha}}{(1+s)^{\alpha}} \frac{ds}{s}; \\ I_2 := |x-y|^{\alpha\beta} \displaystyle\int_1^\infty \frac{1}{\omega(B(x,|x-y|\sqrt{1+s}))} \frac{s^{\alpha\beta/2+\alpha}}{(1+s)^{\alpha}} \frac{ds}{s}. \end{cases}$$

For I_1, it holds

$$I_1 \lesssim \frac{|x-y|^{\alpha\beta}}{\omega(B(x,|x-y|))} \int_0^1 \frac{s^{\alpha\beta/2+\alpha-1}}{(1+s)^{\alpha+\nu/2}} ds \lesssim \frac{|x-y|^{\alpha\beta}}{\omega(B(x,|x-y|))}.$$

For I_2, since $\alpha\beta < \nu$, we obtain

$$\begin{aligned} I_2 &\lesssim \frac{|x-y|^{\alpha\beta}}{\omega(B(x,|x-y|))} \int_1^\infty \frac{1}{(\sqrt{1+s})^{\nu}} \frac{s^{\alpha\beta/2+\alpha}}{(1+s)^{\alpha}} \frac{ds}{s} \\ &\lesssim \frac{|x-y|^{\alpha\beta}}{\omega(B(x,|x-y|))} \int_1^\infty s^{\alpha\beta/2-1-\nu/2} ds \\ &\lesssim \frac{|x-y|^{\alpha\beta}}{\omega(B(x,|x-y|))}. \end{aligned}$$

For (ii), by (i) of Proposition 3.5, we have

$$\begin{aligned} K^L_{\alpha,\beta}(x,y) \lesssim \int_0^\infty & \frac{1}{\omega(B(x,\sqrt{|x-y|^2(s+1)}))} \frac{1}{(|x-y|^2(s+1))^{\alpha}} \\ & \times \left(1+\frac{|x-y|\sqrt{s}}{\rho(x)}\right)^{-N} |x-y|^{\alpha\beta+2\alpha} s^{\alpha\beta/2+\alpha} \frac{ds}{s}. \end{aligned}$$

Note that

$$1+\frac{\sqrt{s}|x-y|}{\rho(x)} \gtrsim \left(1+\frac{|x-y|}{\rho(x)}\right) \min\{1,\sqrt{s}\},$$

which can be deduced easily by considering $s < 1$ and $s \geq 1$. Applying the previous inequality, we obtain, for $\alpha\beta < \nu$,

$$\begin{aligned} K^L_{\alpha,\beta}(x,y) \lesssim & \frac{|x-y|^{\alpha\beta}}{\omega(B(x,|x-y|))} \left(1+\frac{|x-y|}{\rho(x)}\right)^{-N} \\ & \times \left(\int_0^1 s^{\alpha\beta/2+\alpha-1} ds + \int_1^\infty s^{\alpha\beta/2+\alpha-1-\alpha-\nu/2} ds\right) \end{aligned}$$

$$\lesssim \frac{|x-y|^{\alpha\beta}}{\omega(B(x,|x-y|))}\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{-N}.$$

□

It can be seen from Lemma 4.3 that, in order to investigate the behaviors of $I_{\alpha,\beta}$ and $I^L_{\alpha,\beta}$, we only need to study the following fractional operator

$$J_{\alpha,\beta}f(x) := \int_{\mathbb{R}^n} \frac{|x-y|^{\alpha\beta}}{\omega(B(x,|x-y|))} f(y)\omega(y)dy.$$

Now, we will study the boundedness of $J_{\alpha,\beta}$ on $M^\lambda_p(\mathbb{R}^n,\omega)$. Firstly, we need the following lemma about the maximal function M_ω.

Lemma 4.4 ([19, Proposition 3]) *Let ω be a doubling weight. Then for any $\lambda > 0$ and $1 < p \le \infty$, the operator M_ω is bounded on $M^\lambda_p(\mathbb{R}^n,\omega)$.*

As a consequence of Lemma 4.4 and the pointwise inequalities $T^*f(x) \le S^*f(x) \le M_\omega f(x)$, we obtain more boundedness results for the maximal operators T^* and S^*.

Corollary 4.5 *Under the same assumptions of Theorem 4.2, the operators S^* and T^* are bounded on $M^\lambda_p(\mathbb{R}^n,\omega)$ for any $\lambda > 0$ and $1 < p \le \infty$.*

Theorem 4.6 *Let ω be a doubling weight. Given $\beta > 0$, $\alpha \in (0,1)$ and $\lambda > \alpha\beta$, the fractional operator $J_{\alpha,\beta}$ is bounded from $M^\lambda_p(\mathbb{R}^n,\omega)$ to $M^\lambda_s(\mathbb{R}^n,\omega)$ for $1 < p < \lambda/\alpha\beta$ and $1/s = 1/p - \alpha\beta/\lambda$.*

Proof Firstly, we claim that

$$|J_{\alpha,\beta}f(x)| \lesssim \|f\|^{\alpha\beta p/\lambda}_{M^\lambda_p(\mathbb{R}^n,\omega)}(M_\omega f(x))^{1-\alpha\beta p/\lambda}. \tag{4.2}$$

In order to prove (4.2), we adopt the idea which is to get a kind of Hedberg's inequality involving the $M^\lambda_p(\mathbb{R}^n,\omega)$-norm as in [21]. Split $|J_{\alpha,\beta}f(x)| \le I_1 + I_2$, where

$$\begin{cases} I_1 := \displaystyle\int_{B(x,R)} \frac{|x-y|^{\alpha\beta}}{\omega(B(x,|x-y|))}|f(y)|\omega(y)dy; \\ I_2 := \displaystyle\sum_{k=1}^{\infty}\int_{B(x,2^kR)\backslash B(x,2^{k-1}R)} \frac{|x-y|^{\alpha\beta}}{\omega(B(x,|x-y|))}|f(y)|\omega(y)dy. \end{cases}$$

For I_1, it holds

$$I_1 = \sum_{k=0}^{\infty}\int_{|x-y|\simeq 2^{-k}R} \frac{|x-y|^{\alpha\beta}}{\omega(B(x,|x-y|))}|f(y)|\omega(y)dy \lesssim R^{\alpha\beta}M_\omega f(x).$$

For I_2, we also obtain

$$
\begin{aligned}
I_2 &= \sum_{k=1}^{\infty} \int_{|x-y|\simeq 2^k R} \frac{|x-y|^{\alpha\beta}}{\omega(B(x,|x-y|))} |f(y)|\omega(y)dy \\
&\lesssim \sum_{k=1}^{\infty} 2^{k\alpha\beta} R^{\alpha\beta} \frac{1}{\omega(B(x,2^k R))} \Big(\int_{|x-y|\simeq 2^k R} \omega(y)dy \Big)^{1-1/p} \\
&\quad \times \Big(\int_{|x-y|\simeq 2^k R} |f(y)|^p \omega(y)dy \Big)^{1/p} \\
&\lesssim \sum_{k=1}^{\infty} 2^{k\alpha\beta} R^{\alpha\beta} \omega(B(x,2^k R))^{-1/p} \Big(\int_{|x-y|\simeq 2^k R} |f(y)|^p \omega(y)dy \Big)^{1/p} \\
&\lesssim R^{\alpha\beta-\lambda/p} \|f\|_{M_p^\lambda(\mathbb{R}^n,\omega)}.
\end{aligned}
$$

By choosing $R = (M_\omega f(x)/\|f\|_{M_p^\lambda(\mathbb{R}^n,\omega)})^{-p/\lambda}$, we get

$$|J_{\alpha,\beta} f(x)| \lesssim (M_\omega f(x))^{1-\alpha\beta p/\lambda} \|f\|_{M_p^\lambda(\mathbb{R}^n,\omega)}^{\alpha\beta p/\lambda}$$

and

$$\|J_{\alpha,\beta} f\|_{M_s^\lambda(\mathbb{R}^n,\omega)} \lesssim \|(M_\omega f)^{1-\alpha\beta p/\lambda}\|_{M_s^\lambda} \|f\|_{M_p^\lambda(\mathbb{R}^n,\omega)}^{\alpha\beta p/\lambda}.$$

It is easy to see that if ϵ is such that $\epsilon s \geq 1$, then $\|g^\epsilon\|_{M_s^\lambda(\mathbb{R}^n,\omega)} = \|g\|_{M_{\epsilon s}^\lambda(\mathbb{R}^n,\omega)}^\epsilon$. Since $s(1-\alpha\beta p/\lambda) = p$, by the above facts, we obtain

$$\|J_{\alpha,\beta} f\|_{M_s^\lambda(\mathbb{R}^n,\omega)} \lesssim \|M_\omega f\|_{M_p^\lambda(\mathbb{R}^n,\omega)} \|f\|_{M_p^\lambda(\mathbb{R}^n,\omega)}^{\alpha\beta p/\lambda}.$$

Using Lemma 4.4, we can obtain the desired result. □

As a consequence, the following boundedness results for the negative powers of L_0 and L hold.

Theorem 4.7 *Let* $\omega \in A_2 \bigcap RD_\nu \bigcap D_\gamma$ *for some* $2 < \nu \leq \gamma$. *Given* $\beta > 0$ *and* $\alpha \in (0,1)$. *For any* λ *such that* $\alpha\beta < \lambda \leq \nu$, *we have*

(i) $I_{\alpha,\beta}$ *is bounded from* $M_p^\lambda(\mathbb{R}^n,\omega)$ *to* $M_s^\lambda(\mathbb{R}^n,\omega)$ *for* $1 < p < \lambda/\alpha\beta$ *and* $1/s = 1/p - \alpha\beta/\lambda$.

(ii) *Further, for* $V \in RH_q$ *with* $q > \gamma/2$, $I_{\alpha,\beta}^L$ *is bounded from* $M_p^\lambda(\mathbb{R}^n,\omega)$ *to* $M_s^\lambda(\mathbb{R}^n,\omega)$ *for* $1 < p < \lambda/\alpha\beta$ *and* $1/s = 1/p - \alpha\beta/\lambda$.

Then we focus on the mixed operators $I_{\alpha,\beta}^L V^{\sigma/2}$. In this point, we assume $V \in RH_\infty(\omega)$, i.e., there is a positive constant C such that for any ball B,

$$\sup_{x\in B} \omega(x) \leq \frac{C}{\omega(B)} \int_B V(u)\omega(u)du.$$

For $1 < q < +\infty$, we know that $RH_\infty(\omega) \subset RH_q(\omega)$ (see [34]). Under this stronger condition that $V \in RH_\infty(\omega)$, we can prove that the boundedness of $I^L_{\alpha,\beta} V^{\sigma/2}$ and their adjoint operators are the same.

Theorem 4.8 *Assume that $\omega \in A_2 \bigcap RD_\nu \bigcap D_\gamma$ for some $\nu > 2$ and $V \in RH_\infty(\omega)$. Let $\alpha \in (0, 1)$, $\beta > 0$ and $\sigma > 0$ such that $0 < \sigma \leq \alpha\beta < \nu$. For any λ with $\alpha\beta - \sigma < \lambda \leq \nu$,*

- (i) *$I^L_{\alpha,\beta} V^{\sigma/2}$ and $V^{\sigma/2} I^L_{\alpha,\beta}$ are bounded from $M^\lambda_p(\mathbb{R}^n, \omega)$ to $M^\lambda_s(\mathbb{R}^n, \omega)$ for $1 < p < \lambda/(\alpha\beta - \sigma)$ and $1/s = 1/p - (\alpha\beta - \sigma)/\lambda$.*
- (ii) *$I^L_{\alpha,\beta} V^{\sigma/2}$ and $V^{\sigma/2} I^L_{\alpha,\beta}$ are bounded from $M^\lambda_{\lambda/(\alpha\beta-\sigma)}(\mathbb{R}^n, \omega)$ to $M^\lambda_\infty(\mathbb{R}^n, \omega) = L^\infty$.*
- (iii) *$I^L_{\alpha,\beta} V^{\sigma/2}$ and $V^{\sigma/2} I^L_{\alpha,\beta}$ are bounded on $L^p(\mathbb{R}^n, \omega)$ when $\alpha\beta = \sigma$ and $1 < p \leq \infty$.*

Proof We first prove (i). For $0 < \alpha\beta < \nu$, using (ii) of Lemma 4.3, the kernel $K^{L,\sigma}_{\alpha,\beta}$ of $I^L_{\alpha,\beta} V^{\sigma/2}$ is dominated by

$$K^{L,\sigma}_{\alpha,\beta}(x, y) \lesssim \frac{|x-y|^{\alpha\beta}}{\omega(B(x, |x-y|))} \Big(1 + \frac{|x-y|}{\rho(x)}\Big)^{-N} V^{\sigma/2}(y).$$

Since $V \in RH_\infty(\omega)$, taking the ball $B(y, \rho(y))$ in the definition of $\rho(\cdot)$, we easily get $V(y) \lesssim \rho^{-2}(y)$. Moreover, by Lemma 2.1, we also have

$$\rho^{-1}(y) \lesssim \rho^{-1}(x)\Big(1 + |x-y|/\rho(x)\Big)^{N_0}.$$

Then we can deduce that

$$K^{L,\sigma}_{\alpha,\beta}(x, y) \lesssim \frac{|x-y|^{\alpha\beta}\rho^{-\sigma}(x)}{\omega(B(x, |x-y|))} \Big(1 + \frac{|x-y|}{\rho(x)}\Big)^{-(N-N_0\sigma)}. \tag{4.3}$$

Let $\alpha\beta > \sigma$. A direct computation derives

$$\begin{aligned} K^{L,\sigma}_{\alpha,\beta}(x, y) &\lesssim \frac{|x-y|^{\alpha\beta-\sigma}}{\omega(B(x, |x-y|))} \frac{|x-y|^\sigma}{\rho^\sigma(x)} \Big(1 + \frac{|x-y|}{\rho(x)}\Big)^{-(N-N_0\sigma)} \\ &\lesssim \frac{|x-y|^{\alpha\beta-\sigma}}{\omega(B(x, |x-y|))} \Big(1 + \frac{|x-y|}{\rho(x)}\Big)^{-(N-N_0\sigma-\sigma)} \\ &\lesssim \frac{|x-y|^{\alpha\beta-\sigma}}{\omega(B(x, |x-y|))}, \end{aligned}$$

which gives $I^L_{\alpha,\beta} V^{\sigma/2} f(x) \lesssim J_{\alpha\beta-\sigma} f(x) = \int_{\mathbb{R}^n} \frac{|x-y|^{\alpha\beta-\sigma}}{\omega(B(x,|x-y|))} f(y)\omega(y)dy$. Therefore, (i) can be deduced from Theorem 4.6.

It remains to prove (ii). From (4.3), we can write

$$I^L_{\alpha,\beta} V^{\sigma/2} f(x)$$

$$
\begin{aligned}
&\lesssim \int_{\mathbb{R}^n} \frac{|x-y|^{\alpha\beta}\rho^{-\sigma}(x)}{\omega(B(x,|x-y|))}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-(N-N_0\sigma)}|f(y)|\omega(y)dy \\
&= \sum_{k=-\infty}^{+\infty}\int_{B(x,2^k\rho(x))} \frac{|x-y|^{\alpha\beta}\rho^{-\sigma}(x)}{\omega(B(x,|x-y|))}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-(N-N_0\sigma)}|f(y)|\omega(y)dy \\
&\lesssim \sum_{k=-\infty}^{+\infty} 2^{k\sigma}(1+2^k)^{-N+N_0\sigma}\frac{(2^k\rho(x))^{\alpha\beta-\sigma}}{\omega(B(x,2^k\rho(x)))}\int_{B(x,2^k\rho(x))}|f(y)|\omega(y)dy,
\end{aligned}
\tag{4.4}
$$

where N is any positive number, so we can take $N > N_0\sigma$. Since $\alpha\beta > \sigma$ and $p = \lambda/(\alpha\beta - \sigma)$, for each k, we apply Hölder's inequality to obtain

$$
\begin{aligned}
&\frac{(2^k\rho(x))^{\alpha\beta-\sigma}}{\omega(B(x,2^k\rho(x)))}\int_{B(x,2^k\rho(x))}|f(y)|\omega(y)dy \\
&\lesssim \frac{(2^k\rho(x))^{\alpha\beta-\sigma}}{\omega(B(x,2^k\rho(x)))^{(\alpha\beta-\sigma)/\lambda}}\left(\int_{B(x,2^k\rho(x))}|f(y)|^{\lambda/(\alpha\beta-\sigma)}\omega(y)dy\right)^{(\alpha\beta-\sigma)/\lambda} \\
&\lesssim \|f\|_{M^{\lambda}_{\lambda/(\alpha\beta-\sigma)}}.
\end{aligned}
$$

Inserting this estimate into (4.4), we conclude that (ii) holds since the series is convergent.

(iii) When $\alpha\beta = \sigma$, we can deduce that $I^L_{\alpha,\beta}V^{\alpha\beta/2}f(x) \lesssim M_\omega f(x)$. Therefore, the desire results can be deduced from the properties of M_ω.

Finally, all the statements about the adjoint operators are immediate once we notice that their kernels are also bounded by the right hand side of (4.3). □

Remark 4.9 In the above proofs, by (4.3), we can also get

$$
K^{L,\sigma}_{\alpha,\beta}(x,y) \lesssim \frac{|x-y|^{\alpha\beta-\sigma}}{\omega(B(x,|x-y|))}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}.
$$

Therefore, based on Lemma 4.3, we would expect the operators $I^L_{\alpha,\beta}V^{\sigma/2}$ for $\alpha\beta > \sigma$ to behave as $I^L_{\alpha,-(\alpha\beta-\sigma)}$. However, Theorem 4.8 (ii) reveals that mixed operators are slightly better.

Definition 4.10 If there exists a positive number λ such that for some constant C independent of x,

$$
Cr^\lambda \le \omega(B(x,r)), \tag{4.5}
$$

one says that the measure $\omega(x)dx$ is lower-Ahlfors.

Theorem 4.11 *Assume that $\omega \in D_\gamma \bigcap RD_\nu$ for $2 < \nu \le \gamma$, ω is doubling and $V \in RH_q(\omega)$ with $q > \gamma/2$. Let $\alpha \in (0,1)$ and $\beta > 0$ such that $0 \le \sigma \le \alpha\beta < \nu$. If*

ω satisfies (4.5), then

$$\left|I^L_{\alpha,\beta}(V^{\sigma/2}f)(x)\right| \lesssim M^{\alpha\beta-\sigma,(2q/\sigma)'}_{\omega} f(x),$$

where $(2q/\sigma)'$ is the conjugate of $2q/\sigma$.

Proof Let $r = \rho(x)$. Since ω satisfies (4.5), by Lemma 4.3 and Hölder's inequality, we have

$$\begin{aligned}
&\left|I^L_{\alpha,\beta}(V^{\sigma/2}(x)f(x))\right| \\
&\lesssim \sum_{k=-\infty}^{+\infty} \int_{2^{k-1}r\le|x-y|\le 2^k r} \frac{1}{(1+2^k r/\rho(x))^N} \frac{(2^k r)^{\alpha\beta}}{\omega(B(x,2^k r))} V^{\sigma/2}(y)|f(y)|\omega(y)dy \\
&\lesssim \sum_{k=-\infty}^{+\infty} \frac{(2^k r)^{\alpha\beta}}{(1+2^k)^N} \frac{1}{\omega(B(x,2^k r))} \Big[\Big(\int_{|x-y|\le 2^k r} V(y)^q \omega(y)dy\Big)^{\sigma/2q} \\
&\quad \times \Big(\int_{|x-y|\le 2^k r} |f(y)|^{(2q/\sigma)'}\omega(y)dy\Big)^{1/(2q/\sigma)'}\Big] \\
&\lesssim \sum_{k=-\infty}^{+\infty} \frac{(2^k r)^{\alpha\beta}}{(1+2^k)^N} \frac{1}{\omega(B(x,2^k r))^{(\alpha\beta-\sigma)/\lambda}} \\
&\quad \times \Big(\frac{1}{\omega(B(x,2^k r))} \int_{|x-y|\le 2^k r} V(y)\omega(y)dy\Big)^{\sigma/2} M^{\alpha\beta-\sigma,(2q/\sigma)'}_{\omega}(f)(x) \\
&\lesssim \sum_{k=-\infty}^{+\infty} \frac{(2^k r)^{\sigma}}{(1+2^k)^N} \Big(\frac{1}{\omega(B(x,2^k r))} \int_{|x-y|\le 2^k r} V(y)\omega(y)dy\Big)^{\sigma/2} M^{\alpha\beta-\sigma,(2q/\sigma)'}_{\omega}(f)(x).
\end{aligned}$$

For $k \ge 1$, since $V(y)\omega(y)dy$ is a doubling measure and $\omega \in RD_\nu$, we get

$$\begin{aligned}
\frac{(2^k r)^2}{\omega(B(x,2^k r))} \int_{B(x,2^k r)} V(y)\omega(y)dy &\lesssim C_0^k \frac{(2^k r)^2}{\omega(B(x,2^k r))} \int_{B(x,r)} V(y)\omega(y)dy \\
&\lesssim (2^k)^{k_0},
\end{aligned}$$

where $k_0 = 2 - \nu + \log_2 C_0$. For $k \le 0$, Lemma 2.2 implies that

$$\begin{aligned}
\frac{(2^k r)^2}{\omega(B(x,2^k r))} \int_{B(x,2^k r)} V(y)\omega(y)dy &\lesssim \Big(\frac{r}{2^k r}\Big)^{\gamma/q-2} \frac{r^2}{\omega(B(x,r))} \int_{B(x,r)} V(y)\omega(y)dy \\
&\lesssim (2^k)^{2-\gamma/q}.
\end{aligned}$$

Taking N large enough, we deduce that

$$\left|I^L_{\alpha,\beta}(V^{\sigma/2}f)(x)\right| \lesssim M^{\alpha\beta-\sigma,(2q/\sigma)'}_{\omega} f(x).$$

□

By Theorem 4.11 and the duality, we can obtain the following result.

Corollary 4.12 *Assume that $\omega \in D_\gamma \bigcap RD_\nu$ for $2 < \nu \leq \gamma$, ω is doubling and $V \in RH_q(\omega)$ with $q > \gamma/2$. Let $0 \leq \sigma \leq \alpha\beta < \nu$, $\lambda > 0$, $1 < p < +\infty$ and ω satisfy (4.5).*

(i) *For $1 < (2q/\sigma)' < p < \frac{\lambda}{\alpha\beta-\sigma}$ and $1/s = 1/p - (\alpha\beta - \sigma)/\lambda$,*

$$\|I^L_{\alpha,\beta}V^{\sigma/2}f\|_{L^s(\mathbb{R}^n,\omega)} \lesssim \|f\|_{L^p(\mathbb{R}^n,\omega)}.$$

(ii) *For $1 < s < 2q/\sigma$ and $1/s = 1/p - (\alpha\beta - \sigma)/\lambda$,*

$$\|V^{\sigma/2}I^L_{\alpha,\beta}f\|_{L^s(\mathbb{R}^n,\omega)} \lesssim \|f\|_{L^p(\mathbb{R}^n,\omega)}.$$

Definition 4.13 Let $1 \leq p \leq \infty$, $\omega \in A_2$ and $\lambda_1, \lambda_2 \geq 0$. The weighted Morrey space $M_p^{\lambda_1,\lambda_2}(\mathbb{R}^n, \omega)$ is defined as the set of all L^p-locally integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{M_p^{\lambda_1,\lambda_2}(\mathbb{R}^n,\omega)} := \sup_{B=B(x,r)} \left(\frac{r^{\lambda_1}}{\omega(B)^{\lambda_2}}\int_B |f(x)|^p\omega(x)dx\right)^{1/p} < \infty.$$

Theorem 4.14 *Let ω be a A_2-weight such that $\omega \in RD_\nu \bigcap D_\gamma$ with $\nu > 2$ and $V \in RH_q(\omega)$ with $q > \gamma/2$. Let $0 < \sigma \leq \alpha\beta < \nu$. For any $\lambda > 0$ such that $\alpha\beta < \lambda \leq \nu$, $1 < s < 2q/\sigma$ and $1/s = 1/p - (\alpha\beta - \sigma)/\lambda$. Then if ω satisfies (4.5), we have*

$$\|V^{\sigma/2}I^L_{\alpha,\beta}f\|_{M_s^{\lambda s/p,s/p}(\mathbb{R}^n,\omega)} \lesssim \|f\|_{M_p^{\lambda}(\mathbb{R}^n,\omega)}.$$

Proof Picking any $x_0 \in \mathbb{R}^n$ and $r > 0$, we write $f(x) = f_0(x) + \sum_{i=1}^\infty f_i(x)$, where

$$\begin{cases} f_0(x) := \chi_{B(x_0,2r)}f(x);\\ f_i(x) := \chi_{B(x_0,2^{i+1}r)\setminus B(x_0,2^ir)}f(x).\end{cases}$$

Hence, we have

$$\left(\int_{B(x_0,r)} |V^{\sigma/2}I^L_{\alpha,\beta}f(x)|^s\omega(x)dx\right)^{1/s} \leq I_0 + \sum_{i=1}^\infty I_i,$$

where

$$\begin{cases} I_0 := \left(\int_{B(x_0,r)} |V^{\sigma/2}I^L_{\alpha,\beta}f_0(x)|^s\omega(x)dx\right)^{1/s};\\ I_i := \left(\int_{B(x_0,r)} |V^{\sigma/2}I^L_{\alpha,\beta}f_i(x)|^s\omega(x)dx\right)^{1/s}.\end{cases}$$

For I_0, by Corollary 4.12 (ii), we get

$$I_0^s \lesssim \frac{\omega(B(x,r))^\theta}{r^{\lambda\theta}} \|f\|^s_{M_p^\lambda(\mathbb{R}^n,\omega)}, \quad \theta = s/p.$$

For I_i, using Hölder's inequality, the facts that $V \in RH_q(\omega)$ and ω satisfies (4.5), we deduce from Lemma 4.3 that

$$\begin{aligned} I_i^s &\lesssim \int_{B(x_0,r)} V^{s\sigma/2}(x)\omega(x) \\ &\quad \times \Big| \int_{B(x_0,2^{j+1}r)\setminus B(x_0,2^j r)} \frac{(2^i r)^{\alpha\beta}|f(y)|\omega(y)dy}{(1+2^i r/\rho(x_0))^{N/(N_0+1)}\omega(B(x_0,|x_0-y|))} \Big|^s dx \\ &\lesssim \frac{1}{(1+2^i r/\rho(x_0))^{Ns/(N_0+1)}} \frac{(2^i r)^{s\alpha\beta}}{\omega(B(x_0,2^i r))^s} \Big(\int_{B(x_0,2^{i+1}r)} |f(y)|\omega(y)dy \Big)^s \\ &\quad \times \Big(\int_{B(x_0,r)} |V(x)|^{\sigma s/2}\omega(x)dx \Big). \end{aligned}$$

Then by Lemma 2.3, we obtain

$$\begin{aligned} I_i^s &\lesssim \frac{1}{(1+2^i r/\rho(x_0))^{Ns/(N_0+1)}} \frac{(2^i r)^{s\alpha\beta}}{\omega(B(x_0,2^i r))^s} \\ &\quad \times \Big[\Big(\int_{B(x_0,2^{i+1}r)} |f(y)|^p\omega(y)dy \Big)^{1/p} \omega(B(x_0,2^i r))^{1-1/p} \Big]^s \\ &\quad \times \Big(\int_{B(x_0,r)} |V(x)|^q\omega(x)dx \Big)^{\sigma s/2q} \omega(B(x_0,r))^{1-\sigma s/2q} \\ &\lesssim \frac{1}{(1+2^i r/\rho(x_0))^{Ns/(N_0+1)}} \frac{(2^i r)^{s\alpha\beta}}{\omega(B(x_0,2^i r))^{s/p}} \Big(\int_{B(x_0,2^{i+1}r)} |f(y)|^p\omega(y)dy \Big)^{s/p} \\ &\quad \times \omega(B(x_0,r)) \Big(\frac{1}{\omega(B(x_0,r))} \int_{B(x_0,r)} V(x)\omega(x)dx \Big)^{\sigma s/2} \\ &\lesssim \frac{\omega(B(x_0,r))}{(1+2^i r/\rho(x_0))^{N_1}} \frac{1}{(2^i r)^\lambda} \|f\|^s_{M_p^\lambda(\mathbb{R}^n,\omega)}, \end{aligned}$$

where $0 < N_1 < (Ns/(N_0+1) - (\log_2 C_0 + 1)s\sigma/2$ and $1/s = 1/p - (\alpha\beta - \sigma)/\lambda$.

Since $\theta > 1$ and ω satisfies (4.5),

$$\begin{aligned} &\|V^{\sigma/2} I^L_{\alpha,\beta} f\|^s_{M_s^{\lambda s/p, s/p}(\mathbb{R}^n,\omega)} \\ &\lesssim \|f\|^s_{M_p^\lambda(\mathbb{R}^n,\omega)} + \sum_{i=1}^\infty \frac{r^{\lambda(\theta-1)}}{\omega(B(x_0,r))^{\theta-1}} \frac{1}{(1+2^i r/\rho(x_0))^{Ns/(N_0+1)} 2^{i\lambda}} \|f\|^s_{M_p^\lambda(\mathbb{R}^n,\omega)} \\ &\lesssim \|f\|^s_{M_p^\lambda(\mathbb{R}^n,\omega)}. \end{aligned}$$

□

Then similarly to the proof of Theorem 4.14, we can also get

Theorem 4.15 *Let ω be a A_2-weight such that $\omega \in RD_\nu \bigcap D_\gamma$ with $\nu > 2$ and $V \in RH_q(\omega)$, $q > \gamma/2$. Let $0 < \sigma \le \alpha\beta < \nu$. For any $\lambda > 0$ such that $\alpha\beta < \lambda \le \nu$, $2q/(2q-\sigma) < p < \lambda/(\alpha\beta - \sigma)$ and $1/s = 1/p - (\alpha\beta - \sigma)/\lambda$. Then if ω satisfies (4.5), we have*

$$\|I^L_{\alpha,\beta} V^{\sigma/2} f\|_{M_s^{\lambda s/p, s/p}(\mathbb{R}^n,\omega)} \lesssim \|f\|_{M_p^\lambda(\mathbb{R}^n,\omega)}.$$

Proof For any ball $B(x_0, r)$, we can decompose f as follows: $f(x) = f_0(x) + \sum_{i=1}^\infty f_i(x)$, where

$$\begin{cases} f_0(x) := \chi_{B(x_0,2r)}(x) f(x); \\ f_i(x) := \chi_{B(x_0,2^{i+1}r)\setminus B(x_0,2^i r)}(x) f(x). \end{cases}$$

Similarly to the proof of Theorem 4.14, we have

$$\Big(\int_{B(x_0,r)} |I^L_{\alpha,\beta} V^{\sigma/2} f(x)|^s \omega(x)dx\Big)^{1/s} \le I_0 + \sum_{i=1}^\infty I_i,$$

where

$$\begin{cases} I_0 := \Big(\int_{B(x_0,r)} |I^L_{\alpha,\beta} V^{\sigma/2} f_0(x)|^s \omega(x)dx\Big)^{1/s}; \\ I_i := \Big(\int_{B(x_0,r)} |I^L_{\alpha,\beta} V^{\sigma/2} f_i(x)|^s \omega(x)dx\Big)^{1/s}. \end{cases}$$

For I_0, by Corollary 4.12 (i), we get

$$I_0^s \lesssim \frac{\omega(B(x_0,r))^\theta}{r^{\lambda\theta}} \|f\|^s_{M_p^\lambda(\mathbb{R}^n,\omega)}, \quad \theta = s/p.$$

For I_i, using Hölder's inequality, the facts that $V \in RH_q(\omega)$ and ω satisfies (4.5), we deduce from Lemma 4.3 that

$$\begin{aligned} I_i^s &\lesssim \int_{B(x_0,r)} \Big| \int_{B(x_0,2^{j+1}r)\setminus B(x_0,2^j r)} \frac{(2^i r)^{\alpha\beta} |f(y)| V^{\sigma/2}(y)\omega(y)dy}{(1+2^i r/\rho(x_0))^{N/(N_0+1)} \omega(B(x_0,|x_0-y|))} \Big|^s \omega(x)dx \\ &\lesssim \frac{1}{(1+2^i r/\rho(x_0))^{Ns/(N_0+1)}} \frac{(2^i r)^{s\alpha\beta} \omega(B(x_0,r))}{\omega(B(x_0,2^i r))^s} \Big(\int_{B(x_0,2^i r)} |f(y)||V(y)|^{\sigma/2} \omega(y)dy\Big)^s \\ &\lesssim \frac{1}{(1+2^i r/\rho(x_0))^{Ns/(N_0+1)}} \frac{(2^i r)^{s\alpha\beta} \omega(B(x_0,r))}{\omega(B(x_0,2^i r))^s} \Big[\Big(\int_{B(x_0,2^i r)} |f(y)|^p \omega(y)dy\Big)^{1/p} \\ &\quad \times \Big(\int_{B(x_0,2^i r)} |V(y)|^{\sigma p/(2(p-1))} \omega(y)dy\Big)^{(p-1)/p}\Big]^s. \end{aligned}$$

Then by Lemma 2.3, we obtain

$$\begin{aligned}
I_i^s &\lesssim \frac{1}{(1+2^i r/\rho(x_0))^{Ns/(N_0+1)}} \frac{(2^i r)^{s\alpha\beta}\omega(B(x_0,r))}{\omega(B(x_0,2^i r))^{s/p}} \Big(\int_{B(x_0,2^i r)} |f(y)|^p \omega(y)dy\Big)^{s/p} \\
&\quad \times \Big(\frac{1}{\omega(B(x_0,2^i r))}\int_{B(x_0,2^i r)} |V(y)|^{\sigma p/(2(p-1))}\omega(y)dy\Big)^{s(p-1)/p} \\
&\lesssim \frac{1}{(1+2^i r/\rho(x_0))^{Ns/(N_0+1)}} \frac{(2^i r)^{s\alpha\beta}\omega(B(x_0,r))}{\omega(B(x_0,2^i r))^{s/p}} \Big(\int_{B(x_0,2^i r)} |f(y)|^p \omega(y)dy\Big)^{s/p} \\
&\quad \times \Big(\frac{1}{\omega(B(x_0,2^i r))}\int_{B(x_0,2^i r)} |V(y)|\omega(y)dy\Big)^{s\sigma/2} \\
&\lesssim \frac{\omega(B(x_0,r))}{(1+2^i r/\rho(x_0))^{N_2}} \frac{1}{(2^i r)^{\lambda}} \|f\|^s_{M_p^\lambda(\mathbb{R}^n,\omega)},
\end{aligned}$$

where $0 < N_2 < (Ns/(N_0+1) - (\log_2 C_0 + 1)s\sigma/2$ and $1/s = 1/p - (\alpha\beta - \sigma)/\lambda$.

Noting that $\theta > 1$ and ω satisfies (4.5), we obtain

$$\begin{aligned}
&\|I^L_{\alpha,\beta} V^{\sigma/2} f\|^s_{M_s^{\lambda s/p, s/p}(\mathbb{R}^n,\omega)} \\
&\lesssim \|f\|^s_{M_p^\lambda(\mathbb{R}^n,\omega)} + \sum_{i=1}^{\infty} \frac{r^{\lambda(\theta-1)}}{\omega(B(x_0,r))^{\theta-1}} \frac{1}{(1+2^i r/\rho(x_0))^{N_2} 2^{i\lambda}} \|f\|^s_{M_p^\lambda(\mathbb{R}^n,\omega)} \\
&\lesssim \|f\|^s_{M_p^\lambda(\mathbb{R}^n,\omega)}.
\end{aligned}$$

□

5 Boundedness of S^* and $H_{\alpha,\beta}$

In this section, we will investigate the boundedness of operators related to L_0 on Lipschitz type spaces. For this purpose, we first recall the maximal semigroup operator S^* defined as

$$S^* f(x) := \sup_{t>0} \Big| \int_{\mathbb{R}^n} h_{\alpha,t}(x,y) f(y)\omega(y)dy \Big|.$$

Theorem 5.1 *Let ω be a weight in $A_2 \bigcap RD_\nu$ with $\nu > 2$. Assume that $0 < \eta_1 \leq \min\{2\alpha, \eta\}$ and $\eta \in (0,1)$. The maximal operator S^* is bounded on $BMO_0^d(\mathbb{R}^n,\omega) = BMO^d(\mathbb{R}^n,\omega) \bigcap L^\infty$ for any $0 \leq d < \eta_1$.*

Proof We consider only the case $d > 0$; the case $d = 0$ is just the boundedness on L^∞ contained in Theorem 4.2.

Assume that $f \in BMO_0^d(\mathbb{R}^n,\omega)$. As we said, $S^* f(x)$ is finite a.e. $x \in \mathbb{R}^n$. Then for $x, z \in \mathbb{R}^n$,

$$\Big| S^* f(x) - S^* f(z) \Big| \leq \sup_{t>0} \Big| \int_{\mathbb{R}^n} (h_{\alpha,t}(x,y) - h_{\alpha,t}(z,y)) f(y)\omega(y)dy \Big|. \quad (5.1)$$

It should be noted that $S_{\alpha,t}1(x) = S_{\alpha,t}1(z)$. In fact, it follows from [20, Theorem 2.1] and [10, Section 3] that

$$\int_{\mathbb{R}^n} h_t(x, y)w(y)dy = 1.$$

By (1.1), we obtain

$$\int_{\mathbb{R}^n} h_{\alpha,t}(x, y)w(y)dy = \int_0^\infty \eta_t^\alpha(s)\Big(\int_{\mathbb{R}^n} h_s(x, y)w(y)dy\Big)ds = \int_0^\infty \eta_t^\alpha(s)ds = 1.$$

Then we deduce that $S_{\alpha,t}1 \equiv 1$ for any $t > 0$, which implies that $S_{\alpha,t}1(x) = S_{\alpha,t}1(z)$. We conclude that the above integral in the right side of (5.1) is zero when f is constant. Therefore, we may replace $f(y)$ by the difference $f(y) - f(x)$ inside the integral. Then $\Big|S^*f(x) - S^*f(z)\Big| \le I_1 + I_2$, where

$$\begin{cases} I_1 := \sup\limits_{t>0}\Big|\displaystyle\int_{B(x,4|x-z|)} (h_{\alpha,t}(x, y) - h_{\alpha,t}(z, y))(f(y) - f(x))\omega(y)dy\Big|; \\ I_2 := \sup\limits_{t>0}\Big|\displaystyle\int_{(B(x,4|x-z|))^c} (h_{\alpha,t}(x, y) - h_{\alpha,t}(z, y))(f(y) - f(x))\omega(y)dy\Big|. \end{cases}$$

For I_1, it is easy to see that $I_1 \le I_{1,1} + I_{1,2}$, where

$$\begin{cases} I_{1,1} := \sup\limits_{t>0}\Big|\displaystyle\int_{B(x,4|x-z|)} h_{\alpha,t}(x, y)(f(y) - f(x))\omega(y)dy\Big|; \\ I_{1,2} := \sup\limits_{t>0}\Big|\displaystyle\int_{B(x,4|x-z|)} h_{\alpha,t}(z, y)(f(y) - f(x))\omega(y)dy\Big|. \end{cases}$$

The methods proving $I_{1,1}$ and $I_{1,2}$ are similar. We now observe that $B(x, 4|x - z|) \subset B(z, 5|x - z|)$. By $f \in BMO^d(\mathbb{R}^n, \omega)$, we obtain

$$I_{1,1} \lesssim \|f\|_{BMO^d}|x - z|^d \sup_{t>0}\int_{B(x,4|x-z|)} h_{\alpha,t}(x, y)\omega(y)dy \lesssim \|f\|_{BMO^d}|x - z|^d.$$

For I_2, it is obvious that if $y \in (B(x, 4|x - z|))^c$ then $|x - z| \le |x - y|/4$. By the proof of (ii) of Proposition 3.4, we obtain

$$|h_{\alpha,t}(x, y) - h_{\alpha,t}(z, y)| \lesssim \Big(\frac{|x - z|}{t^{1/2\alpha}}\Big)^{\eta_1} \min\Big\{\frac{t^{1+\eta_1/2\alpha}}{\omega(B(x, |x - y|))|x - y|^{2\alpha+\eta_1}}, \frac{1}{\omega(B(x, t^{1/2\alpha}))}\Big\}.$$

If $|x - y| \le t^{1/2\alpha}$, then

$$|h_{\alpha,t}(x, y) - h_{\alpha,t}(z, y)| \lesssim \frac{1}{\omega(B(x, |x - y|))}\Big(\frac{|x - z|}{|x - y|}\Big)^{\eta_1}.$$

If $|x-y|>t^{1/2\alpha}$, then

$$|h_{\alpha,t}(x,y)-h_{\alpha,t}(z,y)|\lesssim \frac{|x-z|^{\eta_1}|x-y|^{2\alpha}}{\omega(B(x,|x-y|))|x-y|^{2\alpha+\eta_1}} = \frac{1}{\omega(B(x,|x-y|))}\Big(\frac{|x-z|}{|x-y|}\Big)^{\eta_1}.$$

From the above arguments, we have

$$|h_{\alpha,t}(x,y)-h_{\alpha,t}(z,y)|\lesssim \frac{1}{\omega(B(x,|x-y|))}\Big(\frac{|x-z|}{|x-y|}\Big)^{\eta_1}.$$

Using $f\in BMO^d(\mathbb{R}^n,\omega)$ again, then

$$\begin{aligned} I_2 &\lesssim \|f\|_{BMO^d}|x-z|^{\eta_1}\sum_{k=2}^{\infty}\int_{2^{k+1}B(x,|x-z|)\backslash 2^kB(x,|x-z|)} \frac{|x-y|^{d-\eta_1}}{\omega(B(x,|x-y|))}\omega(y)dy \\ &\lesssim \|f\|_{BMO^d}|x-z|^{\eta_1}\sum_{k=2}^{\infty}(2^k|x-z|)^{d-\eta_1}. \end{aligned}$$

Since $d<\eta_1$, the convergence of the above sum derives the desired estimate. Finally, using $\|S^*f\|_\infty\lesssim\|f\|_\infty$ again, we complete the proof of the theorem. □

Now we turn our attention to the fractional operators $I_{\alpha,\beta}$. It should be mentioned that in order to deal with these operators acting on functions in $M_p^\lambda(\mathbb{R}^n,\omega)$ with $p\ge\lambda/\alpha\beta$ or in $BMO^d(\mathbb{R}^n,\omega)$, we need not only the kernel size for $0<\alpha\beta<\nu$ in Lemma 4.3 but also the smoothness of $H_{\alpha,\beta}$.

Lemma 5.2 *Let ω be a weight in $A_2\bigcap RD_\nu$ with $\nu>2$. Assume that $0<\eta_1\le\min\{2\alpha,\eta\}$ and $\eta\in(0,1)$. Then for $|x-z|\le|x-y|/4$, we have*

$$|H_{\alpha,\beta}(x,y)-H_{\alpha,\beta}(z,y)|\lesssim\Big(\frac{|x-z|}{|x-y|}\Big)^{\eta_1}\frac{|x-y|^{\alpha\beta}}{\omega(B(x,|x-y|))}. \tag{5.2}$$

Proof To estimate the left hand side above, it only needs to consider

$$\int_0^\infty|h_{\alpha,t}(x,y)-h_{\alpha,t}(z,y)|t^{\beta/2}\frac{dt}{t}.$$

Using (ii) of Proposition 3.4, we deduce

$$\int_0^\infty|h_{\alpha,t}(x,y)-h_{\alpha,t}(z,y)|t^{\beta/2}\frac{dt}{t}=J_1+J_2,$$

where

$$J_1:=\int_{t\le|x-y|^{2\alpha}}\Big(\frac{|x-z|}{t^{1/2\alpha}}\Big)^{\eta_1}\min\Big\{\frac{t^{1+\eta_1/2\alpha}}{\omega(B(x,|x-y|))|x-y|^{2\alpha+\eta_1}},\frac{1}{\omega(B(x,t^{1/2\alpha}))}\Big\}t^{\beta/2}\frac{dt}{t}$$

and

$$J_2 := \int_{t>|x-y|^{2\alpha}} \Big(\frac{|x-z|}{t^{1/2\alpha}}\Big)^{\eta_1} \min\Big\{\frac{t^{1+\eta_1/2\alpha}}{\omega(B(x,|x-y|))|x-y|^{2\alpha+\eta_1}}, \frac{1}{\omega(B(x,t^{1/2\alpha}))}\Big\} t^{\beta/2}\frac{dt}{t}.$$

For J_2, we have

$$\begin{aligned} J_2 &\lesssim \int_{t^{1/\alpha}>|x-y|^2} \frac{1}{\omega(B(x,|x-y|))}\Big(\frac{|x-z|}{t^{1/2\alpha}}\Big)^{\eta_1} t^{\beta/2}\frac{dt}{t} \\ &\lesssim \frac{|x-y|^{\alpha\beta}}{\omega(B(x,|x-y|))}\Big(\frac{|x-z|}{|x-y|}\Big)^{\eta_1}. \end{aligned}$$

For J_1, we obtain

$$\begin{aligned} J_1 &\lesssim \int_{t^{1/\alpha}\le|x-y|^2} \frac{1}{\omega(B(x,|x-y|))|x-y|^{2\alpha}}\Big(\frac{|x-z|}{|x-y|}\Big)^{\eta_1} t^{\beta/2}dt \\ &\lesssim \frac{|x-y|^{\alpha\beta}}{\omega(B(x,|x-y|))}\Big(\frac{|x-z|}{|x-y|}\Big)^{\eta_1}. \end{aligned}$$

□

Below we will state and prove several novel results for the operator $I_{\alpha,\beta}$. Firstly, we are going to modify our operator such that it makes sense for all functions in $M_p^\lambda(\mathbb{R}^n,\omega)$ with $\lambda/\alpha\beta \le p < \lambda/(\alpha\beta-\eta_1)^+$, which means $\lambda/\alpha\beta \le p < \infty$ when $\alpha\beta < \eta_1$ and $\lambda/\alpha\beta \le p < \lambda/(\alpha\beta-\eta_1)$, otherwise. In fact, our definition works on functions in the large spaces $M_1^{\alpha\beta-d}(\mathbb{R}^n,\omega)$ for any $0\le d<\alpha\beta$ and $d<\eta_1$. It is obvious that these spaces contain any of the aforementioned ones since, by Hölder's inequality, $M_p^\lambda(\mathbb{R}^n,\omega)\subset M_1^{d/p}(\mathbb{R}^n,\omega)$ and $d=\alpha\beta-\lambda/p$ satisfies the above conditions for $\lambda/\alpha\beta\le p<\lambda/(\alpha\beta-\eta_1)^+$. Then, we introduce the following operator

$$\widetilde{\mathcal{H}}_{\alpha,\beta} f(x) := \int_{\mathbb{R}^n} (H_{\alpha,\beta}(x,y) - H_{\alpha,\beta}(0,y)\chi_{B_1^c}) f(y)\omega(y)dy$$

for $f\in M_1^{\alpha\beta-d}(\mathbb{R}^n,\omega)$ and $B_1=B(0,1)$. Secondly, we notice that the right hand side gives a locally integrable function. Clearly, it will be enough to show integrability in balls $B(0,R)$ with $R\ge 2$. In fact, we split $\widetilde{\mathcal{H}}_{\alpha,\beta,t} f(x) = I_1(x) + I_2(x)$, where

$$\begin{cases} I_1(x) := \int_{B_1} H_{\alpha,\beta}(x,y) f(y)\omega(y)dy; \\ I_2(x) := \int_{B_1^c} (H_{\alpha,\beta}(x,y) - H_{\alpha,\beta}(0,y)) f(y)\omega(y)dy. \end{cases}$$

Using the Fubini theorem and Lemma 4.3, we obtain

$$\int_{B(0,R)} |I_1(x)|\omega(x)dx \lesssim \int_{B_1} |f(y)|\Big(\int_{B(y,2R)} \frac{|x-y|^{\alpha\beta}}{\omega(B(x,|x-y|))}\omega(x)dx\Big)\omega(y)dy$$

due to the fact $B(0, R) \subset B(y, 2R)$. It is easy to see that $B(y, |x-y|) \subseteq B(x, 2|x-y|)$ and $\omega \in A_2$ imply that

$$\omega(B(y, |x-y|)) \le \omega(B(x, 2|x-y|)) \lesssim \omega(B(x, |x-y|)).$$

The right side of the above inequality is bounded by

$$\begin{aligned}
&\int_{B_1} |f(y)| \left(\sum_{j=-\infty}^{0} \int_{2^j R \le |x-y| < 2^{j+1} R} \frac{|x-y|^{\alpha\beta}}{\omega(B(x, |x-y|))} \omega(x) dx \right) \omega(y) dy \\
&\lesssim \int_{B_1} |f(y)| \left(\sum_{j=-\infty}^{0} \frac{(2^j R)^{\alpha\beta} \omega(B(y, 2^{j+1} R))}{\omega(B(y, 2^j R))} \right) \omega(y) dy \\
&\lesssim R^{\alpha\beta},
\end{aligned}$$

where the above quantity is finite due to the fact that f is locally integrable with respect to the weight ω.

Regarding to $I_2(x)$, we observe that $|I_2(x)| \le I_{2,1}(x) + I_{2,2}(x) + I_{2,3}(x)$, where

$$\begin{cases}
I_{2,1}(x) := \int_{B(0,2R)} H_{\alpha,\beta}(x, y) |f(y)| \omega(y) dy; \\
I_{2,2}(x) := \int_{B(0,2R) \setminus B_1} H_{\alpha,\beta}(0, y) |f(y)| \omega(y) dy; \\
I_{2,3}(x) := \int_{B(0,2R)^c} |H_{\alpha,\beta}(x, y) - H_{\alpha,\beta}(0, y)| |f(y)| \omega(y) dy.
\end{cases}$$

The local integrability of $I_{2,1}(x)$ follows as for $I_1(x)$. Regarding to $I_{2,2}(x)$, we notice that $H_{\alpha,\beta}(0, y) \lesssim \frac{R^{\alpha\beta}}{\omega(B(0,1))}$ for $y \in B(0, 2R) \setminus B_1$ and then $I_{2,2}(x)$ is a finite constant. For $I_{2,3}(x)$, since $x \in B(0, R)$ and $y \in B(0, 2R)^c$, $|y|/2 > |x|$. We apply Lemma 5.2 and $f \in M_1^{\alpha\beta-d}(\mathbb{R}^n, \omega)$ to get

$$\begin{aligned}
I_{2,3}(x) &\lesssim |x|^{\eta_1} \int_{B(0,2R)^c} \frac{|y|^{\alpha\beta-\eta_1}}{\omega(B(0, |y|))} |f(y)| \omega(y) dy \\
&\lesssim |x|^{\eta_1} \sum_{k=1}^{\infty} \frac{(2^k R)^{\alpha\beta-\eta_1}}{\omega(B(0, 2^k R))} \int_{B(0,2^{k+1} R)} |f(y)| \omega(y) dy \\
&\lesssim |x|^{\eta_1} \sum_{k=1}^{\infty} 2^{k(d-\eta_1)} R^{d-\eta_1} \|f\|_{M_1^{\alpha\beta-d}}
\end{aligned}$$

since $d < \eta_1$ the series converges and $I_{2,3}(x)$ is also a locally integrable function with respect to the weight ω.

Then we will establish continuity properties of our modified operator $\widetilde{\mathcal{H}}_{\alpha,\beta}$ in the following theorem.

Theorem 5.3 *Let ω be weight in $A_2 \bigcap RD_\nu$ with $\nu > 2$. Assume that $0 < \eta_1 \leq \min\{2\alpha, \eta\}$ and $\eta \in (0, 1)$. Then, for $0 < \alpha\beta < \nu$, the operator $\widetilde{\mathcal{H}}_{\alpha,\beta}$ maps continuously $M_1^{\alpha\beta-d}(\mathbb{R}^n, \omega)$ into $BMO^d(\mathbb{R}^n, \omega)$ for any given d with $0 \leq d < \min\{\eta_1, \alpha\beta\}$. Furthermore, $\widetilde{\mathcal{H}}_{\alpha,\beta} f$ coincides with $I_{\alpha,\beta} f$ as functions in $BMO^d(\mathbb{R}^n, \omega)$ when f is also compactly supported.*

Proof Using the above arguments, we know that $\widetilde{\mathcal{H}}_{\alpha,\beta} f$ is a locally integrable function and hence it is finite a.e.. Moreover, from a similar argument, it follows that for any given ball $B = B(x_0, r)$, setting $\widetilde{B} = 2B$, we conclude that

$$a_B := \int_{\mathbb{R}^n} (H_{\alpha,\beta}(x_0, y)\chi_{\widetilde{B}^c} - H_{\alpha,\beta}(0, y)\chi_{B_1^c}) f(y)\omega(y)dy$$

is a finite constant. More precisely, take a ball $B^* = B(x_0, R)$ with R large enough such that $2B_1 \bigcup \widetilde{B} \subset B^*$, for example we may choose $R = 2(|x_0| + r + 1)$. Then

$$a_B \leq J_1 + J_2 + J_3,$$

where

$$\begin{cases} J_1 := \displaystyle\int_{B^* \setminus \widetilde{B}} H_{\alpha,\beta}(x_0, y)|f(y)|\omega(y)dy; \\ J_2 := \displaystyle\int_{B^* \setminus B_1} H_{\alpha,\beta}(0, y)|f(y)|\omega(y)dy; \\ J_3 := \displaystyle\int_{B^{*c}} |H_{\alpha,\beta}(x_0, y) - H_{\alpha,\beta}(0, y)||f(y)|\omega(y)dy. \end{cases}$$

For J_1 and J_2, it follows that the kernel is bounded since $2r \leq |x_0 - y| < R$ in J_1 and $2 \leq |y| \leq |x_0| + R$ in J_2. Therefore, the finiteness of J_1 and J_2 can be deduced from the local integrability of f. Regarding J_3, it is easy to see that $|x_0 - y| \geq R > 2|x_0|$. By Lemma 5.2 and similarly to the proof of $I_{2,3}$, we have, for $f \in M_1^{\alpha\beta-d}(\mathbb{R}^n, \omega)$,

$$J_3 \lesssim \|f\|_{M_1^{\alpha\beta-d}(\mathbb{R}^n,\omega)} |x_0|^{\eta_1} R^{\alpha\beta-\eta_1} < \infty.$$

Therefore, we show that for any ball B, $\widetilde{\mathcal{H}}_{\alpha,\beta} f(x) := G_1(x) + G_2(x) + a_B$, where

$$\begin{cases} G_1(x) = \displaystyle\int_{\widetilde{B}} H_{\alpha,\beta}(x, y) f(y)\omega(y)dy; \\ G_2(x) = \displaystyle\int_{\widetilde{B}^c} (H_{\alpha,\beta}(x, y) - H_{\alpha,\beta}(x_0, y)) f(y)\omega(y)dy. \end{cases}$$

We are now in a position to show that $\widetilde{\mathcal{H}}_{\alpha,\beta} f(x)$ belongs to $BMO^d(\mathbb{R}^n, \omega)$. We fix a ball $B = B(x_0, r)$ and use the above expression for that specific ball. For G_1, we integrate $G_1(x)$ over B. Similarly to the above arguments of I_1, we obtain

$$\int_B |G_1(x)|\omega(x)dx \lesssim r^{\alpha\beta} \int_{\widetilde{B}} |f(y)|\omega(y)dy \lesssim \|f\|_{M_1^{\alpha\beta-d}(\mathbb{R}^n,\omega)} r^d \omega(B).$$

Next, in order to calculate the oscillation over B of the remaining terms, we may subtract a constant, for example a_B. Therefore, we only need to control the integral of $|G_2(x)|$. In this manner, applying Lemma 5.2 again, we obtain

$$\begin{aligned}|G_2(x)| &\lesssim |x_0 - x|^{\eta_1} \int_{\widetilde{B}^c} \frac{|x_0 - y|^{\alpha\beta-\eta_1}}{\omega(B(x_0, |x_0 - y|))} |f(y)|\omega(y)dy \\ &\simeq |x - x_0|^{\eta_1} \sum_{k>1} \int_{2^{k+1}B\backslash 2^k B} \frac{|x_0 - y|^{\alpha\beta-\eta_1}}{\omega(B(x_0, |x_0 - y|))} |f(y)|\omega(y)dy \\ &\lesssim \|f\|_{M_1^{\alpha\beta-d}(\mathbb{R}^n,\omega)} r^d\end{aligned}$$

due to $|x - y| \simeq |x_0 - y|$ and $\omega(B(x_0, |x_0 - y|)) \simeq \omega(B(x, |x - y|))$. Averaging with respect to ωdx deduces the desired estimate. Finally, notice that if f is compactly supported, we may take a ball B large enough so that G_2 is zero, which implies that

$$\widetilde{\mathcal{H}}_{\alpha,\beta} f(x) := \int_{\mathbb{R}^n} H_{\alpha,\beta}(x, y) f(y)\omega(y)dy + a_B,$$

and hence $\widetilde{\mathcal{H}}_{\alpha,\beta} f(x)$ equals to $I_{\alpha,\beta} f$ as functions in $BMO^d(\mathbb{R}^n, \omega)$. This completes the proof of Theorem 5.3. □

Remark 5.4 When $\alpha = 1$ and $\omega \equiv 1$, Theorem 5.3 comes back to [19], which implies the boundedness of the modified classical fractional integral of order β from weak-$L^{n/\beta}$ into BMO or, more generally, from weak-L^p into BMO^d for $p \geq n/\beta$ and $d = \beta - p/n < 1$.

In what follows, we study the behavior of $I_{\alpha,\beta}$ on the spaces $BMO^d(\mathbb{R}^n, \omega)$. Via the proof of Theorem 5.1, we have

$$\int_0^\infty \Big(\int_{\mathbb{R}^n} (h_{\alpha,t}(x, y) - h_{\alpha,t}(x_0, y))\omega(y)dy\Big) t^{\beta/2} \frac{dt}{t} = 0.$$

Now, if we take the absolute value inside and reverse the order of integration, it is easy to check that the iterated integral is finite. More precisely,

$$\int_{\mathbb{R}^n} \Big(\int_0^\infty \Big|h_{\alpha,t}(x, y) - h_{\alpha,t}(x_0, y)\Big| t^{\beta/2} \frac{dt}{t}\Big)\omega(y)dy \leq I_1 + I_2,$$

where

$$\begin{cases} I_1 := \int_{B(x,2|x-x_0|)} \Big(\int_0^\infty \Big| h_{\alpha,t}(x,y) - h_{\alpha,t}(x_0,y) \Big| t^{\beta/2} \frac{dt}{t} \Big) \omega(y)dy; \\ I_2 := \int_{B(x,2|x-x_0|)^c} \Big(\int_0^\infty \Big| h_{\alpha,t}(x,y) - h_{\alpha,t}(x_0,y) \Big| t^{\beta/2} \frac{dt}{t} \Big) \omega(y)dy. \end{cases}$$

For I_1, in view of Lemma 4.3, we obtain

$$I_1 \le \int_{B(x,2|x-x_0|)} H_{\alpha,\beta}(x,y)\omega(y)dy + \int_{B(x_0,3|x-x_0|)} H_{\alpha,\beta}(x_0,y)\omega(y)dy \le C.$$

For I_2, using Lemma 5.2, we can deduce that if we assume $\alpha\beta < \eta_1$,

$$I_2 \lesssim |x-x_0|^{\eta_1} \int_{B(x,2|x-x_0|)^c} \frac{|x-y|^{\alpha\beta-\eta_1}}{\omega(B(x,|x-y|))} \omega(y)dy < \infty.$$

Therefore, the order of integration can be reversed to obtain

$$\int_{\mathbb{R}^n} [H_{\alpha,\beta}(x,y) - H_{\alpha,\beta}(x_0,y)]\omega(y)dy = 0, \tag{5.3}$$

and the integral is finite if we assume $\alpha\beta < \eta_1$. Then, given a function $f \in BMO^d(\mathbb{R}^n,\omega)$, $0 \le d < \eta_1$, for some fixed x_0, we define the following operator

$$\ddot{\mathcal{H}}_{\alpha,\beta} f(x) := \int_{\mathbb{R}^n} [H_{\alpha,\beta}(x,y) - H_{\alpha,\beta}(x_0,y)] f(y)\omega(y)dy.$$

Theorem 5.5 *Let ω be a weight in $A_2 \bigcap RD_\nu$ with $\nu > 2$. Assume that $0 < \eta_1 \le \min\{2\alpha, \eta\}$ and $\eta \in (0,1)$. Then, for $0 < \alpha\beta < \eta_1$, the operator $\ddot{\mathcal{H}}_{\alpha,\beta}$ maps continuously $BMO^d(\mathbb{R}^n,\omega)$ into $BMO^{\alpha\beta+d}(\mathbb{R}^n,\omega)$ for any given $d > 0$ such that $0 \le \alpha\beta + d < \eta_1$. Furthermore, when f is compactly supported, $\ddot{\mathcal{H}}_{\alpha,\beta} f$ coincides with $I_{\alpha,\beta} f$ as functions in $BMO^{\alpha\beta+d}(\mathbb{R}^n,\omega)$.*

Proof At first, it should be noted that (5.3) allows us to substitute $f(y)$ by $f(y)-c$ into the integral. Therefore, the definition is independent of the member of the equivalence class.

Secondly, we check that for $f \in BMO^d(\mathbb{R}^n,\omega)$ with $0 < \alpha\beta + d < \eta_1$, it defines a locally integrable function, in fact, it is locally bounded. Then given a ball B and $j \in \mathbb{Z}$, adding and subtracting intermediate averages, we obtain

$$\frac{1}{\omega(2^j B)} \int_{2^j B} |f(x)|\omega(x)dx \le \|f\|_{BMO^d} c(j,d) r^d + |f|_B, \tag{5.4}$$

where for $d > 0$ is either $c(j,d) = 2^{jd}$ when $j > 0$ and $c(j,d) = c$ for $j < 0$ or $c(j,d) = j$ when $d = 0$. Then we show that the integral in the definition converges absolutely for any pair x and x_0. Take $B = B(x, r)$ with $r = 2|x - x_0|$. We split the integral as

$$\int_{\mathbb{R}^n} |H_{\alpha,\beta}(x, y) - H_{\alpha,\beta}(x_0, y)||f(y)|\omega(y)dy = I_1 + I_2,$$

where

$$\begin{cases} I_1 := \int_B |H_{\alpha,\beta}(x, y) - H_{\alpha,\beta}(x_0, y)||f(y)|\omega(y)dy; \\ I_2 := \int_{B^c} |H_{\alpha,\beta}(x, y) - H_{\alpha,\beta}(x_0, y)||f(y)|\omega(y)dy. \end{cases}$$

Furthermore, we write I_1 as $I_1 \leq I_{1,1} + I_{1,2}$, where

$$\begin{cases} I_{1,1} := \int_B H_{\alpha,\beta}(x, y)|f(y)|\omega(y)dy; \\ I_{1,2} := \int_B H_{\alpha,\beta}(x_0, y)|f(y)|\omega(y)dy. \end{cases}$$

By Lemma 4.3 and (5.4), we have

$$\begin{aligned} I_{1,1} &\lesssim r^{\alpha\beta} \sum_{j\leq 0} 2^{j\alpha\beta} \frac{1}{\omega(2^j B)} \int_{2^j B} |f(y)|\omega(y)dy. \\ &\lesssim |x - x_0|^{\alpha\beta} \Big(\|f\|_{BMO^d} |x - x_0|^d + |f|_B \Big). \end{aligned}$$

The proof of $I_{1,2}$ is similar to that of $I_{1,1}$, and so is omitted. Noting that $|f|_{B(x,2|x-x_0|)}$ is a continuous function of x that is also true for the above function and so our original integral is a locally bounded function.

For I_2, in view of Lemma 5.2 and (5.4), we obtain

$$\begin{aligned} I_2 &\lesssim |x - x_0|^{\eta_1} \sum_{j>0} \int_{2^{j+1}B\setminus 2^j B} \frac{|x - y|^{\alpha\beta-\eta_1}}{\omega(B(x, |x - y|))} |f(y)|\omega(y)dy \\ &\lesssim |x - x_0|^{\eta_1} \sum_{j>0} (2^j|x - x_0|)^{\alpha\beta-\eta_1} \frac{1}{\omega(2^j B)} \int_{2^j B} |f(y)|\omega(y)dy \\ &\lesssim |x - x_0|^{\alpha\beta} \Big(|f|_{B(x,2|x-x_0|)} + |x - x_0|^d \|f\|_{BMO^d} \sum_{j>0} c(j, d) 2^{j(\alpha\beta-\eta_1)} \Big), \end{aligned}$$

and the sum is convergent because of $\alpha\beta + d < \eta_1$.

Therefore, we have proved that $\ddot{\mathcal{H}}_{\alpha,\beta} f$ is well defined and it is finite for any x. Moreover, it is independent of the choice of x_0. If we take any another point x_1 in the definition of $\ddot{\mathcal{H}}_{\alpha,\beta}$, the absolute convergence of the integral implies that the difference between the two possible definitions is

$$\int_{\mathbb{R}^n} (H_{\alpha,\beta}(x_1, y) - H_{\alpha,\beta}(x_0, y)) f(y)\omega(y)dy,$$

which gives a finite constant.

Finally, we consider the continuity result. Let x and z be two points and set $B = B(x, 2|x-z|)$. Using (5.3), we can replace f by $f - f_B$ in the definition of the operator. So we obtain

$$\int_{\mathbb{R}^n} |H_{\alpha,\beta}(x, y) - H_{\alpha,\beta}(z, y)||f(y) - f_B|\omega(y)dy = J_1 + J_2,$$

where

$$\begin{cases} J_1 := \displaystyle\int_B |H_{\alpha,\beta}(x, y) - H_{\alpha,\beta}(z, y)||f(y) - f_B|\omega(y)dy; \\ J_2 := \displaystyle\int_{B^c} |H_{\alpha,\beta}(x, y) - H_{\alpha,\beta}(z, y)||f(y) - f_B|\omega(y)dy. \end{cases}$$

Since $|f - f_B| \leq |f - f_{2^j B}| + \sum_{i=2}^{j} |f_{2^i B} - f_{2^{i-1} B}|$, we get the estimate

$$\frac{1}{\omega(2^j B)} \int_{2^j B} |f(x) - f_B|\omega(x)dx \leq c(j, d)\|f\|_{BMO^d}|x - z|^d, \tag{5.5}$$

where for $d > 0$ is either $c(j, d) = 2^{jd}$ when $j > 0$ and $c(j, d) = c$ for $j < 0$ or $c(j, d) = j$ when $d = 0$. Then for J_1, since $B \subset B(z, 3|x - z|)$, we can also get $J_1 \leq J_{1,1} + J_{1,2}$, where

$$\begin{cases} J_{1,1} := \displaystyle\int_{B(x,2|x-z|)} |H_{\alpha,\beta}(x, y)||f(y) - f_B|\omega(y)dy; \\ J_{1,2} := \displaystyle\int_{B(z,3|x-z|)} |H_{\alpha,\beta}(z, y)||f(y) - f_B|\omega(y)dy. \end{cases}$$

Since the proof of $J_{1,2}$ is similar to that of $J_{1,1}$, we only need to prove $J_{1,1}$. By Lemma 4.3 and (5.5), we obtain

$$\begin{aligned} J_{1,1} &\lesssim |x - z|^{\alpha\beta} \sum_{j\leq 0} 2^{j\alpha\beta} \frac{1}{\omega(2^j B)} \int_{2^j B} |f(y) - f_B|\omega(y)dy \\ &\lesssim |x - z|^{\alpha\beta+d}\|f\|_{BMO^d} \sum_{j\leq 0} c(j, d)2^{j\alpha\beta} < \infty. \end{aligned}$$

For J_2, we may apply the smoothness and use (5.5) again as above. In this way, we obtain

$$J_2 \lesssim |x-z|^{\alpha\beta+d}\|f\|_{BMO^d}\sum_{j>0}2^{j(\alpha\beta+d-\eta_1)},$$

and the series is convergent since $\alpha\beta+d<\eta_1$. □

6 Regularity results for operators related to *L*

In this section, we will consider the case of the degenerate Schrödinger operator and the aim of this section is to analyze the behavior of the maximal operator of the semigroup, the fractional integration as well as the mixed operators $I^L_{\alpha,\beta}V^{\sigma/2}$ associated with L.

Lemma 6.1 ([19, Lemma 4]) *Let ω be a doubling weight and $f\in BMO^d_\rho(\mathbb{R}^n,\omega)$.*

(i) *For any critical ball $B=B(x_0,\rho(x_0))$ and $k\geq 0$,*

$$\frac{1}{\omega(2^{-k}B)}\int_{2^{-k}B}|f(y)|\omega(y)dy\leq\|f\|_{BMO^d_\rho}c(k,d)\rho(x_0)^d$$

with $c(k,d)=k$ when $d=0$ and $c(k,d)=c$ when $0<d<1$.

(ii) *For any subcritical ball $B=B(x_0,r)$ with $r\leq\rho(x_0)$ and $k\geq 0$,*

$$\frac{1}{\omega(2^kB)}\int_{2^kB}|f(y)-f_B|\omega(y)dy\leq\|f\|_{BMO^d_\rho}a(k,d)r^d$$

with $a(k,d)=k$ when $d=0$ and $a(k,d)=2^{kd}$ when $0<d<1$.

Given a doubling weight ω, a critical radius function ρ and an index $\alpha\beta\geq 0$, we consider a class of operators called $\alpha\beta$-Schrödinger-Calderón-Zygmund operators with respect to the measure $\omega(x)dx$ (see [28] and [19]). We distinguish the following two cases.

Case 1: $\alpha\beta>0$. T is an integral operator with respect to the measure $\omega(x)dx$, given by a kernel $K(\cdot,\cdot)$ that satisfies the following conditions.

(i) For any $N>0$ there is a constant C_N such that

$$|K(x,y)|\leq C_N\frac{|x-y|^{\alpha\beta}}{\omega(B(x,|x-y|))}\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{-N}.$$

(ii) There exists some $0<\delta<1$ such that for $|x-z|<|x-y|/2$,

$$|K(x,y)-K(z,y)|\lesssim\Big(\frac{|x-z|}{|x-y|}\Big)^{\delta}\frac{|x-y|^{\alpha\beta}}{\omega(B(x,|x-y|))}.$$

Case 2: $\alpha\beta = 0$. T is a linear bounded operator on $L^p(\mathbb{R}^n, \omega)$ for $1 < p < \infty$, which has an associated kernel $K(\cdot, \cdot)$ in the sense that, for any $L^p(\mathbb{R}^n, \omega)$-function with compact support

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y)\omega(y)dy, \quad x \in supp(f)^c.$$

Furthermore, K satisfies conditions (i) and (ii) above with $\alpha\beta = 0$.

After giving this definition, similarly to the arguments in [19], we conclude that T is well defined for functions in $BMO_\rho^d(\mathbb{R}^n, \omega)$. Notice that in both cases, either $\alpha\beta > 0$ or $\alpha\beta = 0$, we may apply the operator T to $f \equiv 1$ since it belongs to $BMO_\rho^d(\mathbb{R}^n, \omega)$, no matter what ρ is. Then we also have

Proposition 6.2 *Let ω be a doubling weight and ρ be a critical radius function. Assume that $\delta_0 = 2 - \gamma/q$, $\eta \in (0, 1)$ and $0 < \delta' < \min\{2\alpha, \eta, \delta_0\}$. Suppose T is an $\alpha\beta$-Schrödinger-Calderón-Zygmund operator with respect to $\omega(x)dx$ that further satisfies the following $T1$-condition:*

There exist $\epsilon > 0$ and a constant C such that for any ball $B = B(x_0, r)$ with $r < \rho(x_0)$,

$$\frac{1}{r^{\alpha\beta}\omega(B)} \int_B |T1(x) - (T1)_B|\omega(x)dx \le C\Big(\frac{r}{\rho(x_0)}\Big)^\epsilon.$$

Then T is bounded from $BMO_\rho^d(\mathbb{R}^n, \omega)$ into $BMO_\rho^{d+\alpha\beta}(\mathbb{R}^n, \omega)$ for any $0 \le d \le \epsilon$ and such that $0 \le \alpha\beta + d < \delta'$.

Remark 6.3 The above result can be also stated in the vector valued setting. Assume that we have a linear operator acting on functions defined on $\mathbb{R}^n$ and taking values in a Banach space $\mathbb{X}$ and it satisfies all the conditions with absolute value replaced by the $\mathbb{X}$-norm, then the conclusion also holds. In [36], for the non-degenerate Schrödinger case, the authors have proved the BMO_ρ^d-boundedness of the maximal operator and square functions in the vector valued setting.

In what follows, we may look at T^* as the L^∞-norm of the vector valued operator $\mathcal{T}f = \{T_{\alpha,t}^L f\}_{t>0}$. So by Theorem 4.2, we can get the boundedness of $\mathcal{T}$ from L^p into $L^p_{\mathbb{X}}$ with $\mathbb{X} = L^\infty$. Moreover, we have

Theorem 6.4 *Let ω be an A_2-weight such that $\omega \in RD_\nu \bigcap D_\gamma$ with $\nu > 2$ and $V \in RH_q(\omega)$ with $q > \gamma/2$. Assume that $\delta_0 = 2 - \gamma/q$ and $\eta \in (0, 1)$.*

(i) *For any $N > 0$, there exists a constant C_N such that*

$$\Big\|K_{\alpha,t}^L(x, y)\Big\|_{\mathbb{X}} \le \frac{C_N}{\omega(B(x, |x-y|))}\Big(1 + \frac{|x-y|}{\rho(x)}\Big)^{-N}.$$

(ii) *For $|x-y|/4 > |x-z|$ and any $0 < \delta' < \delta_1 = \min\{2\alpha, \eta, \delta_0\}$,*

$$\Big\|K_{\alpha,t}^L(x, y) - K_{\alpha,t}^L(z, y)\Big\|_{\mathbb{X}} \lesssim \Big(\frac{|x-z|}{|x-y|}\Big)^{\delta'} \frac{1}{\omega(B(x, |x-y|))}.$$

(iii) *The operator T^* is bounded on $BMO^d_{\mathbb{X},\rho}(\mathbb{R}^n,\omega)$ for any $0 \le d < \delta_1$.*

Proof For (i), using Proposition 3.5 (i), we know that

$$\begin{aligned}&|K^L_{\alpha,t}(x,y)|\\&\le C_N \min\left\{\frac{t^{1+N/\alpha}}{|x-y|^{2\alpha+2N}}\frac{1}{\omega(B(x,|x-y|))},\frac{1}{\omega(B(x,t^{1/2\alpha}))}\right\}\Big(1+\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N}.\end{aligned}$$

If $t^{1/2\alpha} > |x-y|$,

$$\begin{aligned}|K^L_{\alpha,t}(x,y)| &\le \frac{C_N}{\omega(B(x,t^{1/2\alpha}))}\Big(1+\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N}\\&\le \frac{C_N}{\omega(B(x,|x-y|))}\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{-N}.\end{aligned}$$

If $t^{1/2\alpha} \le |x-y|$,

$$\begin{aligned}|K^L_{\alpha,t}(x,y)| &\le C_N\frac{t^{1+N/\alpha}}{|x-y|^{2\alpha+2N}}\frac{1}{\omega(B(x,|x-y|))}\Big(1+\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N}\\&\le \frac{C_N}{\omega(B(x,|x-y|))}\Big(\frac{t^{1/2\alpha}}{|x-y|}\Big)^{2\alpha+N}\Big(\frac{|x-y|}{t^{1/2\alpha}}+\frac{|x-y|}{\rho(x)}\Big)^{-N}\\&\le \frac{C_N}{\omega(B(x,|x-y|))}\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{-N}.\end{aligned}$$

For (ii), by Proposition 3.5 (ii), we can get

$$\begin{aligned}&\Big|K^L_{\alpha,t}(x,y)-K^L_{\alpha,t}(z,y)\Big|\\&\lesssim \min\left\{\frac{t^{1+N/\alpha}|x-z|^{\delta'}}{|x-y|^{2\alpha+2N+\delta'}}\frac{1}{\omega(B(x,|x-y|))},\frac{1}{\omega(B(x,t^{1/2\alpha}))}\Big(\frac{|x-z|}{t^{1/2\alpha}}\Big)^{\delta'}\right\}.\end{aligned}$$

If $t^{1/2\alpha} > |x-y|$,

$$\Big|K^L_{\alpha,t}(x,y)-K^L_{\alpha,t}(z,y)\Big| \lesssim \frac{1}{\omega(B(x,|x-y|))}\Big(\frac{|x-z|}{|x-y|}\Big)^{\delta'}.$$

If $t^{1/2\alpha} \le |x-y|$,

$$\begin{aligned}\Big|K^L_{\alpha,t}(x,y)-K^L_{\alpha,t}(z,y)\Big| &\lesssim \frac{1}{\omega(B(x,|x-y|))}\frac{|x-y|^{2\alpha+2N}|x-z|^{\delta'}}{|x-y|^{2\alpha+2N+\delta'}}\\&\lesssim \frac{1}{\omega(B(x,|x-y|))}\Big(\frac{|x-z|}{|x-y|}\Big)^{\delta'}.\end{aligned}$$

For (iii), we only need to check the $T1$-condition. Let $B = B(x_0,r)$ with $r < \rho(x_0)$ and $x,z \in B$. Denote by B_ρ the ball $B(x_0,2\rho(x_0))$. Let Θ_{B_ρ} be a smooth function with support in $2B_\rho$ and such that $\Theta_{B_\rho} \equiv 1$ on B_ρ, $0 \le \Theta_{B_\rho} \le 1$. The triangle inequality gives $\sup_{t>0}\left|T^L_{\alpha,t}1(x) - T^L_{\alpha,t}1(z)\right| \le I + II + III$, where

$$\begin{cases} I := \sup\limits_{t>0}\left|(T^L_{\alpha,t} - S_{\alpha,t})\Theta_{B_\rho}(x) - (T^L_{\alpha,t} - S_{\alpha,t})\Theta_{B_\rho}(z)\right|; \\ II := \sup\limits_{t>0}\left|S_{\alpha,t}\Theta_{B_\rho}(x) - S_{\alpha,t}\Theta_{B_\rho}(z)\right|; \\ III := \sup\limits_{t>0}\left|T^L_{\alpha,t}(1-\Theta_{B_\rho})(x) - T^L_{\alpha,t}(1-\Theta_{B_\rho})(z)\right|. \end{cases}$$

For I, write $I \le I_1 + I_2$, where

$$\begin{cases} I_1 := \sup\limits_{t>0}\int_{2B}\left|(h_{\alpha,t}(x,y) - K^L_{\alpha,t}(x,y)) - (h_{\alpha,t}(z,y) - K^L_{\alpha,t}(z,y))\right|\omega(y)dy; \\ I_2 := \sup\limits_{t>0}\int_{2B_\rho\backslash 2B}\left|(h_{\alpha,t}(x,y) - K^L_{\alpha,t}(x,y)) - (h_{\alpha,t}(z,y) - K^L_{\alpha,t}(z,y))\right|\omega(y)dy. \end{cases}$$

We further divide the term I_1 as $I_1 \le I_{11} + I_{12}$, where

$$\begin{cases} I_{11} := \sup\limits_{t>0}\int_{2B}\left|h_{\alpha,t}(x,y) - K^L_{\alpha,t}(x,y)\right|\omega(y)dy; \\ I_{12} := \sup\limits_{t>0}\int_{2B}\left|h_{\alpha,t}(z,y) - K^L_{\alpha,t}(z,y)\right|\omega(y)dy. \end{cases}$$

Since the proof of I_{12} is similar to the case of I_{11}, we only give the proof of I_{11}. Using Proposition 3.6 (i), we consider the following two cases.

If $t^{1/2\alpha} \le |x-y|$, then $t^{1/2\alpha} \le |x-y| \le 3r$ and $\rho(x) \simeq \rho(x_0)$ for $x, z \in B$. Therefore, we obtain

$$\begin{aligned} I_{11} &\lesssim \int_{2B}\left(\frac{|x-y|}{\rho(x)}\right)^{\delta_0}\frac{1}{\omega(B(x,\sqrt{t^{1/\alpha}+|x-y|^2}))}\frac{t}{(t^{1/\alpha}+|x-y|^2)^\alpha}\omega(y)dy \\ &\lesssim \left(\frac{r}{\rho(x_0)}\right)^{\delta_0}\int_{\mathbb{R}^n}\frac{1}{\omega(B(x,\sqrt{t^{1/\alpha}+|x-y|^2}))}\frac{t}{(t^{1/\alpha}+|x-y|^2)^\alpha}\omega(y)dy \\ &\lesssim \left(\frac{r}{\rho(x_0)}\right)^{\delta_0}. \end{aligned}$$

If $t^{1/2\alpha} > |x-y|$, we need to discuss the following two cases:

Case 1: $r \ge t^{1/2\alpha}$. We have

$$\begin{aligned} I_{11} &\lesssim \int_{2B}\left(\frac{t^{1/2\alpha}}{\rho(x)}\right)^{\delta_0}\frac{1}{\omega(B(x,\sqrt{t^{1/\alpha}+|x-y|^2}))}\frac{t}{(t^{1/\alpha}+|x-y|^2)^\alpha}\omega(y)dy \\ &\lesssim \left(\frac{r}{\rho(x_0)}\right)^{\delta_0}\int_{\mathbb{R}^n}\frac{1}{\omega(B(x,\sqrt{t^{1/\alpha}+|x-y|^2}))}\frac{t}{(t^{1/\alpha}+|x-y|^2)^\alpha}\omega(y)dy \end{aligned}$$

$$\lesssim \Big(\frac{r}{\rho(x_0)}\Big)^{\delta_0}.$$

Case 2: $r < t^{1/2\alpha}$. Using the reverse doubling condition, we obtain

$$\begin{aligned} I_{11} &\lesssim \int_{2B} \Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{\delta_0} \frac{1}{\omega(B(x, \sqrt{t^{1/\alpha} + |x-y|^2}))} \omega(y)dy \\ &\lesssim \Big(\frac{t^{1/2\alpha}}{\rho(x_0)}\Big)^{\delta_0} \int_{2B} \frac{1}{\omega(B(x,r))} \Big(\frac{r}{t^{1/2\alpha}}\Big)^{\nu} \omega(y)dy \\ &\lesssim \Big(\frac{r}{\rho(x_0)}\Big)^{\delta_0} \end{aligned}$$

since $\delta_0 < \nu$.

For I_2, applying Proposition 3.6 (ii) with $0 < \delta < \min\{\eta, \delta_0\}$, we obtain

$$I_2 \lesssim \Big(\frac{r}{\rho(x_0)}\Big)^{\delta} \int_{2B_\rho \backslash 2B} \frac{1}{\omega(B(x, \sqrt{t^{1/\alpha} + |x-y|^2}))} \frac{t\omega(y)dy}{(t^{1/\alpha} + |x-y|^2)^{\alpha}} \lesssim \Big(\frac{r}{\rho(x_0)}\Big)^{\delta}.$$

Since $0 < \delta < \delta_0$, combining the above estimates derives $I \lesssim (r/\rho(x_0))^{\delta}$.

Next, we estimate the term II. From Theorem 5.1, we obtain the following stronger inequality:

$$\sup_{t>0} \Big| S_{\alpha,t} f(x) - S_{\alpha,t} f(z) \Big| \lesssim \|f\|_{BMO^d} |x-z|^d.$$

Since Θ_{B_ρ} belongs to $BMO^d(\mathbb{R}^n, \omega)$ and it is also a bounded function, then

$$II \lesssim |x-z|^{\delta} \|\Theta_{B_\rho}\|_{BMO^{\delta}}.$$

It is easy to see that $\|\Theta_{B_\rho}\|_{BMO^{\delta}}$ approximates $c/\rho^{\delta}(x_0)$. Then $II \lesssim (r/\rho(x_0))^{\delta}$.

For III, we apply the smoothness inequality of this theorem to get

$$\begin{aligned} III &\lesssim |x-z|^{\delta'} \int_{B_\rho^c} |x-y|^{-\delta'} \frac{1}{\omega(B(x, |x-y|))} \omega(y)dy \\ &\backsimeq |x-z|^{\delta'} \sum_{k=1}^{\infty} \int_{2^k \rho(x_0) \le |x_0 - y| < 2^{k+1} \rho(x_0)} \frac{|x-y|^{-\delta'}}{\omega(B(x, |x-y|))} \omega(y)dy \\ &\lesssim \Big(\frac{|x-z|}{\rho(x_0)}\Big)^{\delta'} \le C\Big(\frac{r}{\rho(x_0)}\Big)^{\delta'}. \end{aligned}$$

Then we conclude that when $\beta = 0$ and $\epsilon = \delta'$,

$$\frac{1}{\omega^2(B)} \int_B \int_B \Big| T^L_{\alpha,t} 1(x) - T^L_{\alpha,t} 1(z) \Big| \omega(z)dz\omega(x)dx \lesssim \Big(\frac{r}{\rho(x_0)}\Big)^{\delta'},$$

where $0 < \delta' < \delta_1 = \min\{2\alpha, \eta, \delta_0\}$. □

Remark 6.5 It should be mentioned that, generally, we obtain an estimate on $BMO^d_{\mathbb{X},\rho}(\mathbb{R}^n, \omega)$ for $\mathcal{T} f$. Nevertheless, as it is easy to see that $\|T^* f\|_{BMO^d_\rho} \le \|\mathcal{T} f\|_{BMO^d_{\mathbb{X},\rho}}$.

Now we deal with the operator $I^L_{\alpha,\beta}$.

Theorem 6.6 *Let ω be an A_2-weight such that $\omega \in RD_\nu \bigcap D_\gamma$ with $\nu > 2$ and $V \in RH_q(\omega)$ with $q > \gamma/2$. Assume that $\delta_0 = 2 - \gamma/q$ and $\eta \in (0, 1)$.*

(i) *For any $N > 0$, there exists a constant C_N such that*

$$|K^L_{\alpha,\beta}(x, y)| \le C_N \frac{|x-y|^{\alpha\beta}}{\omega(B(x, |x-y|))}\Big(1 + \frac{|x-y|}{\rho(x)}\Big)^{-N}.$$

(ii) *For $|x-y|/4 > |x-z|$ and any $0 < \delta' < \delta_1 = \min\{2\alpha, \eta, \delta_0\}$,*

$$|K^L_{\alpha,\beta}(x, y) - K^L_{\alpha,\beta}(z, y)| \lesssim \Big(\frac{|x-z|}{|x-y|}\Big)^{\delta'} \frac{|x-y|^{\alpha\beta}}{\omega(B(x, |x-y|))}.$$

(iii) *The operator $I^L_{\alpha,\beta}$ is bounded from $BMO^d_\rho(\mathbb{R}^n, \omega)$ into $BMO^{\alpha\beta+d}_\rho(\mathbb{R}^n, \omega)$ for any $d \ge 0$ such that $0 < \alpha\beta + d < \delta_1$.*

Proof For (i), the desired result can be seen from Lemma 4.3. For (ii), according to Proposition 3.5, we can get

$$\begin{aligned} &|K^L_{\alpha,\beta}(x, y) - K^L_{\alpha,\beta}(z, y)| \\ &\le \int_0^\infty |K^L_{\alpha,t}(x, y) - K^L_{\alpha,t}(z, y)| t^{\beta/2} \frac{dt}{t} \\ &\lesssim \int_0^\infty \min\Big\{\frac{t^{1+N/\alpha}|x-z|^{\delta'}}{|x-y|^{2\alpha+2N+\delta'}} \frac{1}{\omega(B(x, |x-y|))}, \frac{1}{\omega(B(x, t^{1/2\alpha}))}\Big(\frac{|x-z|}{t^{1/2\alpha}}\Big)^{\delta'}\Big\} t^{\beta/2} \frac{dt}{t}. \end{aligned}$$

Then for $|x-z| \le |x-y|/4$, we can processes as in the proof of Lemma 5.2 with δ' instead of η to obtain

$$|K^L_{\alpha,\beta}(x, y) - K^L_{\alpha,\beta}(z, y)| \lesssim \Big(\frac{|x-z|}{|x-y|}\Big)^{\delta'} \frac{|x-y|^{\alpha\beta}}{\omega(B(x, |x-y|))}.$$

For (iii), we only need to check the $T1$-condition. We further assume that $\alpha\beta < \delta_1$ and pick δ' such that $\alpha\beta < \delta' < \delta_1$. Let $B = B(x_0, r)$ with $r < \rho(x_0)$ and $x, z \in B$. Then we can write $\Big|I^L_{\alpha,\beta}1(x) - I^L_{\alpha,\beta}1(z)\Big| \le I + II + III$, where

$$\begin{cases} I := \Big|(I^L_{\alpha,\beta} - I_{\alpha,\beta})\Theta_{B_\rho}(x) - (I^L_{\alpha,\beta} - I_{\alpha,\beta})\Theta_{B_\rho}(z)\Big|; \\ II := \Big|I_{\alpha,\beta}\Theta_{B_\rho}(x) - I_{\alpha,\beta}\Theta_{B_\rho}(z)\Big|; \\ III := \Big|I^L_{\alpha,\beta}(1 - \Theta_{B_\rho})(x) - I^L_{\alpha,\beta}(1 - \Theta_{B_\rho})(z)\Big|. \end{cases}$$

For I, we denote $D^L_{\alpha,\beta,t}$ as the kernel of the operator $I^L_{\alpha,\beta} - I_{\alpha,\beta}$. Then we have

$$\int_{2B_\rho} \left|D^L_{\alpha,\beta}(x,y) - D^L_{\alpha,\beta}(z,y)\right|\omega(y)dy = I_1 + I_2,$$

where

$$\begin{cases} I_1 := \int_{2B} \left|D^L_{\alpha,\beta}(x,y) - D^L_{\alpha,\beta}(z,y)\right|\omega(y)dy; \\ I_2 := \int_{2B_\rho\setminus 2B} \left|D^L_{\alpha,\beta}(x,y) - D^L_{\alpha,\beta}(z,y)\right|\omega(y)dy. \end{cases}$$

For I_1, we can write $I_1 \le I_{11} + I_{12}$, where

$$\begin{cases} I_{11} := \int_{2B} \left|D^L_{\alpha,\beta}(x,y)\right|\omega(y)dy; \\ I_{12} := \int_{2B} \left|D^L_{\alpha,\beta}(z,y)\right|\omega(y)dy. \end{cases}$$

Since the proofs of I_{11} and I_{12} are similar, we only give the proof of I_{11}. Firstly observe that

$$|D^L_{\alpha,\beta,t}(x,y)| \le \int_0^\infty \left|K^L_{\alpha,t}(x,y) - h_{\alpha,t}(x,y)\right|t^{\beta/2}\frac{dt}{t} = I_3 + I_4,$$

where

$$\begin{cases} I_3 := \int_0^{|x-y|^{2\alpha}} \left|K^L_{\alpha,t}(x,y) - h_{\alpha,t}(x,y)\right|t^{\beta/2}\frac{dt}{t}; \\ I_4 := \int_{|x-y|^{2\alpha}}^\infty \left|K^L_{\alpha,t}(x,y) - h_{\alpha,t}(x,y)\right|t^{\beta/2}\frac{dt}{t}. \end{cases}$$

For I_3, by Proposition 3.6 (ii) with $0 < \delta < \min\{\eta, \delta_0\}$ and the fact that $\rho(x) \simeq \rho(x_0)$, we obtain

$$\begin{aligned} I_3 &\lesssim \frac{|x-y|^\delta}{\rho(x_0)^\delta} \int_0^{|x-y|^{2\alpha}} \frac{1}{\omega(B(x, \frac{\sqrt{t^{1/\alpha}+|x-y|^2}}{|x-y|}|x-y|))} \frac{t^{\beta/2}dt}{(t^{1/\alpha}+|x-y|^2)^\alpha} \\ &\lesssim \frac{|x-y|^\delta}{\rho(x_0)^\delta\omega(B(x,|x-y|))} \int_0^{|x-y|^{2\alpha}} \frac{|x-y|^\nu}{(t^{1/\alpha}+|x-y|^2)^{\nu/2}} \frac{t^{\beta/2}dt}{(t^{1/\alpha}+|x-y|^2)^\alpha} \\ &\lesssim \frac{|x-y|^{\delta+\alpha\beta}}{\rho(x_0)^\delta\omega(B(x,|x-y|))} \end{aligned}$$

and

$$
\begin{aligned}
I_4 &\lesssim \frac{1}{\rho(x_0)^{\delta}\omega(B(x,|x-y|))}\int_{|x-y|^{2\alpha}}^{\infty}\frac{|x-y|^{\nu}}{(t^{1/\alpha}+|x-y|^2)^{\nu/2}}\frac{t^{\beta/2+\delta/2\alpha}dt}{(t^{1/\alpha}+|x-y|^2)^{\alpha}}\\
&\lesssim \frac{|x-y|^{\delta+\alpha\beta}}{\rho(x_0)^{\delta}\omega(B(x,|x-y|))}\int_1^{\infty}\frac{1}{u^{\nu-\alpha\beta-\delta}}\frac{du}{u}\\
&\lesssim \frac{|x-y|^{\delta+\alpha\beta}}{\rho(x_0)^{\delta}\omega(B(x,|x-y|))},
\end{aligned}
$$

where we have used the fact that $\nu > \alpha\beta + \delta$ since $\nu > 2$, $\alpha\beta < \delta' < 1$ and $\delta < 1$.

Therefore, we get

$$
I_{11} \lesssim \frac{1}{\rho(x_0)^{\delta}}\int_{2B}\frac{|x-y|^{\alpha\beta+\delta}}{\omega(B(x,|x-y|))}\omega(y)dy \lesssim r^{\alpha\beta}\Big(\frac{r}{\rho(x_0)}\Big)^{\delta},
$$

where the last inequality follows by splitting the integral domain into the annulus $2^{-k}B \setminus 2^{-(k+1)}B$ and using the doubling property of ω.

For I_2, by Proposition 3.6 (ii) and $\rho(x) \simeq \rho(x_0)$, we obtain

$$
\begin{aligned}
&\Big|D^L_{\alpha,\beta}(x,y) - D^L_{\alpha,\beta}(z,y)\Big|\\
&\le \int_0^{\infty}\Big|(K^L_{\alpha,t}(x,y) - h_{\alpha,t}(x,y)) - (K^L_{\alpha,t}(z,y) - h_{\alpha,t}(z,y))\Big| t^{\beta/2}\frac{dt}{t}\\
&\lesssim \Big(\frac{|x-z|}{\rho(x_0)}\Big)^{\delta}\int_0^{\infty}\frac{1}{\omega(B(x,\sqrt{t^{1/\alpha}+|x-y|^2}))}\frac{1}{(t^{1/\alpha}+|x-y|^2)^{\alpha}}t^{\beta/2}dt\\
&\lesssim \Big(\frac{r}{\rho(x_0)}\Big)^{\delta}\frac{|x-y|^{\alpha\beta}}{\omega(B(x,|x-y|))}.
\end{aligned}
$$

Then

$$
I_2 \lesssim \Big(\frac{r}{\rho(x_0)}\Big)^{\delta}\int_{2B_\rho}\frac{|x-y|^{\alpha\beta}}{\omega(B(x,|x-y|))}\omega(y)dy \lesssim \Big(\frac{r}{\rho(x_0)}\Big)^{\delta-\alpha\beta} r^{\alpha\beta}.
$$

For II, we use Theorem 5.5 for $d = \delta - \alpha\beta$, that certainly satisfies $\alpha\beta + d < \eta$, and with $f = \Theta_{B_\rho}$. Notice that Θ_{B_ρ} is smooth and compactly supported, so $\ddot{\mathcal{H}}_{\alpha,\beta}\Theta_{B_\rho} = I_{\alpha,\beta}\Theta_{B_\rho}$. Therefore,

$$
II \lesssim |x-z|^{\delta}\|\Theta_{B_\rho}\|_{BMO^{\delta-\alpha\beta}} \lesssim r^{\delta}\rho(x_0)^{\alpha\beta-\delta} \lesssim \Big(\frac{r}{\rho(x_0)}\Big)^{\delta-\alpha\beta} r^{\alpha\beta}.
$$

For III, we use the smoothness of the kernel $K^L_{\alpha,\beta}$ in this theorem to obtain

$$
III \lesssim |x-z|^{\delta'}\int_{B^c_\rho}\frac{|x-y|^{\alpha\beta-\delta'}}{\omega(B(x,|x-y|))}\omega(y)dy \lesssim r^{\delta'}\rho(x_0)^{\alpha\beta-\delta'} \lesssim \Big(\frac{r}{\rho(x_0)}\Big)^{\delta'-\alpha\beta} r^{\alpha\beta}.
$$

Combining all the estimates and having in mind that $r < \rho(x_0)$, we have

$$\left|I^L_{\alpha,\beta}1(x) - I^L_{\alpha,\beta}1(z)\right| \lesssim \left(\frac{r}{\rho(x_0)}\right)^{\delta'-\alpha\beta} r^{\alpha\beta}$$

for any $\alpha\beta < \delta' < \delta_1$.

So for $\epsilon = \delta' - \alpha\beta$, $\alpha\beta < \delta' < \delta_1$, and we have

$$\frac{1}{r^{\alpha\beta}\omega^2(B)} \int_B \int_B \left|I^L_{\alpha,\beta}1(x) - I^L_{\alpha,\beta}1(z)\right| \omega(z)dz\omega(x)dx \lesssim \left(\frac{r}{\rho(x_0)}\right)^{\delta'-\alpha\beta}.$$

Finally, we can deduce the desired result from Proposition 6.2. □

For the operator $I^L_{\alpha,\beta}V^{\sigma/2}$, we assume that V satisfies $RH_\infty(\omega)$, which implies that $V(y) \le \rho^{-2}(y)$.

Theorem 6.7 *Let ω be an A_2-weight such that $\omega \in RD_\nu$ with $\nu > 2$ and $V \in RH_\infty(\omega)$. Given $\alpha\beta$ and σ with $\alpha\beta \ge \sigma > 0$. Assume that $\eta \in (0, 1)$.*

(i) *For any $N > 0$, there exists a constant C_N such that*

$$K^{L,\sigma}_{\alpha,\beta}(x, y) = K^L_{\alpha,\beta}(x, y)V^{\sigma/2}(y) \le C_N \frac{|x - y|^{\alpha\beta-\sigma}}{\omega(B(x, |x - y|))}\left(1 + \frac{|x - y|}{\rho(x)}\right)^{-N}.$$

(ii) *For $|x - y|/4 > |x - z|$ and any $0 < \delta' < \delta_1 = \min\{2\alpha, \eta\}$,*

$$|K^{L,\sigma}_{\alpha,\beta}(x, y) - K^{L,\sigma}_{\alpha,\beta}(z, y)| \lesssim \left(\frac{|x - z|}{|x - y|}\right)^{\delta'} \frac{|x - y|^{\alpha\beta-\sigma}}{\omega(B(x, |x - y|))}.$$

(iii) *The operator $I^L_{\alpha,\beta}V^{\sigma/2}$ is bounded from $BMO^d_\rho(\mathbb{R}^n, \omega)$ into $BMO^{d+\alpha\beta-\sigma}_\rho(\mathbb{R}^n, \omega)$ for any $0 \le d \le \sigma$ such that $0 < d + \alpha\beta - \sigma < \delta_1$.*

Proof For (i), when $\alpha\beta \ge \sigma$, in view of (4.3), multiplying and dividing by $|x - y|^\sigma$ and using the decay, we have

$$K^{L,\sigma}_{\alpha,\beta}(x, y) = K^L_{\alpha,\beta}(x, y)V^{\sigma/2}(y) \le C_N \frac{|x - y|^{\alpha\beta-\sigma}}{\omega(B(x, |x - y|))}\left(1 + \frac{|x - y|}{\rho(x)}\right)^{-N}.$$

For (ii), we want to use the smoothness of $K^L_{\alpha,\beta}$, but we first need an improved version of Theorem 6.6 (ii), involving decay at infinity. Notice that for any pair of numbers u and v, we have $|u - v| \le (|u| + |v|)^\theta |u - v|^{1-\theta}$ for any fixed $0 < \theta < 1$. In our case, set $u = K^L_{\alpha,\beta}(x, y)$ and $v = K^L_{\alpha,\beta}(z, y)$. We recall that, by Lemma 4.3(ii),

$$0 \le K^L_{\alpha,\beta}(z, y) \le C_N \frac{|z - y|^{\alpha\beta}}{\omega(B(z, |z - y|))}\left(1 + \frac{|z - y|}{\rho(z)}\right)^{-N}$$

for any positive N and our aim is to check that we may replace z by x on the right hand side provided $|x-z| \le 1/2|x-y|$, so u and v have the same bound. To do so observe that in such case $|x-y| \simeq |z-y|$ and the doubling property of ω gives $\omega(B(x,|x-y|)) \simeq \omega(B(z,|z-y|))$. Besides, from (2.2) and using $|x-z| \le 1/2|x-y|$ again, we get

$$\frac{1}{\rho(z)} \gtrsim \frac{1}{\rho(x)}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N_0/(N_0+1)}.$$

So multiplying by $|z-y|$ and adding the inequality $1 \ge \left(1+\frac{|x-y|}{\rho(x)}\right)^{-N_0/(N_0+1)}$, we have

$$\left(1+\frac{|z-y|}{\rho(z)}\right) \gtrsim \left(1+\frac{|x-y|}{\rho(x)}\right)^{-N_0/(N_0+1)} + \frac{|z-y|}{\rho(x)}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N_0/(N_0+1)}.$$

Then

$$\left(1+\frac{|z-y|}{\rho(z)}\right)^{-N} \lesssim \left(1+\frac{|x-y|}{\rho(x)}\right)^{-\tilde{N}}$$

with $\tilde{N} = N - NN_0/(N_0+1)$, which derives the desired result.

Therefore, inserting the estimates for $|u-v|$ and $|u|+|v|$, we obtain

$$\left|K^L_{\alpha,\beta}(x,y) - K^L_{\alpha,\beta}(z,y)\right| \lesssim \left(\frac{|x-z|}{|x-y|}\right)^{\delta'(1-\theta)} \frac{|x-y|^{\alpha\beta}}{\omega(B(x,|x-y|))}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-\theta N}.$$

Since Theorem 6.6 (ii) is valid for any $\delta' < \delta_1$, by choosing θ small enough and N sufficiently large, we get

$$\left|K^L_{\alpha,\beta}(x,y) - K^L_{\alpha,\beta}(z,y)\right| \lesssim \left(\frac{|x-z|}{|x-y|}\right)^{\delta'} \frac{|x-y|^{\alpha\beta}}{\omega(B(x,|x-y|))}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N} \tag{6.1}$$

for any $0 < \delta' < \delta_1$. Using the previous arguments, we easily obtain the smoothness for $K^{L,\sigma}_{\alpha,\beta}$ in view of the inequality

$$\left|K^{L,\sigma}_{\alpha,\beta}(x,y) - K^{L,\sigma}_{\alpha,\beta}(z,y)\right| \le \left|K^L_{\alpha,\beta}(x,y) - K^L_{\alpha,\beta}(z,y)\right| V^{\sigma/2}(y)$$

and

$$V^{\sigma/2}(y) \lesssim \rho(y)^{-\sigma} \lesssim \rho(x)^{-\sigma}\left(1+\frac{|x-y|}{\rho(x)}\right)^{N_0} \lesssim |x-y|^{-\sigma}\left(1+\frac{|x-y|}{\rho(x)}\right)^{N_0+\sigma}. \tag{6.2}$$

Therefore, choosing $N = N_0 + \sigma$, it follows from the last three inequalities that

$$\left|K_{\alpha,\beta}^{L,\sigma}(x,y) - K_{\alpha,\beta}^{L,\sigma}(z,y)\right| \lesssim \left(\frac{|x-z|}{|x-y|}\right)^{\delta'} \frac{|x-y|^{\alpha\beta-\sigma}}{\omega(B(x,|x-y|))} \tag{6.3}$$

for $0 < \delta' < \delta_1$.

For (iii), we need to check the $T1$-condition. Let $B = B(x_0, r)$ with $r < \rho(x)$ and $x, z \in B$. Then we have

$$\left|I_{\alpha,\beta}^{L} V^{\sigma/2} 1(x) - I_{\alpha,\beta}^{L} V^{\sigma/2} 1(z)\right| \le I + II,$$

where

$$\begin{cases} I := \left|I_{\alpha,\beta}^{L}(V^{\sigma/2}\chi_{5B})(x) - I_{\alpha,\beta}^{L}(V^{\sigma/2}\chi_{5B})(z)\right|; \\ II := \int_{(5B)^c} \left|K_{\alpha,\beta}^{L,\sigma}(x,y) - K_{\alpha,\beta}^{L,\sigma}(z,y)\right| \omega(y)dy. \end{cases}$$

We first estimate I. Notice that in the proof of Theorem 5.3, we just use the size and smoothness of the kernel and the doubling property of the weight. Therefore, a more general result could be obtained for an integral operator with kernel satisfying (5.2) and (4.1). So $I_{\alpha,\beta}^{L}$ can be extended to a bounded operator from $M_1^{\alpha\beta-d}(\mathbb{R}^n, \omega)$ into $BMO^d(\mathbb{R}^n, \omega)$ for $d < \min\{\alpha\beta, \delta_1\}$ since its kernel satisfies the appropriate size and smoothness estimates (see Lemma 4.3(ii) and Theorem 6.6(ii)). Moreover, $V^{\sigma/2}\chi_{5B}$ is a function in $M_1^{\alpha\beta-d}(\mathbb{R}^n, \omega)$ and is compactly supported. In fact, if we take any ball $Q = B(x_1, s)$, then

$$\frac{s^{\alpha\beta-d}}{\omega(Q)} \int_Q V^{\sigma/2}(y)\chi_{5B}(y)\omega(y)dy \lesssim s^{\alpha\beta-d} \rho(x_0)^{-\sigma} \frac{\omega(Q \bigcap 5B)}{\omega(Q)}.$$

Assume $Q \bigcap 5B \neq \emptyset$. If $s \le 5r$, we control the above quantity by $cr^{\alpha\beta-d}\rho(x_0)^{-\sigma}$, having in mind that $\rho(y) \simeq \rho(x_0)$ for $y \in 5B$. Otherwise, $|x_1 - x_0| \le 2s$ and also $Q \subset \tilde{B} = B(x_0, 3s) \subset 5Q$. Since ω is doubling and ν-reverse doubling, so we have

$$\omega(Q) \simeq \omega(\tilde{B}) \ge (s/r)^{\nu}\omega(B) \gtrsim (s/r)^{\alpha\beta-d}\omega(B),$$

which, together with the obvious inequality $\omega(Q \bigcap 5B) \le \omega(5B)$, gives the bound $r^{\alpha\beta-d}\rho(x_0)^{-\sigma}$ when $5r \le s$. In a word, we conclude that $\|V^{\sigma/2}\chi_{5B}\|_{M_1^{\alpha\beta-d}} \lesssim r^{\alpha\beta-d}\rho(x_0)^{-\sigma}$.

Thus we obtain a similar conclusion to Theorem 5.3 for

$$\widetilde{\mathcal{H}}_{\alpha,\beta}^{L}(f)(x) = \int_{\mathbb{R}^n} (K_{\alpha,\beta}^{L}(x,y) - K_{\alpha,\beta}^{L}(0,y)\chi_{B_1^c})(f)(y)\omega(y)dy.$$

By analogy with the conclusion of Theorem 5.3, we can see that

$$I^L_{\alpha,\beta}(V^{\sigma/2}\chi_{5B})(x)=\widetilde{\mathcal{H}}^L_{\alpha,\beta}(V^{\sigma/2}\chi_{5B})(x).$$

Going back to the estimate for I, it follows that

$$I\le 2|I^L_{\alpha,\beta}(V^{\sigma/2}\chi_{5B})(x)|=2\widetilde{\mathcal{H}}^L_{\alpha,\beta}(V^{\sigma/2}\chi_{5B})(x).$$

Similarly to the discussion of Theorem 5.3, we split $\widetilde{\mathcal{H}}^L_{\alpha,\beta}(V^{\sigma/2}\chi_{5B})(x)$ that $\widetilde{\mathcal{H}}^L_{\alpha,\beta}(V^{\sigma/2}\chi_{5B})(x)=G_1(x)+G_2(x)+a_B$, where

$$\begin{cases} G_1(x):=\displaystyle\int_{B(x_0,2r)}K^L_{\alpha,\beta}(x,y)(V^{\sigma/2}\chi_{5B})(y)\omega(y)dy;\\ G_2(x):=\displaystyle\int_{B(x_0,2r)^c}(K^L_{\alpha,\beta}(x,y)-K^L_{\alpha,\beta}(x_0,y))(V^{\sigma/2}\chi_{5B})(y)\omega(y)dy;\\ \quad a_B:=\displaystyle\int_{\mathbb{R}^n}(K^L_{\alpha,\beta}(x_0,y)\chi_{B(x_0,2r)^c}-K^L_{\alpha,\beta}(0,y)\chi_{B_1^c})(V^{\sigma/2}\chi_{5B})(y)\omega(y)dy. \end{cases}$$

For a_B, similarly to the proof in Theorem 5.3, we know that a_B is a finite constant. For $G_1(x)$, since $x\in B(x_0,r)$ and $y\in B(x_0,2r)$, we have $|x-y|\le 3r$. By Lemma 4.3, we obtain

$$\begin{aligned} |G_1(x)|&\le\int_{B(x,3r)}|K^L_{\alpha,\beta}(x,y)||(V^{\sigma/2}\chi_{5B})(y)|\omega(y)dy\\ &\lesssim\int_{B(x,3r)}\frac{|x-y|^{\alpha\beta}}{\omega(B(x,|x-y|))}|(V^{\sigma/2}\chi_{5B})(y)|\omega(y)dy\\ &\lesssim\sum_{j=-\infty}^{0}\int_{2^{j+1}r<|x-y|\le 2^{j+2}r}\frac{(2^jr)^{\alpha\beta}}{\omega(B(x,2^jr))}|(V^{\sigma/2}\chi_{5B})(y)|\omega(y)dy\\ &\lesssim r^d\|V^{\sigma/2}\chi_{5B}\|_{M_1^{\alpha\beta-d}(\mathbb{R}^n,\omega)}. \end{aligned}$$

Therefore, we only need to control the integral of $|G_2(x)|$, whose proof is the same as the one in Theorem 5.3. By (6.1), we obtain

$$\begin{aligned} |G_2(x)|&\lesssim|x_0-x|^{\delta'}\int_{B(x_0,2r)^c}\frac{|x_0-y|^{\alpha\beta-\delta'}}{\omega(B(x_0,|x_0-y|))}|(V^{\sigma/2}\chi_{5B})(y)|\omega(y)dy\\ &\lesssim\|V^{\sigma/2}\chi_{5B}\|_{M_1^{\alpha\beta-d}(\mathbb{R}^n,\omega)}r^d. \end{aligned}$$

In conclusion, we deduce that

$$I\lesssim\|V^{\sigma/2}\chi_{5B}\|_{M_1^{\alpha\beta-d}(\mathbb{R}^n,\omega)}r^d\lesssim r^{\alpha\beta-d}\rho(x_0)^{-\sigma}r^d\simeq r^{\alpha\beta-\sigma}\left(\frac{r}{\rho(x_0)}\right)^{\sigma}.$$

For II, we may use the smoothness of the kernel due to the fact that in our situation $|x-y| \geq 4r \geq 2|x-z|$, but instead of (6.3) we will use a somehow stronger variant, namely,

$$\left|K_{\alpha,\beta}^{L,\sigma}(x,y) - K_{\alpha,\beta}^{L,\sigma}(z,y)\right| \lesssim \frac{1}{\rho(x_0)^\sigma}\left(\frac{|x-z|}{|x-y|}\right)^{\delta'} \frac{|x-y|^{\alpha\beta}}{\omega(B(x,|x-y|))},$$

which can be seen from (6.1) and (6.2) just stopping before the last inequality and using that, in our case, $\rho(x) \simeq \rho(x_0)$. Plugging that estimate into II, we obtain

$$II \lesssim \frac{|x-z|^{\delta'}}{\rho(x_0)^\sigma} \int_{(5B)^c} \frac{|x-y|^{\alpha\beta-\delta'}}{\omega(B(x,|x-y|))}\omega(y)dy.$$

Since the integral in bounded by $Cr^{\alpha\beta-\delta'}$, we get

$$II \lesssim \frac{r^{\alpha\beta}}{\rho(x_0)^\sigma} \lesssim r^{\alpha\beta-\sigma}\left(\frac{r}{\rho(x_0)}\right)^\sigma,$$

which is the same estimate that we obtain for the first term.

In this way we have shown that $T1$-condition holds with $\epsilon = \sigma$. Collecting estimates, we have proved that $I_{\alpha,\beta}^L V^{\sigma/2}$ is an $(\alpha\beta-\sigma)$-Schrödinger-Calderón-Zygmund operator with respect to ωdx and has the smoothness of order δ' for any $0 < \delta' < \delta_1$. Therefore, an application of Proposition 6.2 gives the desired result. □

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Declarations

Conflict of interest The authors declare that there are no conflict of interests.

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