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Static perfect fluid spacetimes on GRW spacetimes

Uday Chand De[1] · **Arpan Sardar**[2]

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Abstract
The present article deals with static perfect fluid spacetimes on generalized Robertson-Walker spacetimes. At first, we show that a generalized Robertson-Walker spacetime with constant scalar curvature as the spatial factor of a static perfect fluid spacetime becomes a perfect fluid spacetime. Next, we prove that under certain restrictions a generalized Robertson-Walker spacetime turns into an Einstein spacetime. As a consequence it is shown that such a spacetime is of Petrov type I, D or O and in case of 4-dimension, the spacetime turns into a Robertson-Walker spacetime.

Keywords Static perfect fluid spacetime · Generalized Robertson-Walker spacetimes · GRW spacetimes · Perfect fluid spacetimes

Mathematics Subject Classification 53C25 · 53C50 · 53Z05

1 Introduction

Let $\tilde{\mathcal{M}}^{n+1} = \mathcal{M}^n \times_f \mathbb{R}$ be a static spacetime with the metric

$$\tilde{g} = g - f^2 dt^2, \tag{1}$$

where $\mathcal{M}^n$ is a Lorentzian manifold with the Lorentzian metric g, $f(> 0)$ being a smooth function on $\mathcal{M}^n$. The Einstein's equations with perfect fluid (PF) as a matter

✉ Uday Chand De
uc_de@yahoo.com

Arpan Sardar
arpansardar51@gmail.com

1 Department of Pure Mathematics, University of Calcutta, 35, B.C. Road, Kolkata, West Bengal 700019, India

2 Department of Mathematics, University of Kalyani, Kalyani, West Bengal 741235, India

field is given by

$$\tilde{\mathcal{S}} - \frac{\tilde{r}}{2}\tilde{g} = -\boldsymbol{\sigma} f^2 dt^2 - \boldsymbol{\rho} g, \tag{2}$$

$\tilde{\mathcal{S}}$ and $\tilde{r}$ indicate the Ricci tensor and the scalar curvature of $\tilde{\mathcal{M}}^{n+1}$, respectively. The smooth functions $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ are respectively named isotropic pressure and energy density of the PF. In modern physics, the static perfect fluid (SPF) spacetime is a fascinating area for study. SPF spacetimes have been studied in [11, 20, 22].

Definition 1 The Lorentzian manifold $\mathcal{M}^n$ is called the spatial factor of a SPF spacetime if the following equations hold:

$$f\mathring{\mathcal{S}} = \mathring{\nabla}^2 f \tag{3}$$

and

$$\triangle f = \left(\frac{n-2}{2(n-1)}r + \frac{n}{n-1}\boldsymbol{\rho}\right) f, \tag{4}$$

where $\mathring{\mathcal{S}} = \mathcal{S} - \frac{r}{n}g$ and $\mathring{\nabla}^2 f = \nabla^2 f - \frac{\triangle f}{n}g$, $\mathcal{S}$ and r indicate the Ricci tensor and the scalar curvature, respectively of $\mathcal{M}^n$.

Remark 1 If $\boldsymbol{\rho} = r = 0$ in (3) and (4), we provide the famous static vacuum Einstein equations:

$$f\mathcal{S} = \nabla^2 f, \ \ \triangle f = 0.$$

Static vacuum Einstein spaces are the particular case of SPF spacetimes. These equations have been investigated by several researchers in theory of relativity [4, 5, 17, 22].

Remark 2 If we take $\boldsymbol{\rho} = -\frac{r}{2}$ in (4), then it is seen

$$\triangle f = -\frac{r}{n-1} f.$$

Substituting this in (3), we obtain the static equation [3]:

$$-\triangle f g + \nabla^2 f - f\mathcal{S} = 0. \tag{5}$$

Equation (5) is named Fischer-Marsden equation [15].

Leandro and Solorzano [24] characterized a half locally conformally flat Riemannian manifold of dimension four obeying (3). More recently, Leandro et al. [23] proved that if the smooth function f fulfills (3) and (4), then isotropic pressure and energy density vanish on the boundary of a Riemannian manifold.

Recently, some researchers studied SPF spacetimes in contact geometry [10], paracontact geometry [32] and almost Kenmotsu manifolds [21] and many others.

Generalized Robertson-Walker (GRW) spacetimes were established by Alias et al. [1, 2]. A GRW spacetime can be written as a warped product $\mathcal{M} = -I \times \varrho^2 \mathcal{M}^*$, dimension of $\mathcal{M}^*$ is $(n-1)$ and $\varrho > 0$ is a scalar function. ϱ is said to be a scale factor or warping function. If the dimension of $\mathcal{M}^*$ is three and of constant curvature, then the spacetime turns into a Robertson-Walker (RW) spacetime. Hence, the GRW spacetime is a natural extension of RW spacetime on which the standard cosmology is modeled. Several researchers [8, 9, 16, 27, 33] and many others have studied GRW spacetimes.

Yano [36] introduced the notion of a torse-forming vector field. A torse-forming vector field on a Lorentzian manifold $(\mathcal{M}^n)$ is defined by

$$\nabla_{X_1} V_1 = \phi X_1 + \pi(X_1) V_1 \tag{6}$$

for all vector field X_1, ϕ is a scalar and π being a 1-form. V_1 is named a concircular vector field [14] for $\pi = 0$.

A unit timelike torse-forming vector field V_1 satisfies the relation:

$$\nabla_{X_1} V_1 = \phi[X_1 + \eta(X_1) V_1]. \tag{7}$$

$\mathcal{M}$ is called a PF spacetime if its non-zero Ricci tensor $\mathcal{S}$ satisfies

$$\mathcal{S} = \alpha_1 g + \alpha_2 \eta \otimes \eta, \tag{8}$$

where α_1, α_2 are scalar fields (not simultaneously zero), ζ is defined by $g(X_1, \zeta) = \eta(X_1)$ for all X_1 and $g(\zeta, \zeta) = -1$. Here ζ is a unit timelike vector (also, named the velocity vector) of the PF spacetime. Every RW spacetime is a PF spacetime [31]. For $n = 4$, the GRW spacetime becomes a PF spacetime if and only if it is a RW spacetime. Several researchers have investigated the physical and geometrical properties of PF spacetimes [6, 13, 34] and many others.

Mantica et al. [28] proved that in a perfect fluid spacetime

$$\alpha_1 = \frac{\boldsymbol{\kappa}(\boldsymbol{\rho} - \boldsymbol{\sigma})}{2-n}, \quad \alpha_2 = \boldsymbol{\kappa}(\boldsymbol{\rho} + \boldsymbol{\sigma}). \tag{9}$$

In a PF spacetime if the state equation $\boldsymbol{\rho} = \boldsymbol{\rho}(\boldsymbol{\sigma})$, then it is said to be isentropic [17]. In particular if $\boldsymbol{\rho} = \boldsymbol{\sigma}$, the PF is named as stiff matter [35].

Chen [8] and Mantica-Molinari [27] have proven the subsequent theorems.

Theorem 1 ([8]) *An n-dimensional ($n \geq 3$) Lorentzian manifold is a GRW-spacetime if and only the manifold admits a timelike concircular vector field.*

Theorem 2 ([27]) *An n-dimensional ($n \geq 3$) Lorentzian manifold is a GRW spacetime if and only if it admits a unit timelike torse-forming vector field, that is also an eigenvector of the Ricci tensor.*

In [27], Mantica and Molinari proved that a GRW spacetime with $div\, C = 0$ (C and 'div' denote the Weyl tensor and divergence, respectively) is a PF spacetime. Here we replace the curvature condition $div\, C = 0$ by SPF spacetime whose spatial factor is the GRW spacetime and acquires the same result. Precisely, we establish that

Theorem 3 *If a GRW spacetime with constant scalar curvature as the spatial factor of a SPF spacetime, then it becomes a perfect fluid spacetime.*

Corollary 1 *A GRW spacetime with constant scalar curvature as the spatial factor of a SPF spacetime represents a stiff matter fluid, provided* $\phi(\zeta f) + \beta = 0$.

Theorem 4 *If a GRW spacetime with constant scalar curvature as the spatial factor of a SPF spacetime, then it becomes an Einstein spacetime, provided* $-\frac{\phi(\zeta f)+\beta}{f}$ *and* $-\frac{f(n-1)(\zeta\phi+\phi^2)+\beta+\phi(\zeta f)}{f}$ *are constants.*

Corollary 2 *Under the same hypothesis in the above theorem, the spacetime is of Petrov type* I, D *or* O *and in case of* 4*-dimension, the spacetime turns into a RW spacetime.*

2 Preliminaries

In this section we prove some propositions for later use.

Proposition 5 *In a GRW spacetime the curvature tensor and Ricci tensor satisfy the relations:*

$$\mathcal{R}(X_1, Y_1)\zeta = (\zeta\phi + \phi^2)[\eta(Y_1)X_1 - \eta(X_1)Y_1] \tag{10}$$

and

$$\mathcal{S}(X_1, \zeta) = (n-1)(\zeta\phi + \phi^2)\eta(X_1). \tag{11}$$

Proof Let us assume that spacetime admit a unit torse-forming vector field ζ. Then from (7), we infer

$$\nabla_{X_1}\zeta = \phi[X_1 + \eta(X_1)\zeta], \tag{12}$$

ϕ is a scalar. Also, from Theorem 2, we have

$$\mathcal{S}(X_1, \zeta) = \psi\eta(X_1), \tag{13}$$

ψ is a non-zero eigenvector.

Now, differentiating (12), we acquire

$$\begin{aligned}\nabla_{Y_1}\nabla_{X_1}\zeta &= (Y_1\phi)[X_1 + \eta(X_1)\zeta] + \phi[\nabla_{Y_1}X_1 + (\nabla_{Y_1}\eta(X_1))\zeta \\ &\quad + \phi(Y_1 + \eta(Y_1)\zeta)\eta(X_1)].\end{aligned} \tag{14}$$

Interchanging X_1 and Y_1 from the above equation, we get

$$\nabla_{X_1}\nabla_{Y_1}\zeta = (X_1\phi)[Y_1 + \eta(Y_1)\zeta] + \phi[\nabla_{X_1}Y_1 + (\nabla_{X_1}\eta(Y_1))\zeta + \phi(X_1 + \eta(X_1)\zeta)\eta(Y_1)]. \tag{15}$$

From equation (12), we provide

$$\nabla_{[X_1,Y_1]}\zeta = \phi\{[X_1, Y_1] + \eta([X_1, Y_1])\zeta\}. \tag{16}$$

Equations (14), (15) and (16) together imply

$$\mathcal{R}(X_1, Y_1)\zeta = (X_1\phi)[Y_1 + \eta(Y_1)\zeta] - (Y_1\phi)[X_1 + \eta(X_1)\zeta] + \phi^2[\eta(Y_1)X_1 - \eta(X_1)Y_1]. \tag{17}$$

Contracting Y_1 from (17), we get

$$\mathcal{S}(X_1, \zeta) = (2-n)(X_1\phi) + (\zeta\phi)\eta(X_1) + (n-1)\phi^2\eta(X_1). \tag{18}$$

Comparing (13) and (18), we infer

$$\psi\eta(X_1) = (2-n)(X_1\phi) + (\zeta\phi)\eta(X_1) + (n-1)\phi^2\eta(X_1). \tag{19}$$

Setting $X_1 = \zeta$ in (19) entails that

$$\psi = (n-1)(\zeta\phi + \phi^2). \tag{20}$$

From the above two equations, we get

$$X_1\phi = -(\zeta\phi)\eta(X_1). \tag{21}$$

Using (21) in (17), we infer

$$\mathcal{R}(X_1, Y_1)\zeta = (\zeta\phi + \phi^2)[\eta(Y_1)X_1 - \eta(X_1)Y_1]. \tag{22}$$

In view of (13) and (20), we get

$$\mathcal{S}(X_1, \zeta) = (n-1)(\zeta\phi + \phi^2)\eta(X_1). \tag{23}$$

This finishes the proof. □

Proposition 6 *In a GRW spacetime, the relation*

$$\{X_1(\zeta\phi + \phi^2)\} + \{\zeta(\zeta\phi + \phi^2)\}\eta(X_1) = 0, \tag{24}$$

holds.

Proof From (10), we get

$$\mathcal{R}(X_1, Y_1)\zeta = (\zeta\phi + \phi^2)[\eta(Y_1)X_1 - \eta(X_1)Y_1]. \tag{25}$$

Now,

$$\begin{aligned}(\nabla_{Z_1} R)(X_1, Y_1)\zeta &= \nabla_{Z_1}\mathcal{R}(X_1, Y_1)\zeta - \mathcal{R}(\nabla_{Z_1}X_1, Y_1)\zeta \\ &\quad -\mathcal{R}(X_1, \nabla_{Z_1}Y_1)\zeta - \mathcal{R}(X_1, Y_1)\nabla_{Z_1}\zeta.\end{aligned} \tag{26}$$

Using (12) and (10) in (26) entails that

$$\begin{aligned}(\nabla_{Z_1} R)(X_1, Y_1)\zeta &= \{Z_1(\zeta\phi + \phi^2)\}[\eta(Y_1)X_1 - \eta(X_1)Y_1] \\ &\quad +\phi(\zeta\phi + \phi^2)[g(Y_1, Z_1)X_1 - g(X_1, Z_1)Y_1] - \phi\mathcal{R}(X_1, Y_1)Z_1.\end{aligned} \tag{27}$$

From the second Bianchi identity, we obtain

$$(\nabla_{Z_1} R)(X_1, Y_1)\zeta + (\nabla_{X_1} R)(Y_1, Z_1)\zeta + (\nabla_{Y_1} R)(Z_1, X_1)\zeta = 0. \tag{28}$$

From the above two equations, we infer

$$\begin{aligned}&[\{Z_1(\zeta\phi + \phi^2)\}\eta(Y_1) - \{Y_1(\zeta\phi + \phi^2)\}\eta(Z_1)]X_1 \\ &\quad +[\{X_1(\zeta\phi + \phi^2)\}\eta(Z_1) - \{Z_1(\zeta\phi + \phi^2)\}\eta(X_1)]Y_1 \\ &\quad +[\{Y_1(\zeta\phi + \phi^2)\}\eta(X_1) - \{X_1(\zeta\phi + \phi^2)\}\eta(Y_1)]Z_1 \\ &\quad -\phi[\mathcal{R}(X_1, Y_1)Z_1 + \mathcal{R}(Y_1, Z_1)X_1 + \mathcal{R}(Z_1, X_1)Y_1] = 0.\end{aligned} \tag{29}$$

Putting $Z_1 = \zeta$ in the foregoing equation gives

$$\begin{aligned}&[\{\zeta(\zeta\phi + \phi^2)\}\eta(Y_1) + \{Y_1(\zeta\phi + \phi^2)\}]X_1 \\ &\quad -[\{X_1(\zeta\phi + \phi^2)\} + \{\zeta(\zeta\phi + \phi^2)\}\eta(X_1)]Y_1 \\ &\quad +[\{Y_1(\zeta\phi + \phi^2)\}\eta(X_1) - \{X_1(\zeta\phi + \phi^2)\}\eta(Y_1)]\zeta \\ &\quad -\phi[\mathcal{R}(X_1, Y_1)\zeta + \mathcal{R}(Y_1, \zeta)X_1 + \mathcal{R}(\zeta, X_1)Y_1] = 0.\end{aligned} \tag{30}$$

From (10), we get

$$\mathcal{R}(\zeta, X_1)Y_1 = (\zeta\phi + \phi^2)[g(X_1, Y_1)\zeta - \eta(Y_1)X_1] \tag{31}$$

and

$$\mathcal{R}(X_1, \zeta)Y_1 = (\zeta\phi + \phi^2)[\eta(Y_1)X_1 - g(X_1, Y_1)\zeta]. \tag{32}$$

Using (10), (31) and (32) in (30) entails that

$$\begin{aligned}&\{\zeta(\zeta\phi + \phi^2)\}[\eta(Y_1)X_1 - \eta(X_1)Y_1] + \{Y_1(\zeta\phi + \phi^2)\}[X_1 + \eta(X_1)\zeta] \\ &\quad -\{X_1(\zeta\phi + \phi^2)\}[Y_1 + \eta(Y_1)\zeta] = 0.\end{aligned} \tag{33}$$

Contracting Y_1 from (33), we infer

$$\{X_1(\zeta\phi + \phi^2)\} + \{\zeta(\zeta\phi + \phi^2)\}\eta(X_1) = 0. \tag{34}$$

Hence the proof is completed. □

Proposition 7 *If* $(\mathcal{M}^n, g)$ *is a SPF spacetime, then*

$$\begin{aligned}\mathcal{R}(X_1, Y_1)Df &= (X_1 f)\mathcal{Q}Y_1 - (Y_1 f)\mathcal{Q}X_1 + f[(\nabla_{X_1}\mathcal{Q})Y_1 - (\nabla_{Y_1}\mathcal{Q})X_1] \\ &\quad +(X_1\boldsymbol{\beta})Y_1 - (Y_1\boldsymbol{\beta})X_1,\end{aligned} \tag{35}$$

where $\boldsymbol{\beta} = -\frac{(r-2\boldsymbol{\rho})}{2(n-1)} f$.

Proof A simple calculation from (2) and (3), we get $\boldsymbol{\sigma} = \frac{r}{2}$ [20, 23]. Hence equations (3) and (4) implies

$$fS = \frac{r - 2\boldsymbol{\rho}}{2(n-1)} fg + \nabla^2 f. \tag{36}$$

Using $\boldsymbol{\beta} = -\frac{(r-2\boldsymbol{\rho})}{2(n-1)} f$ in (36), we get

$$\nabla_{X_1} Df = f\mathcal{Q}X_1 + \boldsymbol{\beta}X_1. \tag{37}$$

Taking covariant derivative of the above equation and after some calculations, we get (35).

This completes the proof. □

3 Main results

3.1 Proof of the Theorem 3

Let a GRW spacetime be the spatial factor of a SPF spacetime. Then from (35), we get

$$\begin{aligned}\mathcal{R}(X_1, Y_1)Df &= (X_1 f)\mathcal{Q}Y_1 - (Y_1 f)\mathcal{Q}X_1 + f[(\nabla_{X_1}\mathcal{Q})Y_1 - (\nabla_{Y_1}\mathcal{Q})X_1] \\ &\quad +(X_1\beta)Y_1 - (Y_1\beta)X_1.\end{aligned} \tag{38}$$

Considering the inner product of (38) with ζ, we obtain

$$\begin{aligned}g(\mathcal{R}(X_1, Y_1)Df, \zeta) &= (X_1 f)\mathcal{S}(Y_1, \zeta) - (Y_1 f)\mathcal{S}(X_1, \zeta) \\ &\quad + f[g((\nabla_{X_1}\mathcal{Q})Y_1, \zeta) - g((\nabla_{Y_1}\mathcal{Q})X_1, \zeta)] \\ &\quad +(X_1\beta)\eta(Y_1) - (Y_1\beta)\eta(X_1).\end{aligned} \tag{39}$$

We assume that $r = constant$. In [11], the authors proved that r is constant if and only if $(\frac{1}{2}r + \boldsymbol{\rho})f$ is constant(=c). Hence, $\boldsymbol{\beta} = -\frac{(r-2\boldsymbol{\rho})}{2(n-1)}f$ implies

$$X_1\beta = -\frac{r}{n-1}(X_1 f). \tag{40}$$

From (11), we acquire

$$\mathcal{Q}\zeta = (n-1)(\zeta\phi + \phi^2)\zeta. \tag{41}$$

The above equation implies

$$\begin{aligned}(\nabla_{X_1}\mathcal{Q})\zeta = {} & (n-1)\{X_1(\zeta\phi + \phi^2)\}\zeta + (n-1)\phi(\zeta\phi + \phi^2)[X_1 + \eta(X_1)\zeta] \\ & -\phi\mathcal{Q}X_1 - (n-1)\phi(\zeta\phi + \phi^2)\eta(X_1)\zeta. \end{aligned} \tag{42}$$

Using (11) and (42) in (39) reveals that

$$\begin{aligned} -g(\mathcal{R}(X_1, Y_1)\zeta, Df) = {} & (n-1)(\zeta\phi + \phi^2)[(X_1 f)\eta(Y_1) - (Y_1 f)\eta(X_1)] \\ & +(n-1)f[\{X_1(\zeta\phi + \phi^2)\}\eta(Y_1) - \{Y_1(\zeta\phi + \phi^2)\}\eta(X_1)] \\ & -\frac{r}{n-1}[(X_1 f)\eta(Y_1) - (Y_1 f)\eta(X_1)]. \end{aligned} \tag{43}$$

Using (10) in the above equation entails that

$$\begin{aligned} & (\zeta\phi + \phi^2)[(Y_1 f)\eta(X_1) - (X_1 f)\eta(Y_1)] \\ & = \left\{(n-1)(\zeta\phi + \phi^2) - \frac{r}{n-1}\right\}[(X_1 f)\eta(Y_1) - (Y_1 f)\eta(X_1)] \\ & \quad +(n-1)f[\{X_1(\zeta\phi + \phi^2)\}\eta(Y_1) - \{Y_1(\zeta\phi + \phi^2)\}\eta(X_1)]. \end{aligned} \tag{44}$$

Setting $Y_1 = \zeta$ in (44) and using (24), we find

$$\left\{n(\zeta\phi + \phi^2) - \frac{r}{n-1}\right\}[X_1 f + (\zeta f)\eta(X_1)] = 0, \tag{45}$$

which implies either $n(\zeta\phi + \phi^2) - \frac{r}{n-1} = 0$ or, $X_1 f + (\zeta f)\eta(X_1) = 0$.
Case I: If $n(\zeta\phi + \phi^2) - \frac{r}{n-1} = 0$, then $r = n(n-1)(\zeta\phi + \phi^2)$, a contradiction, since we take r = constant.
Case II: If $X_1 f + (\zeta f)\eta(X_1) = 0$, then $Df = -(\zeta f)\zeta$. Hence, we get

$$\nabla_{X_1} Df = -\{X_1(\zeta f)\}\zeta - \phi(\zeta f)[X_1 + \eta(X_1)\zeta]. \tag{46}$$

In a SPF spacetime, (37) implies

$$\nabla_{X_1} Df = f\mathcal{Q}X_1 + \beta X_1. \tag{47}$$

In view of (46) and (47), we obtain

$$-\{X_1(\zeta f)\}\eta(Y_1)-\phi(\zeta f)[g(X_1,Y_1)+\eta(X_1)\eta(Y_1)]=f\mathcal{S}(X_1,Y_1)+\beta g(X_1,Y_1). \tag{48}$$

Substituting Y_1 by ζ in (48) yields that

$$\{X_1(\zeta f)\}=\{f(n-1)(\zeta\phi+\phi^2)+\beta\}\eta(X_1). \tag{49}$$

Using (49) in (48) reveals that

$$\begin{aligned} f\mathcal{S}(X_1,Y_1)=-\{\phi(\zeta f)+\beta\}g(X_1,Y_1)-\{f(n-1)(\zeta\phi+\phi^2)+\beta \\ +\phi(\zeta f)\}\eta(X_1)\eta(Y_1), \end{aligned} \tag{50}$$

which represents a perfect fluid spacetime.
This completes the proof. □

3.2 Proof of the Corollary 1

If we take $\phi(\zeta f)+\beta=0$, then (9) and (50) together imply

$$\boldsymbol{\rho}=\boldsymbol{\sigma}, \tag{51}$$

which represents a stiff matter fluid.
This finishes the proof. □

3.3 Proof of the Theorem 4

From (50), we get

$$\mathcal{S}(X_1,Y_1)=ag(X_1,Y_1)+b\eta(X_1)\eta(Y_1), \tag{52}$$

where $a=-\frac{\phi(\zeta f)+\beta}{f}$ and $b=-\frac{f(n-1)(\zeta\phi+\phi^2)+\beta+\phi(\zeta f)}{f}$.
Equation (52) implies

$$\begin{aligned} (\nabla_{X_1}\mathcal{Q})Y_1 &= (X_1a)Y_1+(X_1b)\eta(Y_1)\zeta \\ &\quad +b\phi\{g(X_1,Y_1)\zeta+\eta(Y_1)X_1+2\eta(X_1)\eta(Y_1)\zeta\}. \end{aligned} \tag{53}$$

Considering the inner product of (38) with Df, we find

$$\begin{aligned} g(\mathcal{R}(X_1,Y_1)Df,Df) &= (X_1f)\mathcal{S}(Y_1,Df)-(Y_1f)\mathcal{S}(X_1,Df) \\ &\quad +f[g((\nabla_{X_1}\mathcal{Q})Y_1,Df)-g((\nabla_{Y_1}\mathcal{Q})X_1,Df)] \\ &\quad +(X_1\beta)Y_1f-(Y_1\beta)X_1f. \end{aligned} \tag{54}$$

Using (53) in (54) entails that

$$(X_1 f)\mathcal{S}(Y_1, Df) - (Y_1 f)\mathcal{S}(X_1, Df) + f[(X_1 a)Y_1 + (X_1 b)\eta(Y_1)\zeta - (Y_1 a)X_1$$
$$-(Y_1 b)\eta(X_1)\zeta + b\phi\{\eta(Y_1)X_1 - \eta(X_1)Y_1\}]$$
$$+(X_1\beta)Y_1 f - (Y_1\beta)X_1 f = 0. \tag{55}$$

If we take a = constant and b = constant, and using (52) in (55), we conclude that

$$(X_1 f)[a(Y_1 f) + b(\zeta f)\eta(Y_1)] - (Y_1 f)[a(X_1 f) + b(\zeta f)\eta(X_1)]$$
$$+bf\phi[\eta(Y_1)X_1 - \eta(X_1)Y_1] = 0. \tag{56}$$

Substituting X_1 by ζ in (56) gives

$$b(\zeta f)[Y_1 f + (\zeta f)\eta(Y_1)] + bf\phi[Y_1 + \eta(Y_1)\zeta] = 0. \tag{57}$$

From Case II of the above theorem, we get $Y_1 f + (\zeta f)\eta(Y_1) = 0$. Hence the foregoing equation implies

$$bf\phi[Y_1 + \eta(Y_1)\zeta] = 0, \tag{58}$$

which implies $b = 0$, since f and ϕ are non-zero scalars.
Therefore from (52), we obtain

$$\mathcal{S}(X_1, Y_1) = ag(X_1, Y_1), \tag{59}$$

where $a = -\frac{\phi(\zeta f)+\beta}{f}$. Therefore it represents an Einstein spacetime.
This ends the proof. □

3.4 Proof of the Corollary 2

It is well known that

$$(div\, C)(X_1, Y_1)Z_1 = \frac{\mathbf{n}-3}{\mathbf{n}-2}[\{(\nabla_{X_1}\boldsymbol{S})(Y_1, Z_1) - (\nabla_{Y_1}\boldsymbol{S})(X_1, Z_1)\}$$
$$-\frac{1}{2(\mathbf{n}-1)}\{(X_1\boldsymbol{r})g(Y_1, Z_1) - (Y_1\boldsymbol{r})g(X_1, Z_1)\}], \tag{60}$$

C indicates the Weyl tensor.

From Theorem 3 it follows that a GRW spacetime with constant scalar curvature as the spatial factor of a SPF spacetime becomes an Einstein spacetime. Hence (60) provides $div\, C = 0$.

It is to be noted that in a GRW spacetime, $C(X_1, Y_1)\zeta = 0$ if and only if $div\, C = 0$ [27]. Also, $C(X_1, Y_1)\zeta = 0$ implies the Weyl tensor is purely electric [18]. It is known ([35], p. 73) that if the Weyl tensor is purely electric, then the spacetime is of Petrov type I, D or O.

In 4-dimension, $C(X_1, Y_1)\zeta = 0$ is equivalent to ([25], p. 128)

$$\eta(U_1)C(X_1, Y_1, Z_1, W_1) + \eta(X_1)C(Y_1, U_1, Z_1, W_1) + \eta(Y_1)C(U_1, X_1, Z_1, W_1) = 0, \tag{61}$$

where $\eta(X_1) = g(X_1, \zeta)$ and $C(X_1, Y_1, Z_1, W_1) = g(C(X_1, Y_1)Z_1, W_1)$ for all X_1, Y_1, Z_1, W_1, U_1.
Replacing U_1 by ζ in the foregoing equation, we infer

$$C(X_1, Y_1, Z_1, W_1) = 0, \tag{62}$$

which means that the spacetime is conformally flat.
It is known that a GRW spacetime is conformally flat if and only if the spacetime is a RW spacetime [7].
Thus the proof is completed. □

Remark 3 RW spacetimes with constant scalar curvature were described in [30].

4 Comments

Generalized curvature tensor K^h_{ijk} were introduced by Kobayashi and Nomizu [19], the generalized curvature tensor shares the algebraic properties of the Riemann and the Weyl tensors. In [26], the authors prove that if in a PF spacetime the generalized curvature tensor K^h_{ijk} satisfies the condition

$$\nabla_h K^h_{ijk} = A\nabla_h R^h_{ijk} + B(g_{jk}\nabla_i R - g_{ik}\nabla_j R),$$

where A and B are functions and $A \neq 0$ at any point of the spacetime and the generalized curvature tensor is harmonic, then the spacetime is a GRW spacetime, K^h_{ijk} and R^h_{ijk} being the components of the generalized curvature tensor and curvature tensor, respectively in local coordinates. In general a GRW spacetime is not a PF spacetime. In the current article we obtain the condition under which a GRW spacetime to be a PF spacetime and prove Theorem 3 of our paper.

5 Conclusions

Any stellar model must be spherically symmetric when the classifications of static perfect fluid spacetimes are connected with the fluid ball conjecture. Avez originally attempted the fluid ball hypothesis in 1964, and he began to obtain some results in 1970 and 1980. The conjecture was initially supported by a feasible equation of state that took into account the pressure $\boldsymbol{\rho}$ and density σ [29]. At present, Coutinho et al. [12] offered a straightforward proof of the fluid ball conjecture for the static perfect fluid under the asymptotically flat condition. They did this by proposing a divergence formula.

Some researchers studied SPF spacetimes in contact geometry [10], paracontact geometry [32], almost Kenmotsu manifolds [21] and [11, 20, 22], respectively. In this article, we investigate SPF spacetimes on GRW spacetimes.

Here we show that if a GRW spacetime with constant scalar curvature as the spatial factor of a SPF spacetime, then it becomes a PF spacetime. Also, we prove that under certain restrictions a GRW spacetime with constant scalar curvature as the spatial factor of a SPF spacetime becomes an Einstein spacetime. As a consequence it is shown that such a spacetime is of Petrov type I, D or O and in case of 4-dimension, the spacetime turns into a RW spacetime.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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