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Potential theory for quantum Markov states and other quantum Markov chains

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Abstract

We introduce a potential theory for a class of Quantum Markov Chains whose forward and backward Markov transition operators satisfy a special composition rule. We study the associated recurrence, transient and irreducibility properties and we prove that an irreducible quantum Markov chain is either recurrent or transient. Moreover, we show that our theory applies in many cases such as: quantum random walks, diagonal states, entangled Quantum Markov Chains. A characterization of Entangled Quantum Markov Chains is also given.

Keywords Quantum Markov chains · Potential · Recurrence · Transience

1 Introduction

Potential theory plays an important role in the analysis of classical (see e.g. [10]) and quantum Markov processes ([8, 13, 17, 22]) because it allows one to establish the long time behaviour of the process. Potentials are related with occupation times and the existence of non-trivial potentials characterizes transient regimes. Moreover, they define superharmonic functions and enjoy the useful Riesz decomposition property. It is not clear, however, if this is the case also for Quantum Markov Chains (QMCs) introduced by Accardi [1, 3] where a transition expectation determines two, typically different and non-commuting, Markov transition operators. Notwithstanding, a notion of visit time was introduced in [4, 5] and recurrence (resp. transience) was defined as divergence (resp. finiteness) of the visit time. This approach was also followed later

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in [8]. Visit times are closely related with potential, however, since QMCs are highly more general and non trivial objects than classical Markov processes, the potential obtained in this way does not enjoy characteristic properties such as sub-harmonicity (Definition 2). This undermines the development of a potential theory as rich as the classical one as well as applicability in concrete examples.

In this paper we show that, perhaps surprisingly, one can develop a potential theory for a class of QMCs including diagonal states [6], entangled QMCs [2], quantum Markov states [3] and, in general, those QMCs whose backward and forward transition operator $\mathcal{T}$ and $\mathcal{T}'$ satisfy a special composition rule. The starting point of our analysis is the interpretation of the operator $\mathcal{T}^n \circ \mathcal{T}'$ as the n-step transition operator (formula 3). If $\mathcal{T}$ and $\mathcal{T}'$ satisfy the identity

$$\mathcal{T} \circ \mathcal{T}' \circ \mathcal{T} \circ \mathcal{T}' = \mathcal{T}^2 \circ \mathcal{T}', \tag{1}$$

we can introduce a notion of associated potential enjoying the fundamental properties of classical potentials such as sub-harmonicity (Theorem 1) and Riesz decomposition (Theorem 2). As a result we can apply it in the analysis of transience and recurrence properties.

These properties have been studied in several papers for classical and quantum Markov processes determined by a single Markov operator (see [2, 4, 5, 13, 17, 18, 22] and the references therein). They have also been investigated for more general processes such as open quantum random walks [7, 11] and Quantum Markov Chains in the sense of Gudder [18, 19] that are not Markovian in the strict sense. Our approach does not apply to all these processes but extends the potential theoretic approach to certain non Markovian processes that can be dealt with by two transition operators instead of a single one.

The paper is organized as follows. In Sect. 2, we recall the basic concepts related to the Quantum Markov Chains and we give a motivation for our approach. Moreover, we show that many examples of QMCs belong to the class we are considering. We further introduce, in Sect. 3, the potential for a class of Quantum Markov Chains and we prove its properties: superharmonicity (Theorem 1) and Riesz decomposition (Theorem 2). The study of recurrence and transience by our concept of potential is carried on in Sect. 4 where we prove, in particular, that an irreducible QMC is either recurrent or transient (Theorem 4). In Sect. 5 we apply our results to the so-called entangled QMC [2] and to a QMC associated with a two q-bit model. Finally, in Sect. 6 we collect some final comments and discuss further developments.

2 Quantum Markov chains

Let $\mathcal{M}$ be a von Neumann subalgebra of the algebra $\mathcal{B}(\mathsf{h})$ of all bounded operators on some complex separable Hilbert space h. For each finite set $\Lambda \subset \mathbb{N}^* = \mathbb{N}\backslash\{0\}$, we consider $\mathcal{A}_\Lambda = \otimes_{n\in\Lambda}\mathcal{M}$, where $\otimes$ is the minimal C^*-tensor product (cf. [15], [21]) and $\mathcal{A} = \otimes_{n\in\mathbb{N}^*}\mathcal{M}$ is the inductive limit of $\mathcal{A}_\Lambda$, $\Lambda \subset \mathbb{N}^*$ finite.

We denote by $\mathbb{1}$ the identity operator in $\mathcal{M}$.

A completely positive *normal* unital map $\mathcal{E} : \mathcal{M} \otimes \mathcal{M} \to \mathcal{M}$ is called a *transition expectation* (see [3]). Let ϕ_0 be a given state on $\mathcal{M}$. If $\Lambda = \{i, \ldots, k\} \subset \mathbb{N}^*$, then for all x_j in the j-th copy of $\mathcal{M}$ ($i \le j \le k$) define

$$\phi_\Lambda(x_i \otimes x_{i+1} \otimes \cdots \otimes x_k) = \phi_0\left(\mathcal{E}(\mathbb{1} \otimes \ldots \mathcal{E}\left(x_i \otimes \mathcal{E}\left(x_{i+1} \otimes \cdots \otimes \mathcal{E}(x_k \otimes \mathbb{1})\right)\right)\right)$$

The family of local states $\{\phi_\Lambda \mid \Lambda \subset \mathbb{N}^*, \ |\Lambda| < \infty\}$ satisfies the compatibility condition,

$$\phi_{\Lambda'}\big|_{\mathcal{M}_\Lambda} = \phi_\Lambda, \qquad \text{where } \ \mathcal{M}_\Lambda = \otimes_{j \in \Lambda} \mathcal{M}, \quad \Lambda \subseteq \Lambda'.$$

Then there exists a unique state ϕ on $\mathcal{A}$ such that $\phi\big|_{\mathcal{M}_\Lambda} = \phi_\Lambda$.

Definition 1 The state ϕ is called Quantum Markov Chain (QMC) on $\mathcal{A}$, associated with the pair $(\phi_0, \mathcal{E})$.

With the transition expectation $\mathcal{E}$ we associate two completely positive, identity preserving, normal maps $\mathcal{T}$ and $\mathcal{T}'$ on $\mathcal{M}$ defined by

$$\mathcal{T}(x) = \mathcal{E}\left(\mathbb{1} \otimes x\right), \qquad \mathcal{T}'(x) = \mathcal{E}\left(x \otimes \mathbb{1}\right) \tag{2}$$

which are called respectively the *backward* and *forward Markov transition operators*.

Note that, as explained in [6] Sect. 2, a usual Markov process with associated Markov operator $\mathcal{T}$ on a commutative $\mathcal{M}$ can be viewed as a QMC with $\mathcal{E}(f \otimes g) = f\,\mathcal{T}(g)$ (pointwise product) so that $\mathcal{T}'$ is the identity map.

Remark 1 If $\mathcal{M} = L^\infty(E, \mathcal{F}, \mu)$ for a σ-finite measure μ. A normal state ϕ_0 on $\mathcal{M}$ determines a probability measure on the σ-algebra $\mathcal{F}$. In this case, ϕ_0 is the initial distribution of an E-valued Markov process $(X_n)_{n \ge 0}$ such that, for all $A_1 \in \mathcal{F}$,

$$\phi(1_E \otimes 1_{A_1}) = \mathbb{E}_{\phi_0}\left[(1_E \otimes 1_{A_1})(X_0, X_1)\right] = \mathbb{P}_{\phi_0}\left\{X_1 \in A_1\right\}$$

where 1_{A_1} denotes the indicator function of the set A_1. In addition

$$\mathbb{P}_{\phi_0}\left\{X_1 \in A_1\right\} = \int_E (\mathcal{T} 1_{A_1})(x) \phi_0(\mathrm{d}x) = \phi_0(\mathcal{T} 1_{A_1}) = \phi_0(\mathcal{T}(\mathcal{T}'(1_{A_1}))).$$

In a similar way, if we consider an n-step transition, we find by induction

$$\begin{aligned} \phi\left(1_E \otimes \cdots \otimes 1_{A_n}\right) &= \phi\left(\mathcal{E}\left(\mathbb{1} \otimes \mathcal{E}\left(\mathbb{1} \otimes \cdots \otimes \mathcal{E}(1_{A_n} \otimes \mathbb{1})\right)\right)\right) \\ &= \mathbb{P}_{\phi_0}\left\{X_n \in A_n\right\} \\ &= \phi_0\left(\mathcal{T}^{n-1}(\mathcal{T}'(1_{A_n}))\right) \end{aligned} \tag{3}$$

for any $n \ge 1$ and $A_n \in \mathcal{F}_n$. Therefore $\mathcal{T}^n \circ \mathcal{T}'$ corresponds to the n-step transition operator.
The same interpretation holds for quantum Markov processes determined by a single transition operator $\mathcal{T}$.

There exist important cases where the forward and backward Markov operator satisfy some commutation rule which turns out to be useful for introducing our notion of potential.

2.1 Quantum random walks

We consider a quantum random walk introduced in [6] as a simple example of a QMC. Let $(X, \mathcal{X}, \mu)$ be a measurable space with a σ-finite measure μ, $(U_x)_{x\in X}$ a collection of unitary operators on $L^2(X)$, $(F_x)_{x\in X}$ a collection of Hilbert-Schmidt operators on $L^2(X)$ such that:

1. The maps $x \to U_x$ and $x \to F_x$ are strongly measurable,
2. $\int_X \mathrm{tr}\left(|F_x|^2\right) d\mu(x) = 1$

and define

$$\mathcal{E}(a \otimes b) = \mathrm{tr}_2\left(\int_X \left(U_x^* a\, U_x \otimes F_x b F_x^*\right) d\mu(x)\right) = \int_X \mathrm{tr}\left(|F_x|^2 b\right) U_x^* a\, U_x d\mu(x).$$

The forward and backward Markov operators are given by

$$\mathcal{T}'(a) = \int_X \mathrm{tr}\left(|F_x|^2\right) U_x^* a\, U_x d\mu(x)$$
$$\mathcal{T}(b) = \left(\int_X \mathrm{tr}\left(|F_x|^2 b\right) d\mu(x)\right) \mathbb{1}$$

It turns out that $\mathcal{T}$ is a conditional expectation onto the trivial algebra and the following commutation relation holds $\mathcal{T}' \circ \mathcal{T} = \mathcal{T}$.

2.2 Diagonal states

We now describe QMCs appearing in [5] and called "diagonal states". Let $P = (P_{ij})_{1\le i,j\le d}$ be the stochastic matrix of a classical Markov chain and let $(f_j)_{1\le j\le d}$ be a partition of the identity in $\mathcal{M} = M_d(\mathbb{C})$ of mutually orthogonal rank one projections. Choose an orthonormal basis $(e_j)_{1\le j\le d}$ of $\mathbb{C}^d$ such that $f_j = \left|e_j\right\rangle\left\langle e_j\right|$ for all j. Consider the transition expectation $\mathcal{E} : \mathcal{M} \otimes \mathcal{M} \to \mathcal{M}$ defined by

$$\mathcal{E}\left(a \otimes b\right) = \sum_{h=1}^{d} K_h^* a K_h \mathrm{tr}\left(f_h b\right)$$

where $K_h = \sum_{j=1}^{d} p_{jh}^{1/2} f_j$.

A straightforward computation yields

$$\mathcal{T}(b) = \sum_{h=1}^{d} K_h^* K_h \mathrm{tr}\left(f_h b\right) = \sum_{h,j=1}^{d} p_{jh}\left\langle e_h, b\, e_h\right\rangle \left|e_j\right\rangle\left\langle e_j\right|$$

$$\mathcal{T}'(a) = \sum_{h=1}^{d} K_h^* a K_h = \sum_{h,i,j=1}^{d} p_{ih}^{1/2} p_{jh}^{1/2} \langle e_i, a\, e_j \rangle \, |e_i\rangle \langle e_j| \,.$$

Note that $\mathcal{T}$ maps $M_d(\mathbb{C})$ onto the subalgebra of diagonal matrices and $\mathcal{T}'$ acts as the identity map on this subalgebra. Therefore we have again

$$\mathcal{T}' \circ \mathcal{T} = \mathcal{T}.$$

In other words, $\mathcal{T}'$ acts as the identity map on the range of $\mathcal{T}$.

We shall see later that also the backward and forward Markov operator of the so-called entangled QMCs introduced by Accardi and Fidaleo [2] also satisfy $\mathcal{T}' \circ \mathcal{T} = \mathcal{T}$. A weaker condition turns out to be the key property making potentials for QMCs as useful as those for standard Markov processes both in the commutative and non-commutative cases.

2.3 Quantum Markov states

A quantum Markov state is a quantum Markov chain such that the transition expectation statisfies

$$\mathcal{E}(x \otimes y) = \mathcal{E}(x \otimes \mathcal{E}(y \otimes \mathbb{1})), \qquad \forall x, y \in \mathcal{M} \tag{4}$$

Therefore, for all $x \in \mathcal{M}$, we have

$$\mathcal{T}(x) = \mathcal{E}(\mathbb{1} \otimes x) = \mathcal{E}(\mathbb{1} \otimes \mathcal{E}(x \otimes \mathbb{1})) = \mathcal{T} \circ \mathcal{T}'(x)$$

and hence, by left composition with $\mathcal{T}$,

$$\mathcal{T} \circ \mathcal{T}' \circ \mathcal{T} = \mathcal{T}^2.$$

3 Potential

Inspired by the classical theory of Markov processes [10], and its non commutative counterpart for Quantum Markov Semigroups in [13, 22], in this section we introduce a notion of potential for QMCs developing the definition sketched in [5]. It is well known that the existence of non-trivial potential operators characterises transient regimes for both classical [10] and quantum Markov processes [13].

Definition 2 Let $\mathcal{T}$ be a completely positive, unital map on $\mathcal{M}$. A selfadjoint element x of $\mathcal{M}$ is called $\mathcal{T}$-subharmonic (resp. $\mathcal{T}$-superharmonic) if $\mathcal{T}(x) \geq x$ (resp. $\mathcal{T}(x) \leq x$).

In the sequel, we assume that (1) is satisfied and shall use the quadratic form setting following the book of Kato [20].

Definition 3 Given a positive operator $x \in \mathcal{M}$ we define the *form-potential* of x as the quadratic form $\mathfrak{U}(x)$ on the domain

$$\mathrm{Dom}\,(\mathfrak{U}(x)) = \left\{ u \in \mathsf{h} \,:\, \sum_{n \geq 1} \left\langle u, \mathcal{T}^n(\mathcal{T}'(x))u \right\rangle < \infty \right\}$$

by

$$\mathfrak{U}(x)[u] = \langle u, xu \rangle + \sum_{n \geq 1} \left\langle u, \mathcal{T}^n(\mathcal{T}'(x))u \right\rangle.$$

Note that, for a projection $p \in \mathcal{M}$ the operator

$$\mathcal{T}^{n-1}(\mathcal{T}'(p)) = \mathcal{E}\left(\mathbb{1} \otimes \mathcal{E}\left(\mathbb{1} \otimes \cdots \otimes \mathcal{E}(p \otimes \mathbb{1})\right)\right)$$

appears computing probabilities of visiting the projection p at time n (see formula (3)). Moreover, for quantum Markov processes as in [13, 17, 22], $\mathcal{T}'$ is the identity map and the above definition coincides with the usual one (see [13] Definition 2).

The quadratic form $\mathfrak{U}(x)$ is clearly a symmetric and positive form and, by Theorem 3.13a and Lemma 3.14a p. 461 of [20], it is also closed. Therefore, when it is densely defined, it is represented by a self-adjoint operator (see Theorem 2.1, p. 322, Theorem 2.6, p. 323 and Theorem 2.23 p. 331 of [20]). This motivates the following definition.

Definition 4 A positive $x \in \mathcal{M}$ such that $\mathrm{Dom}\,(\mathfrak{U}(x))$ is dense is called *integrable*. For an integrable x, we denote by $\mathcal{U}(x)$ the self-adjoint operator which represents the quadratic form $\mathfrak{U}(x)$. A positive operator $y \in \mathcal{M}$ is a *potential* if there exists an integrable $x \in \mathcal{M}$ such that $y = \mathcal{U}(x)$.

Note that $\mathrm{Dom}\left(\mathcal{U}(x)^{1/2}\right) = \mathrm{Dom}\,(\mathfrak{U}(x))$ (see [20] Theorem 2.23, p. 331).

We recall that a closed operator X is affiliated with the von Neumann algebra $\mathcal{M}$ if $y\mathrm{Dom}\,(X) \subseteq \mathrm{Dom}\,(X)$ and $yX \subseteq Xy$ for all $y \in \mathcal{M}'$ (the commutant of $\mathcal{M}$). If X is self-adjoint, it is affiliated with $\mathcal{M}$ if and only if its spectral projections belong to $\mathcal{M}$ (see [9] Definition 2.5.7, Lemma 2.5.8 p. 87).

Proposition 1 *For all integrable $x \in \mathcal{M}$, the selfadjoint operator $\mathcal{U}(x)$ is affiliated with $\mathcal{M}$.*

Proof Fix $y \in \mathcal{M}'$ and define $X_n = \sum_{k=1}^{n} \mathcal{T}^k(\mathcal{T}'(x))$, for all $n \geq 1$. Clearly, both X_n and $X_n^{1/2}$ belong to $\mathcal{M}$. Given any $u \in \mathsf{h}$,

$$\sum_{k=1}^{n} \left\langle yu, \mathcal{T}^k(\mathcal{T}'(x))yu \right\rangle = \left\langle yX_n^{1/2}u, yX_n^{1/2}u \right\rangle \leq \|y\|^2 \langle u, X_n u \rangle .$$

As a consequence, if $u \in \mathrm{Dom}\,(\mathfrak{U}(x))$, then

$$\sup_{n \geq 1} \sum_{k=1}^{n} \left\langle yu, \mathcal{T}^k(\mathcal{T}'(x))yu \right\rangle \leq \|y\|^2 \sum_{k=1}^{\infty} \left\langle u, \mathcal{T}^k(\mathcal{T}'(x))u \right\rangle \leq \|y\|^2\, \mathfrak{U}(x)[u].$$

It follows that, if $u \in \mathrm{Dom}\,(\mathfrak{U}(x)) = \mathrm{Dom}\left(\mathcal{U}(x)^{1/2}\right)$, then $yu \in \mathrm{Dom}\,(\mathfrak{U}(x))$.

Now, if $v, u \in \mathrm{Dom}\,(\mathcal{U}(x))$, then $y^*v,\ \ yu \in \mathrm{Dom}\,(\mathfrak{U}(x))$ and

$$\sum_{k=1}^{n} \left\langle y^*v, \mathcal{T}^k(\mathcal{T}'(x))u \right\rangle = \sum_{k=1}^{n} \left\langle \mathcal{T}^k(\mathcal{T}'(x))u, yu \right\rangle$$

so that letting n tend to infinity and using complex polarization, we get

$$\left\langle y^*v, \mathcal{U}(x)u \right\rangle = \langle \mathcal{U}(x)v, yu \rangle$$

namely $\langle v, y\mathcal{U}(x)u\rangle = \langle \mathcal{U}(x)v, yu\rangle$. It follows that $yu \in \mathrm{Dom}\,(\mathcal{U}(x))$ and $\mathcal{U}(x)yu = y\mathcal{U}(x)u$, hence $y\mathcal{U}(x) \subseteq \mathcal{U}(x)y$. $\square$

Potentials defined by a single completely positive map are characterized by simple properties (see, for instance, [17] Theorem 3.3). It is not clear whether this is true in general in the present framework. However, if the forward and backward Markov operator satisfy a weaker form of the composition rule arising from examples in Sects 2.2, 2.1, namely

$$\mathcal{T} \circ \mathcal{T}' \circ \mathcal{T} \circ \mathcal{T}' = \mathcal{T}^2 \circ \mathcal{T}', \tag{5}$$

one can immediately prove by induction that $(\mathcal{T} \circ \mathcal{T}')^m = \mathcal{T}^m \circ \mathcal{T}'$ for all $m \geq 2$. As a consequence, we have the following

Theorem 1 *Let $\mathcal{T}, \mathcal{T}'$ be the backward and forward Markov operators of a QMC with the property (5). A $y \in \mathcal{M}$ is a potential if and only if it is $\mathcal{T} \circ \mathcal{T}'$ superharmonic and $\mathcal{T}^m \circ \mathcal{T}'(y)$ converges strongly to 0 as $m \to \infty$.*

Proof Note that, by the property (5), we have

$$\mathcal{T} \circ \mathcal{T}'\,(\mathcal{U}(x)) = \mathcal{U}(x) - x \leq \mathcal{U}(x).$$

Moreover, for all $m \geq 1$

$$\mathcal{T}^m \circ \mathcal{T}'\,(\mathcal{U}(x)) = \sum_{k=m}^{\infty} \mathcal{T}^k \circ \mathcal{T}'\,(x)$$

and $\mathcal{T}^m \circ \mathcal{T}'(y)$ converges strongly to 0 as $m \to \infty$ because the right-hand side series is strongly convergent.

Conversely, let $y \in \mathcal{M}$ with the above properties and define $x = y - \mathcal{T}(\mathcal{T}'(y))$. For all $n \geq 1$, by (5), we have

$$y - \mathcal{T}(\mathcal{T}'(y)) + \sum_{k=1}^{n} \mathcal{T}^k\left(\mathcal{T}'\left(y - \mathcal{T}(\mathcal{T}'(y))\right)\right) = y - \mathcal{T}^{n+1}(\mathcal{T}(y)).$$

Therefore, taking the limit as $n \to \infty$, we find $y = \mathcal{U}(x)$. $\square$

We can prove also the following version of the Riesz decomposition theorem

Theorem 2 *A $\mathcal{T}\circ\mathcal{T}'$-superharmonic $x\in\mathcal{M}_+$ can be uniquely decomposed as $x=y+z$ where $y\in\mathcal{M}_+$ is a potential and $z\in\mathcal{M}_+$ is a fixed point for $\mathcal{T}\circ\mathcal{T}'$.*

Proof For all $\mathcal{T}\circ\mathcal{T}'$-superharmonic $x\in\mathcal{M}_+$ the sequence $(\left(\mathcal{T}\circ\mathcal{T}'\right)^m(x))_{m\geq 1}$ is decreasing. Therefore, taking the strong limit we can define

$$z=\mathrm{s}-\lim_{m\to\infty}\left(\mathcal{T}\circ\mathcal{T}'\right)^m(x)=\mathrm{s}-\lim_{m\to\infty}\left(\mathcal{T}^m\circ\mathcal{T}'\right)(x).$$

Clearly $\mathcal{T}\circ\mathcal{T}'(z)=z$. Moreover, y is $(\mathcal{T}\circ\mathcal{T}')$-superharmonic and $(\mathcal{T}^m\circ\mathcal{T}')(y)$ converges strongly to 0 as $m\to\infty$ and so it is a potential by Theorem 1.

If $x=y'+z'$ is another decomposition of x into the sum of a potential $y'\in\mathcal{M}_+$ and a $(\mathcal{T}\circ\mathcal{T}')$-fixed point $z'\in\mathcal{M}_+$, then $y-y'=z'-z$ is also a $(\mathcal{T}\circ\mathcal{T}')$-fixed point. Moreover

$$z'-z=(\mathcal{T}\circ\mathcal{T}')^m(z'-z)=(\mathcal{T}\circ\mathcal{T}')^m(y-y')=(\mathcal{T}^m\circ\mathcal{T}')(y-y')$$

and, taking the limit as $m\to+\infty$, $z=z'$ so that also $y=y'$. □

The following result will be useful for producing bounded potentials from integrable operators which are only self-adjoint.

Theorem 3 *For all integrable $x\in\mathcal{M}$, the contraction*

$$y=\mathcal{U}(x)\left(\mathbb{1}+\mathcal{U}(x)\right)^{-1}$$

is $\mathcal{T}\circ\mathcal{T}'$-superharmonic and $(\mathcal{T}^m\circ\mathcal{T}')(y)$ converges strongly to 0 as $m\to\infty$. In particular y is a potential.

Proof For all $n\geq 1$ let $\mathcal{U}_n(x)=x+\sum_{k=1}^n\mathcal{T}^k(\mathcal{T}'(x))$ and note that, for all $m\geq 1$

$$\mathcal{T}^m\circ\mathcal{T}'\left(\mathcal{U}_n(x)\right)=\mathcal{U}_{n+m}(x)-\mathcal{U}_{m-1}(x),\quad \mathcal{U}_0=\mathbb{1}. \tag{6}$$

For $m=1$ it follows that

$$\mathcal{T}\circ\mathcal{T}'\left(\mathcal{U}_n(x)\right)\leq\mathcal{U}_{n+1}(x). \tag{7}$$

Since $\mathcal{T}\circ\mathcal{T}'$ is unital completely positive, and the function $r\to(1+r)^{-1}$ is operator convex on $[0,+\infty[$, we have the inequality

$$\left(\mathbb{1}+\mathcal{T}\circ\mathcal{T}'(\mathcal{U}_n(x))\right)^{-1}\leq\mathcal{T}\circ\mathcal{T}'\left((\mathbb{1}+\mathcal{U}_n(x))^{-1}\right)$$

Note that that $r\to-(1+r)^{-1}$ is operator monotone on $[0,\infty)$. It follows that

$$\left(\mathbb{1}+\mathcal{U}_{n+1}(x)\right)^{-1}\leq\mathcal{T}\circ\mathcal{T}'\left((\mathbb{1}+\mathcal{U}_n(x))\right)^{-1}.$$

It follows that

$$\begin{aligned}\mathcal{T}\circ\mathcal{T}'\left(\mathcal{U}_n(x)\,(\mathbb{1}+\mathcal{U}_n(x))^{-1}\right) &= \mathbb{1}-\mathcal{T}\circ\mathcal{T}'\left((\mathbb{1}+\mathcal{U}_n(x))^{-1}\right)\\ &\le \mathbb{1}-(\mathbb{1}+\mathcal{U}_{n+1}(x))^{-1}\\ &= \mathcal{U}_{n+1}(x)\,(\mathbb{1}+\mathcal{U}_{n+1}(x))^{-1}\,.\end{aligned}$$

Taking the limit as $n\to\infty$, we find $\mathcal{T}\circ\mathcal{T}'(y)\le y$.

Finally, from (6), we have

$$\mathcal{T}^m\circ\mathcal{T}'\left(\mathcal{U}_n(x)\,(\mathbb{1}+\mathcal{U}_n(x))^{-1}\right)\le\mathcal{T}^m\circ\mathcal{T}'(\mathcal{U}_n(x))=\mathcal{U}_{n+m}(x)-\mathcal{U}_n(x)$$

so that, for all $u\in\mathrm{Dom}\,(\mathfrak{U}(x))$,

$$\left\langle u,\mathcal{T}^m\circ\mathcal{T}'\left(\mathcal{U}_n(x)\,(\mathbb{1}+\mathcal{U}_n(x))^{-1}\right)u\right\rangle\le\sum_{k=m}^{n+m}\left\langle u,\mathcal{T}^k\circ\mathcal{T}'(x)u\right\rangle.$$

Taking the limit as $n\to\infty$

$$\left\langle u,\mathcal{T}^m\circ\mathcal{T}'(y)u\right\rangle\le\sum_{k=m}^{\infty}\left\langle u,\mathcal{T}^k\circ\mathcal{T}'(x)u\right\rangle,$$

thus $\left\langle u,\mathcal{T}^m\circ\mathcal{T}'(y)u\right\rangle$ vanishes as m goes to infinity. Since $\mathrm{Dom}\,(\mathfrak{U}(x))$ is dense, and the operators $\mathcal{T}^m(y)$ are uniformly bounded in norm by $\|y\|\le 1$, it follows that $\mathcal{T}^m\circ\mathcal{T}'(y)$ converges strongly to 0 as $m\to\infty$. □

Next results identify two subharmonic projections naturally associated with form potentials with non-zero domain.

Proposition 2 *For all positive $x\in\mathcal{M}$ the orthogonal projection p onto the closure of Dom $(\mathfrak{U}(x))$ belongs to $\mathcal{M}$ and it is $\mathcal{T}\circ\mathcal{T}'$-subharmonic.*

Proof We first check that $p\in\mathcal{M}$. To this end, note that, as in the proof of Proposition 1, for all $y\in\mathcal{M}'$, $n>0$, and $u\in\mathrm{Dom}\,(\mathfrak{U}(x))$ we have

$$\sum_{k=1}^{n}\left\langle yu,\mathcal{T}^k(\mathcal{T}'(x))yu\right\rangle\le\|y\|\sum_{k=1}^{n}\left\langle u,\mathcal{T}^k(\mathcal{T}'(x))u\right\rangle$$

so that, adding $\langle yu,xyu\rangle$ and letting n tendo to infinity, we find $yu\in\mathrm{Dom}\,(\mathfrak{U}(x))$. This implies that $yu=ypu=pypu$ and since $\mathrm{Dom}\,(\mathfrak{U}(x))$ is dense in the range of p, we obtain $yp=pyp$. Considering y^* instead of y we also obtain $y^*p=py^*p$ and, taking the adjoint $py=pyp$. It follows that $py=yp$, namely $p\in\mathcal{M}$ by von Neumann bicommutant theorem.

In order to show that $\mathcal{T} \circ \mathcal{T}'(p) \geq p$ consider a $u \in \mathrm{Dom}\,(\mathfrak{U}(x))$ and note that, since, for all $n \geq 1$ we have

$$\begin{aligned}\mathrm{tr}\left((\mathcal{T} \circ \mathcal{T}')_*(|u\rangle\langle u|)\mathcal{T}^n(\mathcal{T}'(x))\right) &= \mathrm{tr}\left(|u\rangle\langle u|\,\mathcal{T}^{n+1}(\mathcal{T}'(x))\right)\\ &= \left\langle u, \mathcal{T}^{n+1}(\mathcal{T}'(x))u\right\rangle.\end{aligned}$$

The normal state $(\mathcal{T} \circ \mathcal{T}')_*(|u\rangle\langle u|)$ has spectral decomposition $\sum_k \rho_k |u_k\rangle\langle u_k|$ where summation is (obviously) on all k such that $\rho_k > 0$; therefore, summing on n the previous identity we find

$$\begin{aligned}\sum_k \rho_k \sum_{n\geq 1}\left\langle u_k, \mathcal{T}^n(\mathcal{T}'(x))u_k\right\rangle &= \sum_{n\geq 1}\left\langle u, \mathcal{T}^{n+1}(\mathcal{T}'(x))u\right\rangle\\ &= \sum_{n\geq 2}\left\langle u, \mathcal{T}^n(\mathcal{T}'(x))u\right\rangle < \infty.\end{aligned}$$

This shows that $u_k \in \mathrm{Dom}\,(\mathfrak{U}(x))$ for all k and so $p(\mathcal{T} \circ \mathcal{T}')_*(|u\rangle\langle u|) = (\mathcal{T} \circ \mathcal{T}')_*(|u\rangle\langle u|)p = (\mathcal{T} \circ \mathcal{T}')_*(|u\rangle\langle u|)$. It follows that

$$\mathrm{tr}\left(|u\rangle\langle u|\,(\mathcal{T} \circ \mathcal{T}')(p)\right) = \mathrm{tr}\left((\mathcal{T} \circ \mathcal{T}')_*(|u\rangle\langle u|)p\right) = \mathrm{tr}\left((\mathcal{T} \circ \mathcal{T}')_*(|u\rangle\langle u|)\right) = 1$$

and

$$\begin{aligned}0 &= \mathrm{tr}\left(|u\rangle\langle u|\left(p - (\mathcal{T} \circ \mathcal{T}')(p)\right)\right)\\ &= \mathrm{tr}\left(|u\rangle\langle u|\left(p - p(\mathcal{T} \circ \mathcal{T}')(p)p\right)\right).\end{aligned}$$

However, we also have $p(\mathcal{T} \circ \mathcal{T}')(p)p \leq p(\mathcal{T} \circ \mathcal{T}')(\mathbb{1})p \leq p$. Therefore $p(\mathcal{T} \circ \mathcal{T}')(p)p = p$, i.e. $p(\mathcal{T} \circ \mathcal{T}')_*(p^{\perp})p = 0$ so that, by Lemma II.1 [12] $(\mathcal{T} \circ \mathcal{T}')_*(p) \geq p$. □

Proposition 3 *For all positive $x \in \mathcal{M}$ the orthogonal projection p onto $\mathcal{K}(x) = \{u \in Dom\,(\mathfrak{U}(x))\ :\ \mathfrak{U}(x)[u] = 0\}$ is $(\mathcal{T} \circ \mathcal{T}')$-subharmonic.*

Proof Note that for a positive element $x \in \mathcal{M}$, $\mathfrak{U}(x)[u] = 0$ if and only if $\mathcal{U}_n(x)u = 0$ for all $n \geq 0$, where $\mathcal{U}_n(x) = x + \sum_{k=1}^n \mathcal{T}^k(\mathcal{T}'(x))$ for $n \geq 1$ and $\mathcal{U}_0(x) = x$. Fix $n \geq 1$ and let $q_m(n)$ denote the spectral projection of $\mathcal{U}_n(x)$ associated with the interval $]1/m, \|\mathcal{U}_n(x)\|]$ $(m \geq 1)$. Note that $q(n) := l.u.b.\, q_m(n)$ is the projection onto the closure of the range of $\mathcal{U}_n(x)$. From identity (7), one gets

$$\mathcal{T} \circ \mathcal{T}'\,(q_m(n)) \leq m\mathcal{T} \circ \mathcal{T}'\,(\mathcal{U}_n(x)) \leq m\,\mathcal{U}_{n+1}(x).$$

Since $\mathcal{T} \circ \mathcal{T}'\,(q_m(n)) \leq \mathbb{1}$, then we have

$$(\mathcal{T} \circ \mathcal{T}'\,(q_m(n)))^m \leq m\,\mathcal{U}_{n+1}(x)$$

and

$$\mathcal{T} \circ \mathcal{T}'(q_m(n)) \leq m^{1/m} \mathcal{U}_{n+1}(x)^{1/m}$$

By taking $m \to \infty$, we obtain

$$\mathcal{T} \circ \mathcal{T}'(q(n)) \leq q(n+1). \tag{8}$$

Note that the family $q(n)$ is increasing with n and $q = l.u.b.\, q(n) = 1 - p$. Therefore by taking $n \to \infty$ in (8), one gets $\mathcal{T} \circ \mathcal{T}'(p) \geq p$. □

4 Recurrent and transient QMCs

Irreducible classical Markov Chains are recurrent (resp. transient) if and only if they spend an infinite (resp. finite) mean time in bounded regions. The mean visit time, when finite, defines potentials. Therefore transient regimes are characterized by the existence of non trivial (i.e. non-zero or non-infinite) potentials. In this section we show how one can establish recurrence or transience by means of our notion of potential.

We begin by the following preliminary result.

Proposition 4 *The following are equivalent:*

1. *There exists $x \in \mathcal{M}_+$ with $\mathcal{U}(x)$ bounded and $\mathcal{U}(x) > 0$,*
2. *There exists a strictly positive $x \in \mathcal{M}$ such that $\mathcal{U}(x)$ is bounded,*
3. *There exists $x \in \mathcal{M}_+$ with $\mathcal{U}(x)$ self-adjoint and $\mathcal{U}(x) > 0$,*
4. *There exists an increasing family $(p_n)_{n\geq 1}$ of projections in $\mathcal{M}$ such that $\sup_{n\geq 1} p_n = \mathbb{1}$ with $\mathcal{U}(p_n)$ bounded for all n.*

Proof 1. ⇒ 2. Consider

$$y = x + \sum_{m\geq 1} 2^{-m} \mathcal{T}^m \circ \mathcal{T}'(x).$$

Clearly $y \in \mathcal{M}_+$ and $y > 0$ because $\mathcal{U}(x) > 0$. Moreover, by (5),

$$\mathcal{U}(y) \leq \mathcal{U}(x) + \sum_{m\geq 1} 2^{-m} \mathcal{U}(x) = 2\mathcal{U}(x)$$

2. ⇒ 1. Clear from $\mathcal{U}(x) \geq x$.
1. ⇒ 3. Obvious
3. ⇒ 1. From Theorem 3

$$y = \mathcal{U}(x)\,(\mathbb{1} + \mathcal{U}(x))^{-1}$$

is a potential. In particular $y = \mathcal{U}(z) > 0$ where $z = y - \mathcal{T} \circ \mathcal{T}'(y) \in \mathcal{M}_+$. Moreover it is clear that $y = \mathcal{U}(z)$ is bounded.

2. $\Rightarrow$ 4. For all $n \geq 1$ consider the spectral projection p_n of x corresponding to the interval $]1/n, \|x\|]$.

4. $\Rightarrow$ 2. Consider

$$x = \sum_{n \geq 1} 2^{-n} (1 + \|\mathcal{U}(p_n)\|)^{-1} p_n .$$

$\square$

Definition 5 A projection $p \in \mathcal{M}$ is called *transient* if there is a family $(p_i)_{i \in I}$ of projections with $\mathcal{U}(p_i)$ bounded for all i such that $p \leq \bigvee_{i \in I} p_i$.

A QMC is called transient if the identity $\mathbb{1}$ is transient.

Note that the above notion of transience matches the classical one for Markov processes and its non-commutative generalization [13] Definition 3.

Definition 6 A projection p is *recurrent* if, for all u in the range of p either $u \notin \mathrm{Dom}\,(\mathfrak{U}(x))$ or $u \in \mathrm{Dom}\,(\mathfrak{U}(x))$ and $\mathfrak{U}(x)[u] = 0$.

A QMC is called recurrent if every projection is recurrent.

A classical or quantum Markov semigroups $\mathcal{T}$ is called *irreducible* if there exists no non-trivial projection $p \in \mathcal{M}$ which is $\mathcal{T}$-subharmonic. This definition does not seem appropriate in the context of QMCs where one-step transition probabilities to p are computed with $\mathcal{T}'(p)$, and n steps transitions ($n \geq 2$) with $\mathcal{T}^{n-1}(\mathcal{T}'(p))$. The following should be the natural definition

Definition 7 A QMC is called *irreducible* if there exists no non-trivial projection $p \in \mathcal{M}$ such that

$$\mathcal{T}^n(\mathcal{T}'(p)) \leq p$$

for all $n \geq 1$.

Remark 2 . Since $\mathcal{T}^n \circ \mathcal{T}' = (\mathcal{T} \circ \mathcal{T}')^n$, Definition 7 is equivalent that there exists non-trivial $\mathcal{T} \circ \mathcal{T}'$–superhamonic projection.

It is worth noticing here that the above definition, for a QMS, i.e. when $\mathcal{T}'$ is the identity map, coincides with the usual one. It is, however, weaker than Definition 6 of [4] (the projection here is in $\mathcal{M}$, there in $\otimes_{n \geq 1} \mathcal{M}$...).

The following proposition gives a necessary condition for irreducibility.

Proposition 5 *If there exists a non-trivial projection p which is $\mathcal{T}$-subharmonic and $\mathcal{T}'$-subharmonic then the QMC is not irreducible.*

We are now in a position to prove the following

Theorem 4 *An irreducible QMC is either recurrent or transient.*

Proof Consider an irreducible quantum Markov chain and suppose that it is not recurrent. Then there exists a non-zero projection p with $\mathrm{Dom}\,(\mathfrak{U}(p)) \neq \{0\}$. From Proposition 2, the orthogonal projection q_1 on the closure of $\mathrm{Dom}\,(\mathfrak{U}(p))$ belongs to $\mathcal{M}$ and it is $\mathcal{T}\circ\mathcal{T}'$-subharmonic. If we denote by $q_1^c = I - q_1$, then $\mathcal{T}\circ\mathcal{T}'(q_1^c) \leq q_1^c$ and from the commutation rule $\mathcal{T}^n\circ\mathcal{T}'(q_1^c) \leq q_1^c$ for all $n \geq 1$. Since the Markov chain is irreducible, $q_1^c = I$ or $q_1^c = 0$. If $q_1^c = I$, then $q_1 = 0$ and $\mathrm{Dom}\,(\mathfrak{U}(p)) = \{0\}$ which is a contradiction with the fact that $\mathrm{Dom}\,(\mathfrak{U}(p)) \neq \{0\}$. It follows that $q_1 = I$, $\mathrm{Dom}\,(\mathfrak{U}(p))$ is dense in h and p is integrable. Therefore from Theorem 3 the contraction

$$y = \mathcal{U}(p)\,(\mathbb{1} + \mathcal{U}(p))^{-1}$$

is $\mathcal{T}\circ\mathcal{T}'$-superharmonic and it is a potential. In particular, $y = \mathcal{U}(z)$ with $z = y - \mathcal{T}\circ\mathcal{T}'(y)$ is a positive operator. Now our purpose is to prove that $y > 0$. From Proposition 3, the orthogonal projection q_2 onto

$$\mathcal{K}(p) = \{u \in \mathrm{Dom}\,(\mathfrak{U}(p)) : \mathfrak{U}(p)[u] = 0\}$$

is $(\mathcal{T}\circ\mathcal{T}')$-subharmonic. This means $\mathcal{T}^n\circ\mathcal{T}'(q_2^c) \leq q_2^c$ for all $n \geq 1$. Note that the QMC is irreducible. Then we have $q_2^c = 0$ or $q_2^c = I$.

If $q_2^c = 0$, then $q_2 = I$, $\overline{\mathcal{K}(p)} = \mathsf{h} = \overline{\mathrm{Dom}\,(\mathfrak{U}(p))}$. Hence we have

$$\mathfrak{U}(p)[u] = 0 = \langle pu, pu\rangle + \sum_{n\geq 1}\langle u, \mathcal{T}^n\circ\mathcal{T}'(p)u\rangle$$

Therefore $p(u) = 0$, for all $u \in \mathsf{h}$ and this is a contradiction with the fact that $p \neq 0$. Then $q_2^c = I$ and $q_2 = 0$. It follows that $\mathcal{K}(p) = \{0\}$, $\mathcal{U}(p) > 0$ and $y > 0$. Finally from Proposition 4, the QMC is transient. □

5 Applications

5.1 Entangled QMCs

In this section we exhibit another family of Quantum Markov Chains whose forward and backward transition operators satisfy the key property (5). They are essentially a generalization of infinite dimensional entangled Markov chains [16] (see [16] Theorem 4.1 on mean ergodicity of states). A characterization of entangled QMCs is given in the Appendix.

Let I be a countable set and $P = (p_{ij})_{ij\in I}$ be a stochastic matrix. Consider the Hilbert space $\mathsf{h} = \ell^2(I)$ with canonical orthonormal basis $(e_i)_{i\in I}$. It is easy to see that the linear map $V : \mathsf{h} \to \mathsf{h}\otimes\mathsf{h}$ satisfying

$$Ve_i = \sum_{j\in I} p_{ij}^{1/2} e_i \otimes e_j \tag{9}$$

defines an isometry of h into $\mathsf{h} \otimes \mathsf{h}$ so that one can define a transition expectation $\mathcal{E} : \mathcal{B}(\mathsf{h}) \otimes \mathcal{B}(\mathsf{h}) \to \mathcal{B}(\mathsf{h})$ by

$$\mathcal{E}(a \otimes b) = V^*(a \otimes b)V.$$

Note that

$$\begin{aligned} V^* e_h \otimes e_k &= \sum_j \langle e_j, V^* e_h \otimes e_k \rangle e_j \\ &= \sum_j \langle V e_j, e_h \otimes e_k \rangle\, e_j \\ &= \sum_{j,m} p_{jm}^{1/2} \langle e_j \otimes e_m, e_h \otimes e_k \rangle\, e_j \\ &= p_{hk}^{1/2} e_h. \end{aligned}$$

The corresponding forward and backward transition operators are

$$\mathcal{E}(a \otimes \mathbb{1}) = \mathcal{T}'(a) = \sum_{ij} a_{ij} \sum_k p_{ik}^{1/2} p_{jk}^{1/2} |e_i\rangle\langle e_j| = \sum_{ij} a_{ij} \langle r_i, r_j \rangle |e_i\rangle\langle e_j|$$

$$\mathcal{E}(\mathbb{1} \otimes b) = \mathcal{T}(b) = \sum_k \left(\sum_{ij} p_{ki}^{1/2} p_{kj}^{1/2} b_{ij} \right) |e_k\rangle\langle e_k| = \sum_k \langle r_k, b r_k \rangle |e_k\rangle\langle e_k|$$

where r_i denotes the unit vector $\sum_k p_{ik}^{1/2} e_k$ (note that $|\langle r_k, b r_k \rangle| \le \|b\|_\infty \|r_k\|^2 = \|b\|_\infty$ so that $\mathcal{T}$ is a contraction).

Note that $\mathcal{T}$ maps $\mathcal{B}(\mathsf{h})$ onto the maximal abelian subalgebra $\mathcal{D}$ of operators in $\mathcal{B}(\mathsf{h})$ which are diagonal in the given basis. Moreover each operator in $\mathcal{D}$ is a fixed point for $\mathcal{T}'$, therefore one immediately checks the identity

$$\mathcal{T}' \circ \mathcal{T} = \mathcal{T} \tag{10}$$

(as in the case of quantum random walks and diagonal states) and (5) follows by left composition with $\mathcal{T}$ and right composition with $\mathcal{T}'$.

In addition, denoting $\mathcal{P}$ the transition operator on $\ell^\infty(I)$ determined by the stochastic matrix $(p_{ij})_{i,j \in I}$, for all operator $x \in \mathcal{D}$, $x = \sum_j f_j |e_j\rangle\langle e_j|$ one has

$$(\mathcal{T} \circ \mathcal{T}')x = \mathcal{T}x = \sum_j (\mathcal{P} f)_j |e_j\rangle\langle e_j|$$

where $f = (f_j)_{j \in I} \in \ell^\infty(I)$ and, iterating,

$$(\mathcal{T}^m \circ \mathcal{T}')x = \sum_j (\mathcal{P}^m f)_j |e_j\rangle\langle e_j| \tag{11}$$

for all $m \geq 1$. This shows that the action of $\mathcal{T} \circ \mathcal{T}'$ on operators $x \in \mathcal{D}$ is determined by the action of the classical Markov operator $\mathcal{P}$. In fact, the dynamic behaviour of the entangled QMCs is related with the one of the classical Markov chain (see [16] Theorem 4.1 on mean ergodicity of states) as proved by the following

Proposition 6 *If the classical Markov chain with transition operator $\mathcal{P}$ is transient, then also the entangled QMC is transient. Conversely, if the entangled QMC is transient and there exists $x \in \mathcal{M}_+$ with a potential $y = \mathcal{U}(x)$ such that the positive operator $\mathcal{U}(x) - x$ is strictly positive, then the classical Markov chain with transition operator $\mathcal{P}$ is also transient.*

Proof If the classical Markov chain with transition operator $\mathcal{P}$ is transient, then there exists an increasing sequence of projections $(p_n)_{n\geq 1}$ in $\mathcal{D} \subseteq \mathcal{M}$ such that $\sup_{n\geq 1} p_n = \mathbb{1}$ with bounded classical potential

$$\sum_{m\geq 0} \mathcal{P}^m p_n$$

Since $\mathcal{T}'(p_n) = p_n$ and $\mathcal{T}(p_n) = \mathcal{P}p_n$ (see (11)) it follows that also the entangled QMC is transient.

Conversely, suppose that the entangled QMC is transient and let $y = \mathcal{U}(x)$ ($x \in \mathcal{M}_+$) be a strictly positive potential. If, in addition, $\mathcal{U}(x) - x$ is strictly positive, note that $(\mathcal{T} \circ \mathcal{T}')(x)$ is an element of the diagonal algebra $\mathcal{D}$ and satisfies

$$\sum_{m\geq 0} \mathcal{P}^m \left((\mathcal{T} \circ \mathcal{T}')(x)\right) = \sum_{m\geq 0} \mathcal{T}^{m+1} \circ \mathcal{T}'(x) = \mathcal{U}(x) - x.$$

Therefore the positive operator $(\mathcal{T} \circ \mathcal{T}')(x) \in \mathcal{D}$ has a strictly positive potential and the classical Markov chain with transition operator $\mathcal{P}$ is transient. □

Thinking of entangled QMC as a sort of extension of the classical Markov chain with transition operator $\mathcal{P}$, it is not surprising that transience of the latter only implies but is not equivalent to transience of the former. One can find the same phenomenon in several other cases such as the two-dimensional quantum Brownian motion [13] Sect. 6.1, and the quantum Laguerre process considered in [14].

Remark. It is worth noticing that irreducibility of the classical Markov chain with transition matrix P may not imply irreducibility of the associated entangled QMC. Indeed, consider $I = \{0, 1\}$ and $p_{ij} = 1/2$ so that vectors $r_1 = r_2$ are *not* linearly independent and the classical MC is irreducible. Consider now the projection

$$a_{ij} = \frac{(-1)^{i+j}}{2}, \quad \text{i.e.} \quad a = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \geq 0.$$

Clearly $\langle r_i, r_j \rangle = 1$ for all i, j so that $\mathcal{T}'(a) = a$. Since $\langle r_k, a r_k \rangle = 0$ for all k follows that

$$(\mathcal{T} \circ \mathcal{T}')(a) = \mathcal{T}(a) = 0$$

and the QMC is not irreducible.

Diagonal states defined in Sect. 2.2 can be dealt with in the same way, considering an infinite transition matrix P if one wants to find transient QMC.

5.2 QMC associated with two *q*-bits

In this subsection we consider a QMC generalizing the QMC associated with the Heisenberg potential defined in [4] Sect. 6. This will serve as an example of an irreducible QMC which is recurrent and, moreover, satisfies our key condition $\mathcal{T}\circ\mathcal{T}'\circ\mathcal{T}=\mathcal{T}^2$ as in (1) but not in its stronger form $\mathcal{T}'\circ\mathcal{T}\neq\mathcal{T}$ as entangled QMC.

Let $\mathsf{h}=\mathbb{C}^2$, consider self-adjoint operators on $\mathsf{h}\otimes\mathsf{h}$

$$H=\alpha\sigma_1\otimes\sigma_2+\beta\sigma_2\otimes\sigma_3 \tag{12}$$

where $\alpha,\beta\geq 0$. For $\alpha=\beta$ we get the Heisenberg potential as in [4]. Since σ_j and σ_k anticommute for $j\neq k$, the operators $\sigma_1\otimes\sigma_2$ and $\sigma_2\otimes\sigma_3$ commute and

$$\exp\left(\mathrm{i}H\right)=\exp(\mathrm{i}\alpha\,\sigma_1\otimes\sigma_2)\exp(\mathrm{i}\beta\,\sigma_2\otimes\sigma_3).$$

Keeping into account $(\sigma_j\otimes\sigma_k)^2=\mathbb{1}$ we have

$$\mathrm{e}^{\mathrm{i}\alpha\,\sigma_1\otimes\sigma_2}=\cos(\alpha)\mathbb{1}+\mathrm{i}\sin(\alpha)\,\sigma_1\otimes\sigma_2$$

so that

$$\begin{aligned}\mathrm{e}^{\mathrm{i}H}&=(\cos(\alpha)\mathbb{1}+\mathrm{i}\sin(\alpha)\,\sigma_1\otimes\sigma_2)\,(\cos(\beta)\mathbb{1}+\mathrm{i}\sin(\beta)\,\sigma_2\otimes\sigma_3)\\&=\cos(\alpha)\cos(\beta)\mathbb{1}+\mathrm{i}\sin(\alpha)\cos(\beta)\,\sigma_1\otimes\sigma_2\\&\quad+\mathrm{i}\cos(\alpha)\sin(\beta)\,\sigma_2\otimes\sigma_3+\ \sin(\alpha)\sin(\beta)\sigma_3\otimes\sigma_1\end{aligned}$$

As a consequence

$$\begin{aligned}&\mathrm{e}^{-\mathrm{i}H}(a\otimes b)\mathrm{e}^{\mathrm{i}H}\\&\ =\cos^2(\alpha)\cos^2(\beta)\,a\otimes b-\mathrm{i}\sin(\alpha)\cos(\alpha)\cos^2(\beta)(\sigma_1\otimes\sigma_2)(a\otimes b)\\&\quad+\ \cos(\alpha)\cos(\beta)\,(-\mathrm{i}\cos(\alpha)\sin(\beta)\,\sigma_2\otimes\sigma_3+\sin(\alpha)\sin(\beta)\sigma_3\otimes\sigma_1)\ a\otimes b\\&\quad+a\otimes b\ \cos(\alpha)\cos(\beta)\,(\mathrm{i}\sin(\alpha)\cos(\beta)\,\sigma_1\otimes\sigma_2+\mathrm{i}\cos(\alpha)\sin(\beta)\,\sigma_2\otimes\sigma_3)\\&\quad+\ \cos(\alpha)\cos(\beta)\sin(\alpha)\sin(\beta)(a\otimes b)(\sigma_3\otimes\sigma_1)\\&\quad+\ \sin^2(\alpha)\cos^2(\beta)(\sigma_1a\sigma_1)\otimes(\sigma_2b\sigma_2)+\cos^2(\alpha)\sin^2(\beta)(\sigma_2a\sigma_2)\otimes(\sigma_3b\sigma_3)\\&\quad+\ \sin^2(\alpha)\sin^2(\beta)(\sigma_3a\sigma_3)\otimes(\sigma_1b\sigma_1)+\sin(\alpha)\cos(\alpha)\sin(\beta)\cos(\beta)(\sigma_1a\sigma_2)\otimes(\sigma_2b\sigma_3)\\&\quad-\mathrm{i}\sin^2(\alpha)\sin(\beta)\cos(\beta)(\sigma_1a\sigma_3)\otimes(\sigma_2b\sigma_1)\\&\quad+\ \sin(\alpha)\cos(\alpha)\sin(\beta)\cos(\beta)(\sigma_2a\sigma_1)\otimes(\sigma_3b\sigma_2)\\&\quad-\mathrm{i}\sin(\alpha)\cos(\alpha)\sin^2(\beta)(\sigma_2a\sigma_3)\otimes(\sigma_3b\sigma_1)+\mathrm{i}\sin^2(\alpha)\sin(\beta)\cos(\beta)(\sigma_3a\sigma_1)\otimes(\sigma_1b\sigma_2)\\&\quad+\mathrm{i}\sin(\alpha)\cos(\alpha)\sin^2(\beta)(\sigma_3a\sigma_2)\otimes(\sigma_1b\sigma_3)\end{aligned}$$

It follows that

$$\begin{aligned}
&\mathrm{e}^{-\mathrm{i}H}(\mathbb{1}\otimes b)\mathrm{e}^{\mathrm{i}H}\\
&\quad=\cos^2(\alpha)\cos^2(\beta)\,\mathbb{1}\otimes b\\
&\qquad+\mathrm{i}\sin(\alpha)\cos(\alpha)\sin^2(\beta)\,\sigma_1\otimes[b,\sigma_2]+\mathrm{i}\cos^2(\alpha)\sin(\beta)\cos(\beta)\,\sigma_2\otimes[b,\sigma_3]\\
&\qquad+\sin(\alpha)\cos(\alpha)\sin(\beta)\cos(\beta)\,\sigma_3\otimes(b\sigma_1+\sigma_1 b)+\sin^2(\alpha)\sin^2(\beta)\,\mathbb{1}\otimes(\sigma_1 b\sigma_1)\\
&\qquad+\sin^2(\alpha)\cos^2(\beta)\,\mathbb{1}\otimes\sigma_2 b\sigma_2+\cos^2(\alpha)\sin^2(\beta)\,\mathbb{1}\otimes(\sigma_3 b\sigma_3)\\
&\qquad+\mathrm{i}\sin(\alpha)\cos(\alpha)\sin(\beta)\cos(\beta)\,\sigma_3\otimes(\sigma_2 b\sigma_3-\sigma_3 b\sigma_2)\\
&\qquad+\sin^2(\alpha)\sin(\beta)\cos(\beta)\sigma_2\otimes(\sigma_1 b\sigma_2+\sigma_2 b\sigma_1)\\
&\qquad+\sin(\alpha)\cos(\alpha)\sin^2(\beta)\sigma_1\otimes(\sigma_1 b\sigma_3+\sigma_3 b\sigma_1)
\end{aligned}$$

Taking the normalized partial trace $\frac{1}{2}\mathrm{Tr}_2$ we find

$$\mathcal{T}(b)=\mathcal{E}(\mathbb{1}\otimes b)=\frac{1}{2}\left(\mathrm{tr}\,(b)\,\mathbb{1}+\sin(2\alpha)\sin(2\beta)\mathrm{tr}\,(b\sigma_1)\,\sigma_3\right)$$

Note that

$$\mathcal{T}^2(b)=\frac{1}{2}\mathrm{tr}\,(b)\,\mathbb{1} \tag{13}$$

Similarly

$$\begin{aligned}
&\mathrm{e}^{-\mathrm{i}H}(a\otimes\mathbf{1})\mathrm{e}^{\mathrm{i}H}\\
&\quad=\cos^2(\alpha)\cos^2(\beta)\,a\otimes\mathbf{1}\\
&\qquad+\mathrm{i}\sin(\alpha)\cos(\alpha)\cos^2(\beta)\,[a,\sigma_1]\otimes\sigma_2+\mathrm{i}\cos^2(\alpha)\sin(\beta)\cos(\beta)\,[a,\sigma_2]\otimes\sigma_3\\
&\qquad+\sin(\alpha)\sin(\beta)\cos(\alpha)\cos(\beta)(a\sigma_3+\sigma_3 a)\otimes\sigma_1\\
&\qquad+\sin^2(\alpha)\cos^2(\beta)(\sigma_1 a\sigma_1)\otimes\mathbf{1}+\cos^2(\alpha)\sin^2(\beta)(\sigma_2 a\sigma_2)\otimes\mathbf{1}\\
&\qquad+\sin^2(\alpha)\sin^2(\beta)(\sigma_3 a\sigma_3)\otimes\mathbf{1}+\mathrm{i}\sin(\alpha)\cos(\alpha)\sin(\beta)\cos(\beta)(\sigma_1 a\sigma_2)\otimes\sigma_1\\
&\qquad-\sin^2(\alpha)\sin(\beta)\cos(\beta)(\sigma_1 a\sigma_3)\otimes\sigma_3-\mathrm{i}\sin(\alpha)\cos(\alpha)\sin(\beta)\cos(\beta)(\sigma_2 a\sigma_1)\otimes\sigma_1\\
&\qquad-\sin(\alpha)\cos(\alpha)\sin^2(\beta)(\sigma_2 a\sigma_3)\otimes\sigma_2-\sin^2(\alpha)\sin(\beta)\cos(\beta)(\sigma_3 a\sigma_1)\otimes\sigma_3\\
&\qquad-\sin(\alpha)\cos(\alpha)\sin^2(\beta)(\sigma_3 a\sigma_2)\otimes\sigma_2
\end{aligned}$$

Taking the normalized partial trace $\frac{1}{2}\mathrm{Tr}_2$ we get

$$\begin{aligned}
\mathcal{T}'(a)&=\mathcal{E}(a\otimes\mathbb{1})\\
&=\cos^2(\alpha)\cos^2(\beta)a+\sin^2(\alpha)\cos^2(\beta)\sigma_1 a\sigma_1\\
&\quad+\cos^2(\alpha)\sin^2(\beta)\sigma_2 a\sigma_2+\sin^2(\alpha)\sin^2(\beta)\sigma_3 a\sigma_3
\end{aligned}$$

Clearly

$$\mathcal{T}'\circ\mathcal{T}(b)=\frac{1}{2}\mathrm{tr}\,(b)\,\mathbb{1}+\frac{1}{2}\sin(2\alpha)\sin(2\beta)\mathrm{tr}\,(b\sigma_1)\,\mathcal{T}'(\sigma_3)$$

$$= \frac{1}{2}\mathrm{tr}\,(b)\,\mathbb{1} + \frac{1}{8}\sin(4\alpha)\sin(4\beta)\mathrm{tr}\,(b\sigma_1)\,\sigma_3$$

and, since $\mathcal{T}(\sigma_3) = 0$,

$$\mathcal{T} \circ \mathcal{T}' \circ \mathcal{T}(b) = \mathcal{T}^2(b) = \frac{1}{2}\mathrm{tr}\,(b)\,\mathbb{1}$$

but $\mathcal{T}' \circ \mathcal{T} \neq \mathcal{T}$ for almost all choices of α, β.

Proposition 7 *The QMC associated with the two q-bit Hamiltonian (12) is irreducible and recurrent.*

Proof We first prove that it is irreducible. Any non-trivial projection $p \in M_2(\mathbb{C})$ can be written as $p = (\mathbb{1} + u_1\sigma_1 + u_2\sigma_2 + u_3\sigma_3)/2$ with $u_1, u_2, u_3 \in \mathbb{R}$, $|u_1|^2 + |u_2|^2 + |u_3|^2 = 1$. A straightforward computation yields

$$\mathcal{T} \circ \mathcal{T}'(p) = \frac{1}{2}\left(\mathbb{1} + \frac{u_1}{2}\sin(2\alpha)\sin(2\beta)\cos(2\beta)\sigma_3\right)$$

and so $\mathcal{T} \circ \mathcal{T}'(p) \leq p$ implies $u_1 = u_2 = 0$, i.e. $u_3 = \pm 1$, but in this case $\mathcal{T} \circ \mathcal{T}'(p) = \mathbb{1}/2$ and $\mathcal{T} \circ \mathcal{T}'(p) \leq p$ does not hold.
We now check that it is recurrent. By (13) we have

$$\mathcal{T}^2 \circ \mathcal{T}'(p) = \frac{\mathrm{tr}\left(\mathcal{T}'(p)\right)}{2}\mathbb{1} = \frac{1}{2}\mathbb{1}$$

therefore (Definition 3) the series defining the form potential is convergent only for $u = 0$ and the QMC is recurrent. □

6 Conclusion and outlook

We introduced a notion of potential for QMCs whose forward and backward transition operator satisfy the identity (1). Although these processes are not Markov in the strict sense because their transitions are not given by a single transition operator, it is possible to define a notion of potential with all the good properties of potentials determined by a single Markovian operator.

One may guess that any QMC determines "its own" potential. Composition rules of $\mathcal{T}$ and $\mathcal{T}'$ can make it more or less similar to the classical potentials defined by a single Markov operator. However, we do not expect that all QMCs have a "nice" (i.e. satisfying Riesz decomposition etc...) potential. Moreover, $\mathcal{T}', \mathcal{T}$ do not determine a unique $\mathcal{E}$ (consider e.g. $\mathcal{T}, \mathcal{T}'$ as in Sect. 5.2 with two choices of angles $\alpha = \pi/2, \beta = \pi/4$ and $\alpha = \pi/2, \beta = 3\pi/4$). It would be interesting to find a characterization of QMCs with forward and backward Markov transition operator satisfying (1).

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Declarations

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Appendix: characterization of entangled QMCs

Entangled QMCs can be characterized as follows.

Theorem 5 *Let h be a complex separable Hilbert space, $(e_i)_{i\in I}$ an orthonormal basis and $\mathcal{D}$ be the von Neumann algebra of diagonal operators in this basis. Let $\mathcal{E}$: $\mathcal{B}(\mathsf{h}) \otimes \mathcal{B}(\mathsf{h}) \to \mathcal{B}(\mathsf{h})$ be a transition expectation such that*

1. *it is purely generated, i.e. $\mathcal{E}(a\otimes b) = V^*(a\otimes b)V$ for some isometry $V : \mathsf{h} \to \mathsf{h}\otimes\mathsf{h}$,*
2. *$\mathcal{E}(a \otimes \mathbb{1}) = a$ for all $a \in \mathcal{D}$.*

Then there exists a stochastic matrix $(q_{ij})_{i,j\in I}$ and a collection of phases $(e^{\mathrm{i}\,\theta_{ij}})_{i,j\in I}$ such that

$$Ve_i = \sum_j q_{ij}^{1/2} e^{\mathrm{i}\,\theta_{ij}} e_i \otimes e_j$$

and we have also

$$\mathcal{E}(\mathbb{1} \otimes \mathcal{D}) \subseteq \mathcal{D}.$$

Proof By property 2,

$$\sum_j |V^* e_i \otimes e_j\rangle\langle V^* e_i \otimes e_j| = \mathcal{E}(|e_i\rangle\langle e_i| \otimes \mathbb{1}) = |e_i\rangle\langle e_i|$$

It follows that, for all i, we have $V^* e_i \otimes e_j = w_{ij} e_i$ for some complex number w_{ij} with $\sum_j |w_{ij}| = 1$. As a consequence, for all j,

$$\mathcal{E}(\mathbb{1} \otimes |e_j\rangle\langle e_j|) = \sum_k |V^* e_k \otimes e_j\rangle\langle V^* e_k \otimes e_j|) = \sum_k |w_{kj}|^2 |e_k\rangle\langle e_k|.$$

Moreover, we have also

$$\mathbb{1} = \sum_j \mathcal{E}(\mathbb{1} \otimes |e_j\rangle\langle e_j|) = \sum_k \left(\sum_j |w_{kj}|^2 \right) |e_k\rangle\langle e_k|.$$

It follows that $\sum_j |w_{kj}|^2 = 1$ and the matrix $q_{kj} = |w_{kj}|^2$ is stochastic. Defining $\theta_{kj} = -\mathrm{Arg}(w_{kj})$ we find

$$\begin{aligned} Ve_i &= \sum_{h,k} \langle e_h \otimes e_k, Ve_i \rangle e_h \otimes e_k \\ &= \sum_{h,k} \langle V^* e_h \otimes e_k, e_i \rangle e_h \otimes e_k \\ &= \sum_{h,k} \overline{w}_{hk} \langle e_h, e_i \rangle\, e_h \otimes e_k = \sum_k \overline{w}_{ik} e_i \otimes e_k \end{aligned}$$

the conclusion follows. □

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