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Carleson measures and the range of a Cesàro-like operator acting on H^∞

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Abstract
In this paper, we determine the range of a Cesàro-like operator acting on H^∞ by describing characterizations of Carleson type measures on $[0, 1)$. A special case of our result gives an answer to a question posed by P. Galanopoulos, D. Girela and N. Merchán recently.

Keywords Cesàro-like operator · Carleson measure · H^∞ · $BMOA$

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1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. Denote by $H(\mathbb{D})$ the space of functions analytic in $\mathbb{D}$. For $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $H(\mathbb{D})$, the Cesàro operator $\mathcal{C}$ is defined by

$$\mathcal{C}(f)(z) = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} a_k \right) z^n, \quad z \in \mathbb{D}.$$

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See [7, 12, 14, 21, 23, 24] for the investigation of the Cesàro operator acting on some analytic function spaces.

Recently, P. Galanopoulos, D. Girela and N. Merchán [16] considered a Cesàro-like operator $\mathcal{C}_\mu$ on $H(\mathbb{D})$. For nonnegative integer n, let μ_n be the moment of order n of a finite positive Borel measure μ on $[0, 1)$; that is,

$$\mu_n = \int_{[0,1)} t^n d\mu(t).$$

For $f(z) = \sum_{n=0}^{\infty} a_n z^n$ belonging to $H(\mathbb{D})$, the Cesàro-like operator $\mathcal{C}_\mu$ is defined by

$$\mathcal{C}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\mu_n \sum_{k=0}^{n} a_k \right) z^n, \quad z \in \mathbb{D}.$$

If $d\mu(t) = dt$, then $\mathcal{C}_\mu = \mathcal{C}$. In [16, 19], the authors studied the action of $\mathcal{C}_\mu$ on distinct spaces of analytic functions.

We also need to recall some function spaces. For $0 < p < \infty$, H^p denotes the classical Hardy space [13] of those functions $f \in H(\mathbb{D})$ for which

$$\sup_{0<r<1} M_p(r, f) < \infty,$$

where

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}.$$

As usual, denote by H^∞ the space of bounded analytic functions in $\mathbb{D}$. It is well known that H^∞ is a proper subset of the Bloch space $\mathcal{B}$ which consists of those functions $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < \infty.$$

Denote by $\mathrm{Aut}(\mathbb{D})$ the group of Möbius maps of $\mathbb{D}$, namely,

$$\mathrm{Aut}(\mathbb{D}) = \{e^{i\theta}\sigma_a : \ a \in \mathbb{D} \text{ and } \theta \ \text{ is real}\},$$

where

$$\sigma_a(z) = \frac{a - z}{1 - \bar{a}z}, \qquad z \in \mathbb{D}.$$

In 1995 R. Aulaskari, J. Xiao and R. Zhao [2] introduced $\mathcal{Q}_p$ spaces. For $0 \le p < \infty$, a function f analytic in $\mathbb{D}$ belongs to $\mathcal{Q}_p$ if

$$\|f\|_{\mathcal{Q}_p}^2 = \sup_{w \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\sigma_w(z)|^2)^p dA(z) < \infty,$$

where dA is the area measure on $\mathbb{C}$ normalized so that $A(\mathbb{D}) = 1$. $\mathcal{Q}_p$ spaces are Möbius invariant in the sense that

$$\|f\|_{\mathcal{Q}_p} = \|f \circ \phi\|_{\mathcal{Q}_p}$$

for every $f \in \mathcal{Q}_p$ and $\phi \in \mathrm{Aut}(\mathbb{D})$. It was shown in [25] that $\mathcal{Q}_2$ coincides with the Bloch space $\mathcal{B}$. This result was extended in [1] by showing that $\mathcal{Q}_p = \mathcal{B}$ for all $1 < p < \infty$. The space $\mathcal{Q}_1$ coincides with $BMOA$, the set of analytic functions in $\mathbb{D}$ with boundary values of bounded mean oscillation (see [5, 17]). The space $\mathcal{Q}_0$ is the Dirichlet space $\mathcal{D}$. For $0 < p < 1$, the space $\mathcal{Q}_p$ is a proper subset of $BMOA$ and has many interesting properties. See J. Xiao's monographs [26, 27] for the theory of $\mathcal{Q}_p$ spaces.

For $1 \le p < \infty$ and $0 < \alpha \le 1$, the mean Lipschitz space Λ^p_α is the set of those functions $f \in H(\mathbb{D})$ with a non-tangential limit almost everywhere such that $\omega_p(t, f) = O(t^\alpha)$ as $t \to 0$. Here $\omega_p(\cdot, f)$ is the integral modulus of continuity of order p of the function $f(e^{i\theta})$. It is well known (cf. [13, Chapter 5]) that Λ^p_α is a subset of H^p and Λ^p_α consists of those functions $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{\Lambda^p_\alpha} = \sup_{0<r<1} (1-r)^{1-\alpha} M_p(r, f') < \infty.$$

Among these spaces, the spaces $\Lambda^p_{1/p}$ are of special interest. $\Lambda^p_{1/p}$ spaces increase with $p \in (1, \infty)$ in the sense of inclusion and they are contained in $BMOA$ (cf. [10]). By Theorem 1.4 in [4], $\Lambda^p_{1/p} \subseteq \mathcal{Q}_q$ when $1 \le p < 2/(1-q)$ and $0 < q < 1$. In particular, $\Lambda^2_{1/2} \subseteq \mathcal{Q}_q \subseteq \mathcal{B}$ for all $0 < q < \infty$.

Given an arc I of the unit circle $\mathbb{T}$ with arclength $|I|$ (normalized such that $|\mathbb{T}| = 1$), the Carleson box $S(I)$ is given by

$$S(I) = \{r\zeta \in \mathbb{D} : 1 - |I| < r < 1,\ \zeta \in I\}.$$

For $0 < s < \infty$, a positive Borel measure ν on $\mathbb{D}$ is said to be an s-Carleson measure if

$$\sup_{I \subseteq \mathbb{T}} \frac{\nu(S(I))}{|I|^s} < \infty.$$

If ν is a 1-Carleson measure, we write that ν is a Carleson measure characterizing $H^p \subseteq L^p(d\nu)$ for $0 < p < \infty$ (cf. [13]). A positive Borel measure μ on $[0, 1)$ can be seen as a Borel measure on $\mathbb{D}$ by identifying it with the measure $\tilde{\mu}$ defined by

$$\tilde{\mu}(E) = \mu(E \cap [0, 1)),$$

for any Borel subset E of $\mathbb{D}$. Thus μ is an s-Carleson measure on $[0, 1)$ if there is a positive constant C such that

$$\mu([t, 1)) \le C(1-t)^s$$

for all $t \in [0, 1)$. We refer to [8] for the investigation of this kind of measures associated with Hankel measures.

It is known that the Cesàro operator $\mathcal{C}$ is bounded on H^p for all $0 < p < \infty$ (cf. [21, 23, 24]) but this is not true on H^∞. In fact, N. Danikas and A. Siskakis [12] gave that $\mathcal{C}(H^\infty) \nsubseteq H^\infty$ but $\mathcal{C}(H^\infty) \subseteq BMOA$. Later M. Essén and J. Xiao [14] proved that $\mathcal{C}(H^\infty) \subsetneqq \mathcal{Q}_p$ for $0 < p < 1$. Recently, the relation between $\mathcal{C}(H^\infty)$ and a class of Möbius invariant function spaces was considered in [7].

It is quite natural to study $\mathcal{C}_\mu(H^\infty)$. In [16] the authors characterized positive Borel measures μ such that $\mathcal{C}_\mu(H^\infty) \subseteq H^\infty$ and proved that $\mathcal{C}_\mu(H^\infty) \subseteq \mathcal{B}$ if and only if μ is a Carleson measure. Moreover, they showed that if $\mathcal{C}_\mu(H^\infty) \subseteq BMOA$, then μ is a Carleson measure. In [16, p. 20], the authors asked whether or not μ being a Carleson measure implies that $\mathcal{C}_\mu(H^\infty) \subseteq BMOA$. In this paper, by giving some descriptions of s-Carleson measures on $[0, 1)$, for $0 < p < 2$, we show that $\mathcal{C}_\mu(H^\infty) \subseteq \mathcal{Q}_p$ if and only if μ is a Carleson measure, which gives an affirmative answer to their question. We also consider another Cesàro-like operator $\mathcal{C}_{\mu,s}$ and describe the embedding $\mathcal{C}_{\mu,s}(H^\infty) \subseteq X$ in terms of s-Carleson measures, where X is between $\Lambda^p_{1/p}$ and $\mathcal{B}$ for $\max\{1, 1/s\} < p < \infty$.

Throughout this paper, the symbol $A \approx B$ means that $A \lesssim B \lesssim A$. We say that $A \lesssim B$ if there exists a positive constant C such that $A \leq CB$.

2 Positive Borel measures on [0, 1) as Carleson type measures

In this section, we give some characterizations of positive Borel measures on $[0, 1)$ as Carleson type measures.

The following description of Carleson type measures (cf. [9]) is well known.

Lemma A *Suppose $s > 0$, $t > 0$ and μ is a positive Borel measure on $\mathbb{D}$. Then μ is an s-Carleson measure if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^t}{|1-\overline{a}w|^{s+t}} d\mu(w) < \infty. \tag{2.1}$$

For Carleson type measures on $[0, 1)$, we can obtain some descriptions that are different from Lemma A. Now we give the first main result in this section.

Proposition 2.1 *Suppose $0 < t < \infty$, $0 \leq r < s < \infty$ and μ is a finite positive Borel measure on $[0, 1)$. Then the following conditions are equivalent:*

(i) *μ is an s-Carleson measure;*

(ii)

$$\sup_{a\in\mathbb{D}} \int_{[0,1)} \frac{(1-|a|)^t}{(1-x)^r(1-|a|x)^{s+t-r}} d\mu(x) < \infty; \tag{2.2}$$

(iii)

$$\sup_{a\in\mathbb{D}} \int_{[0,1)} \frac{(1-|a|)^t}{(1-x)^r|1-ax|^{s+t-r}} d\mu(x) < \infty. \tag{2.3}$$

Proof $(i) \Rightarrow (ii)$. Let μ be an s-Carleson measure. Fix $a \in \mathbb{D}$ with $|a| \le 1/2$. If $r = 0$, the desired result holds. For $0 < r < s$, using a well-known formula about the distribution function(cf. [15, p.20]), we get

$$
\begin{aligned}
&\int_{[0,1)} \frac{(1-|a|)^t}{(1-x)^r(1-|a|x)^{s+t-r}} d\mu(x) \\
&\approx \int_{[0,1)} \left(\frac{1}{1-x}\right)^r d\mu(x) \\
&\approx r\int_0^\infty \lambda^{r-1}\mu(\{x \in [0,1) : 1-\frac{1}{\lambda} < x\})d\lambda \\
&\lesssim \int_0^1 \lambda^{r-1}\mu([0,1))d\lambda + \int_1^\infty \lambda^{r-1}\mu([1-\frac{1}{\lambda},1))d\lambda \\
&\lesssim 1 + \int_1^\infty \lambda^{r-s-1}d\lambda \lesssim 1.
\end{aligned}
\tag{2.4}
$$

Fix $a \in \mathbb{D}$ with $|a| > 1/2$ and let

$$
S_n(a) = \{x \in [0,1) : 1 - 2^n(1-|a|) \le x < 1\}, \quad n = 1, 2, \cdots .
$$

Let n_a be the minimal integer such that $1 - 2^{n_a}(1-|a|) \le 0$. Then $S_n(a) = [0,1)$ when $n \ge n_a$. If $x \in S_1(a)$, then

$$
1 - |a| \le 1 - |a|x. \tag{2.5}
$$

Also, for $2 \le n \le n_a$ and $x \in S_n(a)\backslash S_{n-1}(a)$, we have

$$
1 - |a|x \ge |a| - x \ge |a| - (1 - 2^{n-1}(1-|a|)) = (2^{n-1}-1)(1-|a|). \tag{2.6}
$$

We write

$$
\begin{aligned}
&\int_{[0,1)} \frac{(1-|a|)^t}{(1-x)^r(1-|a|x)^{s+t-r}} d\mu(x) \\
&= \int_{S_1(a)} \frac{(1-|a|)^t}{(1-x)^r(1-|a|x)^{s+t-r}} d\mu(x) \\
&\quad + \sum_{n=2}^{n_a} \int_{S_n(a)\backslash S_{n-1}(a)} \frac{(1-|a|)^t}{(1-x)^r(1-|a|x)^{s+t-r}} d\mu(x) \\
&=: J_1(a) + J_2(a).
\end{aligned}
$$

If $r = 0$, bearing in mind (2.5), (2.6) and that μ is an s-Carleson measure, it is easy to check that $J_i(a) \lesssim 1$ for $i = 1, 2$. Now consider $0 < t < \infty$ and $0 < r < s < \infty$.

Using (2.5) and some estimates similar to (2.4), we have

$$J_1(a) \lesssim (1-|a|)^{r-s} \int_{S_1(a)} \left(\frac{1}{1-x}\right)^r d\mu(x) \lesssim 1.$$

Note that (2.6) holds, $0 < t < \infty$, $0 < r < s < \infty$ and μ is an s-Carleson measure. Then

$$\begin{aligned}
J_2(a) &\lesssim \sum_{n=2}^{n_a} \frac{(1-|a|)^{r-s}}{2^{n(s+t-r)}} \int_{S_n(a)\setminus S_{n-1}(a)} \left(\frac{1}{1-x}\right)^r d\mu(x) \\
&\lesssim \sum_{n=2}^{n_a} \frac{(1-|a|)^{r-s}}{2^{n(s+t-r)}} \int_0^\infty \lambda^{r-1} \mu\Big(\Big\{x \in [1-2^n(1-|a|),1) : 1-\frac{1}{\lambda} < x\Big\}\Big) d\lambda \\
&\approx \sum_{n=2}^{n_a} \frac{(1-|a|)^{r-s}}{2^{n(s+t-r)}} \Bigg(\int_0^{\frac{1}{2^n(1-|a|)}} \lambda^{r-1} \mu\big([1-2^n(1-|a|),1)\big) d\lambda \\
&\quad + \int_{\frac{1}{2^n(1-|a|)}}^\infty \lambda^{r-1} \mu\Big(\Big[1-\frac{1}{\lambda},1\Big)\Big) d\lambda \Bigg) \\
&\lesssim \sum_{n=2}^{n_a} \frac{(1-|a|)^{r-s}}{2^{n(s+t-r)}} \Bigg(2^{ns}(1-|a|)^s \int_0^{\frac{1}{2^n(1-|a|)}} \lambda^{r-1} d\lambda + \int_{\frac{1}{2^n(1-|a|)}}^\infty \lambda^{r-1-s} d\lambda \Bigg) \\
&\approx \sum_{n=2}^{n_a} \frac{1}{2^{tn}} < \infty.
\end{aligned}$$

Consequently,

$$\sup_{a\in\mathbb{D}} \int_{[0,1)} \frac{(1-|a|)^t}{(1-x)^r(1-|a|x)^{s+t-r}} d\mu(x) < \infty.$$

The implication of $(ii) \Rightarrow (iii)$ is clear.

$(iii) \Rightarrow (i)$. For $r \geq 0$, it is clear that

$$\int_{[0,1)} \frac{(1-|a|)^t}{(1-x)^r|1-ax|^{s+t-r}} d\mu(x) \geq \int_{[0,1)} \frac{(1-|a|)^t}{|1-ax|^{s+t}} d\mu(x)$$

for all $a \in \mathbb{D}$. Combining this with Lemma A, we see that if (2.3) holds, then μ is an s-Carleson measure. □

Remark 1 The condition $0 \leq r < s < \infty$ in Proposition 2.1 can not be changed to $r \geq s > 0$. For example, let $d\mu_1(x) = (1-x)^{s-1}dx$, $x \in [0,1)$. Then μ_1 is an s-Carleson measure but for $r \geq s > 0$,

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{[0,1)} \frac{(1-|a|)^t}{(1-x)^r|1-ax|^{s+t-r}} d\mu_1(x) \\
&\quad \geq \int_0^1 (1-x)^{s-1-r} dx = +\infty.
\end{aligned}$$

Remark 2 μ supported on [0, 1) is essential in Proposition 2.1. For example, consider $0 < t < 1, 0 < r < s < 1$ and $s = r + t$. Set $d\mu_2(w) = |f'(w)|^2(1-|w|^2)^s dA(w)$, $w \in \mathbb{D}$, where $f \in \mathcal{Q}_s \setminus \mathcal{Q}_t$. Note that for $0 < p < \infty$ and $g \in H(\mathbb{D})$, $|g'(w)|^2(1-|w|^2)^p dA(w)$ is a p-Carleson measure if and only if $g \in \mathcal{Q}_p$ (cf. [26]). Hence $d\mu_2$ is an s-Carleson measure. But

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|)^t}{(1-|w|)^r|1-a\overline{w}|^{s+t-r}} d\mu_2(w) \\
&\quad = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(w)|^2 \frac{(1-|a|)^t(1-|w|)^{s-r}}{|1-a\overline{w}|^{s+t-r}} dA(w) \\
&\quad \approx \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(w)|^2(1-|\sigma_a(w)|^2)^t dA(w) = +\infty.
\end{aligned}$$

Before giving the other characterization of Carleson type measures on [0, 1), we need to recall some results.

The following result is Lemma 1 in [20], which generalizes Lemma 3.1 in [18] from $p = 2$ to $1 < p < \infty$.

Lemma B *Let $f \in H(\mathbb{D})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Suppose $1 < p < \infty$ and the sequence $\{a_n\}$ is a decreasing sequence of nonnegative numbers. If X is a subspace of $H(\mathbb{D})$ with $\Lambda^p_{1/p} \subseteq X \subseteq \mathcal{B}$, then*

$$f \in X \iff a_n = O\left(\frac{1}{n}\right).$$

We recall a characterization of s-Carleson measure μ on [0, 1) as follows (cf. [6, Theorem 2.1] or [11, Proposition1]).

Proposition C *Let μ be a finite positive Borel measure on [0, 1) and $s > 0$. Then μ is an s-Carleson measure if and only if the sequence of moments $\{\mu_n\}_{n=0}^{\infty}$ satisfies* $\sup_{n\geq 0}(1+n)^s\mu_n < \infty$.

The following characterization of functions with nonnegative Taylor coefficients in $\mathcal{Q}_p$ is Theorem 2.3 in [3].

Theorem D *Let $0 < p < \infty$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function in $\mathbb{D}$ with $a_n \geq 0$ for all n. Then $f \in \mathcal{Q}_p$ if and only if*

$$\sup_{0\leq r<1} \sum_{n=0}^{\infty} \frac{(1-r)^p}{(n+1)^{p+1}} \left(\sum_{k=0}^{n} (k+1)a_{k+1}(n-k+1)^{p-1} r^{n-k} \right)^2 < \infty.$$

We need the following well-known estimates (cf. [28, Lemma 3.10]).

Lemma E *Let β be any real number. Then*

$$\int_0^{2\pi} \frac{d\theta}{|1 - ze^{-i\theta}|^{1+\beta}} \approx \begin{cases} 1 & \textit{if } \beta < 0, \\ \log \frac{2}{1-|z|^2} & \textit{if } \beta = 0, \\ \frac{1}{(1-|z|^2)^\beta} & \textit{if } \beta > 0, \end{cases}$$

for all $z \in \mathbb{D}$.

For $0 < s < \infty$ and a finite positive Borel measure μ on $[0, 1)$, set

$$f_{\mu,s}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+s)}{\Gamma(s)n!} \mu_n z^n, \quad z \in \mathbb{D}.$$

Now we state the other main result in this section which is inspired by Lemma B and Proposition C.

Proposition 2.2 *Suppose $0 < s < \infty$ and μ is a finite positive Borel measure on $[0, 1)$. Let $1 < p < \infty$ and let X be a subspace of $H(\mathbb{D})$ with $\Lambda^p_{1/p} \subseteq X \subseteq \mathcal{B}$. Then μ is an s-Carleson measure if and only if $f_{\mu,s} \in X$.*

Proof Let μ be an s-Carleson measure. Clearly,

$$f_{\mu,s}(z) = \int_{[0,1)} \frac{1}{(1 - tz)^s} d\mu(t)$$

for any $z \in \mathbb{D}$. For $p > 1$, it follows from the Minkowski inequality and Lemma E that

$$\begin{aligned} M_p(r, f'_{\mu,s}) \leq & s \left(\frac{1}{2\pi} \int_0^{2\pi} \left(\int_{[0,1)} \frac{1}{|1 - tre^{i\theta}|^{s+1}} d\mu(t) \right)^p d\theta \right)^{1/p} \\ \leq & s \int_{[0,1)} \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1 - tre^{i\theta}|^{(s+1)p}} d\theta \right)^{1/p} d\mu(t) \\ \lesssim & \int_{[0,1)} \frac{1}{(1 - tr)^{s+1-\frac{1}{p}}} d\mu(t) \end{aligned}$$

for all $0 < r < 1$. Combining this with Proposition 2.1, we get $f_{\mu,s} \in \Lambda^p_{1/p}$ and hence $f_{\mu,s} \in X$.

On the other hand, let $f_{\mu,s} \in X$. Then $f_{\mu,s} \in \mathcal{Q}_q$ with $q > 1$. By the Stirling formula,

$$\frac{\Gamma(n+s)}{\Gamma(s)n!} \approx (n+1)^{s-1}$$

for all nonnegative integers n. Consequently, by Theorem D we deduce

$$
\begin{aligned}
\infty &> \sum_{n=0}^{\infty} \frac{(1-r)^q}{(n+1)^{q+1}} \left(\sum_{k=0}^{n} (k+2)^s \mu_{k+1} (n-k+1)^{q-1} r^{n-k} \right)^2 \\
&\gtrsim \sum_{n=0}^{\infty} \frac{(1-r)^q}{(4n+1)^{q+1}} \left(\sum_{k=0}^{4n} (k+2)^s \mu_{k+1} (4n-k+1)^{q-1} r^{4n-k} \right)^2 \\
&\gtrsim \sum_{n=0}^{\infty} \frac{(1-r)^q}{(4n+1)^{q+1}} \left(\sum_{k=n}^{2n} (k+2)^s \int_r^1 t^{k+1} d\mu(t) (4n-k+1)^{q-1} r^{4n-k} \right)^2 \\
&\gtrsim \mu^2([r,1))(1-r)^q \sum_{n=0}^{\infty} \frac{r^{8n+2}}{(4n+1)^{q+1}} \left(\sum_{k=n}^{2n} (k+2)^s (4n-k+1)^{q-1} \right)^2 \\
&\gtrsim \mu^2([r,1))(1-r)^q \sum_{n=0}^{\infty} (4n+2)^{2s+q-1} r^{8n+2} \\
&\approx \frac{\mu^2([r,1))}{(1-r)^{2s}}
\end{aligned}
$$

for all $r \in [0, 1)$ which yields that μ is an s-Carleson measure. The proof is complete. □

3 $\mathcal{Q}_p$ spaces and the range of $\mathcal{C}_\mu$ acting on H^∞

In this section, we characterize finite positive Borel measures μ on $[0, 1)$ such that $\mathcal{C}_\mu(H^\infty) \subseteq \mathcal{Q}_p$ for $0 < p < 2$. Descriptions of Carleson measures in Proposition 2.1 play a key role in our proof.

The following lemma is from [22].

Lemma F *Suppose $s > -1$, $r > 0$, $t > 0$ with $r + t - s - 2 > 0$. If $r, t < 2 + s$, then*

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^s}{|1-\overline{a}z|^r |1-\overline{b}z|^t} dA(z) \lesssim \frac{1}{|1-\overline{a}b|^{r+t-s-2}}$$

for all $a, b \in \mathbb{D}$. If $t < 2 + s < r$, then

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^s}{|1-\overline{a}z|^r |1-\overline{b}z|^t} dA(z) \lesssim \frac{(1-|a|^2)^{2+s-r}}{|1-\overline{a}b|^t}$$

for all $a, b \in \mathbb{D}$.

We give our result as follows.

Theorem 3.1 *Suppose $0 < p < 2$ and μ is a finite positive Borel measure on $[0, 1)$. Then $\mathcal{C}_\mu(H^\infty) \subseteq \mathcal{Q}_p$ if and only if μ is a Carleson measure.*

Proof Suppose $\mathcal{C}_\mu(H^\infty) \subseteq \mathcal{Q}_p$. Then $\mathcal{C}_\mu(H^\infty)$ is a subset of the Bloch space. By [16, Theorem 5], μ is a Carleson measure.

Conversely, suppose μ is a Carleson measure and $f \in H^\infty$. Then f is also in the Bloch space $\mathcal{B}$. From Proposition 1 in [16],

$$\mathcal{C}_\mu(f)(z) = \int_{[0,1)} \frac{f(tz)}{1-tz} d\mu(t), \quad z \in \mathbb{D}.$$

Hence for any $z \in \mathbb{D}$,

$$\begin{aligned}
\|\mathcal{C}_\mu(f)\|_{\mathcal{Q}_p} \lesssim \sup_{a\in\mathbb{D}} &\left(\int_{\mathbb{D}} \left(\int_{[0,1)} \frac{|tf'(tz)|}{|1-tz|} d\mu(t)\right)^2 (1-|\sigma_a(z)|^2)^p dA(z)\right)^{\frac{1}{2}} \\
&+ \sup_{a\in\mathbb{D}} \left(\int_{\mathbb{D}} \left(\int_{[0,1)} \frac{|tf(tz)|}{|1-tz|^2} d\mu(t)\right)^2 (1-|\sigma_a(z)|^2)^p dA(z)\right)^{\frac{1}{2}} \\
&\lesssim \|f\|_{\mathcal{B}} \sup_{a\in\mathbb{D}} \left(\int_{\mathbb{D}} \left(\int_{[0,1)} \frac{1}{(1-|tz|)|1-tz|} d\mu(t)\right)^2 (1-|\sigma_a(z)|^2)^p dA(z)\right)^{\frac{1}{2}} \\
&+ \|f\|_{H^\infty} \sup_{a\in\mathbb{D}} \left(\int_{\mathbb{D}} \left(\int_{[0,1)} \frac{1}{|1-tz|^2} d\mu(t)\right)^2 (1-|\sigma_a(z)|^2)^p dA(z)\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.1}$$

Let c be a positive constant such that $2c < \min\{2-p, p\}$. Then

$$(1-|tz|)^2 \geq (1-t)^{2-2c}(1-|z|)^{2c} \tag{3.2}$$

for all $t \in [0, 1)$ and all $z \in \mathbb{D}$. By the Minkowski inequality, (3.2), Lemma F and Proposition 2.1, we get

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \left(\int_{\mathbb{D}} \left(\int_{[0,1)} \frac{1}{(1-|tz|)|1-tz|} d\mu(t)\right)^2 (1-|\sigma_a(z)|^2)^p dA(z)\right)^{\frac{1}{2}} \\
&\leq \sup_{a\in\mathbb{D}} \int_{[0,1)} \left(\int_{\mathbb{D}} \frac{1}{(1-|tz|)^2|1-tz|^2} (1-|\sigma_a(z)|^2)^p dA(z)\right)^{\frac{1}{2}} d\mu(t) \\
&\lesssim \sup_{a\in\mathbb{D}} (1-|a|^2)^{\frac{p}{2}} \int_{[0,1)} \frac{1}{(1-t)^{1-c}} d\mu(t) \Big(\int_{\mathbb{D}} \frac{(1-|z|^2)^{p-2c}}{|1-tz|^2|1-\bar{a}z|^{2p}} dA(z)\Big)^{\frac{1}{2}} \\
&\lesssim \sup_{a\in\mathbb{D}} \int_{[0,1)} \frac{(1-|a|^2)^{\frac{p}{2}}}{(1-t)^{1-c}|1-ta|^{\frac{p}{2}+c}} d\mu(t) < \infty.
\end{aligned} \tag{3.3}$$

Similarly, it follows from Lemma F and Proposition 2.1 that

$$
\begin{aligned}
&\sup_{a\in\mathbb{D}}\left(\int_{\mathbb{D}}\left(\int_{[0,1)}\frac{1}{|1-tz|^2}d\mu(t)\right)^2(1-|\sigma_a(z)|^2)^p dA(z)\right)^{\frac{1}{2}}\\
&\quad\le \sup_{a\in\mathbb{D}}\int_{[0,1)}\left(\int_{\mathbb{D}}\frac{1}{|1-tz|^4}(1-|\sigma_a(z)|^2)^p dA(z)\right)^{\frac{1}{2}}d\mu(t)\\
&\quad\lesssim \sup_{a\in\mathbb{D}}\int_{[0,1)}\frac{(1-|a|^2)^{\frac{p}{2}}}{(1-t^2)^{1-\frac{p}{2}}|1-at|^p}d\mu(t)<\infty.
\end{aligned}
\tag{3.4}
$$

From (3.1), (3.3) and (3.4), we get that $\mathcal{C}_\mu(f)\in\mathcal{Q}_p$. The proof is complete. □

Remark 3 Set $d\mu_0(x) = dx$ on $[0, 1)$. Then $d\mu_0$ is a Carleson measure and $\mathcal{C}_{\mu_0}(1)(z) = \frac{1}{z}\log\frac{1}{1-z}$. Clearly, the function $\mathcal{C}_{\mu_0}(1)$ is not in the Dirichlet space. Thus Theorem 3.1 does not hold when $p = 0$.

Note that $\mathcal{Q}_p = \mathcal{B}$ for any $p > 1$. Theorem 3.1 generalizes Theorem 5 in [16] from the Bloch space $\mathcal{B}$ to all $\mathcal{Q}_p$ spaces. For $p = 1$, Theorem 3.1 gives an answer to a question raised in [16, p. 20]. The proof given here highlights the role of Proposition 2.1. In the next section, we give a more general result where an alternative proof of Theorem 3.1 will be provided.

4 *s*-Carleson measures and the range of another Cesàro-like operator acting on H^{∞}

It is also natural to consider how the characterization of s-Carleson measures in Proposition 2.2 can play a role in the investigation of the range of Cesàro-like operators acting on H^{∞}. We consider this topic by another kind of Cesàro-like operators.

Suppose $0 < s < \infty$ and μ is a finite positive Borel measure on $[0, 1)$. For $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $H(\mathbb{D})$, we define

$$\mathcal{C}_{\mu,s}(f)(z) = \sum_{n=0}^{\infty}\left(\mu_n\sum_{k=0}^{n}\frac{\Gamma(n-k+s)}{\Gamma(s)(n-k)!}a_k\right)z^n,\quad z\in\mathbb{D}.$$

Clearly, $\mathcal{C}_{\mu,1}$ is equal to $\mathcal{C}_\mu$.

Lemma 4.1 *Suppose* $0 < s < \infty$ *and* μ *is a finite positive Borel measure on* $[0, 1)$. *Then*

$$\mathcal{C}_{\mu,s}(f)(z) = \int_{[0,1)}\frac{f(tz)}{(1-tz)^s}d\mu(t)$$

for $f\in H(\mathbb{D})$.

Proof The proof follows from a simple calculation with power series. We omit it. □

We have the following result.

Theorem 4.2 *Suppose* $0 < s < \infty$ *and* μ *is a finite positive Borel measure on* $[0, 1)$. *Let* $\max\{1, \frac{1}{s}\} < p < \infty$ *and let* X *be a subspace of* $H(\mathbb{D})$ *with* $\Lambda^p_{1/p} \subseteq X \subseteq \mathcal{B}$. *Then* $\mathcal{C}_{\mu,s}(H^\infty) \subseteq X$ *if and only if* μ *is an* s*-Carleson measure.*

Proof Let $\mathcal{C}_{\mu,s}(H^\infty) \subseteq X$. Then $\mathcal{C}_{\mu,s}(1) \in X$; that is, $f_{\mu,s} \in X$. It follows from Proposition 2.2 that μ is an s-Carleson measure.

On the other hand, let μ be an s-Carleson measure and $f \in H^\infty$. By Lemma 4.1, we see

$$\mathcal{C}_{\mu,s}(f)'(z) = \int_{[0,1)} \frac{tf'(tz)}{(1-tz)^s} d\mu(t) + \int_{[0,1)} \frac{stf(tz)}{(1-tz)^{s+1}} d\mu(t), \quad z \in \mathbb{D}.$$

Then

$$\begin{aligned}
&\sup_{0<r<1} (1-r)^{1-\frac{1}{p}} \left(\frac{1}{2\pi} \int_0^{2\pi} |\mathcal{C}_{\mu,s}(f)'(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} \\
&\lesssim \|f\|_{\mathcal{B}} \sup_{0<r<1} (1-r)^{1-\frac{1}{p}} \left(\frac{1}{2\pi} \int_0^{2\pi} \left(\int_{[0,1)} \frac{1}{|1-tre^{i\theta}|^s(1-tr)} d\mu(t) \right)^p d\theta \right)^{\frac{1}{p}} \\
&\quad + \|f\|_{H^\infty} \sup_{0<r<1} (1-r)^{1-\frac{1}{p}} \left(\frac{1}{2\pi} \int_0^{2\pi} \left(\int_{[0,1)} \frac{1}{|1-tre^{i\theta}|^{s+1}} d\mu(t) \right)^p d\theta \right)^{\frac{1}{p}}.
\end{aligned} \tag{4.1}$$

Note that $ps > 1$. By the Minkowski inequality, Lemma E and Lemma A, we deduce

$$\begin{aligned}
&\sup_{0<r<1} (1-r)^{1-\frac{1}{p}} \left(\frac{1}{2\pi} \int_0^{2\pi} \left(\int_{[0,1)} \frac{1}{|1-tre^{i\theta}|^s(1-tr)} d\mu(t) \right)^p d\theta \right)^{\frac{1}{p}} \\
&\quad \le \sup_{0<r<1} (1-r)^{1-\frac{1}{p}} \int_{[0,1)} \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1-tre^{i\theta}|^{sp}(1-tr)^p} d\theta \right)^{\frac{1}{p}} d\mu(t) \\
&\quad \lesssim \sup_{0<r<1} (1-r)^{1-\frac{1}{p}} \int_{[0,1)} \frac{1}{(1-tr)^{s+1-\frac{1}{p}}} d\mu(t) < \infty,
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
&\sup_{0<r<1} (1-r)^{1-\frac{1}{p}} \left(\frac{1}{2\pi} \int_0^{2\pi} \left(\int_{[0,1)} \frac{1}{|1-tre^{i\theta}|^{s+1}} d\mu(t) \right)^p d\theta \right)^{\frac{1}{p}} \\
&\quad \lesssim \sup_{0<r<1} (1-r)^{1-\frac{1}{p}} \int_{[0,1)} \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1-tre^{i\theta}|^{(s+1)p}} d\theta \right)^{\frac{1}{p}} d\mu(t) \\
&\quad \lesssim \sup_{0<r<1} (1-r)^{1-\frac{1}{p}} \int_{[0,1)} \frac{1}{(1-tr)^{s+1-\frac{1}{p}}} d\mu(t) < \infty.
\end{aligned} \tag{4.3}$$

From (4.1), (4.2) and (4.3), $\mathcal{C}_{\mu,s}(f) \in \Lambda^p_{1/p}$. Note that $\Lambda^p_{1/p} \subseteq X$. The desired result follows. □

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