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Existence of solutions for some systems of integro-differential equations with transport and superdiffusion

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Abstract

We establish the existence in the sense of sequences of solutions for certain systems of integro-differential equations which involve the drift terms and the square root of the one dimensional negative Laplace operator, on the whole real line or on a finite interval with periodic boundary conditions in the corresponding H^2 spaces. The argument is based on the fixed point technique when the elliptic systems contain first order differential operators with and without Fredholm property. It is proven that, under the reasonable technical conditions, the convergence in L^1 of the integral kernels yields the existence and convergence in H^2 of the solutions. We emphasize that the study of the systems is more complicated than of the scalar case and requires to overcome more cumbersome technicalities.

Keywords Solvability conditions · Non Fredholm operators · Integro-differential systems · Drift terms · Superdiffusion

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1 Introduction

Let us recall that a linear operator L acting from a Banach space E into another Banach space F satisfies the Fredholm property if its image is closed, the dimension of its kernel and the codimension of its image are finite. Consequently, the problem $Lu = f$ is solvable if and only if $\phi_i(f) = 0$ for a finite number of functionals ϕ_i from the dual space F^*. Such properties of the Fredholm operators are broadly used in many methods of the linear and nonlinear analysis.

Elliptic equations in bounded domains with a sufficiently smooth boundary satisfy the Fredholm property if the ellipticity condition, proper ellipticity and Lopatinskii conditions are fulfilled (see e.g. [1, 9, 24, 27]). This is the main result of the theory of linear elliptic problems. In the situation of unbounded domains, these conditions may not be sufficient and the Fredholm property may not be satisfied. For instance, the Laplace operator, $Lu = \Delta u$, in $\mathbb{R}^d$ fails to satisfy the Fredholm property when considered in Hölder spaces, $L : C^{2+\alpha}(\mathbb{R}^d) \to C^{\alpha}(\mathbb{R}^d)$, or in Sobolev spaces, $L : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

Linear elliptic equations in unbounded domains satisfy the Fredholm property if and only if, in addition to the conditions listed above, the limiting operators are invertible (see [28]). In some trivial cases, the limiting operators can be constructed explicitly. For example, if

$$Lu = a(x)u'' + b(x)u' + c(x)u, \quad x \in \mathbb{R},$$

where the coefficients of the operator have limits at infinity,

$$a_{\pm} = \lim_{x \to \pm\infty} a(x), \quad b_{\pm} = \lim_{x \to \pm\infty} b(x), \quad c_{\pm} = \lim_{x \to \pm\infty} c(x),$$

the limiting operators are given by:

$$L_{\pm}u = a_{\pm}u'' + b_{\pm}u' + c_{\pm}u.$$

Since the coefficients are constants, the essential spectrum of the operator, that is the set of complex numbers λ for which the operator $L - \lambda$ does not satisfy the Fredholm property, can be explicitly found by means of the Fourier transform:

$$\lambda_{\pm}(\xi) = -a_{\pm}\xi^2 + b_{\pm}i\xi + c_{\pm}, \quad \xi \in \mathbb{R}.$$

The invertibility of the limiting operators is equivalent to the condition that the origin does not belong to the essential spectrum.

In the case of general elliptic equations, the same assertions hold true. The Fredholm property is satisfied if the essential spectrum does not contain the origin or if the limiting operators are invertible. However, such conditions may not be explicitly written.

In the case of non-Fredholm operators the usual solvability relations may not be applicable and the solvability conditions are, in general, not known. There are certain classes of operators for which the solvability conditions are obtained. We illustrate them with the following example. Consider the equation

$$Lu \equiv \Delta u + au = f \tag{1.1}$$

in $\mathbb{R}^d$, where a is a positive constant. Such operator L coincides with its limiting operators. The homogeneous problem has a nontrivial bounded solution. Thus, the Fredholm property is not satisfied. However, since the operator has constant coefficients, we can apply the Fourier transform and find the solution explicitly. The solvability conditions can be formulated as follows. If $f \in L^2(\mathbb{R}^d)$ and $xf \in L^1(\mathbb{R}^d)$, then there exists a solution of this problem in $H^2(\mathbb{R}^d)$ if and only if

$$\left(f(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\right)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{\sqrt{a}} \quad a.e.$$

(see [35]). Here $S^d_{\sqrt{a}}$ denotes the sphere in $\mathbb{R}^d$ of radius $\sqrt{a}$ centered at the origin. Therefore, though the operator fails to satisfy the Fredholm property, the solvability relations are formulated analogously. However, this similarity is only formal because the range of the operator is not closed.

In the case of the operator with a scalar potential,

$$Lu \equiv \Delta u + a(x)u = f,$$

the Fourier transform is not directly applicable. Nevertheless, the solvability conditions in $\mathbb{R}^3$ can be obtained by a rather sophisticated application of the theory of self-adjoint operators (see [32]). As before, the solvability relations are formulated in terms of the orthogonality to the solutions of the homogeneous adjoint equation. There are several other examples of linear elliptic non Fredholm operators for which the solvability relations can be derived (see [13, 17, 28, 29, 32, 34, 35]).

The solvability conditions play a significant role in the analysis of the nonlinear elliptic equations. In the case of non-Fredholm operators, in spite of some progress in the understanding of the linear problems, there exist only few examples where the nonlinear non-Fredholm operators are analyzed (see [8, 12, 15, 33, 35], [38]). The article [10] is devoted to the studies of the finite and infinite dimensional attractors for evolution equations of mathematical physics. The large time behavior of solutions of a class of fourth-order parabolic equations defined on unbounded domains using the Kolmogorov ε-entropy as a measure was studied in [11]. In [18] the authors consider the attractor for a nonlinear reaction-diffusion system in an unbounded domain in $\mathbb{R}^3$. The articles [19, 26] deal with the understanding of the Fredholm and properness properties of the quasilinear elliptic systems of second order and of the operators of this kind on $\mathbb{R}^N$. In [20] the authors establish the exponential decay and investigate the Fredholm properties in second order quasilinear elliptic systems. In the present work we consider another class of stationary nonlinear systems of equations for which the Fredholm property may not be satisfied:

$$-\sqrt{-\frac{d^2}{dx^2}}u_k + b_k\frac{du_k}{dx} + a_k u_k$$
$$+\int_\Omega G_k(x-y)F_k(u_1(y), u_2(y), ..., u_N(y), y)dy = 0, \quad x \in \Omega, \tag{1.2}$$

where $a_k \geq 0, \ b_k \in \mathbb{R}, \ b_k \neq 0$ are the constants, $1 \leq k \leq N, \ N \geq 2$ and $\Omega \subseteq \mathbb{R}$. Here and throughout the article the vector function

$$u := (u_1, u_2, ..., u_N)^T \in \mathbb{R}^N. \tag{1.3}$$

The nonlocal operator $\sqrt{-\frac{d^2}{dx^2}} : H^1(\Omega) \to L^2(\Omega)$ is defined by means of the spectral calculus and is actively used, for example in the studies of the superdiffusion problems (see e.g. [36, 37] and the references therein). Superdiffusion can be described as a random process of particle motion characterized by the probability density distribution of the jump length. The moments of this density distribution are finite in the case of the normal diffusion, but this is not the case for the superdiffusion. Asymptotic behavior at the infinity of the probability density function determines the value of the power of the negative Laplace operator (see [25]). For the simplicity of the presentation we restrict ourselves to the one dimensional situation (the multidimensional cases are more technical and will be considered in our forthcoming article). The study of the solvability of the integro- differential system (1.2) is more complicated than in the single nonlocal equation case covered in [16]. It requires the use of the Sobolev spaces for the vector functions, which is more cumbersome, especially in the situation on the finite interval with periodic boundary conditions, where we use the constrained subspaces. Moreover, in the argument in our system case we use the auxiliary expressions (5.4), (5.10), (5.27), (5.32) depending on the additional index $1 \leq k \leq N, \ N \geq 2$, which is an extra technicality. In the population dynamics the integro-differential equations describe the models with the intra-specific competition and the nonlocal consumption of resources (see e.g. [2, 4]). We use the explicit form of the solvability conditions and study the existence of solutions of this nonlinear system. The studies of the solutions of the integro-differential problems with the drift terms are crucial for the understanding of the emergence and propagation of patterns in the theory of speciation (see [30]). The solvability of the linear equation containing the Laplacian with the transport term was considered in [34], see also [5]. In the situation when the drift terms are absent, namely when $b_k = 0, \ 1 \leq k \leq N$, the system analogous to (1.2) was discussed in [37] (see also [36]). Verification of biomedical processes with anomalous diffusion, transport and interaction of species was accomplished in [14]. Existence of nontrivial steady states for populations structured with respect to space and a continuous trait was established in [3]. Trend to equilibrium for reaction-diffusion systems arising from complex balanced chemical reaction networks was studied in [7]. The entropy method for generalized Poisson-Nernst-Planck equations was developed in [21].

2 Formulation of the results

The technical assumptions of the present article will be analogical to the ones of Efendiev and Vougalter [16], adapted to the work with vector functions. It is also more difficult to perform the analysis in the Sobolev spaces for vector functions, especially in the system on our finite interval with periodic boundary conditions when the constraints are imposed on some of the components. The nonlinear part of problem (1.2) will satisfy the following regularity conditions.

Assumption 1 Let $1 \leq k \leq N$. Functions $F_k(u, x) : \mathbb{R}^N \times \Omega \to \mathbb{R}$ are satisfying the Caratheodory condition (see [23]), so that

$$\sqrt{\sum_{k=1}^{N} F_k^2(u, x)} \leq K|u|_{\mathbb{R}^N} + h(x) \quad for \quad u \in \mathbb{R}^N, \ x \in \Omega \tag{2.1}$$

with a constant $K > 0$ and $h(x) : \Omega \to \mathbb{R}^+, \quad h(x) \in L^2(\Omega)$. Furthermore, they are Lipschitz continuous functions, so that for any $u^{(1),(2)} \in \mathbb{R}^N, \quad x \in \Omega$:

$$\sqrt{\sum_{k=1}^{N} (F_k(u^{(1)}, x) - F_k(u^{(2)}, x))^2} \leq L|u^{(1)} - u^{(2)}|_{\mathbb{R}^N}, \tag{2.2}$$

with a constant $L > 0$.

Here and further down the norm of a vector function given by (1.3) is:

$$|u|_{\mathbb{R}^N} := \sqrt{\sum_{k=1}^{N} u_k^2}.$$

The work [6] deals with the solvability of a local elliptic problem in a bounded domain in $\mathbb{R}^N$. The nonlinear function involved there was allowed to have a sublinear growth. Note that Assumption 1 is actively used in the proofs of our theorems. We require the sublinear growth for our nonlinear functions, so that the boundedness and the continuity of the nonlinear maps from L^2 to L^2 are equivalent. In order to establish the solvabity of (1.2), we will use the auxiliary system with $1 \leq k \leq N$, namely

$$\sqrt{-\frac{d^2}{dx^2}u_k} - b_k \frac{du_k}{dx} - a_k u_k = \int_\Omega G_k(x - y) F_k(v_1(y), v_2(y), ..., v_N(y), y)dy, \tag{2.3}$$

where $a_k \geq 0, \ b_k \in \mathbb{R}, \ b_k \neq 0$ are the constants. Let us denote

$$(f_1(x), f_2(x))_{L^2(\Omega)} := \int_\Omega f_1(x)\bar{f}_2(x)dx, \tag{2.4}$$

with a slight abuse of notations when these functions are not square integrable, like for instance those involved in orthogonality relations (5.6) below. Indeed, if $f_1(x) \in$

$L^1(\Omega)$ and $f_2(x) \in L^\infty(\Omega)$, then the integral in the right side of (2.4) is well defined. In the first part of the article we consider the situation on the whole real line, $\Omega = \mathbb{R}$, so that the appropriate Sobolev space is equipped with the norm

$$\|\phi\|^2_{H^2(\mathbb{R})} := \|\phi\|^2_{L^2(\mathbb{R})} + \left\|\frac{d^2\phi}{dx^2}\right\|^2_{L^2(\mathbb{R})}. \tag{2.5}$$

Then for a vector function given by (1.3), we have

$$\|u\|^2_{H^2(\mathbb{R},\mathbb{R}^N)} := \sum_{k=1}^{N} \|u_k\|^2_{H^2(\mathbb{R})} = \sum_{k=1}^{N} \left\{ \|u_k\|^2_{L^2(\mathbb{R})} + \left\|\frac{d^2 u_k}{dx^2}\right\|^2_{L^2(\mathbb{R})} \right\}. \tag{2.6}$$

We also use the norm

$$\|u\|^2_{L^2(\mathbb{R},\mathbb{R}^N)} := \sum_{k=1}^{N} \|u_k\|^2_{L^2(\mathbb{R})}.$$

By means of Assumption 1 above, we are not allowed to consider the higher powers of our nonlinearities, than the first one. This is restrictive from the point of view of the applications. But this guarantees that our nonlinear vector function is a bounded and continuous map from $L^2(\Omega, \mathbb{R}^N)$ to $L^2(\Omega, \mathbb{R}^N)$. The main issue for our system of equations above is that in the absence of the drift terms we were dealing with the self-adjoint, non Fredholm operators

$$\sqrt{-\frac{d^2}{dx^2}} - a_k : H^1(\mathbb{R}) \to L^2(\mathbb{R}), \quad a_k \geq 0,$$

which was the obstacle to solve our system (see [36, 37]). The similar situations but in linear problems, both self- adjoint and non self-adjoint containing the differential operators without the Fredholm property have been treated extensively in recent years (see [28, 29, 32, 34, 35]). However, the situation is different when the constants in the drift terms $b_k \neq 0$. For $1 \leq k \leq N$, the operators

$$L_{a,\,b,\,k} := \sqrt{-\frac{d^2}{dx^2}} - b_k \frac{d}{dx} - a_k : \quad H^1(\mathbb{R}) \to L^2(\mathbb{R}), \tag{2.7}$$

with $a_k \geq 0$ and $b_k \in \mathbb{R},\ b_k \neq 0$ contained in the left side of the system of Eq. (2.3) are non-selfadjoint. By means of the standard Fourier transform, it can be trivially obtained that the essential spectra of such operators $L_{a,\,b,\,k}$ are given by

$$\lambda_{a,\,b,\,k}(p) = |p| - a_k - i b_k p, \quad p \in \mathbb{R}.$$

Clearly, for $a_k > 0$ the operators $L_{a,\,b,\,k}$ satisfy the Fredholm property, since their essential spectra do not contain the origin. But when $a_k = 0$, our operators $L_{a,\,b,\,k}$ fail to satisfy the Fredholm property because the origin belongs to their essential spectra. We establish that under the reasonable technical conditions system (2.3) defines a map $T_{a,\ b} : H^2(\mathbb{R}, \mathbb{R}^N) \to H^2(\mathbb{R}, \mathbb{R}^N)$, which is a strict contraction.

Theorem 1 *Let* $\Omega = \mathbb{R},\ N \geq 2,\ 1 \leq l \leq N-1,\ 1 \leq k \leq N,\ b_k \in \mathbb{R},\ b_k \neq 0$ *and* $G_k(x) : \mathbb{R} \to \mathbb{R},\ G_k(x) \in W^{1,1}(\mathbb{R})$ *and Assumption 1 holds.*

(I) *Let* $a_k > 0$ *for* $1 \leq k \leq l$.
(II) *Let* $a_k = 0$ *for* $l+1 \leq k \leq N$, *additionally* $xG_k(x) \in L^1(\mathbb{R})$, *orthogonality conditions* (5.6) *hold and* $2\sqrt{\pi} N_{a,\ b} L < 1$, *where* $N_{a,\ b}$ *is defined in* (5.5) *below. Then the map* $v \mapsto T_{a,b}v = u$ *on* $H^2(\mathbb{R}, \mathbb{R}^N)$ *defined by system* (2.3) *has a unique fixed point* $v^{(a,b)}$, *which is the only solution of the system of Eq.* (1.2) *in* $H^2(\mathbb{R}, \mathbb{R}^N)$.

The fixed point $v^{(a,b)}$ is nontrivial provided that for a certain $1 \leq k \leq N$ the intersection of supports of the Fourier transforms of functions $supp\widehat{F_k(0,x)} \cap supp\widehat{G_k}$ is a set of nonzero Lebesgue measure in $\mathbb{R}$.
Note that in the case (I) of the theorem above, when $a_k > 0$, as distinct part (I) of Assumption 2 of [37] describing the problem without the drift term, the orthogonality conditions are not needed. Let us introduce the sequence of approximate systems of equations related to problem (1.2) on the whole real line, namely

$$-\sqrt{-\frac{d^2}{dx^2}}u_k^{(m)} + b_k\frac{du_k^{(m)}}{dx} + a_k u_k^{(m)} + \int_{-\infty}^{\infty} G_{k,m}(x-y)F_k(u_1^{(m)}(y), u_2^{(m)}(y), ..., u_N^{(m)}(y), y)dy = 0 \tag{2.8}$$

with the constants $a_k \geq 0,\ b_k \in \mathbb{R},\ b_k \neq 0,\ 1 \leq k \leq N$ and $m \in \mathbb{N}$. Each sequence of kernels $\{G_{k,m}(x)\}_{m=1}^{\infty}$ tends to $G_k(x)$ as $m \to \infty$ in the appropriate function spaces discussed below. We establish that, under the given technical conditions, each of systems of Eq. (2.8) has a unique solution $u^{(m)}(x) \in H^2(\mathbb{R}, \mathbb{R}^N)$, limiting system (1.2) possesses a unique solution $u(x) \in H^2(\mathbb{R}, \mathbb{R}^N)$, and $u^{(m)}(x) \to u(x)$ in $H^2(\mathbb{R}, \mathbb{R}^N)$ as $m \to \infty$. This is the so-called *existence of solutions in the sense of sequences*. In such case, the solvability conditions can be formulated for the iterated kernels $G_{k,m}$. They yield the convergence of the kernels in terms of the Fourier transforms (see the Appendix) and, consequently, the convergence of the solutions (Theorems 2, 4). The analogical ideas in the sense of the standard Schrödinger type operators were exploited in [13, 31]. Our second main statement is as follows.

Theorem 2 *Let* $\Omega = \mathbb{R},\ N \geq 2,\ 1 \leq l \leq N-1,\ 1 \leq k \leq N,\ b_k \in \mathbb{R},\ b_k \neq 0,\ m \in \mathbb{N},\ G_{k,m}(x) : \mathbb{R} \to \mathbb{R},\ G_{k,m}(x) \in W^{1,1}(\mathbb{R})$, *so that* $G_{k,m}(x) \to G_k(x)$ *in* $W^{1,1}(\mathbb{R})$ *as* $m \to \infty$. *Let Assumption 1 hold.*

(I) *Let* $a_k > 0$ *for* $1 \leq k \leq l$.
(II) *Let* $a_k = 0$ *for* $l+1 \leq k \leq N$. *Assume that* $xG_{k,m}(x) \in L^1(\mathbb{R}),\ xG_{k,m}(x) \to xG_k(x)$ *in* $L^1(\mathbb{R})$ *as* $m \to \infty$, *orthogonality conditions* (5.12) *are valid along with upper bound* (5.13). *Then each system* (2.8) *possesses a unique solution*

$u^{(m)}(x) \in H^2(\mathbb{R}, \mathbb{R}^N)$, and limiting problem (1.2) admits a unique solution $u(x) \in H^2(\mathbb{R}, \mathbb{R}^N)$, so that $u^{(m)}(x) \to u(x)$ in $H^2(\mathbb{R}, \mathbb{R}^N)$ as $m \to \infty$.

The unique solution $u^{(m)}(x)$ of each system (2.8) is nontrivial provided that for a certain $1 \leq k \leq N$ the intersection of supports of the Fourier images of functions $supp\widehat{F_k(0,x)} \cap supp\widehat{G}_{k,m}$ is a set of nonzero Lebesgue measure in $\mathbb{R}$. Similarly, the unique solution $u(x)$ of limiting system (1.2) does not vanish identically on the real line if $supp\widehat{F_k(0,x)} \cap supp\widehat{G_k}$ is a set of nonzero Lebesgue measure in $\mathbb{R}$ for some $1 \leq k \leq N$.
The second part of the article is devoted to the studies of the analogical system of equations on the finite interval $\Omega = I := [0, 2\pi]$ with periodic boundary conditions. The appropriate function space is given by

$$H^2(I) = \{v(x) : I \to \mathbb{R} \mid v(x), v''(x) \in L^2(I), \quad v(0) = v(2\pi), \quad v'(0) = v'(2\pi)\}.$$

We aim at $u_k(x) \in H^2(I),\ 1 \leq k \leq l$. For the technical purposes, we will use the following auxiliary constrained subspace

$$H_0^2(I) = \{v(x) \in H^2(I) \mid (v(x), 1)_{L^2(I)} = 0\}, \tag{2.9}$$

which is a Hilbert space as well (see e.g. Chapter 2.1 of [22]). The aim is to have $u_k(x) \in H_0^2(I),\ l+1 \leq k \leq N$. Similarly,

$$H_0^1(I) = \{v(x) \in H^1(I) \mid (v(x), 1)_{L^2(I)} = 0\}.$$

The resulting space used to establish the existence in the sense of sequences of solutions $u(x) : I \to \mathbb{R}^N$ of system (1.2) will be the direct sum of the spaces given above, namely

$$H_c^2(I, \mathbb{R}^N) = \oplus_{k=1}^l H^2(I) \oplus_{k=l+1}^N H_0^2(I).$$

The corresponding Sobolev norm is given by

$$\|u\|_{H_c^2(I,\mathbb{R}^N)}^2 := \sum_{k=1}^N \left\{ \|u_k\|_{L^2(I)}^2 + \|u_k''\|_{L^2(I)}^2 \right\},$$

with $u(x) : I \to \mathbb{R}^N$. Another useful norm is

$$\|u\|_{L^2(I,\mathbb{R}^N)}^2 := \sum_{k=1}^N \|u_k\|_{L^2(I)}^2.$$

We establish that system (2.3) in this case defines a map $\tau_{a,b} : H_c^2(I, \mathbb{R}^N) \to H_c^2(I, \mathbb{R}^N)$. This map will be a strict contraction under the stated technical conditions.

Theorem 3 *Let* $\Omega = I,\ N \geq 2,\ 1 \leq l \leq N-1,\ 1 \leq k \leq N,\ b_k \in \mathbb{R},\ b_k \neq 0$ *and* $G_k(x) : I \to \mathbb{R},\ G_k(x) \in C(I),\ \dfrac{dG_k(x)}{dx} \in L^1(I),\ G_k(0) = G_k(2\pi),\ F_k(u, 0) = F_k(u, 2\pi)$ *for* $u \in \mathbb{R}^N$ *and Assumption 1 holds.*

(I) *Let* $a_k > 0$ *for* $1 \leq k \leq l$.
(II) *Let* $a_k = 0$ *for* $l+1 \leq k \leq N$, *orthogonality relations (5.29) hold and* $2\sqrt{\pi}\mathcal{N}_{a,\,b}L < 1$ *with* $\mathcal{N}_{a,\,b}$ *defined in (5.28). Then the map* $\tau_{a,b}v = u$ *on* $H_c^2(I, \mathbb{R}^N)$ *defined by system of Eq. (2.3) has a unique fixed point* $v^{(a,b)}$, *the only solution of system (1.2) in* $H_c^2(I, \mathbb{R}^N)$.

The fixed point $v^{(a,b)}$ is nontrivial on the interval I provided that the Fourier coefficients $G_{k,n}F_k(0, x)_n \neq 0$ for some $1 \leq k \leq N$ and a certain $n \in \mathbb{Z}$.

Remark 1 We use the constrained subspace $H_0^2(I)$ in the direct sum of spaces $H_c^2(I, \mathbb{R}^N)$, such that the Fredholm operators $\sqrt{-\frac{d^2}{dx^2}} - b_k\frac{d}{dx} : H_0^1(I) \to L^2(I)$ for $l+1 \leq k \leq N$ have the trivial kernels.

To show the existence in the sense of sequences of solutions for our integro-differential system of equations on the interval I, we consider the sequence of approximate systems, similarly to the situation on the whole real line with $m \in \mathbb{N},\ 1 \leq k \leq N$ and the constants $a_k \geq 0,\ b_k \in \mathbb{R},\ b_k \neq 0$, so that

$$-\sqrt{-\frac{d^2}{dx^2}}u_k^{(m)} + b_k\frac{du_k^{(m)}}{dx} + a_k u_k^{(m)} + \int_0^{2\pi} G_{k,m}(x-y)F_k(u_1^{(m)}(y), u_2^{(m)}(y), ..., u_N^{(m)}(y), y)dy = 0. \quad (2.10)$$

The final main statement of the article is as follows.

Theorem 4 *Let* $\Omega = I,\ N \geq 2,\ 1 \leq l \leq N-1,\ 1 \leq k \leq N,\ b_k \in \mathbb{R},\ b_k \neq 0,\ m \in \mathbb{N},$

$$G_{k,m}(x) : I \to \mathbb{R},\ G_{k,m}(0) = G_{k,m}(2\pi),\ G_{k,m}(x) \in C(I),\ \frac{dG_{k,m}(x)}{dx} \in L^1(I),$$

so that

$$G_{k,m}(x) \to G_k(x) \quad in \quad C(I), \quad \frac{dG_{k,m}(x)}{dx} \to \frac{dG_k(x)}{dx} \quad in \quad L^1(I) \quad as \quad m \to \infty,$$

$F_k(u, 0) = F_k(u, 2\pi)$ *for* $u \in \mathbb{R}^N$. *Let Assumption 1 hold.*

(I) *Let* $a_k > 0$ *for* $1 \leq k \leq l$.
(II) *Let* $a_k = 0$ *for* $l+1 \leq k \leq N$. *Assume that orthogonality relations (5.34) are valid along with upper bound (5.35). Then each system (2.10) possesses a unique solution* $u^{(m)}(x) \in H_c^2(I, \mathbb{R}^N)$ *and the limiting system of Eq. (1.2) has a unique solution* $u(x) \in H_c^2(I, \mathbb{R}^N)$, *so that* $u^{(m)}(x) \to u(x)$ *in* $H_c^2(I, \mathbb{R}^N)$ *as* $m \to \infty$.

The unique solution $u^{(m)}(x)$ of each system of Eq. (2.10) does not vanish identically on the interval I provided that the Fourier coefficients $G_{k,m,n}F_k(0,x)_n \neq 0$ for some $1 \le k \le N$ and a certain $n \in \mathbb{Z}$. Similarly, the unique solution $u(x)$ of limiting system (1.2) is nontrivial on I if $G_{k,n}F_k(0,x)_n \neq 0$ for a certain $1 \le k \le N$ and some $n \in \mathbb{Z}$.

Remark 2 In the work we deal with the real valued vector functions by means of the assumptions on $F_k(u,x)$, $G_{k,m}(x)$ and $G_k(x)$ contained in the integral terms of the approximate and limiting systems of equations discussed above.

Remark 3 The significance of Theorems 2 and 4 of the article is the continuous dependence of the solutions of our systems with respect to the integral kernels.

Remark 4 Such issues as the spectral properties of the corresponding linearized problems, the stability of the stationary solutions, the generalization of our approaches to the multidimensional spaces will be discussed in our consecutive articles.

3 The whole real line case

Proof of Theorem 1 First we suppose that in the case of $\Omega = \mathbb{R}$ for some $v \in H^2(\mathbb{R},\mathbb{R}^N)$ there exist two solutions $u^{(1),(2)} \in H^2(\mathbb{R},\mathbb{R}^N)$ of system (2.3). Then their difference $w(x) := u^{(1)}(x) - u^{(2)}(x) \in H^2(\mathbb{R},\mathbb{R}^N)$ will satisfy the homogeneous system of equations

$$\sqrt{-\frac{d^2}{dx^2}}w_k - b_k\frac{dw_k}{dx} - a_k w_k = 0, \quad 1 \le k \le N.$$

Since each operator $L_{a,\,b,\,k} : H^1(\mathbb{R}) \to L^2(\mathbb{R})$ defined in (2.7) does not have any nontrivial zero modes, $w(x)$ vanishes identically in $\mathbb{R}$.

Let us choose arbitrarily $v(x) \in H^2(\mathbb{R},\mathbb{R}^N)$ and apply the standard Fourier transform (5.1) to both sides of system (2.3). Thus, we obtain

$$\widehat{u_k}(p) = \sqrt{2\pi}\,\frac{\widehat{G_k}(p)\widehat{f_k}(p)}{|p| - a_k - ib_k p}, \quad p^2\widehat{u_k}(p) = \sqrt{2\pi}\,\frac{p^2\widehat{G_k}(p)\widehat{f_k}(p)}{|p| - a_k - ib_k p}, \quad 1 \le k \le N, \tag{3.1}$$

where $\widehat{f_k}(p)$ stands for the Fourier image of $F_k(v(x),x)$. Evidently, we have the upper bounds

$$|\widehat{u_k}(p)| \le \sqrt{2\pi}N_{a,\,b,\,k}|\widehat{f_k}(p)| \quad and \quad |p^2\widehat{u_k}(p)| \le \sqrt{2\pi}N_{a,\,b,\,k}|\widehat{f_k}(p)|, \quad 1 \le k \le N.$$

Note that $N_{a,\,b,\,k} < \infty$ by virtue of Lemma A1 of the Appendix without any orthogonality conditions if $a_k > 0$ and under orthogonality relation (5.6) for $a_k = 0$. This allows us to derive the estimate from above on the norm

$$\|u\|^2_{H^2(\mathbb{R},\mathbb{R}^N)} = \sum_{k=1}^{N}\{\|\widehat{u_k}(p)\|^2_{L^2(\mathbb{R})} + \|p^2\widehat{u_k}(p)\|^2_{L^2(\mathbb{R})}\}$$

$$\leq 4\pi N_{a,\ b}^2 \sum_{k=1}^{N} \|F_k(v(x), x)\|_{L^2(\mathbb{R})}^2 \tag{3.2}$$

with $N_{a,\ b}$ defined in (5.5). Clearly, the right side of (3.2) is finite via inequality (2.1) of Assumption 1 above since $|v(x)|_{\mathbb{R}^N} \in L^2(\mathbb{R})$. Hence, for an arbitrary $v(x) \in H^2(\mathbb{R}, \mathbb{R}^N)$ there exists a unique solution $u(x) \in H^2(\mathbb{R}, \mathbb{R}^N)$ of system (2.3) with its Fourier image given by (3.1) and the map $T_{a,b} : H^2(\mathbb{R}, \mathbb{R}^N) \to H^2(\mathbb{R}, \mathbb{R}^N)$ is well defined. This allows us to choose arbitrary $v^{(1),(2)}(x) \in H^2(\mathbb{R}, \mathbb{R}^N)$, so that their images $u^{(1),(2)} = T_{a,b}v^{(1),(2)} \in H^2(\mathbb{R}, \mathbb{R}^N)$. By means of (2.3), we have for $1 \leq k \leq N$

$$\sqrt{-\frac{d^2}{dx^2}}u_k^{(1)} - b_k\frac{du_k^{(1)}}{dx} - au_k^{(1)} = \int_{-\infty}^{\infty} G_k(x-y)F_k(v_1^{(1)}(y), v_2^{(1)}(y), ..., v_N^{(1)}(y), y)dy, \tag{3.3}$$

$$\sqrt{-\frac{d^2}{dx^2}}u_k^{(2)} - b_k\frac{du_k^{(2)}}{dx} - au_k^{(2)} = \int_{-\infty}^{\infty} G_k(x-y)F_k(v_1^{(2)}(y), v_2^{(2)}(y), ..., v_N^{(2)}(y), y)dy. \tag{3.4}$$

We apply the standard Fourier transform (5.1) to both sides of systems (3.3) and (3.4). This yields for $1 \leq k \leq N$

$$\widehat{u_k^{(1)}}(p) = \sqrt{2\pi}\frac{\widehat{G_k}(p)\widehat{f_k^{(1)}}(p)}{|p| - a_k - ib_k p}, \quad p^2\widehat{u_k^{(1)}}(p) = \sqrt{2\pi}\frac{p^2\widehat{G_k}(p)\widehat{f_k^{(1)}}(p)}{|p| - a_k - ib_k p},$$
$$\widehat{u_k^{(2)}}(p) = \sqrt{2\pi}\frac{\widehat{G_k}(p)\widehat{f_k^{(2)}}(p)}{|p| - a_k - ib_k p}, \quad p^2\widehat{u_k^{(2)}}(p) = \sqrt{2\pi}\frac{p^2\widehat{G_k}(p)\widehat{f_k^{(2)}}(p)}{|p| - a_k - ib_k p}.$$

Here $\widehat{f_k^{(1)}}(p)$ and $\widehat{f_k^{(2)}}(p)$ denote the Fourier transforms of $F_k(v^{(1)}(x), x)$ and $F_k(v^{(2)}(x), x)$ respectively. Evidently, we have the upper bounds

$$\left|\widehat{u_k^{(1)}}(p) - \widehat{u_k^{(2)}}(p)\right| \leq \sqrt{2\pi}N_{a,\ b,\ k}\left|\widehat{f_k^{(1)}}(p) - \widehat{f_k^{(2)}}(p)\right|,$$
$$\left|p^2\widehat{u_k^{(1)}}(p) - p^2\widehat{u_k^{(2)}}(p)\right| \leq \sqrt{2\pi}N_{a,\ b,\ k}\left|\widehat{f_k^{(1)}}(p) - \widehat{f_k^{(2)}}(p)\right|,$$

where $1 \leq k \leq N$. This allows us to derive the inequality for the norms

$$\|u^{(1)} - u^{(2)}\|_{H^2(\mathbb{R},\mathbb{R}^N)}^2 \leq 4\pi N_{a,\ b}^2 \sum_{k=1}^{N} \|F_k(v^{(1)}(x), x) - F_k(v^{(2)}(x), x)\|_{L^2(\mathbb{R})}^2.$$

Obviously, $v_k^{(1),(2)}(x) \in H^2(\mathbb{R}) \subset L^\infty(\mathbb{R})$ due to the Sobolev embedding. By means of (2.2) of Assumption 1 we easily derive

$$\|T_{a,b}v_1 - T_{a,b}v_2\|_{H^2(\mathbb{R},\mathbb{R}^N)} \le 2\sqrt{\pi}N_{a,\,b}L\|v_1 - v_2\|_{H^2(\mathbb{R},\mathbb{R}^N)}. \tag{3.5}$$

The constant in the right side of (3.5) is less than via the one of our assumptions. By virtue of the Fixed Point Theorem, there exists a unique vector function $v^{(a,b)} \in H^2(\mathbb{R},\mathbb{R}^N)$, such that $T_{a,b}v^{(a,b)} = v^{(a,b)}$. This is the only solution of the system of Eq. (1.2) in $H^2(\mathbb{R},\mathbb{R}^N)$. Suppose $v^{(a,b)}(x)$ vanishes identically in $\mathbb{R}$. This will be a contradiction to our assumption that for some $1 \le k \le N$ the Fourier transforms of $G_k(x)$ and $F_k(0,x)$ are nontrivial on a set of nonzero Lebesgue measure on the real line. □

We turn our attention to establishing the existence in the sense of sequences of the solutions for our system of integro-differential equation on $\mathbb{R}$.

Proof of Theorem 2 By means of the result of Theorem 1 above, each system of equations (2.8) admits a unique solution $u^{(m)}(x) \in H^2(\mathbb{R},\mathbb{R}^N),\ m \in \mathbb{N}$. Limiting system (1.2) has a unique solution $u(x) \in H^2(\mathbb{R},\mathbb{R}^N)$ by virtue of Lemma A2 below along with Theorem 1. We apply the standard Fourier transform (5.1) to both sides of problems (1.2) and (2.8). This gives us for $1 \le k \le N,\ m \in \mathbb{N}$

$$\widehat{u_k}(p) = \sqrt{2\pi}\frac{\widehat{G_k}(p)\widehat{\varphi_k}(p)}{|p| - a_k - ib_k p}, \quad \widehat{u_k^{(m)}}(p) = \sqrt{2\pi}\frac{\widehat{G_{k,m}}(p)\widehat{\varphi_{k,m}}(p)}{|p| - a_k - ib_k p}. \tag{3.6}$$

Here $\widehat{\varphi_k}(p)$ and $\widehat{\varphi_{k,m}}(p)$ denote the Fourier transforms of $F_k(u(x), x)$ and $F_k(u^{(m)}(x), x)$ respectively. Evidently,

$$\left|\widehat{u_k^{(m)}}(p) - \widehat{u_k}(p)\right| \le \sqrt{2\pi}\left\|\frac{\widehat{G_{k,m}}(p)}{|p| - a_k - ib_k p} - \frac{\widehat{G_k}(p)}{|p| - a_k - ib_k p}\right\|_{L^\infty(\mathbb{R})}|\widehat{\varphi_k}(p)|$$
$$+\sqrt{2\pi}\left\|\frac{\widehat{G_{k,m}}(p)}{|p| - a_k - ib_k p}\right\|_{L^\infty(\mathbb{R})}|\widehat{\varphi_{k,m}}(p) - \widehat{\varphi_k}(p)|.$$

Hence,

$$\|u_k^{(m)} - u_k\|_{L^2(\mathbb{R})} \le \sqrt{2\pi}\left\|\frac{\widehat{G_{k,m}}(p)}{|p| - a_k - ib_k p} - \frac{\widehat{G_k}(p)}{|p| - a_k - ib_k p}\right\|_{L^\infty(\mathbb{R})}\|F_k(u(x), x)\|_{L^2(\mathbb{R})}$$
$$+\sqrt{2\pi}\left\|\frac{\widehat{G_{k,m}}(p)}{|p| - a_k - ib_k p}\right\|_{L^\infty(\mathbb{R})}\|F_k(u^{(m)}(x), x) - F_k(u(x), x)\|_{L^2(\mathbb{R})}.$$

By means of inequality (2.2) of Assumption 1 above we arrive at

$$\sqrt{\sum_{k=1}^N \|F_k(u^{(m)}(x), x) - F_k(u(x), x)\|_{L^2(\mathbb{R})}^2} \le L\|u^{(m)}(x) - u(x)\|_{L^2(\mathbb{R},\mathbb{R}^N)}. \tag{3.7}$$

Obviously, $u_k^{(m)}(x),\ u_k(x) \in H^2(\mathbb{R}) \subset L^\infty(\mathbb{R})$ for $1 \le k \le N,\ m \in \mathbb{N}$ via the Sobolev embedding. We derive

$$\|u^{(m)}(x) - u(x)\|^2_{L^2(\mathbb{R},\mathbb{R}^N)}$$
$$\le 4\pi \sum_{k=1}^N \left\| \frac{\widehat{G_{k,m}}(p)}{|p| - a_k - ib_k p} - \frac{\widehat{G_k}(p)}{|p| - a_k - ib_k p} \right\|^2_{L^\infty(\mathbb{R})} \|F_k(u(x), x)\|^2_{L^2(\mathbb{R})}$$
$$+ 4\pi \left[N_{a,\ b}^{(m)} \right]^2 L^2 \|u^{(m)}(x) - u(x)\|^2_{L^2(\mathbb{R},\mathbb{R}^N)},$$

so that via (5.13), we have $\|u^{(m)}(x) - u(x)\|^2_{L^2(\mathbb{R},\mathbb{R}^N)} \le$

$$\le \frac{4\pi}{\varepsilon(2-\varepsilon)} \sum_{k=1}^N \left\| \frac{\widehat{G_{k,m}}(p)}{|p| - a_k - ib_k p} - \frac{\widehat{G_k}(p)}{|p| - a_k - ib_k p} \right\|^2_{L^\infty(\mathbb{R})} \|F_k(u(x), x)\|^2_{L^2(\mathbb{R})}.$$

Upper bound (2.1) of Assumption 1 above gives us that $F_k(u(x), x) \in L^2(\mathbb{R}),\ 1 \le k \le N$ for $u(x) \in H^2(\mathbb{R}, \mathbb{R}^N)$. Thus,

$$u^{(m)}(x) \to u(x), \quad m \to \infty \tag{3.8}$$

in $L^2(\mathbb{R}, \mathbb{R}^N)$ by means of the result of Lemma A2 of the Appendix. Clearly, for $1 \le k \le N,\ m \in \mathbb{N}$ we have

$$p^2 \widehat{u_k}(p) = \sqrt{2\pi} \frac{p^2 \widehat{G_k}(p) \widehat{\varphi_k}(p)}{|p| - a_k - ib_k p}, \quad p^2 \widehat{u_k^{(m)}}(p) = \sqrt{2\pi} \frac{p^2 \widehat{G_{k,m}}(p) \widehat{\varphi_{k,m}}(p)}{|p| - a_k - ib_k p}.$$

Hence, we obtain

$$\left| p^2 \widehat{u_k^{(m)}}(p) - p^2 \widehat{u_k}(p) \right| \le \sqrt{2\pi} \left\| \frac{p^2 \widehat{G_{k,m}}(p)}{|p| - a_k - ib_k p} - \frac{p^2 \widehat{G_k}(p)}{|p| - a_k - ib_k p} \right\|_{L^\infty(\mathbb{R})} |\widehat{\varphi_k}(p)|$$
$$+ \sqrt{2\pi} \left\| \frac{p^2 \widehat{G_{k,m}}(p)}{|p| - a_k - ib_k p} \right\|_{L^\infty(\mathbb{R})} |\widehat{\varphi_{k,m}}(p) - \widehat{\varphi_k}(p)|.$$

Using inequality (3.7), we arrive at

$$\left\| \frac{d^2 u_k^{(m)}}{dx^2} - \frac{d^2 u_k}{dx^2} \right\|_{L^2(\mathbb{R})}$$
$$\le \sqrt{2\pi} \left\| \frac{p^2 \widehat{G_{k,m}}(p)}{|p| - a_k - ib_k p} - \frac{p^2 \widehat{G_k}(p)}{|p| - a_k - ib_k p} \right\|_{L^\infty(\mathbb{R})} \|F_k(u(x), x)\|_{L^2(\mathbb{R})}$$
$$+ \sqrt{2\pi} \left\| \frac{p^2 \widehat{G_{k,m}}(p)}{|p| - a_k - ib_k p} \right\|_{L^\infty(\mathbb{R})} L \|u^{(m)}(x) - u(x)\|_{L^2(\mathbb{R},\mathbb{R}^N)}.$$

By virtue of the result of Lemma A2 of the Appendix along with (3.8), we establish that

$$\frac{d^2u^{(m)}}{dx^2} \to \frac{d^2u}{dx^2} \quad in \quad L^2(\mathbb{R}, \mathbb{R}^N), \quad m \to \infty.$$

Definition (2.6) of the norm implies that $u^{(m)}(x) \to u(x)$ in $H^2(\mathbb{R}, \mathbb{R}^N)$ as $m \to \infty$.

We suppose that the unique solution $u^{(m)}(x)$ of the system of Eq. (2.8) studied above vanishes on the whole real line for some $m \in \mathbb{N}$. This will contradict to our assumption above that for some $1 \le k \le N$ the Fourier transforms of $G_{k,m}(x)$ and $F_k(0, x)$ are nontrivial on a set of nonzero Lebesgue measure on the real line. The similar reasoning holds for the unique solution $u(x)$ of limiting system of Eq. (1.2). □

4 The problem on the finite interval

Proof of Theorem 3 Evidently, each operator contained in the left side of the system of Eq. (2.3)

$$\mathcal{L}_{a,\, b,\, k} := \sqrt{-\frac{d^2}{dx^2}} - b_k \frac{d}{dx} - a_k : \quad H^1(I) \to L^2(I), \quad 1 \le k \le l \tag{4.1}$$

with the constants $a_k > 0,\ b_k \in \mathbb{R},\ b_k \neq 0$ is Fredholm, non-selfadjoint. Its set of eigenvalues is given by

$$\lambda_{a,b,k}(n) = |n| - a_k - i b_k n, \quad n \in \mathbb{Z}. \tag{4.2}$$

Its eigenfunctions are the standard Fourier harmonics $\frac{e^{inx}}{\sqrt{2\pi}}$, $n \in \mathbb{Z}$. When $a_k = 0$, we will exploit the analogous ideas in the constrained subspace (2.9) instead of $H^2(I)$. Clearly, the eigenvalues of each operator $\mathcal{L}_{a,\, b,\, k}$ are simple, as distinct from the analogical situation without the drift term, when the eigenvalues corresponding to $n \neq 0$ are two-fold degenerate (see [37]).

Let us suppose that for a certain $v(x) \in H_c^2(I, \mathbb{R}^N)$ there exist two solutions $u^{(1),(2)}(x) \in H_c^2(I, \mathbb{R}^N)$ of system (2.3) with $\Omega = I$. Then the vector function $w(x) := u^{(1)}(x) - u^{(2)}(x) \in H_c^2(I, \mathbb{R}^N)$ will satisfy the homogeneous system of equations

$$\sqrt{-\frac{d^2}{dx^2}} w_k - b_k \frac{dw_k}{dx} - a_k w_k = 0, \quad 1 \le k \le N.$$

Because the operator $\mathcal{L}_{a,\, b,\, k} : H^1(I) \to L^2(I)$ with $a_k > 0$ for $1 \le k \le l$ discussed above does not have any nontrivial zero modes, we obtain that $w(x)$ is trivial in I.

Let us choose an arbitrary $v(x) \in H_c^2(I, \mathbb{R}^N)$ and apply the Fourier transform (5.23) to system (2.3) studied on the interval I. This gives us

$$u_{k,n} = \sqrt{2\pi}\frac{G_{k,n} f_{k,n}}{|n| - a_k - ib_k n},$$
$$n^2 u_{k,n} = \sqrt{2\pi}\frac{n^2 G_{k,n} f_{k,n}}{|n| - a_k - ib_k n}, \quad 1 \le k \le N, \quad n \in \mathbb{Z}, \tag{4.3}$$

where $f_{k,n} := F_k(v(x), x)_n$. We easily obtain the estimates from above

$$|u_{k,n}| \le \sqrt{2\pi}\mathcal{N}_{a,\ b,\ k}|f_{k,n}|, \quad |n^2 u_{k,n}| \le \sqrt{2\pi}\mathcal{N}_{a,\ b,\ k}|f_{k,n}|.$$

Clearly, $\mathcal{N}_{a,\ b,\ k} < \infty$ under the stated conditions via the result of Lemma A3 of the Appendix. Therefore,

$$\|u\|^2_{H_c^2(I,\mathbb{R}^N)} = \sum_{k=1}^{N}\left[\sum_{n=-\infty}^{\infty}|u_{k,n}|^2 + \sum_{n=-\infty}^{\infty}|n^2 u_{k,n}|^2\right]$$
$$\le 4\pi\mathcal{N}^2_{a,\ b}\sum_{k=1}^{N}\|F_k(v(x), x)\|^2_{L^2(I)} \tag{4.4}$$

with $\mathcal{N}_{a,\ b}$ defined in (5.28). Evidently, the right side of (4.4) is finite via inequality (2.1) of Assumption 1 for $|v(x)|_{\mathbb{R}^N} \in L^2(I)$. Thus, for an arbitrary $v(x) \in H_c^2(I, \mathbb{R}^N)$ there exists a unique $u(x) \in H_c^2(I, \mathbb{R}^N)$, which satisfies system (2.3) and its Fourier image is given by (4.3). Therefore, the map $\tau_{a,b} : H_c^2(I, \mathbb{R}^N) \to H_c^2(I, \mathbb{R}^N)$ is well defined.

We consider any $v^{(1),(2)}(x) \in H_c^2(I, \mathbb{R}^N)$, so that their images under the map mentioned above $u^{(1),(2)} = \tau_{a,b}v^{(1),(2)} \in H_c^2(I, \mathbb{R}^N)$. By means of (2.3), we have for $1 \le k \le N$ that

$$\sqrt{-\frac{d^2}{dx^2}u_k^{(1)} - b_k\frac{du_k^{(1)}}{dx}} - a_k u_k^{(1)}$$
$$= \int_0^{2\pi} G_k(x-y)F_k(v_1^{(1)}(y), v_2^{(1)}(y), ..., v_N^{(1)}(y), y)dy, \tag{4.5}$$
$$\sqrt{-\frac{d^2}{dx^2}u_k^{(2)} - b_k\frac{du_k^{(2)}}{dx}} - a_k u_k^{(2)}$$
$$= \int_0^{2\pi} G_k(x-y)F_k(v_1^{(2)}(y), v_2^{(2)}(y), ..., v_N^{(2)}(y), y)dy, \tag{4.6}$$

where $a_k \ge 0,\ b_k \in \mathbb{R},\ b_k \ne 0$ are the constants. By virtue of Fourier transform (5.23) applied to both sides of the systems of Eqs. (4.5) and (4.6), we easily derive for $1 \le k \le N,\ n \in \mathbb{Z}$ that

$$u_{k,n}^{(1)} = \sqrt{2\pi}\frac{G_{k,n} f_{k,n}^{(1)}}{|n| - a_k - ib_k n}, \quad u_{k,n}^{(2)} = \sqrt{2\pi}\frac{G_{k,n} f_{k,n}^{(2)}}{|n| - a_k - ib_k n},$$

$$n^2 u_{k,n}^{(1)} = \sqrt{2\pi}\frac{n^2 G_{k,n} f_{k,n}^{(1)}}{|n| - a_k - ib_k n}, \quad n^2 u_{k,n}^{(2)} = \sqrt{2\pi}\frac{n^2 G_{k,n} f_{k,n}^{(2)}}{|n| - a_k - ib_k n},$$

with $f_{k,n}^{(j)} := F_k(v^{(j)}(x), x)_n, \ j = 1, 2$. Thus,

$$|u_{k,n}^{(1)} - u_{k,n}^{(2)}| \le \sqrt{2\pi}\mathcal{N}_{a,\,b}|f_{k,n}^{(1)} - f_{k,n}^{(2)}|, \quad |n^2(u_{k,n}^{(1)} - u_{k,n}^{(2)})| \le \sqrt{2\pi}\mathcal{N}_{a,\,b}|f_{k,n}^{(1)} - f_{k,n}^{(2)}|.$$

Hence, we estimate the norm as

$$\|u^{(1)} - u^{(2)}\|_{H_c^2(I,\mathbb{R}^N)}^2 = \sum_{k=1}^{N}\Big[\sum_{n=-\infty}^{\infty} |u_{k,n}^{(1)} - u_{k,n}^{(2)}|^2 + \sum_{n=-\infty}^{\infty} |n^2(u_{k,n}^{(1)} - u_{k,n}^{(2)})|^2\Big]$$
$$\le 4\pi\mathcal{N}_{a,\,b}^2 \sum_{k=1}^{N} \|F_k(v^{(1)}(x), x) - F_k(v^{(2)}(x), x)\|_{L^2(I)}^2.$$

Evidently, $v_k^{(1),(2)}(x) \in H^2(I) \subset L^\infty(I), \ 1 \le k \le N$ by means of the Sobolev embedding. Using inequality (2.2) of Assumption 1, we derive

$$\|\tau_{a,b}v^{(1)} - \tau_{a,b}v^{(2)}\|_{H_c^2(I,\mathbb{R}^N)} \le 2\sqrt{\pi}\mathcal{N}_{a,\,b}L\|v^{(1)} - v^{(2)}\|_{H_c^2(I,\mathbb{R}^N)}. \tag{4.7}$$

The constant in the right side of estimate (4.7) is less than due to the one of our assumptions. Then the Fixed Point Theorem gives us the existence and uniqueness of a vector function $v^{(a,b)} \in H_c^2(I, \mathbb{R}^N)$ satisfying $\tau_{a,b}v^{(a,b)} = v^{(a,b)}$. This is the only solution of system (1.2) in $H_c^2(I, \mathbb{R}^N)$. If we suppose that $v^{(a,b)}(x)$ vanishes identically in I, we will obtain the contradiction to our condition that $G_{k,n}F_k(0, x)_n \neq 0$ for a certain $1 \le k \le N$ and some $n \in \mathbb{Z}$. □

Let us turn our attention to establishing the final main result of the work.

Proof of Theorem 4 Obviously, the limiting kernels $G_k(x), \ 1 \le k \le N$ are periodic as well on our interval I (see the argument of Lemma A4 of the Appendix). Each system (2.10) admits a unique solution $u^{(m)}(x) \in H_c^2(I, \mathbb{R}^N), \ m \in \mathbb{N}$ by means of the result of Theorem 3 above. The limiting system of equations (1.2) has a unique solution $u(x) \in H_c^2(I, \mathbb{R}^N)$ due to Lemma A4 below along with Theorem 3.

We apply Fourier transform (5.23) to both sides of systems (1.2) and (2.10). This gives us

$$u_{k,n} = \sqrt{2\pi}\frac{G_{k,n}\varphi_{k,n}}{|n| - a_k - ib_k n}, \quad u_{k,n}^{(m)} = \sqrt{2\pi}\frac{G_{k,m,n}\varphi_{k,n}^{(m)}}{|n| - a_k - ib_k n}, \tag{4.8}$$

with $1 \le k \le N, \ n \in \mathbb{Z}, \ m \in \mathbb{N}$. Here $\varphi_{k,n}$ and $\varphi_{k,n}^{(m)}$ stand for the Fourier images of $F_k(u(x), x)$ and $F_k(u^{(m)}(x), x)$ respectively under transform (5.23). We have a trivial estimate from above

$$|u_{k,n}^{(m)} - u_{k,n}| \le \sqrt{2\pi}\left\| \frac{G_{k,m,n}}{|n| - a_k - ib_k n} - \frac{G_{k,n}}{|n| - a_k - ib_k n}\right\|_{l^\infty} |\varphi_{k,n}|$$
$$+ \sqrt{2\pi}\left\| \frac{G_{k,m,n}}{|n| - a_k - ib_k n}\right\|_{l^\infty} |\varphi_{k,n}^{(m)} - \varphi_{k,n}|.$$

Thus,

$$\|u_k^{(m)} - u_k\|_{L^2(I)} \le \sqrt{2\pi}\left\| \frac{G_{k,m,n}}{|n| - a_k - ib_k n} - \frac{G_{k,n}}{|n| - a_k - ib_k n}\right\|_{l^\infty} \|F_k(u(x), x)\|_{L^2(I)}$$
$$+ \sqrt{2\pi}\left\| \frac{G_{k,m,n}}{|n| - a_k - ib_k n}\right\|_{l^\infty} \|F_k(u^{(m)}(x), x) - F_k(u(x), x)\|_{L^2(I)}.$$

Inequality (2.2) of Assumption 1 above implies that

$$\sqrt{\sum_{k=1}^N \|F_k(u^{(m)}(x), x) - F_k(u(x), x)\|_{L^2(I)}^2} \le L\|u^{(m)}(x) - u(x)\|_{L^2(I,\mathbb{R}^N)}. \tag{4.9}$$

Evidently, $u_k^{(m)}(x), u_k(x) \in H^2(I) \subset L^\infty(I),\ 1 \le k \le N$ due to the Sobolev embedding. Obviously,

$$\|u^{(m)}(x) - u(x)\|_{L^2(I,\mathbb{R}^N)}^2$$
$$\le 4\pi \sum_{k=1}^N \left\| \frac{G_{k,m,n}}{|n| - a_k - ib_k n} - \frac{G_{k,n}}{|n| - a_k - ib_k n}\right\|_{l^\infty}^2 \|F_k(u(x), x)\|_{L^2(I)}^2$$
$$+ 4\pi \left[\mathcal{N}_{a,\, b}^{(m)}\right]^2 L^2 \|u^{(m)}(x) - u(x)\|_{L^2(I,\mathbb{R}^N)}^2.$$

Hence, we arrive at $\|u^{(m)}(x) - u(x)\|_{L^2(I,\mathbb{R}^N)}^2 \le$

$$\le \frac{4\pi}{\varepsilon(2-\varepsilon)} \sum_{k=1}^N \left\| \frac{G_{k,m,n}}{|n| - a_k - ib_k n} - \frac{G_{k,n}}{|n| - a_k - ib_k n}\right\|_{l^\infty}^2 \|F_k(u(x), x)\|_{L^2(I)}^2.$$

Evidently, $F_k(u(x), x) \in L^2(I),\ 1 \le k \le N$ for $u(x) \in H_c^2(I, \mathbb{R}^N)$ via inequality (2.1) of Assumption 1. By means of the result of Lemma A4 of the Appendix we derive that

$$u^{(m)}(x) \to u(x), \quad m \to \infty \tag{4.10}$$

in $L^2(I, \mathbb{R}^N)$. Clearly,

$$|n^2 u_{k,n}^{(m)} - n^2 u_{k,n}| \le \sqrt{2\pi}\left\| \frac{n^2 G_{k,m,n}}{|n| - a_k - ib_k n} - \frac{n^2 G_{k,n}}{|n| - a_k - ib_k n}\right\|_{l^\infty} |\varphi_{k,n}|$$

$$+\sqrt{2\pi}\left\|\frac{n^2 G_{k,m,n}}{|n|-a_k-ib_kn}\right\|_{l^\infty}|\varphi_{k,n}^{(m)}-\varphi_{k,n}|.$$

Using (4.9) we arrive at

$$\left\|\frac{d^2u_k^{(m)}}{dx^2}-\frac{d^2u_k}{dx^2}\right\|_{L^2(I)}\leq\sqrt{2\pi}\left\|\frac{n^2G_{k,m,n}}{|n|-a_k-ib_kn}-\frac{n^2G_{k,n}}{|n|-a_k-ib_kn}\right\|_{l^\infty}\|F_k(u(x),x)\|_{L^2(I)}$$
$$+\sqrt{2\pi}\left\|\frac{n^2G_{k,m,n}}{|n|-a_k-ib_kn}\right\|_{l^\infty}L\|u^{(m)}(x)-u(x)\|_{L^2(I,\mathbb{R}^N)}.$$

By virtue of Lemma A4 along with (4.10), we obtain that $\frac{d^2u^{(m)}}{dx^2}\to\frac{d^2u}{dx^2}$ as $m\to\infty$ in $L^2(I,\mathbb{R}^N)$. Therefore, $u^{(m)}(x)\to u(x)$ in the $H_c^2(I,\mathbb{R}^N)$ norm as $m\to\infty$.

Suppose that $u^{(m)}(x)$ is trivial in the interval I for a certain $m\in\mathbb{N}$. This will yield a contradiction to our assumption that $G_{k,m,n}F_k(0,x)_n\neq 0$ for a certain $1\leq k\leq N$ and some $n\in\mathbb{Z}$. The analogical reasoning holds for the solution $u(x)$ of the limiting system of Eq. (1.2). □

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Declarations

Conflict of interest The authors declare that they have no competing interests as defined by Springer, or other interests that might be perceived to influence the results and/or discussion reported in this paper.

Appendix

Let $G_k(x)$ be a function, $G_k(x):\mathbb{R}\to\mathbb{R}$, for which we denote its standard Fourier transform using the hat symbol as

$$\widehat{G_k}(p):=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}G_k(x)e^{-ipx}dx,\quad p\in\mathbb{R}. \tag{5.1}$$

Clearly,

$$\|\widehat{G_k}(p)\|_{L^\infty(\mathbb{R})}\leq\frac{1}{\sqrt{2\pi}}\|G_k(x)\|_{L^1(\mathbb{R})} \tag{5.2}$$

and $G_k(x) = \dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{G_k}(q)e^{iqx}dq,\ x \in \mathbb{R}$. By means of (5.2), we have

$$\|p\widehat{G_k}(p)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2\pi}}\left\|\frac{dG_k(x)}{dx}\right\|_{L^1(\mathbb{R})}. \tag{5.3}$$

For the technical purposes we will use the auxiliary quantities

$$N_{a,\ b,\ k} := max\left\{\left\|\frac{\widehat{G_k}(p)}{|p| - a_k - ib_k p}\right\|_{L^\infty(\mathbb{R})},\quad \left\|\frac{p^2\widehat{G_k}(p)}{|p| - a_k - ib_k p}\right\|_{L^\infty(\mathbb{R})}\right\}, \tag{5.4}$$

where $a_k \geq 0,\ b_k \in \mathbb{R},\ b_k \neq 0$ are the constants, $1 \leq k \leq N,\ N \geq 2$. Under the assumptions of Lemma A1 below, all the quantities (5.4) will be finite, so that

$$N_{a,\ b} := max_{1\leq k\leq N} N_{a,\ b,\ k} < \infty. \tag{5.5}$$

The auxiliary lemmas below are the adaptations of the ones proved in [16] in order to study the single integro-differential equation with drift and superdiffusion, analogical to system (1.2). Let us provide them for the convenience of the readers.

Lemma A1 *Let $N \geq 2,\ 1 \leq k \leq N,\ b_k \in \mathbb{R},\ b_k \neq 0$ and $G_k(x) : \mathbb{R} \to \mathbb{R},\ G_k(x) \in W^{1,1}(\mathbb{R})$ and $1 \leq l \leq N-1$.*

a) Let $a_k > 0$ for $1 \leq k \leq l$. Then $N_{a,\ b,\ k} < \infty$.
b) Let $a_k = 0$ for $l+1 \leq k \leq N$ and additionally $xG_k(x) \in L^1(\mathbb{R})$. Then $N_{0,\ b,\ k} < \infty$ if and only if

$$(G_k(x), 1)_{L^2(\mathbb{R})} = 0 \tag{5.6}$$

is valid.

Proof First of all, it can be trivially checked that in both cases a) and b) of the lemma, under our assumptions the expressions

$$\frac{p^2\widehat{G_k}(p)}{|p| - a_k - ib_k p} \in L^\infty(\mathbb{R}), \quad 1 \leq k \leq N. \tag{5.7}$$

Evidently, the functions $\dfrac{p}{|p| - a_k - ib_k p}$ are bounded and $p\widehat{G_k}(p) \in L^\infty(\mathbb{R})$ via inequality (5.3) above, which yields (5.7). We turn our attention to establishing the result of the part a) of our lemma. Let us estimate the expressions

$$\frac{|\widehat{G_k}(p)|}{\sqrt{(|p| - a_k)^2 + b_k^2 p^2}}, \quad 1 \leq k \leq l. \tag{5.8}$$

Clearly, the numerator of (5.8) can be bounded from above via (5.2) and the denominator in (5.8) can be trivially estimated below by a finite, positive constant, so that

$$\left|\frac{\widehat{G_k}(p)}{|p| - a_k - ib_k p}\right| \le C\|G_k(x)\|_{L^1(\mathbb{R})} < \infty$$

as assumed. Here and below C will stand for a finite, positive constant. This implies that under the given conditions, if $a_k > 0$ we have $N_{a,\ b,\ k} < \infty$. In the cases of $a_k = 0$, we will use that

$$\widehat{G_k}(p) = \widehat{G_k}(0) + \int_0^p \frac{d\widehat{G_k}(s)}{ds} ds.$$

Thus,

$$\frac{\widehat{G_k}(p)}{|p| - ib_k p} = \frac{\widehat{G_k}(0)}{|p| - ib_k p} + \frac{\int_0^p \frac{d\widehat{G_k}(s)}{ds} ds}{|p| - ib_k p}. \tag{5.9}$$

Using definition (5.1) of the standard Fourier transform, we easily obtain

$$\left|\frac{d\widehat{G_k}(p)}{dp}\right| \le \frac{1}{\sqrt{2\pi}}\|xG_k(x)\|_{L^1(\mathbb{R})}.$$

Hence,

$$\left|\frac{\int_0^p \frac{d\widehat{G_k}(s)}{ds} ds}{|p| - ib_k p}\right| \le \frac{\|xG_k(x)\|_{L^1(\mathbb{R})}}{\sqrt{2\pi(1 + b_k^2)}} < \infty$$

due to our assumptions. Therefore, the expression in the left side of (5.9) is bounded if and only if $\widehat{G_k}(0) = 0$, which is equivalent to orthogonality relation (5.6). $\square$

We introduce the following technical expressions, which will help us to study systems (2.8).

$$N_{a,\ b,\ k}^{(m)} := max\left\{\left\|\frac{\widehat{G_{k,m}}(p)}{|p| - a_k - ib_k p}\right\|_{L^\infty(\mathbb{R})}, \quad \left\|\frac{p^2\widehat{G_{k,m}}(p)}{|p| - a_k - ib_k p}\right\|_{L^\infty(\mathbb{R})}\right\}, \tag{5.10}$$

where $a_k \ge 0,\ b_k \in \mathbb{R},\ b_k \ne 0$ are the constants, $1 \le k \le N,\ N \ge 2$ and $m \in \mathbb{N}$. Under the conditions of Lemma A2 below, expressions (5.10) will be finite. This will enable us to define

$$N_{a,\ b}^{(m)} := max_{1 \le k \le N} N_{a,\ b,\ k}^{(m)} < \infty \tag{5.11}$$

with $m \in \mathbb{N}$. We have the following technical proposition.

Lemma A2 *Let* $m \in \mathbb{N},\ N \geq 2,\ 1 \leq k \leq N,\ b_k \in \mathbb{R},\ b_k \neq 0$ *and* $G_{k,m}(x) : \mathbb{R} \to \mathbb{R},\ G_{k,m}(x) \in W^{1,1}(\mathbb{R})$*, so that* $G_{k,m}(x) \to G_k(x)$ *in* $W^{1,1}(\mathbb{R})$ *as* $m \to \infty$ *and* $1 \leq l \leq N-1$.

(a) *Let* $a_k > 0$ *for* $1 \leq k \leq l$.
(b) *Let* $a_k = 0$ *for* $l+1 \leq k \leq N$ *and in addition* $xG_{k,m}(x) \in L^1(\mathbb{R})$*, so that* $xG_{k,m}(x) \to xG_k(x)$ *in* $L^1(\mathbb{R})$ *as* $m \to \infty$ *and*

$$(G_{k,m}(x), 1)_{L^2(\mathbb{R})} = 0, \quad m \in \mathbb{N} \tag{5.12}$$

is valid. Let in addition

$$2\sqrt{\pi} N_{a,\ b}^{(m)} L \leq 1 - \varepsilon \tag{5.13}$$

for all $m \in \mathbb{N}$ *as well with a certain fixed* $0 < \varepsilon < 1$*. Then, for all* $1 \leq k \leq N$*, we have*

$$\frac{\widehat{G_{k,m}}(p)}{|p| - a_k - ib_k p} \to \frac{\widehat{G_k}(p)}{|p| - a_k - ib_k p}, \quad m \to \infty, \tag{5.14}$$

$$\frac{p^2\widehat{G_{k,m}}(p)}{|p| - a_k - ib_k p} \to \frac{p^2\widehat{G_k}(p)}{|p| - a_k - ib_k p}, \quad m \to \infty \tag{5.15}$$

in $L^\infty(\mathbb{R})$*, so that*

$$\left\| \frac{\widehat{G_{k,m}}(p)}{|p| - a_k - ib_k p} \right\|_{L^\infty(\mathbb{R})} \to \left\| \frac{\widehat{G_k}(p)}{|p| - a_k - ib_k p} \right\|_{L^\infty(\mathbb{R})}, \quad m \to \infty, \tag{5.16}$$

$$\left\| \frac{p^2\widehat{G_{k,m}}(p)}{|p| - a_k - ib_k p} \right\|_{L^\infty(\mathbb{R})} \to \left\| \frac{p^2\widehat{G_k}(p)}{|p| - a_k - ib_k p} \right\|_{L^\infty(\mathbb{R})}, \quad m \to \infty. \tag{5.17}$$

Furthermore,

$$2\sqrt{\pi} N_{a,\ b} L \leq 1 - \varepsilon. \tag{5.18}$$

Proof By means of inequality (5.2), we easily obtain for $1 \leq k \leq N$ that

$$\|\widehat{G_{k,m}}(p) - \widehat{G_k}(p)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2\pi}} \|G_{k,m}(x) - G_k(x)\|_{L^1(\mathbb{R})} \to 0, \quad m \to \infty \tag{5.19}$$

due to the one of our assumptions. Evidently, (5.16) and (5.17) will trivially follow from the statements of (5.14) and (5.15) respectively by virtue of the standard triangle inequality.

We use the fact that the functions $\frac{p}{|p| - a_k - ib_k p} \in L^\infty(\mathbb{R})$ along with the analog of bound (5.3). This yields

$$\left|\frac{p^2\widehat{G_{k,m}}(p)}{|p| - a_k - ib_k p} - \frac{p^2\widehat{G_k}(p)}{|p| - a_k - ib_k p}\right| \le C\|p[\widehat{G_{k,m}}(p) - \widehat{G_k}(p)]\|_{L^\infty(\mathbb{R})}$$
$$\le \frac{C}{\sqrt{2\pi}}\left\|\frac{dG_{k,m}(x)}{dx} - \frac{dG_k(x)}{dx}\right\|_{L^1(\mathbb{R})}.$$

Thus,

$$\left\|\frac{p^2\widehat{G_{k,m}}(p)}{|p| - a_k - ib_k p} - \frac{p^2\widehat{G_k}(p)}{|p| - a_k - ib_k p}\right\|_{L^\infty(\mathbb{R})} \le \frac{C}{\sqrt{2\pi}}\left\|\frac{dG_{k,m}(x)}{dx} - \frac{dG_k(x)}{dx}\right\|_{L^1(\mathbb{R})} \to 0$$

as $m \to \infty$ via the one of our assumptions, so that (5.15) is valid. Let us establish (5.14) in the situation a) when $a_k > 0$. For that purpose we need to consider

$$\frac{|\widehat{G_{k,m}}(p) - \widehat{G_k}(p)|}{\sqrt{(|p| - a_k)^2 + b_k^2 p^2}}, \quad 1 \le k \le l. \tag{5.20}$$

Evidently, the denominator in fraction (5.20) can be bounded from below by a positive constant and the numerator in (5.20) can be estimated from above by means of (5.19). Hence,

$$\left\|\frac{\widehat{G_{k,m}}(p)}{|p| - a_k - ib_k p} - \frac{\widehat{G_k}(p)}{|p| - a_k - ib_k p}\right\|_{L^\infty(\mathbb{R})} \le C\|G_{k,m}(x) - G_k(x)\|_{L^1(\mathbb{R})} \to 0$$

as $m \to \infty$ due to the one of the assumptions, so that (5.14) is valid in the case a) of the lemma. Then we turn our attention to proving (5.14) in the situation b) when $a_k = 0$. In this case orthogonality conditions (5.12) are valid as assumed. We easily derive that the analogical statements will hold in the limit. Evidently,

$$|(G_k(x), 1)_{L^2(\mathbb{R})}| = |(G_k(x) - G_{k,m}(x), 1)_{L^2(\mathbb{R})}| \le \|G_{k,m}(x) - G_k(x)\|_{L^1(\mathbb{R})} \to 0$$

as $m \to \infty$ by virtue of the one of our assumptions. Thus,

$$(G_k(x), 1)_{L^2(\mathbb{R})} = 0, \quad l + 1 \le k \le N \tag{5.21}$$

is valid. Obviously, we have

$$\widehat{G_k}(p) = \widehat{G_k}(0) + \int_0^p \frac{d\widehat{G_k}(s)}{ds}ds, \quad \widehat{G_{k,m}}(p) = \widehat{G_{k,m}}(0) + \int_0^p \frac{d\widehat{G_{k,m}}(s)}{ds}ds,$$

with $l+1 \le k \le N,\ m \in \mathbb{N}$. Formulas (5.21) and (5.12) imply that

$$\widehat{G_k}(0)=0, \quad \widehat{G_{k,m}}(0)=0, \quad l+1 \le k \le N, \quad m \in \mathbb{N}.$$

Hence,

$$\left|\frac{\widehat{G_{k,m}}(p)}{|p|-ib_kp}-\frac{\widehat{G_k}(p)}{|p|-ib_kp}\right| = \left|\frac{\int_0^p \left[\frac{d\widehat{G_{k,m}}(s)}{ds}-\frac{d\widehat{G_k}(s)}{ds}\right]ds}{|p|-ib_kp}\right|. \tag{5.22}$$

Using the definition of the standard Fourier transform (5.1) we easily derive

$$\left|\frac{d\widehat{G_{k,m}}(p)}{dp}-\frac{d\widehat{G_k}(p)}{dp}\right| \le \frac{1}{\sqrt{2\pi}}\|xG_{k,m}(x)-xG_k(x)\|_{L^1(\mathbb{R})}.$$

This allows us to obtain the estimate from above on the right side of (5.22) as

$$\frac{\|xG_{k,m}(x)-xG_k(x)\|_{L^1(\mathbb{R})}}{\sqrt{2\pi(1+b_k^2)}},$$

such that

$$\left\|\frac{\widehat{G_{k,m}}(p)}{|p|-ib_kp}-\frac{\widehat{G_k}(p)}{|p|-ib_kp}\right\|_{L^\infty(\mathbb{R})} \le \frac{\|xG_{k,m}(x)-xG_k(x)\|_{L^1(\mathbb{R})}}{\sqrt{2\pi(1+b_k^2)}} \to 0, \quad m \to \infty$$

as assumed. Therefore, (5.14) is valid in the case b) of the lemma when $a_k = 0$. Evidently, under the stated conditions we have

$$N_{a,\ b,\ k} < \infty, \quad N_{a,\ b,\ k}^{(m)} < \infty, \quad m \in \mathbb{N}, \quad 1 \le k \le N, \quad a_k \ge 0, \quad b_k \in \mathbb{R}, \quad b_k \ne 0$$

by means of the result of Lemma A1 above. We have inequalities (5.13). An trivial limiting argument using (5.16) and (5.17) gives us (5.18). □

Consider the function $G_k(x) : I \to \mathbb{R}$, so that $G_k(0) = G_k(2\pi)$. Its Fourier transform on our finite interval is given by

$$G_{k,n} := \int_0^{2\pi} G_k(x)\frac{e^{-inx}}{\sqrt{2\pi}}dx, \quad n \in \mathbb{Z}, \tag{5.23}$$

such that $G_k(x) = \sum_{n=-\infty}^{\infty} G_{k,n}\frac{e^{inx}}{\sqrt{2\pi}}$. Obviously, the upper bound

$$\|G_{k,n}\|_{l^\infty} \le \frac{1}{\sqrt{2\pi}}\|G_k(x)\|_{L^1(I)} \tag{5.24}$$

is valid. Evidently, if our function is continuous on the interval I, we have the estimate from above

$$\|G_k(x)\|_{L^1(I)} \le 2\pi \|G_k(x)\|_{C(I)}. \tag{5.25}$$

The upper bound

$$\|nG_{k,n}\|_{l^\infty} \le \frac{1}{\sqrt{2\pi}} \left\| \frac{dG_k(x)}{dx} \right\|_{L^1(I)} \tag{5.26}$$

trivially comes from (5.24). Analogously to the whole real line case, we define

$$\mathcal{N}_{a,\ b,\ k} := max\left\{ \left\| \frac{G_{k,n}}{|n| - a_k - ib_k n} \right\|_{l^\infty}, \quad \left\| \frac{n^2 G_{k,n}}{|n| - a_k - ib_k n} \right\|_{l^\infty} \right\}, \tag{5.27}$$

where $a_k \ge 0,\ b_k \in \mathbb{R},\ b_k \ne 0$ are the constants, $1 \le k \le N,\ N \ge 2$. Let $\mathcal{N}_{0,\ b,\ k}$ denote (5.27) when a_k vanishes. Under the conditions of Lemma A3 below, the expressions $\mathcal{N}_{a,\ b,\ k}$ will be finite. This will enable us to introduce

$$\mathcal{N}_{a,\ b} := max_{1 \le k \le N} \mathcal{N}_{a,\ b,\ k} < \infty. \tag{5.28}$$

We have the following elementary statement.

Lemma A3 *Let* $N \ge 2,\ 1 \le k \le N,\ b_k \in \mathbb{R},\ b_k \ne 0,\ 1 \le l \le N-1$ *and* $G_k(x) : I \to \mathbb{R},\ G_k(x) \in C(I),\ \frac{dG_k(x)}{dx} \in L^1(I),\ G_k(0) = G_k(2\pi)$.

(a) *Let* $a_k > 0$ *for* $1 \le k \le l$. *Then* $\mathcal{N}_{a,\ b,\ k} < \infty$.
(b) *If* $a_k = 0$ *for* $l+1 \le k \le N$ *then* $\mathcal{N}_{0,\ b,\ k} < \infty$ *if and only if the orthogonality relation*

$$(G_k(x), 1)_{L^2(I)} = 0 \tag{5.29}$$

holds.

Proof It can be easily checked that in both cases (a) and (b) of our lemma under the given conditions we have

$$\frac{n^2 G_{k,n}}{|n| - a_k - ib_k n} \in l^\infty, \quad 1 \le k \le N. \tag{5.30}$$

Clearly, $\frac{n}{|n| - a_k - ib_k n} \in l^\infty$ and $nG_{k,n} \in l^\infty$ via inequality (5.26) along with the one of the stated assumptions. Hence (5.30) is valid.

Let us establish the statement of the part a) of the lemma. For that purpose, we need to consider the expression

$$\frac{|G_{k,n}|}{\sqrt{(|n| - a_k)^2 + b_k^2 n^2}}, \quad 1 \le k \le l. \tag{5.31}$$

Evidently, the denominator in (5.31) can be easily bounded from below by a positive constant. The numerator in (5.31) can be trivially estimated from above by means of (5.24) along with (5.25). Hence, $\mathcal{N}_{a,\ b,\ k} < \infty$ in the case when $a_k > 0$. Let us demonstrate the validity of the result of the lemma in the situation when $a_k = 0$. Obviously,

$$\left|\frac{G_{k,n}}{|n| - ib_k n}\right|, \quad l+1 \le k \le N$$

is bounded if and only if $G_{k,0} = 0$. This is equivalent to orthogonality condition (5.29). In this case we easily arrive at for $l+1 \le k \le N$ that

$$\left|\frac{G_{k,n}}{|n| - ib_k n}\right| \le \frac{1}{\sqrt{2\pi}|n|}\frac{\|G_k(x)\|_{L^1(I)}}{\sqrt{1+b_k^2}} \le \sqrt{2\pi}\frac{\|G_k(x)\|_{C(I)}}{\sqrt{1+b_k^2}} < \infty$$

by virtue of (5.24) and (5.25) under our assumptions. □

In order to study the systems of Eq. (2.10), we will use

$$\mathcal{N}^{(m)}_{a,\ b,\ k} := max\left\{\left\|\frac{G_{k,m,n}}{|n| - a_k - ib_k n}\right\|_{l^\infty}, \quad \left\|\frac{n^2 G_{k,m,n}}{|n| - a_k - ib_k n}\right\|_{l^\infty}\right\}, \tag{5.32}$$

where $a_k \ge 0,\ b_k \in \mathbb{R},\ b_k \neq 0$ are the constants, $1 \le k \le N,\ N \ge 2$ and $m \in \mathbb{N}$. Under the assumptions of Lemma A4 below, we have that all $\mathcal{N}^{(m)}_{a,\ b,\ k} < \infty$. This will allow us to introduce

$$\mathcal{N}^{(m)}_{a,\ b} = max_{1 \le k \le N}\mathcal{N}^{(m)}_{a,\ b,\ k}, \quad m \in \mathbb{N}. \tag{5.33}$$

We conclude the work with the following auxiliary proposition.

Lemma A4 *Let* $m \in \mathbb{N},\ N \ge 2,\ 1 \le k \le N,\ b_k \in \mathbb{R},\ b_k \neq 0,\ 1 \le l \le N-1$ *and*

$$G_{k,m}(x) : I \to \mathbb{R}, \quad G_{k,m}(x) \in C(I), \quad \frac{dG_{k,m}(x)}{dx} \in L^1(I), \quad G_{k,m}(0) = G_{k,m}(2\pi),$$

and

$$G_{k,m}(x) \to G_k(x) \ \ in \ \ C(I), \quad \frac{dG_{k,m}(x)}{dx} \to \frac{dG_k(x)}{dx} \ \ in \ \ L^1(I)$$

as $m \to \infty$.

(a) *Let* $a_k > 0$ *for* $1 \le k \le l$.
(b) *Let* $a_k = 0$ *for* $l+1 \le k \le N$ *and in addition*

$$(G_{k,m}(x), 1)_{L^2(I)} = 0, \quad m \in \mathbb{N}. \tag{5.34}$$

We also assume that

$$2\sqrt{\pi}\mathcal{N}_{a,\ b}^{(m)}L \le 1-\varepsilon \tag{5.35}$$

is valid for all $m \in \mathbb{N}$ *as well with some fixed* $0<\varepsilon<1$. *Then, for all* $1 \le k \le N$, *we have*

$$\frac{G_{k,m,n}}{|n|-a_k-ib_k n} \to \frac{G_{k,n}}{|n|-a_k-ib_k n}, \quad m \to \infty, \tag{5.36}$$

$$\frac{n^2 G_{k,m,n}}{|n|-a_k-ib_k n} \to \frac{n^2 G_{k,n}}{|n|-a_k-ib_k n}, \quad m \to \infty \tag{5.37}$$

in l^∞, *so that*

$$\left\|\frac{G_{k,m,n}}{|n|-a_k-ib_k n}\right\|_{l^\infty} \to \left\|\frac{G_{k,n}}{|n|-a_k-ib_k n}\right\|_{l^\infty}, \quad m \to \infty, \tag{5.38}$$

$$\left\|\frac{n^2 G_{k,m,n}}{|n|-a_k-ib_k n}\right\|_{l^\infty} \to \left\|\frac{n^2 G_{k,n}}{|n|-a_k-ib_k n}\right\|_{l^\infty}, \quad m \to \infty. \tag{5.39}$$

Furthermore, the estimate

$$2\sqrt{\pi}\mathcal{N}_{a,\ b}L \le 1-\varepsilon \tag{5.40}$$

holds.

Proof Obviously, under the stated assumptions, the limiting kernels $G_k(x),\ 1 \le k \le N$ are periodic as well. Indeed, we easily obtain

$$|G_k(0)-G_k(2\pi)| \le |G_k(0)-G_{k,m}(0)| + |G_{k,m}(2\pi)-G_k(2\pi)| \le 2\|G_{k,m}(x)-G_k(x)\|_{C(I)} \to 0$$

as $m \to \infty$ as assumed. Thus, $G_k(0)=G_k(2\pi),\ 1 \le k \le N$. By virtue of (5.24) along with (5.25) we arrive at

$$\|G_{k,m,n}-G_{k,n}\|_{l^\infty} \le \frac{1}{\sqrt{2\pi}}\|G_{k,m}-G_k\|_{L^1(I)} \le \sqrt{2\pi}\|G_{k,m}-G_k\|_{C(I)} \to 0, \quad m \to \infty \tag{5.41}$$

due to the one of our assumptions. It can be trivially checked that the statements of (5.36) and (5.37) will imply (5.38) and (5.39) respectively via the triangle inequality. Using (5.26), we obtain the estimate from above

$$\left\|\frac{n^2 G_{k,m,n}}{|n|-a_k-ib_k n}-\frac{n^2 G_{k,n}}{|n|-a_k-ib_k n}\right\|_{l^\infty} \le \frac{1}{\sqrt{2\pi}}\left\|\frac{n}{|n|-a_k-ib_k n}\right\|_{l^\infty}$$

$$\left\| \frac{dG_{k,m}(x)}{dx} - \frac{dG_k(x)}{dx} \right\|_{L^1(I)},$$

which tends to zero as $m \to \infty$ as assumed, so that (5.37) is valid. Let us establish (5.36) in the situation a) when $a_k > 0$. For that purpose, we need to treat

$$\frac{|G_{k,m,n} - G_{k,n}|}{\sqrt{(|n| - a_k)^2 + b_k^2 n^2}}, \quad 1 \le k \le l. \tag{5.42}$$

Obviously, the denominator of (5.42) can be bounded from below by a positive constant and the numerator estimated from above via (5.41). This gives us (5.36) for $a_k > 0$.

Let us demonstrate the validity of (5.36) in the case case b) when $a_k = 0$. By means of the one of the given assumptions, we have orthogonality conditions (5.34). It can be trivially checked that the analogical relations holds in the limit. Indeed,

$$|(G_k(x), 1)_{L^2(I)}| = |(G_k(x) - G_{k,m}(x), 1)_{L^2(I)}| \le 2\pi \|G_{k,m}(x) - G_k(x)\|_{C(I)} \to 0, \quad m \to \infty$$

via the one of our assumptions. Thus,

$$(G_k(x), 1)_{L^2(I)} = 0, \quad l + 1 \le k \le N.$$

This is equivalent to $G_{k,0} = 0, \ l + 1 \le k \le N$. Evidently, $G_{k,m,0} = 0, \ l + 1 \le k \le N, \ m \in \mathbb{N}$ by virtue of orthogonality condition (5.34). Using (5.41), we easily obtain that

$$\left| \frac{G_{k,m,n} - G_{k,n}}{|n| - ib_k n} \right| \le \frac{\sqrt{2\pi} \|G_{k,m}(x) - G_k(x)\|_{C(I)}}{\sqrt{1 + b_k^2}}.$$

Since the norm in the right side of this estimate from above tends to zero as $m \to \infty$, (5.36) holds in the case when $a_k = 0$ as well. Clearly, under the stated assumptions we have

$$\mathcal{N}_{a,\,b,\,k} < \infty, \quad \mathcal{N}_{a,\,b,\,k}^{(m)} < \infty, \quad m \in \mathbb{N}, \quad 1 \le k \le N, \quad a_k \ge 0, \quad b_k \in \mathbb{R}, \quad b_k \ne 0$$

by virtue of the result of our Lemma A3 above. We assume the validity of upper bound (5.35). A simple limiting argument using (5.38) and (5.39) gives us (5.40). □

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