Check for
updates

Lipschitz regularity of energy-minimal mappings between doubly connected Riemann surfaces

David Kalaj[1]

Abstract

Let M and N be doubly connected Riemann surfaces with $\mathscr{C}^{1,\alpha}$ boundaries and with nonvanishing conformal metrics σ and $\wp$ respectively, and assume that $\wp$ is a smooth metric with bounded Gauss curvature $\mathcal{K}$ and finite area. Assume that $\mathcal{H}^\wp(M, N)$ is the class of all $\mathscr{W}^{1,2}$ homeomorphisms between M and N and assume that $\mathcal{E}^\wp :$ $\mathcal{H}^\wp(M, N) \to \mathbf{R}$ is the Dirichlet-energy functional, where $\overline{\mathcal{H}}^\wp(M, N)$ is the closure of $\mathcal{H}^\wp(M, N)$ in $\mathscr{W}^{1,2}(M, N)$. By using a result of Iwaniec, Kovalev and Onninen in Iwaniec et al. (Duke Math J 162(4):643–672, 2013) that the minimizer, is locally Lipschitz, we prove that the minimizer, of the energy functional $\mathcal{E}^\wp$, which is not a diffeomorphism in general, is a globally Lipschitz mapping of M onto N. Note that, this result is new also for flat Riemann surfaces, i.e. for the planar domains furnished with the Euclidean metric.

Keywords Dirichlet energy · Riemann surfaces · Minimal energy · Harmonic mapping · Conformal modulus

Mathematics Subject Classification Primary 58E20; Secondary 30C62 · 31A05

Contents

✉ David Kalaj
  davidk@ucg.ac.me

[1] Faculty of Natural Sciences and Mathematics, University of Montenegro, Cetinjski put b.b., 81000 Podgorica, Montenegro

Ⓑ Birkhäuser

1 Introduction

The primary goal of this paper is to study the Lipschitz behavior of stationary deformations of the Dirichlet energy of mappings between doubly-connected Riemann surfaces. The existence part has been proved. More precisely, Iwaniec, Koh, Kovalev, and Onninen in [8] proved that there exists so-called deformation f that maps a double connected domain $\mathbb{X}$ onto a doubly connected domain $\mathbb{Y}$ in the complex plane and which minimizes the Dirichlet energy integral throughout the class of deformations $\mathfrak{D}(\mathbb{X}, \mathbb{Y})$ which contains the class of all Sobolev $\mathscr{W}^{1,2}(\mathbb{X})$ homeomorphisms. The minimizer is a harmonic diffeomorphism, provided that the conformal moduli satisfy the relation $\mathrm{Mod}(\mathbb{X}) \leq \mathrm{Mod}(\mathbb{Y})$. Moreover, the Dirichlet energy is invariant under conformal change of the original domain. This is why the original domain can be chosen to be equal to $\mathbb{X} = \mathbb{A}(r, R) := \{z : r < |z| < R\}$. The Hopf's differential defined by $\mathrm{Hopf}(f) = f_z \overline{f}_z$ has very special form for so-called stationary deformations namely

$$\mathrm{Hopf}(f)(z) = \frac{c}{z^2}, \quad z \in \mathbb{A}(r, R).$$

Later this result has been generalized by the author in [14] for certain metrics $\wp$ satisfying some general conditions in $\mathbb{Y}$.

In this case Hopf's differential is defined by

$$\mathrm{Hopf}(f) = \wp^2(f(z)) f_z \overline{f}_z$$

and has a very special form for a so-called stationary deformation namely

$$\mathrm{Hopf}(f)(z) = \frac{c}{z^2}, \quad z \in \mathbb{A}(r, R),$$

see Sect. 2.1 below.

Further, in [9], Iwaniec, Kovalev, and Onninen proved that every stationary deformation is locally Lipschitz in the domain.

In [12] the author proved that if $f : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ is a ρ-harmonic diffeomorphic minimizer, and $\partial\mathbb{X}, \partial\mathbb{Y} \in \mathscr{C}^2$ then f is Lipschitz continuous up to the boundary.

In [15] the author and Lamel proved that a minimizer of the Euclidean Dirichlet energy that is a diffeomorphic surjection of $\mathbb{X}$ onto $\mathbb{Y}$ has smooth extension up to the boundary. More precisely, they proved that if $f : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ is a Euclidean diffeomorphic minimizer, so that $\partial\mathbb{X}, \partial\mathbb{Y} \in \mathscr{C}^{1,\alpha}$, then the $f \in \mathscr{C}^{1,\alpha'}(\overline{\mathbb{X}})$, where $\alpha' = \alpha$, if $\mathrm{Mod}(\mathbb{X}) \geq \mathrm{Mod}(\mathbb{Y})$ and $\alpha' = \alpha/(2 + \alpha)$ if $\mathrm{Mod}(\mathbb{X}) < \mathrm{Mod}(\mathbb{Y})$.

In [13] the author extended the main result in [15] and proved the following extension of the Kellogg theorem. Every diffeomorphic minimiser of Dirichlet energy of Sobolev mappings between doubly connected Riemannian surfaces $(\mathbb{X}, \sigma)$ and $(\mathbb{Y}, \rho)$ having $\mathscr{C}^{n,\alpha}$ boundary, $0 < \alpha < 1$, is $\mathscr{C}^{n,\alpha}$ up to the boundary, provided the metric ρ is smooth enough. Here n is a positive integer. It is crucial that every diffeomorphic minimizer of Dirichlet energy is a harmonic mapping with a very special Hopf differential and this fact is used in the proof.

1.1 Harmonic mappings between Riemann surfaces

Let $M = (\mathbb{X}, \sigma)$ and $N = (\mathbb{Y}, \wp)$ be Riemann surfaces with metrics σ and $\wp$, respectively, where $\mathbb{X}$ and $\mathbb{Y}$ are planar domains. If a mapping $f : M \to N$, is C^2, then f is said to be harmonic (to avoid confusion we will sometimes say $\wp$-harmonic) if

$$f_{z\bar{z}} + (\log \wp^2)_w \circ f \cdot f_z f_{\bar{z}} = 0, \tag{1.1}$$

where z and w are the local parameters on M and N respectively. Also f satisfies (1.1) if and only if its Hopf differential

$$\mathrm{Hopf}(f) = \wp^2 \circ f f_z \overline{f_{\bar{z}}} \tag{1.2}$$

is the holomorphic quadratic differential on M. Let

$$|\partial f|^2 := \frac{\wp^2(f(z))}{\sigma^2(z)} \left| \frac{\partial f}{\partial z} \right|^2 \quad \text{and} \quad |\bar{\partial} f|^2 := \frac{\wp^2(f(z))}{\sigma^2(z)} \left| \frac{\partial f}{\partial \bar{z}} \right|^2$$

where $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$ are standard complex partial derivatives. The $\wp$-Jacobian is defined by

$$J(f) := |\partial f|^2 - |\partial \bar{f}|^2.$$

If u is sense preserving, then the $\wp$-Jacobian is positive. The Hilbert-Schmidt norm of differential df is the square root of the energy $e(f)$ and is defined by

$$|df| = \sqrt{2|\partial f|^2 + 2|\partial \bar{f}|^2}. \tag{1.3}$$

For $g : M \mapsto N$ the $\wp$-*Dirichlet energy* is defined by

$$\mathcal{E}^{\wp}[g] = \int_M |dg|^2 dV_\sigma, \tag{1.4}$$

where ∂g, and $\bar{\partial} g$ are the partial derivatives taken with respect to the metrics ϱ and σ, and dV_σ, is the volume element on (M, σ), which in local coordinates takes the form $\sigma^2(w)du \wedge dv$, $w = u + iv$. Assume that the energy integral of f is bounded. Then a stationary point f of the corresponding functional where the homotopy class of f is the range of this functional is a harmonic mapping. In order to derive the equation (1.1), we have to use the outer variation, namely: $g_t = f + th$, $t \in (-\epsilon, \epsilon)$, where h has a compact support in $\mathbb{X}$. Recall that $M = (\mathbb{X}, \sigma)$. For the last definition and some important properties of harmonic maps see [10, Chapter 8]. We refer to Sect. 2.1 below, for type another variation, the so-called inner variation, where a different class of stationary mappings is derived.

It follows from the definition that, if a is conformal and f is harmonic, then $f \circ a$ is harmonic. Moreover if b is conformal, $b \circ f$ is also harmonic but with respect to (possibly) an another metric $\wp_1$.

Notice that the harmonicity and Dirichlet energy do not depend on metric σ on a domain so we will assume from now on $\sigma(z) \equiv 1$. This is why throughout this paper $M = (\mathbb{X}, \mathbf{1})$ and $N = (\mathbb{Y}, \wp)$ will be doubly connected domains in the complex plane $\mathbb{C}$ (possibly unbounded), where $\mathbf{1}$ is the Euclidean metric. Moreover $\wp$ is a nonvanishing smooth metric defined in $\mathbb{Y}$ with bounded Gauss curvature $\mathcal{K}$ where

$$\mathcal{K}(z) = -\frac{\Delta \log \wp(z)}{\wp^2(z)}, \tag{1.5}$$

(we put $\kappa := \sup_{z \in \mathbb{Y}} |\mathcal{K}(z)| < \infty$) and with a finite area defined by

$$\mathcal{A}(\wp) = \int_{\mathbb{Y}} \wp^2(w) du dv, \quad w = u + iv.$$

We call a metric ρ an admissible one if there is a constant $C_\wp > 0$ so that

$$|\nabla \wp(w)| \le C_\wp \wp(w), \quad w \in \mathbb{Y} \ i.e. \ \nabla \log \wp \in L^\infty(\mathbb{Y}) \tag{1.6}$$

which means that $\wp$ is so-called approximately analytic function (c.f. [5]).

Assume that the domain of $\wp$ is the unit disk $\mathbf{D} := \{z : |z| < 1\} \subset \mathbf{C}$. From (1.6) and boundedness of $\wp$, it follows that it is Lipschitz, and so it is continuous up to the boundary. Again by using (1.6), the function $\psi(t) = \wp(te^{i\beta})$, $0 < t < 1$, $\beta \in [0, 2\pi]$ satisfies the differential inequalities $-C_\wp \le \partial_t \log \psi(t) \le C_\wp$, which by integrating in $[0, t]$ imply that $\psi(0)e^{-C_\wp t} \le \psi(t) \le \psi(0)e^{C_\wp t}$. Therefore under the above conditions there holds the double inequality

$$0 < \wp(0)e^{-C_\wp} \le \wp(w) \le \wp(0)e^{C_\wp} < \infty, \quad w \in \mathbf{D}. \tag{1.7}$$

A similar inequality to (1.7) can be proved for $\mathbb{Y}$ instead of $\mathbf{D}$. The Euclidean metric ($\wp \equiv 1$) is an admissible metric. The Riemannian metric defined by $\wp(w) = 1/(1 + |w|^2)^2$ is admissible as well. The Hyperbolic metric $h(w) = 1/(1 - |w|^2)^2$ is not an admissible metric on the unit disk neither on the annuli $\mathbb{A}(r, 1) := \{z : r < |z| < 1\}$, but it is admissible in $\mathbb{A}(r, R) := \{z : r < |z| < R\}$, where $0 < r < R < 1$. We call such a metric *allowable* one (cf. [1, P. 11]). If $\wp$ is a given metric in $\mathbb{Y}$, we conventionally extend it to be equal to 0 in $\partial\mathbb{Y}$. As we already pointed out, we will study the minimum of Dirichlet integral of mappings between certain sets. We refer to the introduction of [8] and references therein for a good setting of this problem and some connection with the theory of nonlinear elasticity. Notice first that a change of variables $w = f(z)$ in (1.4) yields

$$\mathcal{E}^\wp[f] = 2 \int_{\mathbb{X}} \wp^2(f(z)) J_f(z) \, dz + 4 \int_{\mathbb{X}} \wp^2(f(z))|f_{\bar{z}}|^2 dz \ge 2\mathcal{A}(\wp) \tag{1.8}$$

where J_f is the Jacobian determinant and $\mathcal{A}(\wp)$ is the area of $\mathbb{Y}$ and $dz := dx \wedge dy$ is the area element w.r. to Lebesgue measure on the complex plane. A conformal mapping of $f : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$; that is, a homeomorphic solution of the Cauchy-Riemann system $f_{\bar{z}} = 0$, would be an obvious choice for the minimizer of (1.8). For arbitrary multiply connected domains there is no such mapping.

Any energy-minimal diffeomorphism satisfies Euler-Lagrange equation, since one can perform first variations while preserving the diffeomorphism property. However, in the case of Euclidean metric $\wp \equiv 1$, the existence of a harmonic diffeomorphism between certain sets does not imply the existence of an energy-minimal one, see [8, Example 9.1]. Example 9.1 in [8] has been constructed with help of affine self-mappings of the complex plane. For a general metric $\wp$, affine transformations are not harmonic, thus we cannot produce a similar example.

1.2 Statement of results

Assume that $\mathbb{X}$ and $\mathbb{Y}$ are doubly-connected domains, whose boundary components are Jordan curves, and let $\wp$ be an admissible metric in $\mathbb{Y}$.

Assume that $\mathscr{W}_\wp^{1,2}(\mathbb{X}, \mathbb{Y})$ is the class of mappings that belongs to $\mathscr{W}_{\text{loc}}^{1,2}$ and satisfy the inequality

$$\int_\mathbb{X} \wp^2(f(z))(|f_z|^2 + |f_{\bar{z}}|^2)dxdy + \int_\mathbb{X} \wp^2(f(z))|f(z)|^2 dxdy < \infty.$$

Assume that $\mathcal{H}^\wp(\mathbb{X}, \mathbb{Y}) \subset \mathscr{W}_\wp^{1,2}(\mathbb{X}, \mathbb{Y})$ is the class of homeomorphic mappings between $\mathbb{X}$ and $\mathbb{Y}$, that maps the inner boundary onto inner boundary and outer boundary onto the outer boundary.

Let $\overline{\mathcal{H}}^\wp(\mathbb{X}, \mathbb{Y})$ be the closure of $\mathcal{H}^\wp(\mathbb{X}, \mathbb{Y})$ in the strong topology of $\mathscr{W}_\wp^{1,2}(\mathbb{X}, \mathbb{Y})$.

Now in view of (1.7), we can see that

$$\mathscr{W}_\wp^{1,2}(\mathbb{X}, \mathbb{Y}) = \mathscr{W}^{1,2}(\mathbb{X}, \mathbb{Y})$$

and

$$\mathcal{H}^\wp(\mathbb{X}, \mathbb{Y}) = \mathcal{H}(\mathbb{X}, \mathbb{Y}),$$

where the class $\mathcal{H}(\mathbb{X}, \mathbb{Y})$ stands for the corresponding class for the case $\wp = 1$, i.e. when $\wp$ is the Euclidean metric.

Then by [6, eq. 2.4] (or [7]), we have that $\overline{\mathcal{H}}(\mathbb{X}, \mathbb{Y})$ coincides with the closure in the weak topology of $\mathcal{H}(\mathbb{X}, \mathbb{Y})$ of $\mathscr{W}^{1,2}(\mathbb{X}, \mathbb{Y})$ and as well as with the weak limit of homeomorphisms $h_j : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ in the Sobolev space $\mathscr{W}^{1,2}(\mathbb{X}, \mathbb{Y})$, provided that the target annulus is Lipschitz regular.

The main result of this paper is the following extension of the main result in [12].

Theorem 1.1 *Suppose that $\mathbb{X}$ and $\mathbb{Y}$ are doubly connected domains in $\mathbb{C}$ with $\mathscr{C}^{1,\alpha}$ boundaries. In other words, the boundaries of $\mathbb{X}$ and $\mathbb{Y}$ are two pairs of smooth Jordan curves. Let $\wp$ be an admissible metric in $\mathbb{Y}$. Then there exists a mapping $w \in \overline{\mathcal{H}}^\wp(\mathbb{X}, \mathbb{Y})$*

that minimizes $\wp$-energy throughout the class $\mathcal{H}^\wp(\mathbb{X}, \mathbb{Y})$ and it is Lipschitz continuous up to the boundary of $\mathbb{X}$. For $\mathrm{Mod}(\mathbb{X}) \leq \mathrm{Mod}(\mathbb{Y})$, the minimizer w is a ρ-harmonic diffeomorphism of $\mathbb{X}$ onto $\mathbb{Y}$.

Remark 1.2 In contrast to the main result obtained in [12], in Theorem 1.1 the minimizer is not necessarily a diffeomorphism, and such a minimizer exists almost always except in some degenerate cases (see Proposition 2.2 below). Note that this result is also new for $\wp \equiv 1$ i.e. for the Euclidean metric.

The minimizer of the energy is not always a homeomorphism. In fact, for the minimizer to be a homeomorphism, the domains $\mathbb{X}$ and $\mathbb{Y}$ need to satisfy some conditions. As it is said before the condition $\mathrm{Mod}(\mathbb{X}) \leq \mathrm{Mod}(\mathbb{Y})$ that the minimizer is a certain harmonic diffeomorphism of $\mathbb{X}$ onto $\mathbb{Y}$. The example below (Example 1.3) is taken from [14] for general radial metric. It shows that the minimizer of the energy is not always a homeomorphism between $\mathbb{X}$ and $\mathbb{Y}$. For a corresponding example for the Euclidean metric see [8]. The same result can be stated for somehow more general setting under the additional condition $\mathrm{Mod}(\mathbb{X}) \leq \mathrm{Mod}(\mathbb{Y})$, namely we can minimize the Dirichlet energy throughout the class of all deformations $\mathfrak{D}^\wp(\mathbb{X}, \mathbb{Y}) \supseteq \mathcal{H}(\mathbb{X}, \mathbb{Y})$, where $\mathfrak{D}^\wp(\mathbb{X}, \mathbb{Y})$ is the so-called class of deformations [8, 14]. In that case, the minimizer is a harmonic diffeomorphism. We believe that the same conclusion of Theorem 1.1 still holds if we replace Sobolev homeomorphisms by deformations, without imposing the inequality between moduli of annuli. Indeed the only important thing we need for the proof is that the minimizer has a continuous extension up to the boundary, and we were able to get such behavior for the limit of homeomorphisms (see the last part of Proposition 2.2).

Example 1.3 Assume that $\wp(w) = \rho(|w|)$ is a radial metric defined in the annulus $\Omega^* = \mathbb{A}(1, R)$. Assume that $\Omega = \mathbb{A}(1, r)$. Choose

$$r < r_\diamond = \exp\left(\int_1^R \frac{\rho(y)dy}{\sqrt{y^2\rho^2(y) - R^2\rho^2(R)}}\right).$$

Then $r < r_\diamond < 1$.

Further for $\gamma = -\delta^2\rho^2(\delta)$, we have well-defined function

$$q_\diamond(s) = \exp\left(\int_\sigma^s \frac{dy}{\sqrt{y^2 - \delta^2\rho^2(\delta)\varrho^2}}\right), \quad \delta \leq s \leq \sigma. \tag{1.9}$$

Then the infimum $\mathrm{E}^\rho(\Omega, \Omega^*)$ is realized by a non-injective deformation $h \colon \Omega \xrightarrow{\text{onto}} \Omega^*$

$$h(z) = \begin{cases} R\frac{z}{|z|} & \text{for } r < |z| \leq r_\diamond \\ h_\diamond(z) & \text{for } r_\diamond \leq |z| < 1 \end{cases}$$

where $h_\diamond(z) = p_\diamond(|z|)e^{it} = (q_\diamond)^{-1}(|z|)e^{it}$, $z = |z|e^{it}$ and $q_\diamond$ is defined in (1.9) below

Here the radial projection $z \mapsto Rz/|z|$ hammers $A(r, r_\diamond)$ onto the circle $|z| = R$ while the critical ρ-Nitsche mapping $h_\diamond$ takes $A(r_\diamond, 1)$ homeomorphically onto Ω^*.

2 Preliminary results

2.1 Stationary mappings

We call a mapping $h \in \overline{\mathcal{H}}^{\wp}(\mathbb{X}, \mathbb{Y})$ *stationary* if

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{E}^{\wp}[h \circ \phi_t^{-1}] = 0 \tag{2.1}$$

for every family of diffeomorphisms $t \to \phi_t \colon \mathbb{X} \to \mathbb{X}$ which depends smoothly on the parameter $t \in \mathbb{R}$ and satisfy $\phi_0 = $ id. The latter mean that the mapping $\mathbb{X} \times [0, \epsilon_0] \ni (z, t) \to \phi_t(z) \in \mathbb{X}$ is a smooth mapping for some $\epsilon_0 > 0$. We now have.

Lemma 2.1 [14] *Let* $\mathbb{X} = A(r, R)$ *be a circular annulus,* $0 < r < R < \infty$, *and assume that* $\mathbb{Y}$ *is a doubly connected domain. If* $h \in \overline{\mathcal{H}}^{\wp}(\mathbb{X}, \mathbb{Y})$ *is a stationary mapping, then*

$$\wp^2(h(z))h_z\overline{h_{\overline{z}}} \equiv \frac{c}{z^2} \quad \text{in } \mathbb{X} \tag{2.2}$$

where $c \in \mathbb{R}$ *is a constant. Moreover* $c \geq 0$ *if* $\mathrm{Mod}(\mathbb{X}) \leq \mathrm{Mod}(\mathbb{Y})$ *and* $c < 0$ *if* $\mathrm{Mod}(\mathbb{X}) > \mathrm{Mod}(\mathbb{Y})$.

Notice that the corresponding lemma in [14] is for deformations, but the proof works as well for the class $\overline{\mathcal{H}}^{\wp}(\mathbb{X}, \mathbb{Y}) \subset \mathfrak{D}^{\wp}(\mathbb{X}, \mathbb{Y})$. We can also prove this by following the lines of the corresponding Proposition 3.4 in [6]. The only difference is that the corresponding Hopf differential contains $\wp^2(f(z))$. For the definition of $\mathfrak{D}^{\wp}(\mathbb{X}, \mathbb{Y})$ and related concepts we refer to [8, 14].

Now we have the following general result

Proposition 2.2 *Let* $\mathbb{X}$ *and* $\mathbb{Y}$ *be bounded doubly connected planar domains, such that* $\mathbb{Y}$ *is a Lipschitz domain. Assume that the boundary components of* $\mathbb{X}$ *do not degenerate into points.*

(1) There exists $h \in \overline{\mathcal{H}}^{\wp}(\mathbb{X}, \mathbb{Y})$ *such that*

$$\mathcal{E}^{\wp}[h] = E^{\wp}(\mathbb{X}, \mathbb{Y}) = \inf\{\mathcal{E}^{\wp}[h] : h \in \mathcal{H}^{\wp}(\mathbb{X}, \mathbb{Y})\}.$$

(2) If $\mathrm{Mod}(\mathbb{X}) \leq \mathrm{Mod}(\mathbb{Y})$, *then* h *is a* ρ-*harmonic diffeomorphism of* $\mathbb{X}$ *onto* $\mathbb{Y}$.
(3) The mapping h

    (a) is a stationary mapping with respect to the definition (2.1)
    (b) has a continuous extension up to the boundary of $\mathbb{X}$.

The first statement follows from [6, Theorem 1.1.] and [6, Lemma 2.9], in view of (1.7). The second statement is already proved in [14]. The last statement of the previous theorem follows from the inequality [6, Eq. 2.7]:

$$|f(x) - f(y)|^2 \leq \frac{C(\mathbb{X}, \mathbb{Y})}{\log \left(e + \frac{\text{diam}(\mathbb{X})}{|x-y|}\right)} \int_{\mathbb{X}} |Df|^2 dz, \quad x, y \in \mathbb{X}. \qquad (2.3)$$

The fact that $\int_{\mathbb{X}} |Df|^2 dz$ is finite follows from (1.7) and

$$\int_{\mathbb{X}} \wp^2(f)|Df|^2 dz < \infty.$$

3 Proof of the main result

We need the following important result concerning the local Lipschitz character of certain mappings proved by Iwaniec et al. [9].

Proposition 3.1 *Let $h \in \mathcal{W}^{1,2}(\mathbb{X})$ be a mapping with nonnegative Jacobian. Suppose that the Hopf product $h_z \overline{h_{\bar{z}}}$ is bounded and Hölder continuous. Then h is locally Lipschitz but not necessarily $\mathscr{C}^1$-smooth.*

We need also the following three results. To formulate the first one, let us define the class of $(K, K')-$ quasiconformal mappings, where $K \geq 1$, $K' \geq 0$. A mapping $f : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$, between two planar domains $\mathbb{X}$ and $\mathbb{Y}$ is called $(K, K')-$ quasiconformal if f is continuous, $f \in \mathcal{W}_{\text{loc}}^{1,2}(\mathbb{X})$ and if $|f_z|^2 + |f_{\bar{z}}|^2 \leq KJ(f, z) + K'$ almost everywhere in $\mathbb{X}$.

Lemma 3.2 *Every sense-preserving solution of Hopf equation so that $\text{Hopf}(f)$ is bounded that maps $\mathbb{A}(1, R)$ into $\mathbb{Y}$, mapping the inner/outer boundary to inner/outer boundary is (K, K') quasiconformal, where*

$$K = 1 \text{ and } K' = \sup_{z \in \mathbb{A}(1,R)} \frac{|c|}{|z|^2 \wp^2(f)}.$$

Proof We have

$$|Df(z)|^2 = |f_z|^2 + |f_{\bar{z}}|^2$$
$$\leq (|f_z|^2 - |f_{\bar{z}}|^2) + 2|f_{\bar{z}}|^2$$
$$\leq (|f_z|^2 - |f_{\bar{z}}|^2) + 2|f_{\bar{z}} f_z|.$$

Since

$$|f_{\bar{z}} f_z| = \frac{\text{Hopf}(f)}{|\wp(f(z))|^2} = \frac{|c|}{|z|^2 \wp^2(f)} \leq K'.$$

This implies the claim. $\qquad \square$

Further, we formulate the following local property of (K, K')-quasiconformal mappings.

Proposition 3.3 [4, Theorem 12.3] *Let $K > 1$, $K' \geq 0$ and assume that $f : \mathbb{X} \to \mathbf{C}$ is a (K, K')-quasiconformal mapping so that $|f(z)| \leq M$, $z \in \mathbb{X}$ and assume that $\mathbb{X}' \Subset \mathbb{X}$ and let $d = \text{dist}(\mathbb{X}', \partial\mathbb{X})$. Then there is a constant $C = C(K)$ so that*

$$|f(z) - f(z')| \leq C(K)(M + d\sqrt{K'})|z - z'|^\beta, \ z, z' \in \mathbb{X}'$$

where

$$\beta = K - \sqrt{K^2 - 1}.$$

If $K = 1$, then the above theorem can be formulated for K_1 instead of K, where $K_1 > 1$ is an arbitrary constant. For example for $K_1 = 5/4$, for which we get $\beta = K_1 - \sqrt{K_1^2 - 1} = 1/2$.

We also need the following lemma

Lemma 3.4 [14] *Assume that $\mathbb{X}$ and $\mathbb{Y}$ are doubly connected domains with $\mathscr{C}^{1,\alpha}$ boundaries, and assume that $a \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{A}(1, R)$ and $b \colon \mathbb{A}(1, \rho) \xrightarrow{\text{onto}} \mathbb{Y}$ are univalent conformal mappings and define $\wp_1(w) = \wp(b(w))|b'(w)|$, $w \in \mathbb{A}(1, \rho)$. Then*

(a) $\mathcal{E}^\wp[b \circ f \circ a] = \mathcal{E}^{\wp_1}[f]$ provided that one of the two sides exist.
(b) $b \circ f \circ a \in \mathcal{H}^\wp(\mathbb{X}, \mathbb{Y})$ if and only if $f \in \mathcal{H}^{\wp_1}(\mathbb{A}(1, R), \mathbb{A}(1, \rho))$.
(c) $\wp$ is an admissible metric if and only if $\wp_1$ is an admissible metric.
(d) $b \circ f \circ a$ is $\wp$-harmonic if and only if f is $\wp_1$-harmonic.

In conclusion, $b \circ f \circ a$ is a minimizer if and only if f is a minimizer.

Proof of Theorem 1.1 The existence part has been discussed in Proposition 2.2. We will apply a self-improving argument. In view of Lemma 3.4, we can assume that $\mathbb{X} = \mathbb{A}(1, R)$ for a constant $R > 1$. Assume that $f : \mathbb{A}(1, R) \to \mathbb{Y}$, has the following Hopf differential

$$\wp^2(f(z))f_z \bar{f}_z = \frac{c}{z^2}.$$

Let $\Psi : \mathbb{Y} \to \mathbb{A}(1, \rho)$ be a conformal diffeomorphism and let $\Phi = \Psi^{-1}$.

Then $F(z) = \Psi \circ f : \mathbb{A}(1, R) \to \mathbb{A}(1, \rho)$ is a minimizer of $\wp_1$-energy, between $\mathbb{A}(1, R)$ and $\mathbb{A}(1, \rho)$, where $\wp_1(\zeta) = \wp(\Phi(\zeta))|\Phi'(\zeta)|$ and

$$\text{Hopf}(F)(z) = \wp_1^2(F(z))F_z\bar{F}_z = \frac{c}{z^2}. \tag{3.1}$$

Further for

$$\tilde{F}(z) = \begin{cases} F(z), & 1 < |z| \leq R; \\ \rho^2/\overline{F(R^2/\bar{z})}, & R \leq |z| < R^2 \\ 1/\overline{F(1/\bar{z})}, & 1/R \leq |z| \leq 1, \end{cases}$$

let

$$\tilde{\wp}(w) = \begin{cases} \wp_1(w), & 1 \le |w| \le \rho; \\ \wp_1(\rho^2/\bar{w}), & \rho < |w| < \rho^2, \\ \wp_1(1/\bar{w}), & 1/\rho < |w| \le 1. \end{cases}$$

Observe that $\tilde{F}$ is continuous. Namely, if $F(Re^{it}) = \rho e^{is}$, then

$$\rho^2/\overline{F(R^2/(Re^{-it}))} = \rho e^{is}.$$

Thus $\tilde{F} : \mathbb{A}(1/R, R^2) \to \mathbb{A}(1/\rho, \rho^2)$ is continuous and belongs to the same class as F, which mean it is in $\mathscr{W}^{1,2}$. Then by direct calculation, we get

$$\mathrm{Hopf}(\tilde{F}) = \tilde{\wp}^2(\tilde{F}(z))\tilde{F}_z\overline{\tilde{F}}_z = \begin{cases} \frac{c}{z^2}, & 1 < |z| \le R; \\ \frac{\rho^4}{|F(R^2/\bar{z})|^4}\frac{c}{z^2}, & R \le |z| < R^2. \\ \frac{1}{|F(1/\bar{z})|^4}\frac{c}{z^2}, & 1/R \le |z| < 1. \end{cases}$$

Let us demonstrate for example

$$\mathrm{Hopf}(\tilde{F}) = \frac{1}{|F(1/\bar{z})|^4}\frac{c}{z^2}$$

for $1/R \le |z| < 1$.

We have

$$\tilde{F}_z(z) = \frac{d}{dz}\left(\frac{1}{\overline{F(\frac{1}{\bar{z}})}}\right) = \frac{1}{\overline{F^2(\frac{1}{\bar{z}})}}\overline{F_z\left(\frac{1}{\bar{z}}\right)}\frac{1}{z^2}$$

and

$$\overline{\tilde{F}}_z(z) = \frac{d}{dz}\left(\frac{1}{F(\frac{1}{\bar{z}})}\right) = \frac{1}{F^2(\frac{1}{\bar{z}})}\overline{F_z\left(\frac{1}{\bar{z}}\right)}\frac{1}{z^2}.$$

Then from (3.1) we get

$$\tilde{\wp}^2(\tilde{F}(z))\tilde{F}_z(z)\overline{\tilde{F}}_z(z) = \frac{1}{|F(\frac{1}{\bar{z}})|^4}\frac{cz^2}{z^4} = \frac{1}{|F(\frac{1}{\bar{z}})|^4}\frac{c}{z^2}.$$

From the Kellogg theorem for conformal mappings, we know that Φ' is $\mathscr{C}^\alpha$ up to the boundary of $\mathbb{Y}$. This implies in particular that the function

$$\tilde{F}_z\overline{\tilde{F}}_z = \frac{\mathrm{Hopf}(\tilde{F})}{\tilde{\wp}^2(\tilde{F}(z))}$$

is bounded in $\mathbb{A}(1/R, R^2)$. From Lemma 3.2, in view of Proposition 2.2, we obtain that $\tilde{F}$ is $(1, \tilde{K})$-quasiconformal. Now Proposition 3.3 implies that F is $\frac{1}{2}$-Hölder continuous in $\mathbb{A}(1, R)$. More precisely

$$|\tilde{F}(z) - \tilde{F}(z')| \le C \left(\rho^2 + \frac{R^2 - 1}{R^2} \sqrt{\tilde{K}} \right) |z - z'|^{1/2}.$$

So

$$|F(z) - F(z')| \le C \left(\rho^2 + \frac{R^2 - 1}{R^2} \sqrt{\tilde{K}} \right) |z - z'|^{1/2}.$$

This implies that $\tilde{F}_z \overline{\tilde{F}_z}$ is $\alpha \cdot 1/2$-Hölder continuous as a composition of two mappings which are α- and $1/2$-Hölder continuous respectively in $\mathbb{A}(1/R, R^2)$. Now Theorem 3.1 implies that $\tilde{F}$ is locally Lipschitz. In particular F is Lipschitz in $\overline{\mathbb{A}(1, R)} \Subset \mathbb{A}(1/R, R^2)$. Therefore $f : \mathbb{A}(1, R) \to \mathbb{Y}$ is Lipschitz continuous. Now Lemma 3.4 and Kellogg theorem for conformal mappings imply the claim. This finishes the proof of the main result. □

Data availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The author declares that there is no conflicts of interest.

References

1. Ahlfors, L.V.: Lectures on Quasiconformal Mappings, 2nd edn. With supplemental chapters by C. J. Earle, I. Kra, M. Shishikura and J. H. Hubbard. University Lecture Series, 38. American Mathematical Society, Providence (2006)
2. Astala, K., Iwaniec, T., Martin, G.: Elliptic partial differential equations and quasiconformal mappings in the plane. Princeton University Press, NJ (2009)
3. Evans, L.C.: A new proof of local $C^{1,\alpha}$ regularity for solutions of certain degenerate elliptic P.D.E. J. Differ. Equ. **45**, 356–373 (1982)
4. Gilbarg, D., Trudinger, N.: Elliptic Partial Differential Equations of Second Order, 2 Edn. Springer 1977 (1983)
5. Heinz, E.: On certain nonlinear elliptic differential equations and univalent mappings. J. Anal. Math. **5**, 197–272 (1957)
6. Iwaniec, T., Onninen, J.: Mappings of least Dirichlet energy and their Hopf differentials. Arch. Ration. Mech. Anal. **209**(2), 401–453 (2013)
7. Iwaniec, T., Onninen, J.: Limits of Sobolev homeomorphisms. J. Eur. Math. Soc. (JEMS) **19**(2), 473–505 (2017)
8. Iwaniec, T., Koh, N.-T., Kovalev, L.V., Onninen, J.: Existence of energy-minimal diffeomorphisms between doubly connected domains. Invent. Math. **186**, 667–707 (2011)
9. Iwaniec, T., Onninen, J., Kovalev, L.: Lipschitz regularity for inner-variational equations. Duke Math. J. **162**(4), 643–672 (2013)
10. Jost, J.: Riemannian Geometry and Geometric Analysis, 7th edn. Universitext. Springer, Cham. ISBN 978-3-319-61859-3/pbk; 978-3-319-61860-9/ebook). xiv, 697 p. (2017)

11. Jost, J.: Minimal surfaces and Teichmüller theory. In: Yau, S.-T. (Ed.), Tsing Hua Lectures On Geometry and Analysis, Taiwan, 1990–91, pp. 149–211. International Press, Cambridge (1997)
12. Kalaj, D.: Lipschitz property of minimisers between double connected surfaces. J. Geom. Anal. **30**, 4150–4165 (2020)
13. Kalaj, D.: Kellogg's Theorem for Diffeomophic Minimisers of Dirichlet Energy Between Doubly Connected Riemann Surfaces, Calc. Var. Partial Differ. Equ. 61, No. 5, Paper No. 168, 26 p. (2022)
14. Kalaj, D.: Energy-minimal diffeomorphisms between doubly connected Riemann surfaces. Calc. Var. Partial. Differ. Equ. **51**, 465–494 (2014)
15. Kalaj, D., Lamel, B.: Minimisers and Kellogg's theorem. Math. Ann. **377**(3–4), 1643–1672 (2020)
16. Kalaj, D., Mateljevic, M.: (K, K')-quasiconformal harmonic mappings. Potential Anal. **36**(1), 117–135 (2012)