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Some characterizations of slice regular Lipschitz type spaces

José Oscar González-Cervantes[1] · **Daniel González-Campos**[2] · **Juan Bory-Reyes**[3]

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Abstract
We give some characterizations of Lipschitz type spaces of slice regular functions in the unit ball of the skew field of quaternions with prescribed modulus of continuity.

Keywords Quaternionic slice regular functions · Lipschitz type spaces · Schwarz-Pick lemma · Equivalent norms

Mathematics Subject Classification 30G30 · 30G35 · 35R11 · 51F30

1 Introduction

The quaternionic valued functions of a quaternionic variable, often referred to as slice regular functions, was born in [1, 2]. This class of functions, which would somehow resemble the classical theory of holomorphic functions of one complex variable, has been studied extensively in the last years, see [2–8] and the references given there.

It was shown in [9, 10] some characterization of generalized Lipschitz type spaces of holomorphic functions with prescribed behavior near the unit circle centered at the origin, determined by a regular majorant in terms of the moduli of their members. Rather surprisingly, several authors attempted to extend the aforementioned charac-

✉ Juan Bory-Reyes
juanboryreyes@yahoo.com

José Oscar González-Cervantes
jogc200678@gmail.com

Daniel González-Campos
daniel_uz13@hotmail.com

1 ESFM-Instituto Politécnico Nacional, 07338 Mexico City, Mexico

2 SEPI-ESFM-Instituto Politécnico Nacional, 07338 Mexico City, Mexico

3 SEPI-ESIME-Zacatenco-Instituto Politécnico Nacional, 07338 Mexico City, Mexico

terizations to (holomorphic) smoothness spaces of both complex and vector-valued functions, see [11–16] and the references therein.

This paper is devoted to establish some analogous results of Dyakonov's paper [9] for the theory of slice regular quaternionic functions. The main results are Proposition 3.5 and Corollary 3.15 of Proposition 3.14 as well as Corollary 3.18 of Proposition 3.17.

2 Preliminaries

2.1 Antecedents in standard complex analysis

For the convenience of the reader, we recall the relevant material from [9, 10] and provide some additional notations and terminology, thus making our exposition self-contained.

Let $\mathbb{D}$ stand for the unit disc in the complex plane $\mathbb{C}$, $\mathbb{S}^1$ be the unit circle and $\overline{\mathbb{D}} := \mathbb{D} \cup \mathbb{S}^1$. The algebra $Hol(\mathbb{D})$ consists of those holomorphic functions on $\mathbb{D}$ that are continuous up to $\mathbb{S}^1$.

A continuous function $\omega : [0, 2] \to \mathbb{R}_+$ with $\omega(0) = 0$ will be called a regular majorant if $\omega(t)$ is increasing, $\dfrac{\omega(t)}{t}$ is decreasing for $t \in [0, 2]$ and such that

$$\int_0^x \frac{\omega(t)}{t} dt + x \int_x^2 \frac{\omega(t)}{t^2} dt \le C\omega(x), \quad 0 < x < 2.$$

Here and subsequently C stands for a positive real constant, not necessarily the same at each occurrence. When necessary, we will use subscripts to differentiate several constants.

Given a regular majorant the Lipschitz type space, denoted by $\Lambda_\omega(\mathbb{D})$, consists (by definition) of all complex valued functions f defined on $\mathbb{D}$ such that

$$|f(z) - f(\zeta)| \le C\omega(|z - \zeta|), \quad \forall z, \zeta \in \mathbb{D}.$$

The class $\Lambda_\omega(\mathbb{S}^1)$, is defined similarly.

Let us state the main results of [9] as Theorems A and B, the proofs of which were considerably shorted in [10].

Theorem A *Let ω be a regular majorant. A function f holomorphic in $\mathbb{D}$ is in $\Lambda_\omega(\mathbb{D})$ if and only if so is its modulus $|f|$.*

If $f \in Hol(\mathbb{D})$, then $|f|$ is a subharmonic function, hence the Poisson integral of $|f|$, denoted by $P[|f|]$, is equal to the smallest harmonic majorant in $\mathbb{D}$. In particular, $P[|f|] - |f| \ge 0$ in $\mathbb{D}$.

Theorem B *Let ω be a regular majorant, $f \in Hol(\mathbb{D})$, and assume the boundary function of $|f|$ belongs to $\Lambda_\omega(\mathbb{S}^1)$. Then f is in $\Lambda_w(\mathbb{D})$ if and only if*

$$P[|f|](z) - |f(z)| \le C\omega(1 - |z|).$$

The following notation will be needed

$$\|f\|_{\Lambda_\omega(\mathbb{D})} = \sup\{\frac{|f(z)-f(\zeta)|}{\omega(|z-\zeta|)} \mid z,\zeta \in \mathbb{D},\ z \neq \zeta\},\quad \forall f \in C(\overline{\mathbb{D}},\mathbb{C}).$$

The notation $\mathfrak{A} \asymp \mathfrak{B}$ means that there exist positive constants C_1 and C_2 such that $C_1\mathfrak{A} \leq \mathfrak{B} \leq C_2\mathfrak{A}$.

Let ω be a majorant and $f \in Hol(\mathbb{D}) \cap C(\overline{\mathbb{D}},\mathbb{C})$. We introduce the following notations:

$$\begin{aligned} N_1(f) &:= \|\, |f|\, \|_{\Lambda_\omega(\mathbb{S}^1)} + \sup\left\{\frac{P[|f|](z) - |f|(z)}{\omega(1-|z|)} \mid z \in \mathbb{D}\right\},\\ N_2(f) &:= \|\, |f|\, \|_{\Lambda_\omega(\mathbb{S}^1)} + \sup\left\{\frac{|\,|f|(\zeta) - |f|(r\zeta)\,|}{\omega(1-r)} \mid \zeta \in \mathbb{S}^1,\ 0 \leq r < 1\right\},\\ N_3(f) &:= \|\, |f|\, \|_{\Lambda_\omega(\overline{\mathbb{D}})}. \end{aligned}$$

In particular, we have:

1. If ω and ω^2 are regular majorants then

$$\|f\|_{\Lambda_\omega(\mathbb{D})} \asymp \sup\left\{\frac{\{P[|f|^2](z) - |f(z)|^2\}^{\frac{1}{2}}}{\omega(1-|z|)} \mid z \in \mathbb{D}\right\}. \tag{2.1}$$

2. If ω is a regular majorant then

$$\|f\|_{\Lambda_\omega(\mathbb{D})} \asymp N_1(f) \asymp N_2(f) \asymp N_3(f), \tag{2.2}$$

for any $f \in Hol(\mathbb{D}) \cap C(\overline{\mathbb{D}},\mathbb{C})$.

2.2 Brief introduction to slice regular functions

A quaternion is given by $q = x_0 + x_1e_1 + x_2e_2 + x_3e_3$ where x_0, x_1, x_2, x_3 are real values and the imaginary units satisfy: $e_1^2 = e_2^2 = e_3^2 = -1$, $e_1e_2 = -e_2e_1 = e_3$, $e_2e_3 = -e_3e_2 = e_1$, $e_3e_1 = -e_1e_3 = e_2$. The skew field of quaternions is denoted by $\mathbb{H}$. The sets $\{e_1, e_2, e_3\}$ and $\{1, e_1, e_2, e_3\}$ are called the standard basis of $\mathbb{R}^3$ and $\mathbb{H}$, respectively. The vector part of $q \in \mathbb{H}$ is $\mathbf{q} = x_1e_1 + x_2e_2 + x_3e_3$ and its real part is $q_0 = x_0$. The quaternionic conjugation of q is $\bar{q} = q_0 - \mathbf{q}$ and its norm is $\|q\| := \sqrt{x_0^2 + x_1^2 + x_2^3 + x_3^2} = \sqrt{q\bar{q}} = \sqrt{\bar{q}q}$.

By abuse of notation, the unit open ball in $\mathbb{H}$ will be denoted by $\mathbb{D}^4 := \{q \in \mathbb{H} \mid \|q\| < 1\}$ so will the unit spheres in $\mathbb{R}^3$ (in $\mathbb{H}$) by $\mathbb{S}^2 := \{\mathbf{q} \in \mathbb{R}^3 \mid \|\mathbf{q}\| = 1\}$ ($\mathbb{S}^3 := \{q \in \mathbb{H} \mid \|q\| = 1\}$), respectively.

The quaternionic structure allows us to see that $\mathbf{i}^2 = -1$, for every $\mathbf{i} \in \mathbb{S}^2$. Then $\mathbb{C}(\mathbf{i}) := \{x + \mathbf{i}y;\ \mid x, y \in \mathbb{R}\} \cong \mathbb{C}$ as fields, and any $q \in \mathbb{H} \setminus \mathbb{R}$ may be rewritten by

$x + \mathbf{I}_q y$ where $x, y \in \mathbb{R}$ and $\mathbf{I}_q := \|\mathbf{q}\|^{-1}\mathbf{q} \in \mathbb{S}^2$; i.e., $q \in \mathbb{C}(\mathbf{I}_q)$. Note that $q \in \mathbb{R}$ belongs to every complex plane.

Given $u \in \mathbb{S}^3$, the mapping $\mathbf{q} \mapsto u\mathbf{q}\bar{u}$ for all $\mathbf{q} \in \mathbb{R}^3$ is a quaternionic rotation that preserves $\mathbb{R}^3$, see [17, 18]. For any $\mathbf{i} \in \mathbb{S}^2$ we will write

$$\mathbb{D}_{\mathbf{i}} := \mathbb{D}^4 \cap \mathbb{C}(\mathbf{i})$$

and

$$\mathbb{S}_{\mathbf{i}} := \mathbb{S}^2 \cap \mathbb{C}(\mathbf{i}).$$

Now, we recall few aspects of the slice regular functions theory of [4–6, 8, 19].

Definition 2.1 Let $\Omega \subset \mathbb{H}$ be an open domain. A real differentiable function $f : \Omega \to \mathbb{H}$ is called (left) slice regular function on Ω if

$$\overline{\partial}_{\mathbf{i}} f \mid_{\Omega\cap\mathbb{C}(\mathbf{i})} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathbf{i}\frac{\partial}{\partial y}\right) f \mid \mid_{\Omega\cap\mathbb{C}(\mathbf{i})} = 0 \ \text{ on } \Omega_{\mathbf{i}} := \Omega \cap \mathbb{C}(\mathbf{i}),$$

for all $\mathbf{i} \in \mathbb{S}^2$ and its derivative, or Cullen derivative, see [1], is given by

$$f' = \partial_{\mathbf{i}} f \mid_{\Omega\cap\mathbb{C}(\mathbf{i})} = \frac{\partial}{\partial x} f \mid_{\Omega\cap\mathbb{C}(\mathbf{i})} = \partial_x f \mid_{\Omega\cap\mathbb{C}(\mathbf{i})} .$$

Let $\mathcal{SR}(\Omega)$ denote the right linear space of slice regular functions on Ω.

Definition 2.2 A set $U \subset \mathbb{H}$ is called axially symmetric if $x + \mathbf{i}y \in U$ with $x, y \in \mathbb{R}$, then $\{x + \mathbf{j}y \mid \mathbf{j} \in \mathbb{S}^2\} \subset U$ and $U \cap \mathbb{R} \neq \emptyset$. A domain $U \subset \mathbb{H}$ is called slice domain, or s-domain, if $U_{\mathbf{i}} = U \cap \mathbb{C}(\mathbf{i})$ is a domain in $\mathbb{C}(\mathbf{i})$ for all $\mathbf{i} \in \mathbb{S}^2$.

Let $\Omega \subset \mathbb{H}$ an axially symmetric s-domain. A function $f \in \mathcal{SR}(\Omega)$ is said to be intrinsic if $f(q) = \overline{f(\bar{q})}$ for all $q \in \Omega$. The real linear space of intrinsic slice regular functions on Ω will be denoted by $\mathcal{N}(\Omega)$, see [7, 19]. We will denote by Z_f the set of zeroes of function f.

Theorem 2.3 *Let $\Omega \subset \mathbb{H}$ be an axially symmetric s-domain and $f \in \mathcal{SR}(\Omega)$.*

1. *(Splitting Property) For every $\mathbf{i}, \mathbf{j} \in \mathbb{S}$, orthogonal to each other, there exist holomorphic functions $F, G : \Omega_{\mathbf{i}} \to \mathbb{C}(\mathbf{i})$ such that $f_{|\Omega_{\mathbf{i}}} = F + G\mathbf{j}$ on $\Omega_{\mathbf{i}}$, see [4].*
2. *(Representation Formula) For every $q = x + \mathbf{I}_q y \in \Omega$ with $x, y \in \mathbb{R}$ and $\mathbf{I}_q \in \mathbb{S}^2$ the following identity holds*

$$f(x + \mathbf{I}_q y) = \frac{1}{2}[f(x + \mathbf{i}y) + f(x - \mathbf{i}y)] + \frac{1}{2}\mathbf{I}_q\mathbf{i}[f(x - \mathbf{i}y) - f(x + \mathbf{i}y)],$$

for all $\mathbf{i} \in \mathbb{S}^2$, see [5].

In the case of $\Omega = \mathbb{D}^4$ and given $f, g \in \mathcal{SR}(\mathbb{D}^4)$ there exist two sequences of quaternions (a_n) and (b_n) such that

$$f(q) = \sum_{n=0}^{\infty} q^n a_n, \; g(q) = \sum_{n=0}^{\infty} q^n b_n.$$

The product $f * g$ is defined as $f * g(q) := \sum_{n=0}^{\infty} q^n \sum_{k=0}^{n} a_k b_{n-k}$ for all $q \in \mathbb{D}^4$.

For $f(q) \neq 0$ the following property holds

$$f * g(q) = f(q)g(f(q)^{-1}qf(q)),$$

see [4]. What is more, if f^s has no zeroes, the $*$-inverse of f is given by

$$f^{-*} = \frac{1}{f^s} * f^c$$

and

$$(f^{-*})' = -f^{-*} * f' * f^{-*},$$

where $f^c(q) := \sum_{n=0}^{\infty} q^n \overline{a_n}$ for all $q \in \mathbb{D}^4$ and $f^s := f * f^c = f^c * f$, see [4, 20, 21].

3 Main results

Definition 3.1 Let ω be a regular majorant and $\mathbf{i} \in \mathbb{S}^2$. The set of all functions $f : \mathbb{D}^4 \to \mathbb{H}$ such that

$$\|f(x) - f(y)\| \leq C\omega(\|x - y\|), \quad \forall x, y \in \mathbb{D}_{\mathbf{i}}$$

will be denoted by ${}_{\mathbf{i}}\Lambda_{\omega}(\mathbb{D}^4)$.

We write ${}_{\mathbf{i}}\Lambda_{\omega}(\mathbb{S}^3)$ for the set of all functions $f : \mathbb{S}^3 \to \mathbb{H}$ such that

$$\|f(x) - f(y)\| \leq C\omega(\|x - y\|), \quad \forall x, y \in \mathbb{S}_{\mathbf{i}}.$$

The norm of a function $f \in {}_{\mathbf{i}}\Lambda_{\omega}(\mathbb{D}^4)$ is defined as

$$\|f\|_{{}_{\mathbf{i}}\Lambda_{\omega}(\mathbb{D}^4)} = \sup\left\{\frac{\|f(x) - f(y)\|}{\omega(\|x - y\|)} \;\middle|\; x, y \in \mathbb{D}_{\mathbf{i}},\; x \neq y\right\}.$$

Definition 3.2 Let ω_1, ω_2 be regular majorants and $\mathbf{i}, \mathbf{j} \in \mathbb{S}^2$ orthogonal to each other. We write ${}_{\mathbf{i}}\Lambda_{\omega_1,\omega_2}(\mathbb{D}^4)$ for the set of all functions $f : \mathbb{D}^4 \to \mathbb{H}$ such that

$$\|f_k(x) - f_k(y)\| \leq C_k\omega_k(\|x - y\|), \quad \forall x, y \in \mathbb{D}_{\mathbf{i}},$$

for $k = 1, 2$, where $f\mid_{\mathbb{D}_{\mathbf{i}}} = f_1 + f_2\mathbf{j}$ with $f_1, f_2 : \mathbb{D}_{\mathbf{i}} \to \mathbb{C}(\mathbf{i})$. The set ${}_{\mathbf{i}}\Lambda_{\omega_1,\omega_2}(\mathbb{S}^3)$ consists of all $f : \mathbb{S}^3 \to \mathbb{H}$ such that

$$\|f_k(x) - f_k(y)\| \leq C_k\omega_k(\|x - y\|), \quad \forall x, y \in \mathbb{S}_{\mathbf{i}},$$

for $k = 1, 2$, where $f\mid_{\mathbb{D}_{\mathbf{i}}} = f_1 + f_2\mathbf{j}$ with $f_1, f_2 : \mathbb{S}_{\mathbf{i}} \to \mathbb{C}(\mathbf{i})$. For $f \in {}_{\mathbf{i}}\Lambda_{\omega_1,\omega_2}(\mathbb{D}^4)$ we define

$$\|f\|^2_{{}_{\mathbf{i}}\Lambda_{\omega_1,\omega_2}(\mathbb{D}^4)} = \sup\left\{\frac{\|f_1(x) - f_1(y)\|^2}{\omega_1(\|x - y\|)^2} + \frac{\|f_2(x) - f_2(y)\|^2}{\omega_2(\|x - y\|)^2} \mid x, y \in \mathbb{D}_{\mathbf{i}},\ x \neq y\right\}.$$

Given $\mathbf{i} \in \mathbb{S}^2$, the $\mathbf{i}$-Poisson integral of $u \in C(\mathbb{S}_{\mathbf{i}}, \mathbb{R})$ is

$$P_{\mathbf{i}}[u](q) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{\mathbf{i}t})\frac{1 - \|q\|^2}{\|q - e^{\mathbf{i}t}\|^2}dt, \quad q \in \mathbb{D}^4.$$

Remark 3.3 1. Let $f \in \mathbb{D}^4 \to \mathbb{H}$ and $f = f_1 + f_2\mathbf{j}$ on $\mathbb{D}_{\mathbf{i}}$ with $f_1, f_2 : \mathbb{D}_{\mathbf{i}} \to \mathbb{D}_{\mathbf{i}}$. Then

$$2f_1 = f - \mathbf{i}f\mathbf{i},\ \ 2f_2\mathbf{j} = f + \mathbf{i}f\mathbf{i},\ \text{ on } \mathbb{D}_{\mathbf{i}},$$

where $\mathbf{i}, \mathbf{j} \in \mathbb{S}^2$ are orthogonal to each other.
If $\mathbf{j}'$ is another orthogonal vector to $\mathbf{i}$ and $f = g_1 + g_2\mathbf{j}'$ on $\mathbb{D}_{\mathbf{i}}$ then $f_1 = g_1$, $f_2 = -g_2\mathbf{j}'\overline{\mathbf{j}}$ and due to the usage of the quaternionic norm in the previous definitions we see that theses do not depends of the choose of $\mathbf{j}$, since

$$\|f_2(x) - f_2(y)\| = \|g_2(x) - g_2(y)\|, \quad \forall x, y \in \mathbb{D}_{\mathbf{i}}.$$

2. Let $\omega, \omega_1, \omega_2$ regular majorants and $f : \mathbb{D}^4 \to \mathbb{H}$ with $f\mid_{\mathbb{D}_{\mathbf{i}}} = f_1 + f_2\mathbf{j}$, where $f_1, f_2 : \mathbb{D}_{\mathbf{i}} \to \mathbb{C}(\mathbf{i})$, with $\mathbf{i}, \mathbf{j} \in \mathbb{S}^2$ are orthogonal to each other. Due to inequalities

$$\left.\begin{array}{l}\|f_1(x) - f_1(y)\|\\ \|f_2(x) - f_2(y)\|\end{array}\right\} \leq \|f(x) - f(y)\| \leq \|f_1(x) - f_1(y)\| + \|f_2(x) - f_2(y)\|,$$

for all $x, y \in \mathbb{D}_{\mathbf{i}}$, we get that

$${}_{\mathbf{i}}\Lambda_{\omega,\omega}(\mathbb{D}^4) = {}_{\mathbf{i}}\Lambda_{\omega}(\mathbb{D}^4)$$

and

$${}_{\mathbf{i}}\Lambda_{\omega_1,\omega_2}(\mathbb{D}^4) \subset {}_{\mathbf{i}}\Lambda_{\omega_1+\omega_2}(\mathbb{D}^4).$$

Similar relationships are obtained for ${}_{\mathbf{i}}\Lambda_{\omega}(\mathbb{S}^3)$ and ${}_{\mathbf{i}}\Lambda_{\omega_1,\omega_2}(\mathbb{S}^3)$.

Definition 3.4 The symbol $G\Lambda_\omega(\mathbb{D}^4)$ stands for the set of all quaternionic-valued functions f defined on $\mathbb{D}^4$ such that

$$\|f(x) - f(y)\| \le C\omega(\|x-y\|), \quad \forall x, y \in \mathbb{D}^4.$$

Meanwhile, $G\Lambda_\omega(\mathbb{S}^3)$ denotes the set of all quaternionic-valued functions f defined on $\mathbb{S}^3$ such that

$$\|f(x) - f(y)\| \le C\omega(\|x-y\|), \quad \forall x, y \in \mathbb{S}^3.$$

Proposition 3.5 *Let ω_1, ω_2 regular majorants and $\mathbf{i} \in \mathbb{S}^2$. Then*

$${}_{\mathbf{i}}\Lambda_{\omega_1,\omega_2}(\mathbb{D}^4) \cap \mathcal{SR}(\mathbb{D}^4) \ \subset \ G\Lambda_{\omega_1+\omega_2}(\mathbb{D}^4) \cap \mathcal{SR}(\mathbb{D}^4) \ \subset \ {}_{\mathbf{i}}\Lambda_{\omega_1+\omega_2}(\mathbb{D}^4) \cap \mathcal{SR}(\mathbb{D}^4).$$

Proof The relationship $G\Lambda_{\omega_1+\omega_2}(\mathbb{D}^4) \cap \mathcal{SR}(\mathbb{D}^4) \subset {}_{\mathbf{i}}\Lambda_{\omega_1+\omega_2}(\mathbb{D}^4) \cap \mathcal{SR}(\mathbb{D}^4)$ is a direct consequence of Definition 3.4.

On the other hand, we shall see that given $f \in {}_{\mathbf{i}}\Lambda_{\omega_1,\omega_2}(\mathbb{D}^4) \cap \mathcal{SR}(\mathbb{D}^4)$ there exists a constant $L > 0$ such that

$$\|f(p) - f(q)\| \le L \sum_{k=1}^{2} \omega_k(\|p-q\|), \quad \forall p, q \in \mathbb{D}^4.$$

Given $p, q \in \mathbb{D}^4$ consider the following cases:

1. Suppose that $\mathbf{p}$ and $\mathbf{q}$ are both the zero vector. By the Splitting Property we get

$$\begin{aligned}\|f(p) - f(q)\| &\le \|f_1(p) - f_1(q)\| + \|f_2(p) - f_2(q)\| \\ &\le C_3\left(\omega_1(\|p-q\|) + \omega_2(\|p-q\|)\right),\end{aligned}$$

   where $C_3 = \max\{C_1, C_2\}$.
2. Suppose $\mathbf{p}$ is not the zero vector while $\mathbf{q}$ is. Consider $z = p_0 + \mathbf{i}|\mathbf{p}|$ and $\zeta = q = \bar{\zeta}$. Combining the Representation Formula with Splitting Property we obtain

$$\begin{aligned}2\|f(p) - f(q)\| &= \left\| \left(1 - \frac{\mathbf{p}}{\|\mathbf{p}\|}\mathbf{i}\right)(f(z) - f(\zeta)) + \left(1 + \frac{\mathbf{p}}{\|\mathbf{p}\|}\mathbf{i}\right)(f(\bar{z}) - f(\bar{\zeta}))\right\| \\ &\le 2\|f(z) - f(\zeta)\| + 2\|f(\bar{z}) - f(\bar{\zeta})\| \\ &\le 2\sum_{k=1}^{2}(\|f_k(z) - f_k(\zeta)\| + \|f_k(\bar{z}) - f_k(\bar{\zeta})\|) \\ &\le 4C_3\left(\omega_1(\|p-q\|) + \omega_2(\|p-q\|)\right).\end{aligned}$$

   where $\|z - \zeta\| = \|p - q\|$ is used.

3. Consider $p, q \in \mathbb{D}^4$ such that neither $\mathbf{p}$ nor $\mathbf{q}$ is the zero vector. Set $z = p_0 + \mathbf{i}|\mathbf{p}|$ and $\zeta = q_0 + \mathbf{i}|\mathbf{q}|$. Representation Formula gives

$$
\begin{aligned}
&2\|f(p) - f(q)\| \\
&= \left\| \left\{ \left(1 - \frac{\mathbf{p}}{\|\mathbf{p}\|}\mathbf{i}\right) f(z) + \left(1 + \frac{\mathbf{p}}{\|\mathbf{p}\|}\mathbf{i}\right) f(\bar{z}) - \left(1 - \frac{\mathbf{q}}{\|\mathbf{q}\|}\mathbf{i}\right) f(\zeta) - \left(1 + \frac{\mathbf{q}}{\|\mathbf{q}\|}\mathbf{i}\right) f(\bar{\zeta}) \right\} \right\| \\
&= \left\| \left\{ \left(1 - \frac{\mathbf{p}}{\|\mathbf{p}\|}\mathbf{i}\right) (f(z) - f(\zeta)) + \left(1 + \frac{\mathbf{p}}{\|\mathbf{p}\|}\mathbf{i}\right) (f(\bar{z}) - f(\bar{\zeta})) \right. \right. \\
&\quad \left. \left. + \left(\frac{\mathbf{p}}{\|\mathbf{p}\|}\mathbf{i} - \frac{\mathbf{q}}{\|\mathbf{q}\|}\mathbf{i}\right) (f(\bar{\zeta}) - f(\zeta)) \right\} \right\| \\
&\leq 2\|f(z) - f(\zeta)\| + 2\|f(\bar{z}) - f(\bar{\zeta})\| + \left\| \frac{\mathbf{p}}{\|\mathbf{p}\|} - \frac{\mathbf{q}}{\|\mathbf{q}\|} \right\| \|f(\bar{\zeta}) - f(\zeta)\| \\
&\leq 4C_3 \sum_{k=1}^{2} \omega_k(\|z - \zeta\|) + \left\| \frac{\mathbf{p}}{\|\mathbf{p}\|} - \frac{\mathbf{q}}{\|\mathbf{q}\|} \right\| C_3 \sum_{k=1}^{2} \omega_k(\|\bar{\zeta} - \zeta\|).
\end{aligned}
$$

Note that $\|z - \zeta\| = \sqrt{(p_0 - q_0)^2 + (\|\mathbf{p}\| - \|\mathbf{q}\|)^2} \leq \|p - q\|$ and as ω_1 and ω_2 are increasing functions then

$$
2\|f(p) - f(q)\| \leq 4C_3 \left\{ \sum_{k=1}^{2} \left(\omega_k(\|p - q\|) + \frac{1}{4} \left\| \frac{\mathbf{p}}{\|\mathbf{p}\|} - \frac{\mathbf{q}}{\|\mathbf{q}\|} \right\| \omega_k(2\|\mathbf{q}\|) \right) \right\}. \tag{3.1}
$$

If $2\|\mathbf{q}\| \leq \|p - q\|$ then $\omega_k(2\|\mathbf{q}\|) \leq \omega_k(\|p - q\|)$, for $k = 1, 2$, and

$$
\|f(p) - f(q)\| \leq 3C_3 \sum_{k=1}^{2} \omega_k(\|p - q\|).
$$

On the other hand, if $\|p - q\| < 2\|\mathbf{q}\|$, from (3.1), we get

$$
\begin{aligned}
\frac{\|f(p) - f(q)\|}{\sum_{k=1}^{2} \omega_k(\|p - q\|)} &\leq 2C_3 \left\{ 1 + \frac{1}{4} \left\| \frac{\mathbf{p}}{\|\mathbf{p}\|} - \frac{\mathbf{q}}{\|\mathbf{q}\|} \right\| \frac{\sum_{k=1}^{2} \omega_k(2\|\mathbf{q}\|)}{\sum_{k=1}^{2} \omega_k(\|p - q\|)} \right\} \\
&\leq 2C_3 \left\{ 1 + \frac{1}{4} \left\| \frac{\mathbf{p}}{\|\mathbf{p}\|} - \frac{\mathbf{q}}{\|\mathbf{q}\|} \right\| \sum_{k=1}^{2} \frac{\omega_k(2\|\mathbf{q}\|)}{\omega_k(\|p - q\|)} \right\} \\
&\leq 2C_3 \left\{ 1 + \frac{1}{4} \left\| \frac{\mathbf{p}}{\|\mathbf{p}\|} - \frac{\mathbf{q}}{\|\mathbf{q}\|} \right\| \frac{2\|\mathbf{q}\|}{\|p - q\|} \sum_{k=1}^{2} \frac{\frac{\omega_k(2\|\mathbf{q}\|)}{2\|\mathbf{q}\|}}{\frac{\omega_k(\|p - q\|)}{\|p - q\|}} \right\}
\end{aligned}
$$

As $\dfrac{\omega_k(t)}{t}$ is decreasing, for $k = 1, 2$, and $\|p - q\| < 2\|\mathbf{q}\|$ then

$$\sum_{k=1}^{2} \frac{\dfrac{\omega_k(2\|\mathbf{q}\|)}{2\|\mathbf{q}\|}}{\dfrac{\omega_k(\|p-q\|)}{\|p-q\|}} \leq 2$$

and

$$\|f(p) - f(q)\| \leq 2C_3 \left\{ 1 + \left\| \frac{\mathbf{p}}{\|\mathbf{p}\|} - \frac{\mathbf{q}}{\|\mathbf{q}\|} \right\| \frac{\|\mathbf{q}\|}{\|p-q\|} \right\} \sum_{k=1}^{2} \omega_k(\|p-q\|).$$

It is easily seen that

$$\begin{aligned} \| \frac{\mathbf{p}}{\|\mathbf{p}\|} - \frac{\mathbf{q}}{\|\mathbf{q}\|} \| \frac{\|\mathbf{q}\|}{\|p-q\|} &\leq \left\| \frac{\mathbf{p}}{\|\mathbf{p}\|} - \frac{\mathbf{p}}{\|\mathbf{q}\|} + \frac{\mathbf{p}}{\|\mathbf{q}\|} - \frac{\mathbf{q}}{\|\mathbf{q}\|} \right\| \frac{\|\mathbf{q}\|}{\|p-q\|} \\ &\leq \frac{|\|\mathbf{q}\| - \|\mathbf{p}\||}{\|\mathbf{p}\|\|\mathbf{q}\|} \frac{\|\mathbf{pq}\|}{\|p-q\|} + \frac{\|\mathbf{p}-\mathbf{q}\|}{\|p-q\|} \leq 2, \end{aligned}$$

and

$$\|f(p) - f(q)\| \leq 6C_3 \sum_{k=1}^{2} \omega_k(\|p-q\|),$$

which completes the proof by choosing $L = 6C_3$. □

Remark 3.6 We have proved more, namely that for $\omega_1 = \omega_2 = \omega$ we are lead to

$${}_{\mathbf{i}}\Lambda_{\omega}(\mathbb{D}^4) \cap \mathcal{SR}(\mathbb{D}^4) = G\Lambda_{\omega}(\mathbb{D}^4) \cap \mathcal{SR}(\mathbb{D}^4).$$

Therefore, every slice regular function space associated to a majorant on a fix slice, it is also associated to the same majorant on the four-dimensional unit ball and reciprocally.

We proceed to describe some algebraic properties of the previously introduced functions sets.

Proposition 3.7 *Set* $\mathbf{i} \in \mathbb{S}^2$.

1. *Given a regular majorant* ω*, the sets* ${}_{\mathbf{i}}\Lambda_{\omega}(\mathbb{D}^4) \cap \mathcal{SR}(\mathbb{D}^4)$ *and* $G\Lambda_{\omega}(\mathbb{D}^4) \cap \mathcal{SR}(\mathbb{D}^4)$ *are quaternionic right linear spaces.*
2. *Let* ω_1, ω_2 *be two regular majorants and let* $f, g \in {}_{\mathbf{i}}\Lambda_{\omega_1,\omega_2}(\mathbb{D}^4)$*. For every* $a \in \mathbb{H}$ *we have* $f + g \in {}_{\mathbf{i}}\Lambda_{\omega_1,\omega_2}(\mathbb{D}^4)$ *and*

$$fa \in {}_{\mathbf{i}}\Lambda_{\|a_1\|\omega_1 + \|a_2\|\omega_2,\ \|a_2\|\omega_1 + \|a_1\|\omega_2}(\mathbb{D}^4),$$

where $a = a_1 + a_2\mathbf{j}$ *with* $a_1, a_2 \in \mathbb{C}(\mathbf{i})$ *and* $\mathbf{j}$ *is orthogonal to* $\mathbf{i}$.

Proof 1. Given $f, g \in {}_{\mathbf{i}}\Lambda_{\omega}(\mathbb{D}^4) \cap \mathcal{SR}(\mathbb{D}^4)$ and $a \in \mathbb{H}$ we see that

$$\begin{aligned}\|(fa+g)(x)-(fa+g)(y)\| &\le \|a\|\|f(x)-f(y)\|+\|g(x)-g(y)\| \\ &\le C\omega(\|x-y\|), \quad \forall x, y \in \mathbb{D}_{\mathbf{i}}.\end{aligned}$$

Similar inequalities are used to see that $G\Lambda_{\omega}(\mathbb{D}^4)\cap\mathcal{SR}(\mathbb{D}^4)$ is a quaternionic right linear space.

2. Given $f, g \in {}_{\mathbf{i}}\Lambda_{\omega_1,\omega_2}(\mathbb{D}^4)$ we have $f+g \in {}_{\mathbf{i}}\Lambda_{\omega_1,\omega_2}(\mathbb{D}^4)$ following a similar computation to the above. Denote $a = a_1 + a_2\mathbf{j}$, $f\mid_{\mathbb{D}_{\mathbf{i}}} = f_1 + f_2\mathbf{j}$, $g\mid_{\mathbb{D}_{\mathbf{i}}} = g_1 + g_2\mathbf{j}$ with $\mathbf{j} \in \mathbb{S}^2$ orthogonal to $\mathbf{i}$ and $a_1, a_2 \in \mathbb{C}(\mathbf{i})$ and $f_1, f_2, g_1, g_2 \in Hol(\mathbb{D}_{\mathbf{i}})$. We obtain that

$$fa\mid_{\mathbb{D}_{\mathbf{i}}} = (f_1a_1 - f_2\bar{a}_2) + (f_1a_2 + f_2\bar{a}_1)\mathbf{j}$$

and

$$\begin{aligned}\|(f_1a_1-f_2\bar{a}_2)(x)-(f_1a_1-f_2\bar{a}_2)(y)\| &\le C_1(\omega_1(\|x-y\|)\|a_1\| \\ &\quad + \omega_2(\|x-y\|)\|a_2\|),\end{aligned}$$

$$\begin{aligned}\|(f_1a_2+f_2\bar{a}_1)(x)-(f_1a_2+f_2\bar{a}_1)(y)\| &\le C_2(\omega_1(\|x-y\|)\|a_2\| \\ &\quad + \omega_2(\|x-y\|)\|a_1\|).\end{aligned}$$

for all $x, y \in \mathbb{D}_{\mathbf{i}}$.
Note that picking out $\max\{\|a_1\|, \|a_2\|\}$ we can prove that

$${}_{\mathbf{i}}\Lambda_{\|a_1\|\omega_1+\|a_2\|\omega_2,\ \|a_2\|\omega_1+\|a_1\|\omega_2}(\mathbb{D}^4) \subset {}_{\mathbf{i}}\Lambda_{\omega_1+\omega_2}(\mathbb{D}^4).$$

□

Corollary 3.8 *Let ω_1, ω_2 be two regular majorants and $f \in {}_{\mathbf{i}}\Lambda_{\omega}(\mathbb{D}^4)\cap\mathcal{N}(\mathbb{D}^4)$. Then*

$$\|f\|_{{}_{\mathbf{i}}\Lambda_{\omega_1}(\mathbb{D}^4)} = \|f\|_{{}_{\mathbf{k}}\Lambda_{\omega_1}(\mathbb{D}^4)} = \|f\|_{{}_{\mathbf{i}}\Lambda_{\omega_1,\omega_2}(\mathbb{D}^4)},$$

for all $\mathbf{k} \in \mathbb{S}^2$.

Proof Note that given $f \in \mathcal{N}(\Omega)$ there exists a sequence of real numbers $(a_n)_{n=0}^{\infty}$, see [7, 19], such that $f(q) = \sum_{n=0}^{\infty} q^n a_n$ for all $q \in \mathbb{D}^4$. Therefore for all $u \in \mathbb{S}^3$ one has that

$$\begin{aligned}\|f(x)-f(y)\| &= \|u\left(\sum_{n=0}^{\infty} x^n a_n - \sum_{n=0}^{\infty} y^n a_n\right)\bar{u}\| = \|\sum_{n=0}^{\infty}(ux\bar{u})^n a_n - \sum_{n=0}^{\infty}(uy\bar{u})^n a_n\| \\ &= \|f(ux\bar{u}) - f(uy\bar{u})\|.\end{aligned}$$

Choosing $u \in \mathbb{S}^3$ such that $u\mathbf{i}\bar{u} = \mathbf{k}$ one obtains the first equality and for the second one we see that $f\mid_{\mathbb{D}_{\mathbf{i}}} = f\mid_{\mathbb{D}_{\mathbf{i}}} + 0\mathbf{j}$, i.e., $f_1 = f\mid_{\mathbb{D}_{\mathbf{i}}}$ and $f_2 = 0$ in Definition 3.2. □

Remark 3.9 Note that if $f \in {}_{\mathbf{i}}\Lambda_\omega(\mathbb{D}^4) \cap C(\mathbb{D}_\mathbf{i}, \mathbb{H})$, then $f\mid_{\mathbb{D}_\mathbf{i}}$ can be extended to a continuous function on $\overline{\mathbb{D}_\mathbf{i}}$. Similarly, if $f \in G\Lambda_\omega(\mathbb{D}^4) \cap C(\mathbb{D}^4, \mathbb{H})$, then f can be extended to a continuous function on $\overline{\mathbb{D}^4}$.

Now, we shall extend [9, Theorems 1 and 2] to slice regular function theory.

Proposition 3.10 *1. Set* $\mathbf{i} \in \mathbb{S}^2$ *and* $f \in \mathcal{SR}(\mathbb{D}^4) \cap C(\overline{\mathbb{D}^4}, \mathbb{H})$. *Let* ω *and* ω^2 *regular majorants. Then*

$$\|f\|^2_{{}_{\mathbf{i}}\Lambda_\omega(\mathbb{D}^4)} \asymp \sup\left\{\frac{P[\|f_1\|^2](x) - \|f_1(x)\|^2}{\omega(1-\|x\|)^2} \mid x \in \mathbb{D}_\mathbf{i}\right\}$$

$$+ \sup\left\{\frac{P[\|f_2\|^2](x) - \|f_2(x)\|^2}{\omega(1-\|x\|)^2} \mid x \in \mathbb{D}_\mathbf{i}\right\},$$

where $f\mid_{\mathbb{D}_\mathbf{i}} = f_1 + f_2\mathbf{j}$ *with* $\mathbf{j} \in \mathbb{S}^2$ *orthogonal to* $\mathbf{i}$ *and* $f_1, f_2 \in Hol(\mathbb{D}_\mathbf{i})$.

2. *Set* $\mathbf{i} \in \mathbb{S}^2$ *and* $f \in \mathcal{SR}(\mathbb{D}^4) \cap C(\overline{\mathbb{D}^4}, \mathbb{H})$. *Let* ω *be a regular majorant. Then*

$$\|f\|^2_{{}_{\mathbf{i}}\Lambda_\omega(\mathbb{D}^4)} \asymp N_1(f_1)^2 + N_1(f_2)^2 \asymp N_2(f_1)^2 + N_2(f_2)^2 \asymp N_3(f_1)^2 + N_3(f_2)^2,$$

where $f\mid_{\mathbb{D}_\mathbf{i}} = f_1 + f_2\mathbf{j}$ *with* $\mathbf{j} \in \mathbb{S}^2$ *orthogonal to* $\mathbf{i}$ *and* $f_1, f_2 \in Hol(\mathbb{D}_\mathbf{i})$.

3. *Set* $\mathbf{i}, \mathbf{k} \in \mathbb{S}^2$ *and consider the regular majorants* $\omega, \omega_1, \omega_2$.
   (a) *If* $f \in {}_{\mathbf{i}}\Lambda_\omega(\mathbb{D}^4) \cap \mathcal{SR}(\mathbb{D}^4) \cap C(\overline{\mathbb{D}^4}, \mathbb{H})$, *then*

$$\|f\|_{{}_{\mathbf{i}}\Lambda_\omega(\mathbb{D}^4)} \le 2\|f\|_{{}_{\mathbf{k}}\Lambda_\omega(\mathbb{D}^4)} \le 4\|f\|_{{}_{\mathbf{i}}\Lambda_\omega(\mathbb{D}^4)}.$$

   (b) *If* $f \in {}_{\mathbf{i}}\Lambda_{\omega,\omega}(\mathbb{D}^4)$, *then*

$$\|f\|_{{}_{\mathbf{i}}\Lambda_{\omega,\omega}(\mathbb{D}^4)} = \|f\|_{{}_{\mathbf{i}}\Lambda_\omega(\mathbb{D}^4)}.$$

   If $f \in {}_{\mathbf{i}}\Lambda_{\omega_1,\omega_2}(\mathbb{D}^4)$ *then*

$$\|f\|_{{}_{\mathbf{i}}\Lambda_{\omega_1+\omega_2}(\mathbb{D}^4)} \le \|f\|_{{}_{\mathbf{i}}\Lambda_{\omega_1,\omega_2}(\mathbb{D}^4)}.$$

Proof 1. By (2.1) and the fact that $\alpha \asymp \beta$ and $\delta \asymp \gamma$ imply $\alpha^2 + \delta^2 \asymp \beta^2 + \gamma^2$. Also, the application of inequalities

$$\|f\|^2_{{}_{\mathbf{i}}\Lambda_\omega(\mathbb{D}^4)} \le \|f_1\|^2_{{}_{\mathbf{i}}\Lambda_\omega(\mathbb{D}_\mathbf{i})} + \|f_2\|^2_{{}_{\mathbf{i}}\Lambda_\omega(\mathbb{D}_\mathbf{i})} \le 2\|f\|^2_{{}_{\mathbf{i}}\Lambda_\omega(\mathbb{D}^4)}.$$

2. By (2.2) and the properties stated above.
3. Fact (a) follows from direct computations and the idea of the ensuing Fact (b) is the following:

$$\|f\|^2_{{}_{\mathbf{i}}\Lambda_{\omega,\omega}(\mathbb{D}^4)} = \sup\left\{\frac{\|f_1(x)-f_1(y)\|^2}{\omega(\|x-y\|)^2} + \frac{\|f_2(x)-f_2(y)\|^2}{\omega(\|x-y\|)^2} \mid x, y \in \mathbb{D}_\mathbf{i},\ x \neq y\right\}$$

$$= \sup\left\{\frac{\|f(x)-f(y)\|^2}{\omega(\|x-y\|)^2} \ \mid \ x, y \in \mathbb{D}_{\mathbf{i}}, \ x \neq y\right\} = \|f\|^2_{{}_{\mathbf{i}}\Lambda_\omega(\mathbb{D}^4)}.$$

□

Corollary 3.11 1. *Set* $\mathbf{i} \in \mathbb{S}^2$ *and* $f \in \mathcal{N}(\mathbb{D}^4) \cap C(\overline{\mathbb{D}^4}, \mathbb{H})$. *If* ω *and* ω^2 *are regular majorant, then*

$$\sup\left\{\frac{P[\|f\mid_{\mathbb{D}_{\mathbf{i}}}\|^2](x) - \|f\mid_{\mathbb{D}_{\mathbf{i}}}(x)\|^2}{\omega(1-\|x\|)} \ \mid \ x \in \mathbb{D}_{\mathbf{i}}\right\} \asymp \|f\|_{{}_{\mathbf{i}}\Lambda_\omega(\mathbb{D}^4)}.$$

2. *Suppose* $\mathbf{i} \in \mathbb{S}^2$, $f \in \mathcal{N}(\mathbb{D}^4) \cap C(\overline{\mathbb{D}^4}, \mathbb{H})$ *and* ω *a regular majorant. Then*

$$\|f\|^2_{{}_{\mathbf{i}}\Lambda_\omega(\mathbb{D}^4)} \asymp N_1(f\mid_{\mathbb{D}_{\mathbf{i}}}) \asymp N_2(f\mid_{\mathbb{D}_{\mathbf{i}}}) \asymp N_3(f\mid_{\mathbb{D}_{\mathbf{i}}}),$$

Proof Both facts follow from $f_1 = f_1\mid_{\mathbb{D}_{\mathbf{i}}}$ and $f_2 = 0$ in Definition 3.2 since $f \in \mathcal{N}(\mathbb{D}^4)$ and $f(q) = \sum_{n=0}^{\infty} q^n a_n$ for all $q \in \mathbb{D}^4$ iff $a_n \in \mathbb{R}$ for all n, see [7, 19]. □

As the function sets given in Definitions 3.1, 3.2 and 3.4 depend of unit vectors the following proposition shows some relationships between them.

Proposition 3.12 *Set* $\mathbf{i} \in \mathbb{S}^2$ *and consider the regular majorants* $\omega, \omega_1, \omega_2$.

1. *If* $f \in {}_{\mathbf{i}}\Lambda_\omega(\mathbb{D}^4)$ *then* $\|f\|, \|f \pm \mathbf{i} f \mathbf{i}\| \in {}_{\mathbf{i}}\Lambda_\omega(\mathbb{D}^4)$.
2. ${}_{\mathbf{i}}\Lambda_\omega(\mathbb{D}^4) = {}_{\mathbf{i}}\Lambda_{\omega,\omega}(\mathbb{D}^4)$.
3. ${}_{\mathbf{i}}\Lambda_{\omega_1,\omega_2}(\mathbb{D}^4) \subset {}_{\mathbf{i}}\Lambda_{\omega_1+\omega_2}(\mathbb{D}^4)$.
4. ${}_{\mathbf{i}}\Lambda_\omega(\mathbb{D}^4) \cap \mathcal{SR}(\mathbb{D}^4) = {}_{\mathbf{j}}\Lambda_\omega(\mathbb{D}^4) \cap \mathcal{SR}(\mathbb{D}^4)$.

Proof 1. Given $f \in {}_{\mathbf{i}}\Lambda_\omega(\mathbb{D}^4)$ set $f = f_1 + f_2\mathbf{j}$ on $\mathbb{D}_{\mathbf{i}}$, where $\mathbf{j} \in \mathbb{S}^2$ is orthogonal to $\mathbf{i}$ and $f_1, f_2 : \mathbb{D}_{\mathbf{i}} \to \mathbb{D}_{\mathbf{i}}$. Then we see that

$$2f_1 = f - \mathbf{i} f \mathbf{i}, \ \ 2f_2\mathbf{j} = f + \mathbf{i} f \mathbf{i}, \ \text{ on } \ \mathbb{D}_{\mathbf{i}}. \tag{3.2}$$

From inequalities

$$\begin{aligned} |\,\|f(x)\| - \|f(y)\|\,| &\leq \|f(x) - f(y)\|, \\ \max\{|\,\|f_1(x)\| - \|f_1(y)\|\,|, \ |\,\|f_2(x)\| - \|f_2(y)\|\,|\} &\leq \|f(x) - f(y)\|, \end{aligned}$$

for all $x, y \in \mathbb{D}_{\mathbf{i}}$, it follows that $\|f\|, \|f \pm \mathbf{i} f \mathbf{i}\| \in {}_{\mathbf{i}}\Lambda_\omega(\mathbb{D}^4)$.

2. and 3. With the previous notation the identity

$$\|f(x) - f(y)\|^2 = \|f_1(x) - f_1(y)\|^2 + \|f_2(x) - f_2(y)\|^2, \quad \forall x, y \in \mathbb{D}_{\mathbf{i}}, \tag{3.3}$$

holds, allowing us to see that

$${}_{\mathbf{i}}\Lambda_\omega(\mathbb{D}^4) = {}_{\mathbf{i}}\Lambda_{\omega,\omega}(\mathbb{D}^4)$$

and

$$_{\mathbf{i}}\Lambda_{\omega_1,\omega_2}(\mathbb{D}^4) \subset {}_{\mathbf{i}}\Lambda_{\omega_1+\omega_2}(\mathbb{D}^4).$$

4. Given $f \in {}_{\mathbf{i}}\Lambda_{\omega}(\mathbb{D}^4) \cap \mathcal{SR}(\mathbb{D}^4)$ and $\mathbf{j} \in \mathbb{S}^2$, according to the Representation Formula, we have for every $x_0 + x_1\mathbf{j}$, $y_0 + y_1\mathbf{j} \in \mathbb{D}_{\mathbf{j}}$ with $x_0, x_1, y_0, y_1 \in \mathbb{R}$

$$\begin{aligned}
&\|f(x_0 + x_1\mathbf{j}) - f(y_0 + y_1\mathbf{j})\| \\
&= \frac{1}{2}\|(1 - \mathbf{ji})(f(x) - f(y)) + (1 + \mathbf{ji})(f(\bar{x}) - f(\bar{y}))\| \\
&\leq \|f(x) - f(y)\| + \|f(\bar{x}) - f(\bar{y})\| \leq 2C\omega(\|x - y\|) \\
&\leq 2C\omega(\|(x_0 + x_1\mathbf{j}) - (y_0 + y_1\mathbf{j})\|),
\end{aligned}$$

where $x = x_0 + x_1\mathbf{i}$ and $y = y_0 + y_1\mathbf{i}$.

□

Remark 3.13 Repeating the computations presented in Propositions 3.7 and 3.12 enables us to see that ${}_{\mathbf{i}}\Lambda_{\omega}(\mathbb{S}^3)$, ${}_{\mathbf{i}}\Lambda_{\omega_1,\omega_2}(\mathbb{S}^3)$ and $G\Lambda_{\omega}(\mathbb{S}^3)$ have similar properties of ${}_{\mathbf{i}}\Lambda_{\omega}(\mathbb{D}^4)$, ${}_{\mathbf{i}}\Lambda_{\omega_1,\omega_2}(\mathbb{D}^4)$ and $G\Lambda_{\omega}(\mathbb{D}^4)$, respectively, and that is why they are omitted.

Let ω, ω_1 and ω_2 be regular majorant. The next propositions characterize the elements of $\mathcal{SR}(\mathbb{D}^4) \cap {}_{\mathbf{i}}\Lambda_{\omega}(\mathbb{D}^4)$ and of $\mathcal{SR}(\mathbb{D}^4) \cap {}_{\mathbf{i}}\Lambda_{\omega_1,\omega_2}(\mathbb{D}^4)$, which extend results contained in [9, 10].

Proposition 3.14 1. *Set $f \in \mathcal{SR}(\mathbb{D}^4)$. Then $f \in {}_{\mathbf{i}}\Lambda_{\omega}(\mathbb{D}^4)$ if and only if*

$$\|f'(x) \pm \mathbf{i}f'(x)\mathbf{i}\| \leq C\frac{\omega(1 - \|x\|)}{1 - \|x\|},$$

for C independent of $x \in \mathbb{D}_{\mathbf{i}}$.

2. *Set $f \in \mathcal{SR}(\mathbb{D}^4) \cap C(\overline{\mathbb{D}^4}, \mathbb{H})$. Then*

$$\frac{1}{2}(1 - \|x\|)\|f'(x) \pm \mathbf{i}f'(x)\mathbf{i}\| + \|f(x) \pm \mathbf{i}f(x)\mathbf{i}\| \leq 2M_x,$$

for C independent of $x \in \mathbb{D}_{\mathbf{i}}$, where

$$M_x = \sup\{\|f(y)\| \ \mid \ \|y - x\| \leq 1 - \|x\|, \ y \in \mathbb{D}_{\mathbf{i}}\}.$$

3. *Set $f \in \mathcal{SR}(\mathbb{D}^4)$. Then $f \in {}_{\mathbf{i}}\Lambda_{\omega}(\mathbb{D}^4)$ if and only if*

$$\|f'(x)\| \leq C\frac{\omega(1 - \|x\|)}{1 - \|x\|},$$

for C independent of $x \in \mathbb{D}_{\mathbf{i}}$.

4. *Set* $f \in \mathcal{SR}(\mathbb{D}^4) \cap {}_{\mathbf{i}}\Lambda_{\omega_1,\omega_2}(\mathbb{D}^4)$. *Then*

$$\|f'(x)\| \le C\left(\frac{\sqrt{\omega_1(1-\|x\|)^2+\omega_2(1-\|x\|)^2}}{(1-\|x\|)}\right),$$

for C *independent of* $x \in \mathbb{D}_{\mathbf{i}}$.

5. *Set* $f \in \mathcal{SR}(\mathbb{D}^4) \cap C(\overline{\mathbb{D}^4}, \mathbb{H})$. *Then*

$$\begin{aligned}&\frac{1}{4}(1-\|x\|)^2\|f'(x)\|^2+\|f(x)\|^2\\ &\quad\le (\|x\|-1)(\,\|f_1'(x)\|\|f_1(x)\|+\|f_2'(x)\|\|f_2(x)\|\,)+M_{1,x}^2+M_{2,x}^2,\end{aligned}$$

for $x \in \mathbb{D}_{\mathbf{i}}$, *where* $f\mid_{\mathbb{D}_{\mathbf{i}}} = f_1 + f_2\mathbf{j}$ *with* $f_1, f_2 \in Hol(\mathbb{D}_{\mathbf{i}}) \cap C(\overline{\mathbb{D}}_{\mathbf{i}}, \mathbb{C}(\mathbf{i}))$ *and*

$$M_{k,x} = \sup\{\|f_k(y)\| \mid \|y-x\| \le 1-\|x\|,\ y \in \mathbb{D}_{\mathbf{i}}\},$$

for $k = 1, 2$.

6. *Consider* $f \in \mathcal{SR}(\mathbb{D}^4) \cap C(\overline{\mathbb{D}^4}, \mathbb{H})$. *Let us introduce one more piece of notation:*

$$M = \sup\{\|f(w)\| \mid w \in \mathbb{D}_{\mathbf{i}}\}.$$

If there exists $\mathbf{i} \in \mathbb{S}^2$ *and a regular majorant* ω *such that*

$$\|M^2 - \overline{f(x)}f(\tilde{x})\| \le C(1+\|x\|)\omega(1-\|x\|),$$

for C *independent of* $x \in \mathbb{D}_{\mathbf{i}} \setminus Z_{f'}$, *where*

$$\begin{aligned}\tilde{x} &= (\overline{f(x)})^{-1}T_g(f'(x)^{-1}xf'(x))\overline{f(x)},\\ g(x) &= 1-\overline{f(x)} * f(x),\end{aligned}$$

where $T_g(q) = (g^c(q))^{-1}qg^c(q)$ *for all* $q \in \mathbb{D}^4$ *such that* $g^s(q) \ne 0$, *then*

$$\|f'(x)\| \le \frac{C}{M}\frac{\omega(1-\|x\|)}{1-\|x\|};$$

that is, $f \in {}_{\mathbf{i}}\Lambda_\omega(\mathbb{D}^4)$, *see Fact 3. of the present proposition.*

Proof 1. With the Splitting Property in mind, consider $f_1, f_2 \in Hol(\mathbb{D}_{\mathbf{i}})$ such that $f\mid_{\mathbb{D}_{\mathbf{i}}} = f_1 + f_2\mathbf{j}$ where $\mathbf{j} \in \mathbb{S}^2$ is orthogonal to $\mathbf{i}$. From Fact 2. of Proposition 3.12 one has that $f \in {}_{\mathbf{i}}\Lambda_\omega(\mathbb{D}^4)$ if and only if $f_1, f_2 \in \Lambda_{\omega,\omega}(\mathbb{D}_{\mathbf{i}})$, that is,

$$\|f'(x) \pm \mathbf{i}f'(x)\mathbf{i}\| \le C\frac{\omega(1-\|x\|)}{1-\|x\|},$$

for C independent of $x \in \mathbb{D}_{\mathbf{i}}$, where we use [10, Lemma 1]. The result is obtained from (3.2) applied to f'.

2. Splitting Property implies that $f\mid_{\mathbb{D}_{\mathbf{i}}}= f_1+f_2\mathbf{j}$ where $f_1, f_2 \in Hol(\mathbb{D}_{\mathbf{i}})$ and $\mathbf{j}\in\mathbb{S}^2$ is orthogonal to $\mathbf{i}$. From Fact 2. of Proposition 3.12 one has that $f \in {}_{\mathbf{i}}\Lambda_{\omega}(\mathbb{D}^4)$ if and only if $f_1, f_2 \in \Lambda_{\omega,\omega}(\mathbb{D}_{\mathbf{i}})$. Applying [10, Lemma 2] to the complex components of $f\mid_{\mathbb{D}_{\mathbf{i}}}$ and using (3.2) in f and f' to finish the proof.
3. It follows from Fact 1. commbining with the following consequence of the parallelogram identity:

$$4\|f'(x)\|^2 = \|f'(x)+\mathbf{i}f'(x)\mathbf{i}\|^2 + \|f'(x)-\mathbf{i}f'(x)\mathbf{i}\|^2,$$

for all $x \in \mathbb{D}_{\mathbf{i}}$.
4. Kipping in mind the Splitting Property, set $f_1, f_2 \in Hol(\mathbb{D}_{\mathbf{i}})$ such that $f\mid_{\mathbb{D}_{\mathbf{i}}}= f_1+f_2\mathbf{j}$ where $\mathbf{j}\in\mathbb{S}^2$ is orthogonal to $\mathbf{i}$. From Fact 2. of Proposition 3.12 one has that $f \in {}_{\mathbf{i}}\Lambda_{\omega}(\mathbb{D}^4)$ if and only if $f_1, f_2 \in \Lambda_{\omega,\omega}(\mathbb{D}_{\mathbf{i}})$, i.e.,

$$\|f_k'(x)\| \leq C\frac{\omega_k(1-\|x\|)}{1-\|x\|},$$

for C independent of $x\in\mathbb{D}_{\mathbf{i}}$ and for $k=1,2$, see [10, Lemma 1]. Applying (3.3) to f' yields

$$\|f'(x)\|^2 \leq C^2\left(\frac{\omega_1(1-\|x\|)^2+\omega_2(1-\|x\|)^2}{(1-\|x\|)^2}\right).$$

5. Given $f\mid_{\mathbb{D}_{\mathbf{i}}}= f_1+f_2\mathbf{j}$ with $f_1, f_2 \in Hol(\mathbb{D}_{\mathbf{i}})\cap C(\bar{\mathbb{D}}_{\mathbf{i}}, \mathbb{C}(\mathbf{i}))$ from [10, Lemma 2] we see that

$$\frac{1}{2}(1-\|x\|)\|f_k'(x)\| + \|f_k(x)\| \leq M_{k,x},$$

for $x\in\mathbb{D}_{\mathbf{i}}$, where $M_{k,x} = \sup\{\|f_k(y)\| \mid \|y-x\|\leq 1-\|x\|,\ y\in\mathbb{D}_{\mathbf{i}}\}$ for $k=1,2$.
Therefore

$$\frac{1}{4}(1-\|x\|)^2\|f_k'(x)\|^2 + (1-\|x\|)\|f_k'(x)\|\|f_k(x)\| + \|f_k(x)\|^2 \leq M_{k,x}^2,$$

for $k=1,2$. Adding terms in the previous inequalities and using (3.3) applied to f and f', the main result follows.
6. Let $f \in \mathcal{SR}(\mathbb{D}^4)\cap C(\overline{\mathbb{D}^4}, \mathbb{H})$. With the notation $F(x) = \dfrac{f(x)}{M}$ for all $x\in\mathbb{D}^4$ rewrite the hypothesis as follows:

$$\|1-\overline{F(x)}F(\tilde{x})\| \leq \frac{C}{M^2}(1+\|x\|)\omega(1-\|x\|),$$

for C independent of x, or equivalently

$$\frac{\|1-\overline{F(x)}F(\tilde{x})\|}{1-\|x\|^2} \leq \frac{C}{M^2}\,\frac{\omega(1-\|x\|)}{1-\|x\|}.$$

Equation (3.10) of [22, Theorem 3.7 (Schwarz-Pick lemma)] shows that if $f : \mathbb{D}^4 \to \mathbb{D}^4$ is a slice regular function and $q_0 \in \mathbb{D}^4$ implies that

$$\|\partial_c f * (1 - f(q_0) * f(q))^{-*}\|_{|q_0} \leq \frac{1}{1-|q_0|^2}.$$

One can also see [23]. This finally yields

$$\|F'(x)\| \leq \frac{C}{M^2}\frac{\omega(1-\|x\|)}{1-\|x\|},$$

that is

$$\|f'(x)\| \leq \frac{C}{M}\frac{\omega(1-\|x\|)}{1-\|x\|}$$

and Fact 6. is proved

□

Corollary 3.15 1. *Set* $f \in \mathcal{SR}(\mathbb{D}^4)$. *Then* $f \in {}_{\mathbf{i}}\Lambda_{\omega}(\mathbb{D}^4)$ *if and only if*

$$\|f'(q)\| \leq C\frac{\omega(1-\|q\|)}{1-\|q\|},$$

for some C *independent of the choice of* $q \in \mathbb{D}^4$.

2. *Set* $f \in \mathcal{SR}(\mathbb{D}^4) \cap C(\overline{\mathbb{D}^4}, \mathbb{H})$ *then*

$$\begin{aligned}
&\frac{1}{2}(1-\|q\|)\|f'(q)\| + \|f(q)\| \\
&\quad\leq \ \sup\left\{\|f(y)\| \ \middle|\ \sqrt{(\Re y - q_0)^2 + (\Im y - |\mathbf{q}|)^2} \leq 1-\|q\|,\ y \in \mathbb{D}_{\mathbf{i}}\right\} \\
&\qquad + \sup\left\{\|f(y)\| \ \middle|\ \sqrt{(\Re y - q_0)^2 + (\Im y + |\mathbf{q}|)^2} \leq 1-\|q\|,\ y \in \mathbb{D}_{\mathbf{i}}\right\}.
\end{aligned}$$

3. *If* $f \in \mathcal{SR}(\mathbb{D}^4) \cap {}_{\mathbf{i}}\Lambda_{\omega_1,\omega_2}(\mathbb{D}^4)$ *then*

$$\|f'(q)\| \leq C\left(\frac{\omega_1(1-\|q\|) + \omega_2(1-\|q\|)}{(1-\|q\|)}\right),$$

for C *independent of the choice of* $q \in \mathbb{D}^4$.

Proof 1. The sufficient condition is a direct consequence of the previous proposition and

$$\|f'(x) \pm \mathbf{i} f'(x)\mathbf{i}\| \leq 2\|f'(x)\|, \quad \forall x \in \mathbb{D}_{\mathbf{i}}.$$

On the other hand, let $f \in {}_{\mathbf{i}}\Lambda_{\omega}(\mathbb{D}^4) \cap \mathcal{SR}(\mathbb{D}^4)$ and $q \in \mathbb{D}^4$ such that $\mathbf{q}$ is not the vector zero. Applying Representation Formula we deduce that

$$\begin{aligned}2\|f'(q)\| &\leq 2\|f'(x)\| + 2\|f'(\bar{x})\| \leq \|f'(x) + \mathbf{i} f'(x)\mathbf{i}\| + \|f'(x) - \mathbf{i} f'(x)\mathbf{i}\| \\ &\quad + \|f'(\bar{x}) + \mathbf{i} f'(\bar{x})\mathbf{i}\| + \|f'(\bar{x}) - \mathbf{i} f'(\bar{x})\mathbf{i}\| \\ &\leq 4C\frac{\omega(1-\|x\|)}{1-\|x\|} = 4C\frac{\omega(1-\|q\|)}{1-\|q\|},\end{aligned}$$

where $x = q_0 + \mathbf{i}|\mathbf{q}| \in \mathbb{D}_{\mathbf{i}}$.

2. For fixed $q \in \mathbb{D}^4$ such that $\mathbf{q}$ is not the zero vector, let $x = x_0 + \mathbf{i}|\mathbf{q}| \in \mathbb{D}_{\mathbf{i}}$. By the Representation Formula we obtain

$$\begin{aligned}&\frac{1}{2}(1-\|q\|)\|f'(q)\| + \|f(q)\| \\ &\leq \left(\frac{1}{2}(1-\|x\|)\|f'(x)\| + \|f(x)\|\right) + \left(\frac{1}{2}(1-\|\bar{x}\|)\|f'(\bar{x})\| + \|f(\bar{x})\|\right) \\ &\leq \left(\frac{1}{4}(1-\|x\|)\|f'(x) + \mathbf{i} f'(x)\mathbf{i}\| + \frac{1}{2}\|f(x) + \mathbf{i} f(x)\mathbf{i}\|\right) \\ &\quad + \left(\frac{1}{4}(1-\|x\|)\|f'(x) - \mathbf{i} f'(x)\mathbf{i}\| + \frac{1}{2}\|f(x) - \mathbf{i} f(x)\mathbf{i}\|\right) \\ &\quad + \left(\frac{1}{4}(1-\|\bar{x}\|)\|f'(\bar{x}) + \mathbf{i} f'(\bar{x})\mathbf{i}\| + \frac{1}{2}\|f(\bar{x}) + \mathbf{i} f(\bar{x})\mathbf{i}\|\right) \\ &\quad + \left(\frac{1}{4}(1-\|\bar{x}\|)\|f'(\bar{x}) - \mathbf{i} f'(\bar{x})\mathbf{i}\| + \frac{1}{2}\|f(\bar{x}) - \mathbf{i} f(\bar{x})\mathbf{i}\|\right) \\ &\leq M_x + M_{\bar{x}},\end{aligned}$$

where

$$M_x = \left\{\|f(y)\| \;\middle|\; \sqrt{(\Re y - q_0)^2 + (\Im y - |\mathbf{q}|)^2} \leq 1 - \|q\|,\ y \in \mathbb{D}_{\mathbf{i}}\right\}.$$

$$M_{\bar{x}} = \left\{\|f(y)\| \;\middle|\; \sqrt{(\Re y - q_0)^2 + (\Im y + |\mathbf{q}|)^2} \leq 1 - \|q\|,\ y \in \mathbb{D}_{\mathbf{i}}\right\}.$$

3. It follows in a similar way like Fact 1.

□

Here are some properties of $P_{\mathbf{i}}$ and its dependence on $\mathbf{i}$.

Proposition 3.16 *Let* $\mathbf{i} \in \mathbb{S}^2$. *The following items hold*

1. *Given* $r \in \mathbb{S}^3$ *write* $T_r(q) := rq\bar{r}$ *for all* $q \in \mathbb{D}^4$. *Then*

$$P_{T_r(\mathbf{i})}[u](q) = P_{\mathbf{i}}[u \circ T_r](T_r^{-1}(q)),$$

for all $u \in C^1(\mathbb{S}_{T_r(\mathbf{i})}, \mathbb{R})$.

2. *Given* $f \in C(\overline{\mathbb{D}^4}, \mathbb{H})$ *we have on* $\mathbb{D}_{\mathbf{i}}$ *that*

$$P_{\mathbf{i}}[\|f \pm \mathbf{i}f\mathbf{i}\|] \leq 2P_{\mathbf{i}}[\|f\|] \leq P_{\mathbf{i}}[\|f - \mathbf{i}f\mathbf{i}\|] + P_{\mathbf{i}}[\|f + \mathbf{i}f\mathbf{i}\|].$$

3. *Given* $f \in \mathcal{SR}(\mathbb{D}^4) \cap C(\overline{\mathbb{D}^4}, \mathbb{H})$ *and* $\mathbf{j} \in \mathbb{S}^2$ *it follows that*

$$\frac{1}{2\pi}\int_0^{2\pi} \|(x - e^{\mathbf{j}t})^{-*2} * f(e^{\mathbf{j}t})\|(1 - \|x\|^2)dt \leq 2P_{\mathbf{i}}[\|f\|](x),$$

for all $x \in \mathbb{D}_{\mathbf{i}}$.

Proof 1. Since $e^{r\mathbf{i}\bar{r}t} = re^{\mathbf{i}t}\bar{r}$ we have

$$P_{T_r(\mathbf{i})}[u](q) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{r\mathbf{i}\bar{r}t})\frac{1 - \|q\|^2}{\|q - e^{r\mathbf{i}\bar{r}t}\|^2}dt,$$

and it may be concluded that

$$P_{T_r(\mathbf{i})}[u](q) = \frac{1}{2\pi}\int_0^{2\pi} u \circ T_r(e^{\mathbf{i}t})\frac{1 - \|\bar{r}qr\|^2}{\|\bar{r}qr - e^{\mathbf{i}t}\|^2}dt.$$

2. It is due to (3.2) and (3.3)
3. Let $x \in \mathbb{D}_{\mathbf{i}}$ and $\mathbf{j} \in \mathbb{S}^2$. According to the Representation Formula and the established continuity we have

$$\begin{aligned}(x - e^{\mathbf{j}t})^{-*2}) * f(e^{\mathbf{j}t}) = \frac{1}{2}[&(1 + \mathbf{ji})(x - e^{-\mathbf{i}t})^{-2}f(e^{-\mathbf{i}t}) \\ &+(1 - \mathbf{ji})(x - e^{\mathbf{i}t})^{-2}f(e^{\mathbf{i}t})].\end{aligned}$$

Therefore

$$\begin{aligned}\|(x - e^{\mathbf{j}t})^{-*2}) * f(e^{\mathbf{j}t})\|(1 - \|x\|^2) = {}&\|x - e^{-\mathbf{i}t}\|^{-2}\|f(e^{-\mathbf{i}t})\|(1 - \|x\|^2) \\ &+ \|x - e^{\mathbf{i}t}\|^{-2}\|f(e^{\mathbf{i}t})\|(1 - \|x\|^2)\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi} \|(x - e^{\mathbf{j}t})^{-*2} * f(e^{\mathbf{j}t})\|(1 - \|x\|^2)dt \\ &\quad\leq \frac{1}{2\pi}\int_0^{2\pi} \|x - e^{-\mathbf{i}t}\|^{-2}\|f(e^{-\mathbf{i}t})\|(1 - \|x\|^2)dt\end{aligned}$$

$$+\frac{1}{2\pi}\int_0^{2\pi}\|x-e^{\mathbf{i}t}\|^{-2}\|f(e^{\mathbf{i}t})\|(1-\|x\|^2)dt$$
$$=2P_{\mathbf{i}}[\|f\|](x).$$

□

Proposition 3.17 *Set* $f\in\mathcal{SR}(\mathbb{D}^4)\cap C(\overline{\mathbb{D}^4})$.

1. *Let* ω *be a regular majorant such that* $\|f\|\in{}_{\mathbf{i}}\Lambda_\omega(\mathbb{S}^3)$. *Then* $f\in{}_{\mathbf{i}}\Lambda_\omega(\mathbb{D}^4)$ *if and only if*

$$P_{\mathbf{i}}(\|f\pm\mathbf{i}f\mathbf{i}\|)(x)-\|f(x)\pm\mathbf{i}f(x)\mathbf{i}\|\leq C\omega(1-\|x\|),\quad\forall x\in\mathbb{D}_{\mathbf{i}}.$$

2. *Let* ω_1, ω_2 *be two regular majorant such that* $\|f\|\in{}_{\mathbf{i}}\Lambda_{\omega_1,\omega_2}(\mathbb{S}^3)$. *Then* $f\in{}_{\mathbf{i}}\Lambda_{\omega_1,\omega_2}(\mathbb{D}^4)$ *if and only if*

$$P_{\mathbf{i}}(\|f-\mathbf{i}f\mathbf{i}\|)(x)-\|f(x)-\mathbf{i}f(x)\mathbf{i}\|\leq C\omega_1(1-\|x\|),$$
$$P_{\mathbf{i}}(\|f+\mathbf{i}f\mathbf{i}\|)(x)-\|f(x)+\mathbf{i}f(x)\mathbf{i}\|\leq C\omega_2(1-\|x\|),\quad\forall x\in\mathbb{D}_{\mathbf{i}}.$$

Proof 1. Combining (3.2) with Proposition 3.12 and Theorem B of Section 2 completes the proof.
2. It follows from identities (3.2) and Theorem B of Section 2.

□

Corollary 3.18 *Consider* $f\in\mathcal{SR}(\mathbb{D}^4)\cap C(\overline{\mathbb{D}^4})$.

1. *Let* ω *be a regular majorant such that* $\|f\|\in{}_{\mathbf{i}}\Lambda_\omega(\mathbb{S}^3)$. *If* $f\in{}_{\mathbf{i}}\Lambda_\omega(\mathbb{D}^4)$ *and* $q\in\mathbb{D}^4$ *satisfies*

$$\langle q,e^{\mathbf{i}t}\rangle\leq q_0\cos t\pm|\mathbf{q}|\sin t,\quad\forall t\in[0,2\pi],$$

then

$$P_{\mathbf{i}}(\|f\|)(q)-2\|f(q_0\pm\mathbf{i}|\mathbf{q}|)\|\leq C\omega(1-\|q\|).$$

2. *Let* ω_1, ω_2 *be two regular majorant such that* $\|f\|\in{}_{\mathbf{i}}\Lambda_{\omega_1,\omega_2}(\mathbb{S}^3)$. *If* $f\in{}_{\mathbf{i}}\Lambda_{\omega_1,\omega_2}(\mathbb{D}^4)$ *and* $q\in\mathbb{D}^4$ *holds*

$$\langle q,e^{\mathbf{i}t}\rangle\leq q_0\cos t\pm|\mathbf{q}|\sin t,\quad\forall t\in[0,2\pi],$$

then

$$P_{\mathbf{i}}(\|f\|)(q)-2\|f(q_0\pm\mathbf{i}|\mathbf{q}|)\|\leq C\left(\omega_1(1-\|q\|)+\omega_2(1-\|q\|)\right).$$

Proof 1. Let $x=q_0\pm\mathbf{i}|\mathbf{q}|$. It follows easily that $|q-e^{\mathbf{i}}|\geq|x-e^{\mathbf{i}t}|$ for all $t\in[0,2\pi]$. Of course,

$$2P_{\mathbf{i}}(\|f\|)(q)\leq 2P_{\mathbf{i}}(\|f\|)(x)\leq P_{\mathbf{i}}(\|f+\mathbf{i}f\mathbf{i}\|)(x)+P_{\mathbf{i}}(\|f-\mathbf{i}f\mathbf{i}\|)(x)$$

$$\leq \|f(x)+\mathbf{i}f(x)\mathbf{i}\|+\|f(x)-\mathbf{i}f(x)\mathbf{i}\|+2C\omega(1-\|x\|).$$

Therefore

$$P_{\mathbf{i}}(\|f\|)(q)\leq 2\|f(q_0\pm \mathbf{i}|\mathbf{q}|)\|+C\omega(1-\|q\|).$$

2. Similar arguments to those above.

□

4 Conclusions and future works

In summary, characterizations of the Lipschitz type spaces of slice regular functions in the unit ball of the skew-field of quaternions with prescribed modulus of continuity, despite the non-commutativity of quaternions, are presented. The main results go on to the global case. Importantly, the present findings suggest the possibility to extend the study to the theory of slice monogenic functions associated to Clifford algebras, as a good starting point for further research.

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