Analysis and Mathematical Physics (2022) 12:51
https://doi.org/10.1007/s13324-022-00649-x

Cesàro-like operators acting on spaces of analytic functions

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Received: 2 November 2021 / Revised: 10 January 2022 / Accepted: 14 January 2022 /
Published online: 17 March 2022

Abstract

Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$. If μ is a finite positive Borel measure on the interval $[0, 1)$ and f is an analytic function in $\mathbb{D}$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{D})$, we define

$$\mathcal{C}_\mu(f)(z) = \sum_{n=0}^{\infty} \mu_n \left(\sum_{k=0}^{n} a_k \right) z^n, \quad z \in \mathbb{D},$$

where, for $n \geq 0$, μ_n denotes the n-th moment of the measure μ, that is, $\mu_n = \int_{[0,1)} t^n d\mu(t)$. In this way, $\mathcal{C}_\mu$ becomes a linear operator defined on the space $\mathrm{Hol}(\mathbb{D})$ of all analytic functions in $\mathbb{D}$. We study the action of the operators $\mathcal{C}_\mu$ on distinct spaces of analytic functions in $\mathbb{D}$, such as the Hardy spaces H^p, the weighted Bergman spaces A^p_α, $BMOA$, and the Bloch space $\mathcal{B}$.

This research is supported in part by a grant from "El Ministerio de Economía y Competitividad", Spain (PGC2018-096166-B-I00) and by grants from la Junta de Andalucía (FQM-210 and UMA18-FEDERJA-002) .

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1 Introduction and main results

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc in the complex plane $\mathbb{C}$ and let $\mathrm{Hol}(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$ endowed with the topology of uniform convergence in compact subsets.

If $0 < r < 1$ and $f \in \mathrm{Hol}(\mathbb{D})$, we set

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right)^{1/p}, \quad 0 < p < \infty,$$
$$M_\infty(r, f) = \sup_{|z|=r} |f(z)|.$$

For $0 < p \leq \infty$, the Hardy space H^p consists of those $f \in \mathrm{Hol}(\mathbb{D})$ such that

$$\|f\|_{H^p} \stackrel{\text{def}}{=} \sup_{0<r<1} M_p(r, f) < \infty.$$

We refer to [13] for the notation and results regarding Hardy spaces.

Let dA denote the area measure on $\mathbb{D}$, normalized so that the area of $\mathbb{D}$ is 1. Thus $dA(z) = \frac{1}{\pi}\, dx\, dy = \frac{1}{\pi}\, r\, dr\, d\theta$. For $0 < p < \infty$ and $\alpha > -1$ the weighted Bergman space A^p_α consists of those $f \in \mathrm{Hol}(\mathbb{D})$ such that

$$\|f\|_{A^p_\alpha} \stackrel{\text{def}}{=} \left(\int_{\mathbb{D}} |f(z)|^p \, dA_\alpha(z) \right)^{1/p} < \infty,$$

where $dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$. We refer to [14,25,39] for the notation and results about Bergman spaces.

The space $BMOA$ consists of those functions $f \in H^1$ whose boundary values have bounded mean oscillation on $\partial\mathbb{D}$. We refer to [16] for the theory of $BMOA$-functions.

Finally, we recall that a function $f \in \mathrm{Hol}(\mathbb{D})$ is said to be a Bloch function if

$$\|f\|_{\mathcal{B}} \stackrel{\text{def}}{=} |f(0)| + \sup_{z\in\mathbb{D}} (1 - |z|^2)|f'(z)| < \infty.$$

The space of all Bloch functions is denoted by $\mathcal{B}$. A classical reference for the theory of Bloch functions is [2]. Let us recall that

$$H^\infty \subsetneq BMOA \subsetneq \mathcal{B}, \qquad BMOA \subsetneq \bigcap_{0<p<\infty} H^p, \qquad \mathcal{B} \subsetneq A^p_\alpha \ (p > 0,\ \alpha > -1).$$

The Cesàro operator $\mathcal{C}$ is defined over the space of all complex sequences as follows: If $(a) = \{a_k\}_{k=0}^\infty$ is a sequence of complex numbers then

$$\mathcal{C}\left((a)\right) = \left\{ \frac{1}{n+1} \sum_{k=0}^{n} a_k \right\}_{n=0}^{\infty}.$$

The operator $\mathcal{C}$ is known to be bounded from ℓ^p to ℓ^p for $1 < p < \infty$. In fact, the sharp inequalities

$$\|\mathcal{C}\left((a)\right)\|_p \leq \frac{p}{p-1}\|(a)\|_p, \quad (a) \in \ell^p, \quad 1 < p < \infty,$$

were proved by Hardy [21] and Landau [29] (see also [24, Theorem 326, p.239]).

Identifying any given function $f \in \mathrm{Hol}(\mathbb{D})$ with the sequence $\{a_k\}_{k=0}^{\infty}$ of its Taylor coefficients, the Cesàro operator $\mathcal{C}$ becomes a linear operator from $\mathrm{Hol}(\mathbb{D})$ into itself as follows:

If $f \in \mathrm{Hol}(\mathbb{D})$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ $(z \in \mathbb{D})$, then

$$\mathcal{C}(f)(z) = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} a_k \right) z^n, \quad z \in \mathbb{D}.$$

The Cesàro operator is bounded on H^p for $0 < p < \infty$. For $1 < p < \infty$, this follows from a result of Hardy on Fourier series [22] together with the M. Riesz's theorem on the conjugate function [13, Theorem 4.1]. Siskakis [33] used semigroups of composition operators to give an alternative proof of this result and to extend it to $p = 1$. A direct proof of the boundedness on H^1 was given by Siskakis in [34]. Miao [31] dealt with the case $0 < p < 1$. Stempak [36] gave a proof valid for $0 < p \leq 2$ and Andersen [1] provided another proof valid for all $p < \infty$.

In this paper we associate to every positive finite Borel measure on $[0, 1)$ a certain operator $\mathcal{C}_\mu$ acting on $\mathrm{Hol}(\mathbb{D})$ which is a natural generalization of the classical Cesàro operator $\mathcal{C}$.

If μ is a positive finite Borel measure on $[0, 1)$ and n is a non-negative integer, we let μ_n denote the moment of order n of μ, that is,

$$\mu_n = \int_{[0,1)} t^n \, d\mu(t), \quad n = 0, 1, 2, \ldots.$$

If $f \in \mathrm{Hol}(\mathbb{D})$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{D})$, we define $\mathcal{C}_\mu(f)$ as follows

$$\mathcal{C}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\mu_n \sum_{k=0}^{n} a_k \right) z^n, \quad z \in \mathbb{D}.$$

It is clear that $\mathcal{C}_\mu$ is a well defined linear operator $\mathcal{C}_\mu : \mathrm{Hol}(\mathbb{D}) \to \mathrm{Hol}(\mathbb{D})$. When μ is the Lebesgue measure on $[0, 1)$, the operator $\mathcal{C}_\mu$ reduces to the classical Cesàro operator $\mathcal{C}$.

Our main objective in this work is to characterize those positive finite Borel measures μ on $[0, 1)$ such that the operator $\mathcal{C}_\mu$ is bounded or compact on classical subspaces of $\mathrm{Hol}(\mathbb{D})$ such as the Hardy spaces H^p, the weighted Bergman spaces A^p_α, and the spaces $BMOA$ and $\mathcal{B}$.

Measures of Carleson type will play a basic role in the sequel. If $I \subset \partial\mathbb{D}$ is an interval, $|I|$ will denote the length of I. The *Carleson square* $S(I)$ is defined as

$$S(I)=\left\{re^{it}:\, e^{it}\in I,\quad 1-\frac{|I|}{2\pi}\le r<1\right\}.$$

If $s>0$ and μ is a positive Borel measure on $\mathbb{D}$, we shall say that μ is an s-Carleson measure if there exists a positive constant C such that

$$\mu\left(S(I)\right)\le C|I|^s,\quad \text{for any interval} I\subset\partial\mathbb{D}.$$

If μ satisfies $\mu\left(S(I)\right)=\mathrm{o}\left(|I|^s\right)$, as $|I|\to 0$, then we say that μ is a *vanishing s-Carleson measure*.

A 1-Carleson measure, respectively, a vanishing 1-Carleson measure, will be simply called a Carleson measure, respectively, a vanishing Carleson measure.

We recall that Carleson [7] proved that $H^p \subset L^p(d\mu)$ $(0<p<\infty)$, if and only if μ is a Carleson measure (see [13, Chapter 9]).

Following [38], if μ is a positive Borel measure on $\mathbb{D}$, $0\le\alpha<\infty$, and $0<s<\infty$, we say that μ is an α-logarithmic s-Carleson measure if there exists a positive constant C such that

$$\frac{\mu\left(S(I)\right)\left(\log\frac{2}{|I|}\right)^{\alpha}}{|I|^s}\le C,\quad \text{for any interval} I\subset\partial\mathbb{D}.$$

If $\mu\left(S(I)\right)\left(\log\frac{2}{|I|}\right)^{\alpha}=\mathrm{o}\left(|I|^s\right)$, as $|I|\to 0$, we say that μ is a vanishing α-logarithmic s-Carleson measure.

A measure μ on $[0,1)$ can be seen as a measure on $\mathbb{D}$ with support contained in the radius $[0,1)$. In this way, a positive Borel measure μ on $[0,1)$ is an s-Carleson measure if and only if there exists a positive constant C such that

$$\mu\left([t,1)\right)\le C(1-t)^s,\quad 0\le t<1,$$

and we have similar statements for vanishing s-Carleson measures, for α-logarithmic s-Carleson measures, and for vanishing α-logarithmic s-Carleson measures.

Among other, we shall prove the following results.

Theorem 1 *Suppose that* $1\le p<\infty$ *and let* μ *be a positive finite Borel measure on* $[0,1)$. *Then the following conditions are equivalent.*

(i) The measure μ *is a Carleson measure.*
(ii) The operator $\mathcal{C}_\mu$ *is bounded from* H^p *into itself.*

Theorem 2 *Suppose that* $1\le p<\infty$ *and let* μ *be a positive finite Borel measure on* $[0,1)$. *Then the following conditions are equivalent.*

(i) The measure μ *is a vanishing Carleson measure.*
(ii) The operator $\mathcal{C}_\mu$ *is compact from* H^p *into itself.*

Danikas and Siskakis [12] observed that $\mathcal{C}(H^\infty) \not\subset H^\infty$ and $\mathcal{C}(BMOA) \not\subset BMOA$ and studied the action of the the Cesàro operator on these spaces. We will devote Sects. 3.3 and 5 to study the Cesàro-like operators $\mathcal{C}_\mu$ acting on these spaces. Let us just mention here the following result.

Theorem 3 *Let μ be a positive finite Borel measure on $[0, 1)$. Then the following conditions are equivalent.*

(i) The measure μ is a 1-logarithmic 1-Carleson measure.
(ii) The operator $\mathcal{C}_\mu$ is bounded from $BMOA$ into itself.
(iii) The operator $\mathcal{C}_\mu$ is bounded from the Bloch space $\mathcal{B}$ into itself.

Section 3 will be devoted to present the proofs of Theorem 1 and Theorem 2 as well as some further results concerning the action of the operators $\mathcal{C}_\mu$ on Hardy spaces. Section 4 will deal with the action of the operators $\mathcal{C}_\mu$ on Bergman spaces and, as we have already mentioned, Sect. 5 will be devoted to study the operators $\mathcal{C}_\mu$ acting on $BMOA$, the Bloch space, and some related spaces. In particular, Sect. 5 will include a proof of Theorem 3 and the substitute of this result concerning compactness.

In Sect. 2 we shall give two alternative representations of the operator $\mathcal{C}_\mu$, one of them is an integral representation and the other one involves the convolution with a fixed analytic function in $\mathbb{D}$. We shall also introduce a related operator which will be denoted T_μ and which will play a basic role in the proofs of some of our results.

Throughout the paper, if μ is a finite positive Borel measure on $[0, 1)$, for $n \geq 0$, μ_n will denote the moment of order n of μ. Also, we shall be using the convention that $C = C(p, \alpha, q, \beta, \dots)$ will denote a positive constant which depends only upon the displayed parameters $p, \alpha, q, \beta \dots$ (which sometimes will be omitted) but not necessarily the same at different occurrences. Furthermore, for two real-valued functions K_1, K_2 we write $K_1 \lesssim K_2$, or $K_1 \gtrsim K_2$, if there exists a positive constant C independent of the arguments such that $K_1 \leq CK_2$, respectively $K_1 \geq CK_2$. If we have $K_1 \lesssim K_2$ and $K_1 \gtrsim K_2$ simultaneously, then we say that K_1 and K_2 are equivalent and we write $K_1 \asymp K_2$.

Let us close this section noticing that, since the subspaces X of $\mathrm{Hol}(\mathbb{D})$ we shall be dealing with are Banach spaces continuously embedded in $\mathrm{Hol}(\mathbb{D})$, to prove that the operator $\mathcal{C}_\mu$ (or T_μ, to be defined below) is bounded on X it suffices to show that it maps X into X by appealing to the closed graph theorem.

2 Alternative representations of $\mathcal{C}_\mu$ and a related operator

A simple calculation with power series gives the following integral representation of the operators $\mathcal{C}_\mu$.

Proposition 1 *If μ is a positive finite Borel measure on $[0, 1)$ and $f \in \mathrm{Hol}(\mathbb{D})$ then*

$$\mathcal{C}_\mu(f)(z) = \int_{[0,1)} \frac{f(tz)}{1 - tz}\, d\mu(t), \quad z \in \mathbb{D}. \tag{1}$$

Next we shall give another expression for $\mathcal{C}_\mu(f)$ involving the convolution of analytic functions. If f and g are two analytic functions in the unit disc,

$$f(z)=\sum_{n=0}^{\infty}a_nz^n,\quad g(z)=\sum_{n=0}^{\infty}b_nz^n,\quad z\in\mathbb{D},$$

the convolution $f\star g$ of f and g is defined by

$$f\star g(z)=\sum_{n=0}^{\infty}a_nb_nz^n,\quad z\in\mathbb{D}.$$

Lemma 1 *Let μ be a positive finite Borel measure on $[0,1)$ and set*

$$F(z)=\sum_{n=0}^{\infty}\mu_nz^n,\quad z\in\mathbb{D}.$$

If $f\in \mathrm{Hol}(\mathbb{D})$ and

$$g(z)=\frac{f(z)}{1-z},\quad z\in\mathbb{D},$$

then $\mathcal{C}_\mu(f)=F\star g$.

The proof is elementary and will be omitted.

The following result regarding the radial measures μ we are considering will be used in our work.

Lemma 2 *Let μ be a finite positive Borel measure on the interval $[0,1)$ and, for $n\geq 0$, let μ_n denote the moment of order n of μ.*

(i) μ is a Carleson measure if and only if $\mu_n=\mathrm{O}(\frac{1}{n})$.
(ii) μ is a vanishing Carleson measure if and only if $\mu_n=\mathrm{o}(\frac{1}{n})$.
(iii) μ is a 1-logarithmic 1-Carleson measure if and only if $\mu_n=\mathrm{O}(\frac{1}{n\log n})$.
(iv) μ is a vanishing 1 logarithmic 1-Carleson measure if and only if $\mu_n=\mathrm{o}(\frac{1}{n\log n})$.

Proof (i) is Proposition 8 of [8] and (ii) follows with a similar argument. Lemma 2. 7 of [19] gives one implication of (iii) and the other one follows from the from the simple inequality

$$\mu\left(\left[1-\frac{1}{n},1\right)\right)\lesssim\int_{\left[1-\frac{1}{n},1\right)}t^n\,d\mu(t)\leq\mu_n.$$

Finally, (iv) can be proved with an argument similar the the one used to prove (iii). ◻

Now we define a new operator operator T_μ associated to μ which will be important in our work because it will become the adjoint of $\mathcal{C}_\mu$ in distinct instances.

If μ is a finite positive Borel measure on $[0, 1)$ and $f \in \mathrm{Hol}(\mathbb{D})$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($z \in \mathbb{D}$) we set

$$T_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=n}^{\infty} \mu_k a_k \right) z^n,$$

whenever the right hand side makes sense and defines an analytic function in $\mathbb{D}$.

Clearly, the operator T_μ is not defined over the whole space $\mathrm{Hol}(\mathbb{D})$. We have the following result.

Proposition 2 *Let μ is a finite positive Borel measure on* $[0, 1)$.

(a) If P is a polynomial then $T_\mu(P)$ is well defined and it also a polynomial.
(b) If μ is a Carleson measure then T_μ is well defined on H^1.

Proof (a) is clear. To prove (b) we use the fact that if μ is a Carleson measure then $\mu_n = \mathrm{O}\left(n^{-1}\right)$ (see Lemma 2). This and Hardy's inequality [13, p. 48] shows that if $f \in H^1$, $f(z) = \sum_{k=1}^{\infty} a_k z^k$, then there exists $C > 0$ such that

$$\sum_{k=n}^{\infty} \mu_k |a_k| \le C \sum_{k=n}^{\infty} \frac{|a_k|}{k+1} \le C\pi \|f\|_{H^1}$$

for all n. Clearly, this implies (b). □

It is well known that, for $1 < p < \infty$, the dual of H^p is identifiable with H^q, $\frac{1}{p} + \frac{1}{q} = 1$, with the pairing

$$< f, g >_{H^p} = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} d\theta = \sum_{n=0}^{\infty} a_n \overline{b_n}$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^p$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n \in H^q$ (see [13, Theorem 7.3]).

Similarly, if $1 < p < \infty$ and $\alpha > 1$, the dual of A^p_α is identifiable with A^q_α with the pairing

$$< f, g >_{p,\alpha} = \int_{\mathbb{D}} f(z) \overline{g(z)} dA_\alpha(z) = \sum_{n=0}^{\infty} c_{n,\alpha} a_n \overline{b_n},$$

where

$$c_{n,\alpha} = \frac{n!\,\Gamma(2+\alpha)}{\Gamma(n+2+\alpha)}, \quad n = 0, 1, 2, \dots,$$

and $f(z) = \sum_{n=0}^{\infty} a_n z^n \in A^p_\alpha$, $g(z) = \sum_{n=0}^{\infty} b_n z^n \in A^q_\alpha$ (see [25, Theorem 1.16 and p. 5]). A simple calculation gives the following result.

Proposition 3 *Let μ be a positive finite Borel measure on* $[0, 1)$.

(i) If $1 < p < \infty$, $f \in H^p$, and g is a polynomial then

$$< \mathcal{C}_\mu(f), g >_{H^p} = < f, T_\mu(g) >_{H^p} .$$

(ii) If $1 < p < \infty$, $\alpha > -1$, $f \in A^p_\alpha$, and g is a polynomial then

$$< \mathcal{C}_\mu(f), g >_{p,\alpha} = < f, T_\mu(g) >_{p,\alpha} .$$

Proposition 3, together with the fact that the polynomials are dense in all the spaces H^p $(p < \infty)$ and A^p_α $(p < \infty,\ \alpha > -1)$, readily implies the following result.

Proposition 4 *Suppose that $1 < p < \infty$ and let μ be a positive finite Borel measure on* $[0, 1)$. *Let q be the conjugate exponent of p, that is, $\frac{1}{p} + \frac{1}{q} = 1$.*

(i) If $\mathcal{C}_\mu$ is a bounded operator from H^p into itself, then there exists a positive constant C such that

$$\|T_\mu(P)\|_{H^q} \le C\|P\|_{H^q}$$

for every polynomial P. Consequently, T_μ extends to a bounded linear operator from H^q into itself. This extension, which will be also denoted by T_μ, is the adjoint of $\mathcal{C}_\mu$.

(ii) Suppose that $\alpha > -1$. If $\mathcal{C}_\mu$ is a bounded operator from A^p_α into itself, then there exists a positive constant C such that

$$\|T_\mu(P)\|_{A^q_\alpha} \le C\|P\|_{A^q_\alpha}$$

for every polynomial P. Consequently, T_μ extends to a bounded linear operator from A^q_α into itself. This extension, which will be also denoted by T_μ, is the adjoint of $\mathcal{C}_\mu$.

3 The operators $\mathcal{C}_\mu$ acting on Hardy spaces

In this section we shall study the action of the operators $\mathcal{C}_\mu$ on Hardy spaces.

We shall use complex interpolation to prove some of our results. Let us refer to [39, Chapter 2] for the terminology and basic results concerning complex interpolation.

If X_0 and X_1 are two compatible Banach spaces then, for $0 < \theta < 1$, $(X_0, X_1)_\theta$ stands for the space obtained by the complex method of interpolation of Calderón [5]. It is well known (see [6,26,32]) that if $1 \le p_0, p_1 \le \infty$, $0 < \theta < 1$, and $1/p = (1-\theta)/p_0 + \theta/p_1$, then

$$(H^{p_0}, H^{p_1})_\theta = H^p. \tag{2}$$

In particular,

$$H^p = (H^2, H^1)_\theta, \quad \text{if } 1 < p < 2 \text{ and } \theta = \frac{2}{p} - 1. \tag{3}$$

3.1 Proof of Theorem 1

We shall split it in several cases.

Proof of the implication (i) ⇒ (ii) when $p = 1$. Assume that μ is a Carleson measure and take $f \in H^1$. Set

$$g(z) = \frac{f(z)}{1-z}, \quad z \in \mathbb{D},$$

and

$$t_k = 1 - \frac{1}{2^k}, \quad k = 0, 1, 2, \ldots$$

Using the integral representation on $\mathcal{C}_\mu$, we see that, for $0 < r < 1$,

$$\begin{aligned}
M_1\left(r, \mathcal{C}_\mu(f)\right) &= \frac{1}{2\pi}\int_0^{2\pi}\left|\int_{[0,1)}\frac{f(rte^{i\theta})}{1-rte^{i\theta}}\,d\mu(t)\right|\,d\theta \\
&\le \frac{1}{2\pi}\int_0^{2\pi}\int_{[0,1)}\left|g(rte^{i\theta})\right|\,d\mu(t)\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{k=1}^{\infty}\int_{[t_{k-1},t_k)}\left|g(rte^{i\theta})\right|\,d\mu(t)\right)\,d\theta \\
&\le \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{k=1}^{\infty}\left[\sup_{0\le t\le t_k}\left|g(rte^{i\theta})\right|\right]\right)\mu\left([t_{k-1},t_k]\right)\,d\theta.
\end{aligned}$$

Since μ is a Carleson measure, $\mu\left([t_{k-1}, t_k]\right) \lesssim \frac{1}{2^k}$. Using this, the Hardy-Littlewood maximal theorem [13, Theorem 1.9], the fact that integral means $M_1(s, g)$ increase with s, and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
M_1\left(r, \mathcal{C}_\mu(f)\right) &\le \sum_{k=1}^{\infty}\frac{1}{2^k}\left(\frac{1}{2\pi}\int_0^{2\pi}\left[\sup_{0\le t\le t_k}\left|g(rte^{i\theta})\right|\right]\,d\theta\right) \\
&\lesssim \sum_{k=1}^{\infty}\frac{1}{2^k}M_1(rt_k, g) \\
&\lesssim \sum_{k=1}^{\infty}\frac{1}{2^k}2^k\int_{t_k}^{t_{k+1}} M_1(rt, g)\,dt
\end{aligned}$$

$$\lesssim \int_0^r \frac{1}{2\pi} \int_0^{2\pi} \left| g(te^{i\theta}) \right| d\theta\, dt$$
$$\lesssim \int_0^r \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{f(te^{i\theta})}{1 - te^{i\theta}} \right| d\theta\, dt$$
$$\lesssim \int_0^r M_2(t, f) \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|1 - te^{i\theta}|^2}\, d\theta \right)^{1/2} dt$$
$$\lesssim \int_0^r M_2(t, f)(1 - t)^{-1/2}\, dt.$$

Making the change of variables $t = rs$ in the last integral and setting $f_r(z) = f(rz)$ $(z \in \mathbb{D})$, it follows that

$$M_1\left(r, \mathcal{C}_\mu(f)\right) \lesssim \int_0^1 M_2(sr, f)(1 - sr)^{-1/2}\, ds$$
$$= \int_0^1 M_2(s, f_r)(1 - sr)^{-1/2}\, ds \le \int_0^1 M_2(s, f_r)(1 - s)^{-1/2}\, ds.$$

Using a result of Hardy and Littlewood [23] (see also [34]) we see that

$$\int_0^1 M_2(s, f_r)(1 - s)^{-1/2}\, dt \lesssim \|f_r\|_{H^1}.$$

Then it follows that

$$M_1\left(r, \mathcal{C}_\mu(f)\right) \lesssim M_1(r, f). \tag{4}$$

This implies that $\mathcal{C}_\mu(f) \in H^1$ and that $\|\mathcal{C}_\mu(f)\|_{H^1} \lesssim \|f\|_{H^1}$. □

Proof of the implication (i) ⇒ (ii) when $p = 2$. Assume that μ is a Carleson measure and take $f \in H^2$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{D})$. Using [8, Proposition 1] we see that $|\mu_n| \lesssim \frac{1}{n+1}$. Using this, the definition of $\mathcal{C}_\mu(f)$, and the fact that the Cesàro operator is bounded on H^2, it follows that

$$\|\mathcal{C}_\mu(f)\|_{H^2}^2 = \sum_{n=0}^{\infty} \mu_n^2 \left| \sum_{k=0}^{n} a_k \right|^2 \lesssim \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} \left| \sum_{k=0}^{n} a_k \right|^2$$
$$= \|\mathcal{C}(f)\|_{H^2}^2 \lesssim \|f\|_{H^2}^2.$$

□

Proof of the implication (i) ⇒ (ii) for $1 < p < 2$. Since (i) ⇒ (ii) when $p = 1$ and $p = 2$, the fact that (i) ⇒ (ii) when $1 < p < 2$ follows using (3) and Theorem 2. 4 of [39]. □

To prove the remaining case, that is, the implication (i) ⇒ (ii) for $2 < p < \infty$ we shall use ideas of Andersen [1]. Actually, our next argument works for $1 < p < \infty$.

Proof of the implication (i) ⇒ (ii) for $1 < p < \infty$. Assume that μ is a Carleson measure, $1 < p < \infty$, and $f \in H^p$.

For $0 < r < 1$, set

$$K_{r,\mu}(\theta,\varphi) = \int_{[0,1)} \frac{(1+t)(1-t)}{|1-te^{i\varphi}|^2(1-tre^{i\theta})}\,d\mu(t), \quad \theta,\varphi \in [-\pi,\pi].$$

Arguing just as in [1, p. 621], using Fubini's theorem, we have that

$$\mathcal{C}_\mu(f)(re^{i\theta}) = \int_{-\pi}^{\pi} K_{r,\mu}(\theta,\varphi) f(re^{i(\theta+\varphi)})\,d\varphi. \tag{5}$$

Now, letting $\{t_k\}_{k=0}^{\infty}$ be as above, using the fact that $\mu([t_k,t_{k+1})) \lesssim \frac{1}{2^k}$, and simple estimates, we obatin

$$\begin{aligned}
|K_{r,\mu}(\theta,\varphi)| &\le 2\int_{[0,1)} \frac{1-t}{|1-te^{i\varphi}|^2|1-tre^{i\theta}|}\,d\mu(t)\\
&\lesssim \int_{[0,1)} \frac{1-t}{[(1-t)^2+\varphi^2][(1-t)^2+\theta^2]^{1/2}}\,d\mu(t)\\
&\lesssim \sum_{k=0}^{\infty}\int_{t_k}^{t_{k+1}} \frac{1-t}{[(1-t)^2+\varphi^2][(1-t)^2+\theta]^{1/2}}\,d\mu(t)\\
&\lesssim \sum_{k=0}^{\infty}\frac{1}{2^k}\frac{\frac{1}{2^k}}{\left[\left(\frac{1}{2^{k+1}}\right)^2+\varphi^2\right]\left[\left(\frac{1}{2^{k+1}}\right)^2+\theta^2\right]^{1/2}}\\
&\lesssim \sum_{k=0}^{\infty}\int_{t_k}^{t_{k+1}} \frac{\frac{1}{2^k}}{\left[\left(\frac{1}{2^{k+1}}\right)^2+\varphi^2\right]\left[\left(\frac{1}{2^{k+1}}\right)^2+\theta^2\right]^{1/2}}\,dt\\
&\lesssim \int_0^1 \frac{1-t}{[(1-t)^2+\varphi^2][(1-t)^2+\theta^2]^{1/2}}\,dt\\
&= \int_0^1 \frac{x}{[x^2+\varphi^2][x^2+\theta^2]^{1/2}}\,dx
\end{aligned}$$

Then, using Lemma 2.1 of [1], we see that for all $\theta,\varphi \in (-\pi,\pi)\setminus\{0\}$ and $r \in (0,1)$, we have

$$|K_{r,\mu}(\theta,\varphi)| \lesssim \frac{H(\varphi/\theta)}{|\theta|},$$

where

$$H(s) = \frac{\log(2+1/|s|)}{1+|s|}, \quad s \ne 0.$$

Using this and (5) it follows that

$$|\mathcal{C}_\mu(f)(re^{i\theta})| \lesssim \int_{-\pi}^{\pi} \frac{H(\varphi/\theta)}{|\theta|} |f(re^{i(\theta+\varphi)})|\, d\varphi, \quad \theta \in (-\pi,\pi)\setminus\{0\},\ 0<r<1.$$

Then the argument in p. 622 of [1] yields that

$$M_p(r, \mathcal{C}_\mu(f)) \lesssim M_p(r, f) \tag{6}$$

and, hence $\|\mathcal{C}_\mu(f)\|_{H^p} \lesssim \|f\|_{H^p}$. □

Proof of the implication (ii) ⇒ (i) for $1 \le p \le 2$. Suppose that $1 \le p \le 2$ and that $\mathcal{C}_\mu$ is bounded on H^p. Recall that, for $\alpha > 0$,

$$\frac{1}{(1-z)^\alpha} = \sum_{n=0}^{\infty} a_n(\alpha) z^n, \quad z \in \mathbb{D}$$

where

$$a_n(\alpha) \asymp n^{\alpha-1}. \tag{7}$$

For $0 < a < 1$, set

$$f_a(z) = \left(\frac{1-a^2}{(1-az)^2}\right)^{1/p} = (1-a^2)^{1/p} \sum_{n=0}^{\infty} a_n(2/p) a^n z^n, \quad z \in \mathbb{D}.$$

We have that

$$f_a \in H^p \text{ and } \|f_a\|_{H^p} = 1, \quad 0 < a < 1.$$

Since $\mathcal{C}_\mu$ is bounded on H^p, we have

$$\|\mathcal{C}_\mu(f_a)\|_{H^p}^p \lesssim 1. \tag{8}$$

Now

$$\mathcal{C}_\mu(f_a)(z) = (1-a^2)^{1/p} \sum_{n=0}^{\infty} \mu_n \left(\sum_{k=0}^{n} a_k(2/p) a^k\right) z^n, \quad z \in \mathbb{D}.$$

Using the fact that $1 \le p \le 2$, [13, Theorem 6.2], (7), and the fact that the sequence $\{\mu_n\}$ is decreasing, we obtain

$$
\begin{aligned}
&(1-a)\mu_N^p \sum_{n=0}^{N}(n+1)^{p-2}\left(\sum_{k=0}^{n} k^{\frac{2}{p}-1}a^k\right)^p \\
&\le (1-a)\sum_{n=0}^{N}(n+1)^{p-2}\mu_n^p\left(\sum_{k=0}^{n} k^{\frac{2}{p}-1}a^k\right)^p \\
&\le (1-a)\sum_{n=0}^{\infty}(n+1)^{p-2}\mu_n^p\left(\sum_{k=0}^{n} k^{\frac{2}{p}-1}a^k\right)^p \\
&\asymp (1-a^2)\sum_{n=0}^{\infty}(n+1)^{p-2}\mu_n^p\left(\sum_{k=0}^{n} a_k(2/p)a^k\right)^p \\
&\lesssim \|\mathcal{C}_\mu(f_a)\|_{H^p}^p,
\end{aligned}
$$

for every positive integer N and every $a \in (0,1)$. Taking $a = 1 - \frac{1}{N}$ and using the fact that $\mathcal{C}_\mu$ is bounded on H^p, we obtain

$$
\frac{\mu_N^p}{N}\sum_{n=0}^{N}(n+1)^{p-2}\left(\sum_{k=0}^{n} k^{\frac{2}{p}-1}\right)^p \asymp \mu_N^p N^p \lesssim \|\mathcal{C}_\mu(f_a)\|_{H^p}^p \lesssim \|f_a\|_{H^p}. \tag{9}
$$

This and (8) imply that $\mu_N \lesssim \frac{1}{N}$. Using again Lemma 2, this yields that μ is a Carleson measure. □

Proof of the implication (ii) ⇒ (i) for $2 \le p < \infty$. Suppose that $2 < p < \infty$ and that $\mathcal{C}_\mu$ is a bounded operator on H^p. Let q be the conjugate exponent of p, that is, $\frac{1}{p} + \frac{1}{q} = 1$. Bearing in mind Proposition 2 and Proposition 3, we see that the operator T_μ, initially defined over polynomials, extends to a bounded operator on H^q.

For $0 < a < 1$ and $N \in \mathbb{N}$, set

$$
f_a(z) = \left(\frac{1-a^2}{(1-az)^2}\right)^{1/q} = (1-a^2)^{1/q}\sum_{n=0}^{\infty} a_n(2/q)a^n z^n, \quad z \in \mathbb{D},
$$

$$
f_{a,N}(z) = (1-a^2)^{1/q}\sum_{n=0}^{N} a_n(2/q)a^n z^n, \quad z \in \mathbb{D}.
$$

We have that for all $a \in (0,1)$, $f_a \in H^q$ and $\|f_a\|_{H^q} = 1$. Since T_μ is bounded on H^q, it follows that

$$
\|T_\mu(f_a)\|_{H^q} \lesssim 1 \tag{10}
$$

Also, for every a, $f_{a,N} \to f_a$, as $N \to \infty$ in H^q and uniformly on compact subsets of $\mathbb{D}$. Now, $T_\mu\left(f_{a,N}\right)(z) = (1-a^2)^{1/q}\sum_{n=0}^{N}\left(\sum_{k=n}^{N}\mu_k a_k(2/q)a^k\right) z^n$ $(z \in \mathbb{D})$ and then, using that $1 < q < 2$ and [13, Theorem 6.2], we have that

$$(1-a)\sum_{n=1}^{N}(n+1)^{q-2}\left(\sum_{k=n}^{N}\mu_k a_k(2/q)a^k\right)^q \lesssim \|T_\mu\left(f_{a,N}\right)\|_{H^q}^q.$$

Letting N tend to ∞, we obtain

$$(1-a)\sum_{n=1}^{\infty}(n+1)^{q-2}\left(\sum_{k=n}^{\infty}\mu_k a_k(2/q)a^k\right)^q \lesssim \|T_\mu\left(f_a\right)\|_{H^q}^q.$$

Taking $a = 1 - \frac{1}{N}$ and letting $[N/2]$ denote the largest integer less than or equal to $N/2$, we obtain

$$\begin{aligned}\|T_\mu\left(f_a\right)\|_{H^q}^q &\gtrsim (1-a)\sum_{n=1}^{N}(n+1)^{q-2}\left(\sum_{k=n}^{N}\mu_k a_k(2/q)a^k\right)^q \\ &\gtrsim \frac{\mu_N^q}{N}\sum_{n=1}^{N}n^{q-2}\left(\sum_{k=n}^{N}k^{\frac{2}{q}-1}\right)^q \gtrsim \frac{\mu_N^q}{N}\sum_{n=1}^{[N/2]}n^{q-2}\left(\sum_{k=[N/2]}^{N}k^{\frac{2}{q}-1}\right)^q \\ &\asymp \frac{\mu_N^q}{N}\sum_{n=1}^{[N/2]}n^{q-2}\left(N^{2/q}\right)^q \asymp \mu_N^q N^q. \end{aligned} \tag{11}$$

Using (10), it follows that $\mu_N \lesssim \frac{1}{N}$ and then Lemma 2 implies that μ is a Carleson measure. □

3.2 Proof of Theorem 2

Proof Let us start with the implication (ii) $\Rightarrow$ (i). We shall consider the cases $1 \le p \le 2$ and $2 < p < \infty$ separately.

Suppose first that $1 \le p \le 2$ and $\mathcal{C}_\mu$ is compact from H^p into itself. As in the proof of Theorem 1, for $0 < a < 1$, set

$$f_a(z) = \left(\frac{1-a^2}{(1-az)^2}\right)^{1/p}, \quad z \in \mathbb{D}.$$

We have that $\|f_a\|_{H^p} = 1$ for all a and, also, $f_a \to 0$, as $a \to 1$, uniformly on compact subsets of $\mathbb{D}$. Hence, $\|\mathcal{C}_\mu(f_a)\|_{H^p} \to 0$, as $a \to 1$. But in the course of the proof of the implication (ii) $\Rightarrow$ (i) of Theorem 1, we obtained that $\mu_N N \lesssim \|\mathcal{C}_\mu(f_a)\|_{H^p}$ for $a = 1 - \frac{1}{N}$ (see (9)). Then it follows that $\mu_N = \mathrm{o}\left(\frac{1}{N}\right)$ and this implies that μ is a vanishing Carleson measure.

Suppose now that $2 < p < \infty$ and $\mathcal{C}_\mu$ is compact from H^p into itself. By Theorem 1, μ is a Carleson measure and then it follows that the operator T_μ is well defined on H^q ($\frac{1}{p} + \frac{1}{q} = 1$) and it is the adjoint of $\mathcal{C}_\mu$. For $0 < a < 1$, set $f_a(z) = \left(\frac{1-a^2}{(1-az)^2}\right)^{1/q}$, ($z \in \mathbb{D}$). We have that $\|f_a\|_{H^q} = 1$ for all a and, also, $f_a \to 0$, as $a \to 1$, uniformly on compact subsets of $\mathbb{D}$. By Schauder's theorem [10, p. 174], T_μ is a compact operator from H^q into itself and, hence, $\|T_\mu(f_a)\|_{H^q} \to 0$. In the course of the proof of the implication (ii) $\Rightarrow$ (i) of Theorem 1, we obtained that $\mu_N N \lesssim \|T_\mu(f_a)\|_{H^q}$ for $a = 1 - \frac{1}{N}$ (see (11)). Then it follows that $\mu_N = o\left(\frac{1}{N}\right)$ and, hence, μ is a vanishing Carleson measure.

To prove the other implication we shall consider the cases $p = 2$, $p = 1$, $1 < p < 2$, and $2 < p < \infty$ separately.

Let us start with the case $p = 2$. So assume that μ is a vanishing Carleson measure and let $\{f_n\}$ be a sequence of functions in H^2 with $\|f_n\|_{H^2} \le 1$, for all n, and such that $f_n \to 0$, uniformly on compact subsets of $\mathbb{D}$.

Since μ is a vanishing Carleson measure $\mu_k = o\left(\frac{1}{k}\right)$, as $k \to \infty$. Say

$$\mu_k = \frac{\varepsilon_k}{k+1}, \quad k = 0, 1, 2, \dots .$$

Then $\{\varepsilon_k\} \to 0$. Say that, for every n,

$$f_n(z) = \sum_{k=0}^{\infty} a_k^{(n)} z^k, \quad z \in \mathbb{D}.$$

Since the Cesàro operator $\mathcal{C}$ is bounded on H^2, there exists $M > 0$ such that

$$\|\mathcal{C}(f_n)\|_{H^2}^2 \le M, \quad \text{for all } n. \tag{12}$$

Take $\varepsilon > 0$ and next take a natural number N such that

$$k \ge N \quad \Rightarrow \quad \varepsilon_k^2 < \frac{\varepsilon}{2M}.$$

We have

$$\begin{aligned}
\|\mathcal{C}_\mu(f_n)\|_{H^2}^2 &= \sum_{k=0}^{\infty} \mu_k^2 \left| \sum_{j=0}^{k} a_j^{(n)} \right|^2 \\
&= \sum_{k=0}^{N} \mu_k^2 \left| \sum_{j=0}^{k} a_j^{(n)} \right|^2 + \sum_{k=N+1}^{\infty} \frac{\varepsilon_k^2}{(k+1)^2} \left| \sum_{j=0}^{k} a_j^{(n)} \right|^2 \\
&\le \sum_{k=0}^{N} \mu_k^2 \left| \sum_{j=0}^{k} a_j^{(n)} \right|^2 + \frac{\varepsilon}{2M} \sum_{k=0}^{\infty} \frac{1}{(k+1)^2} \left| \sum_{j=0}^{k} a_j^{(n)} \right|^2 \\
&= \sum_{k=0}^{N} \mu_k^2 \left| \sum_{j=0}^{k} a_j^{(n)} \right|^2 + \frac{\varepsilon}{2M} \|\mathcal{C}(f_n)\|_{H^2}^2
\end{aligned}$$

$$\leq \sum_{k=0}^{N} \mu_k^2 \left| \sum_{j=0}^{k} a_j^{(n)} \right|^2 + \frac{\varepsilon}{2}.$$

Now, since $f_n \to 0$, uniformly on compact subsets of $\mathbb{D}$, it follows that

$$\sum_{k=0}^{N} \mu_k^2 \left| \sum_{j=0}^{k} a_j^{(n)} \right|^2 \to 0, \quad \text{as } n \to \infty.$$

Then it follows that that there exist $n_0 \in \mathbb{N}$ such that $\|\mathcal{C}_\mu(f_n)\|_{H^2}^2 < \varepsilon$ for all $n \geq n_0$. So, we have proved that $\|\mathcal{C}_\mu(f_n)\|_{H^2}^2 \to 0$. The compactness of $\mathcal{C}_\mu$ on H^2 follows.

Let us move to the case $p = 1$. Assume that μ is a vanishing Carleson measure and let $\{f_n\}$ be a sequence of functions in H^1 with $\|f_n\|_{H^1} \leq 1$, for all n, and such that $f_n \to 0$, uniformly on compact subsets of $\mathbb{D}$.

Set

$$g_n(z) = \frac{f_n(z)}{1 - z}, \quad z \in \mathbb{D}, \quad n \in \mathbb{N},$$

and

$$t_k = 1 - \frac{1}{2^k}, \quad k = 0, 1, 2, \ldots$$

As in the proof of the implication (i) $\Rightarrow$ (ii) in Theorem 1 when $p = 1$ we see that, for $0 < r < 1$ and $n \in \mathbb{N}$,

$$M_1\left(r, \mathcal{C}_\mu(f_n)\right) \leq \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{k=1}^{\infty} \left[\sup_{0 \leq t \leq t_k} \left| g_n(rte^{i\theta}) \right| \right] \right) \mu\left([t_{k-1}, t_k]\right) \, d\theta$$

and, hence,

$$\|\mathcal{C}_\mu(f_n)\|_{H^1} \leq \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{k=1}^{\infty} \left[\sup_{0 \leq t \leq t_k} \left| g_n(te^{i\theta}) \right| \right] \right) \mu\left([t_{k-1}, t_k]\right) \, d\theta. \tag{13}$$

Since μ is a vanishing Carleson measure $\mu\left([t_{k-1}, t_k]\right) = \mathrm{o}(2^{-k})$ and, hence, we have

$$\mu\left([t_{k-1}, t_k]\right) = \frac{\varepsilon_k}{2^k}, \quad \text{where } \varepsilon_k \geq 0 \text{and } \{\varepsilon_k\} \to 0.$$

On the other hand, looking at the proof of Theorem 1, we see that there exists $C > 0$ such that

$$\sum_{k=1}^{\infty} \int_{t_k}^{t_{k+1}} M_1(t, g_n) dt \leq C \|f_n\|_{H^1} \leq C, \quad n \in \mathbb{N}. \tag{14}$$

Take $\varepsilon > 0$ and then take $N \in \mathbb{N}$ so that $\varepsilon_k \leq \frac{\varepsilon}{2CK}$, for all $k \geq N$, where K is the constant in the Hardy-Littlewood maximal estimate

$$\frac{1}{2\pi}\int_0^{2\pi}\left[\sup_{0<t<1}|F(te^{i\theta})\right]d\theta \leq K\|F\|_{H^1}.$$

Using (13) we see that

$$\|\mathcal{C}_\mu(f_n)\|_{H^1} \leq I(n) + II(n),$$

where

$$\begin{aligned}
I(n) &= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{k=1}^{N}\left[\sup_{0\leq t\leq t_k}\left|g_n(te^{i\theta})\right|\right]\right)\mu\left([t_{k-1},t_k]\right)d\theta,\\
II(n) &= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{k=N+1}^{\infty}\left[\sup_{0\leq t\leq t_k}\left|g_n(te^{i\theta})\right|\right]\right)\mu\left([t_{k-1},t_k]\right)d\theta.
\end{aligned}$$

Using (14), we obtain

$$\begin{aligned}
II(n) &\leq \sum_{k=N+1}^{\infty}\frac{\varepsilon_k}{2^k}\frac{1}{2\pi}\int_0^{2\pi}\left[\sup_{0\leq t\leq t_k}\left|g_n(te^{i\theta})\right|\right]d\theta\\
&\leq \frac{\varepsilon}{2C}\sum_{k=1}^{\infty}\frac{1}{2^k}M_1(t_k,g_n)\\
&\leq \frac{\varepsilon}{2C}\sum_{k=1}^{\infty}\frac{1}{2^k}\int_{t_k}^{t_{k+1}}M_1(t,g_n)dt\\
&\leq \frac{\varepsilon}{2}.
\end{aligned}$$

Since $f_n \to 0$, uniformly on compact subsets of $\mathbb{D}$, it is clear that $I(n) \to 0$, as $n \to \infty$. Then it follows that there exists $n_0 \in \mathbb{N}$ such that $\|\mathcal{C}_\mu(f_n)\|_{H^1} < \varepsilon$ whenever $n \geq n_0$. Thus,we have shown that $\|\mathcal{C}_\mu(f_n)\|_{H^1} \to 0$, as $n \to \infty$ and the compactness of $\mathcal{C}_\mu$ on H^1 follows.

To deal with the cases $1 < p < 2$ and $2 < p < \infty$, we use again complex interpolation.

Suppose first that $1 < p < 2$ and μ is a vanishing Carleson measure. Recall that

$$H^p = (H^2, H^1)_\theta, \text{ with } \theta = \frac{2}{p} - 1.$$

We have also that if $2 < s < \infty$ then

$$H^2 = \left(H^s, H^1\right)_\alpha$$

for a certain $\alpha \in (0,1)$, namely, $\alpha = \left(\frac{1}{2} - \frac{1}{s}\right) / \left(1 - \frac{1}{s}\right)$. Since H^2 is reflexive, and $\mathcal{C}_\mu$ is compact from H^2 into H^2 and from H^1 into H^1, Theorem 10 of [11] gives that and $\mathcal{C}_\mu$ is compact from H^p into H^p.

Suppose now that $2 < p < \infty$ and μ is a vanishing Carleson measure. Let q be conjugate exponent of p. Take q_1 with $1 < q_1 < q < 2$. We have that T_μ is compact from H^2 into itself and continuous from H^{q_1} into H^{q_1}. Also, $H^q = (H^2, H^{q_1})_\theta$ for a certain $\theta \in (0,1)$. Then, Theorem 10 of [11] gives that and T_μ is compact from H^q into H^q and, hence, $\mathcal{C}_\mu$ is compact from H^p into itself. □

3.3 The operators $\mathcal{C}_\mu$ acting on H^∞

For the constant function 1 we have

$$\mathcal{C}(1)(z) = \frac{1}{z}\log\frac{1}{1-z} = \sum_{n=0}^{\infty}\frac{z^n}{n+1}, \quad z \in \mathbb{D}.$$

Consequently, $\mathcal{C}(H^\infty) \not\subset H^\infty$.

If μ is positive finite Borel measure on $[0,1)$ then

$$\mathcal{C}_\mu(1)(z) = \int_{[0,1)}\frac{d\mu(t)}{1-tz} = \sum_{n=0}^{\infty}\mu_n z^n, \quad z \in \mathbb{D}.$$

So, it follows that

$$\mathcal{C}_\mu(1) \in H^\infty \;\Leftrightarrow\; \int_{[0,1]}\frac{d\mu(t)}{1-t} < \infty \;\Leftrightarrow\; \sum_{n=0}^{\infty}\mu_n < \infty.$$

This easily implies the following result.

Theorem 4 *Let μ be positive finite Borel measure on* $[0,1)$. *Then the following conditions are equivalent.*

(i) $\mathcal{C}_\mu$ *is a bounded operator from* H^∞ *into itself.*
(ii) $\int_{[0,1]}\frac{d\mu(t)}{1-t} < \infty$.
(iii) $\sum_{n=0}^{\infty}\mu_n < \infty$.

Danikas and Siskakis [12] proved that

$$\mathcal{C}(H^\infty) \subset BMOA \subset \mathcal{B}.$$

We extend this result obtaining a characterization of those positive finite Borel measure μ on $[0,1)$ for which $\mathcal{C}_\mu(H^\infty) \subset \mathcal{B}$.

Theorem 5 *Let μ be positive finite Borel measure on* $[0,1)$. *Then the following conditions are equivalent*

(i) $\mathcal{C}_\mu$ *is a bounded operator from* H^∞ *into the Bloch space* $\mathcal{B}$.
(ii) μ *is a Carleson measure.*

Proof Let us start with the implication (i) $\Rightarrow$ (ii). So, assume that $\mathcal{C}_\mu(H^\infty) \subset \mathcal{B}$. Then $\mathcal{C}_\mu(1) \in \mathcal{B}$, but, as we have seen above

$$\mathcal{C}_\mu(1)(z) = \sum_{n=0}^{\infty} \mu_n z^n, \quad z \in \mathbb{D},$$

and then, using the fact that the sequence $\{\mu_n\}$ is a decreasing sequence of nonnegative numbers and Lemma B, we see that $\mu_n = \mathrm{O}\left(\frac{1}{n}\right)$ which is equivalent to saying that μ is a Carleson measure.

Let us turn now to prove the other implication. So, assume that μ is a Carleson measure and take $f \in H^\infty$. Using the integral representation of $\mathcal{C}_\mu$ we see that

$$\mathcal{C}_\mu(f)'(z) = \int_{[0,1)} \frac{tf'(tz)}{1-tz}\, d\mu(t) + \int_{[0,1)} \frac{tf(tz)}{(1-tz)^2}\, d\mu(t), \quad z \in \mathbb{D}.$$

Hence, using that $f \in H^\infty \subset \mathcal{B}$, we obtain

$$\begin{aligned} \left|\mathcal{C}_\mu(f)'(z)\right| &\le \int_{[0,1)} \frac{|f'(tz)|}{|1-tz|}\, d\mu(t) + \int_{[0,1)} \frac{|f(tz)|}{|1-tz|^2}\, d\mu(t) \\ &\lesssim \int_{[0,1)} \frac{d\mu(t)}{(1-|tz|)^2}, \quad z \in \mathbb{D}. \end{aligned} \tag{15}$$

Take $z \in \mathbb{D}$ and set $r = |z|$. Set also

$$\phi(t) = \mu([0,t)) - \mu([0,1)) = -\mu([t,1)), \quad 0 \le t < 1.$$

Integrating by parts and using the fact that μ is a Carleson measure, we obtain

$$\begin{aligned} \int_{[0,1)} \frac{d\mu(t)}{(1-|tz|)^2} &= \int_{[0,1)} \frac{d\mu(t)}{(1-tr)^2} = \mu([0,1)) + 2r\int_0^1 \frac{\mu([t,1))}{(1-tr)^3}\, dt \\ &\lesssim \mu([0,1)) + \int_0^1 \frac{1-t}{(1-tr)^3}\, dt \\ &= \mu([0,1)) + \int_0^r \frac{1-t}{(1-tr)^3}\, dt + \int_r^1 \frac{1-t}{(1-tr)^3}\, dt \\ &\lesssim \mu([0,1)) + \int_0^r \frac{1}{(1-t)^2}\, dt + \frac{1}{(1-r)^3}\int_r^1 (1-t)\, dt \\ &\lesssim \frac{1}{1-r}. \end{aligned}$$

This and (15) yield that $\mathcal{C}_\mu(f) \in \mathcal{B}$. $\square$

It is natural to ask whether or not μ being a Carleson measure implies that $\mathcal{C}_\mu(H^\infty) \subset BMOA$. We do not know the answer to this question.

4 The operators $\mathcal{C}_\mu$ acting on Bergman spaces

The boundedness of the Cesàro operator on Bergman spaces was studied in [1] and [35] where the following result was proved.

Theorem A *If* $p > 0$ *and* $\alpha > -1$*, then the Cesàro operator is bounded from* A^p_α *into itself.*

In the course of our proof of Theorem 1, we proved that if μ is a Carleson measure, $1 \le p < \infty$, and $f \in H^p$, then $M_p(r, \mathcal{C}_\mu(f) \lesssim M_p(r, f)$ (see (4) and (6)). This readily yields that that if μ is a Carleson measure, $1 \le p < \infty$, and $\alpha > -1$, then $\mathcal{C}_\mu$ is bounded from A^p_α into itself.

For $p > 1$ we shall give a different proof of this result and we shall also prove that the converse is true. Hence, our work in particular will lead to a new proof of the boundedness of the classical Cesàro operator on the spaces A^p_α ($1 < p < \infty$, $\alpha > -1$).

Theorem 6 *Suppose that* $1 < p < \infty$ *and* $\alpha > -1$. *Let* μ *be a positive finite Borel measure on* $[0, 1)$.*Then the following conditions are equivalent.*

(i) The measure μ *is a Carleson measure.*
(ii) The operator $\mathcal{C}_\mu$ *is bounded from* A^p_α *into itself.*

Let us collect several results which will be needed in the proof of Theorem 6.

Let us start recalling the given $1 \le p \le \infty$ and $0 < \alpha \le 1$, the mean Lipschitz space Λ^p_α consists of those functions f analytic in $\mathbb{D}$ having a non-tangential limit almost everywhere for which $\omega_p(\delta, f) = O(\delta^\alpha)$, as $\delta \to 0$, where $\omega_p(., f)$ is the integral modulus of continuity of order p of the boundary values $f(e^{i\theta})$ of f. A classical result of Hardy and Littlewood [23] (see also Chapter 5 of [13]) asserts that for $1 \le p \le \infty$ and $0 < \alpha \le 1$, we have that $\Lambda^p_\alpha \subset H^p$ and

$$\Lambda^p_\alpha = \left\{ f \text{ analytic in } \mathbb{D} : M_p(r, f') = \mathrm{O}\left(\frac{1}{(1-r)^{1-\alpha}}\right), \quad \text{as } r \to 1 \right\}.$$

The space Λ^p_α is a Banach space with the norm $\|\cdot\|_{p,\alpha}$ given by

$$\|f\|_{p,\alpha} = |f(0)| + \sup_{0 \le r < 1} (1-r)^{1-\alpha} M_p(r, f').$$

Of special interest are the spaces $\Lambda^p_{1/p}$ since they lie in the border of continuity. Indeed, if $1 < p < \infty$ and $\alpha > \frac{1}{p}$ then each $f \in \Lambda^p_\alpha$ has a continuous extension to the closed unit disc. This is not true for $\alpha = \frac{1}{p}$. This follows easily noticing that the function $f(z) = \log(1-z)$ belongs to $\Lambda^p_{1/p}$ for all $p \in (1, \infty)$. Cima and Petersen proved in [9] that $\Lambda^2_{1/2} \subset BMOA$ and this result was generalized by Bourdon, Shapiro and Sledd who proved in [4] that

$$\Lambda^p_{1/p} \subset BMOA, \quad 1 < p < \infty.$$

This was shown to be sharp in a very strong sense in [3].

The following result of Merchán [30, Lemma 1] (see also [18, Theorem 2] and [17, Theorem 2]) will be needed in our work.

Lemma B *Let $f \in \mathrm{Hol}(\mathbb{D})$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($z \in \mathbb{D}$). Suppose that $1 < p < \infty$ and that the sequence $\{a_n\}$ is a decreasing sequence of nonnegative numbers. If $1 < p < \infty$ and X is a subspace of $\mathrm{Hol}(\mathbb{D})$ with $\Lambda^p_{1/p} \subset X \subset \mathcal{B}$, then*

$$f \in X \quad \Leftrightarrow \quad a_n = \mathrm{O}\left(\frac{1}{n}\right).$$

We shall also use some results on pointwise multipliers and coefficient multipliers of Bergman spaces and Hardy spaces.

Let us start recalling that for $g \in \mathrm{Hol}(\mathbb{D})$, the multiplication operator M_g is defined by

$$M_g(f)(z) \stackrel{\text{def}}{=} g(z)f(z), \quad f \in \mathrm{Hol}(\mathbb{D}),\ z \in \mathbb{D}.$$

If X and Y are two spaces of analytic functions in $\mathbb{D}$ (which will always be assumed to be Banach or F-spaces continuously embedded in $\mathrm{Hol}(\mathbb{D})$) and $g \in \mathrm{Hol}(\mathbb{D})$ then g is said to be a *pointwise multiplier* from X to Y if $M_g(X) \subset Y$. The space of all multipliers from X to Y will be denoted by $M(X, Y)$. Using the closed graph theorem we see that if $g \in M(X, Y)$ then M_g is a bounded operator from X into Y. The following result is a particular case of Theorem C of [37].

Theorem C *Suppose that $1 < p < \infty$ and $\alpha > -1$. Then*

$$M\left(A^p_\alpha, A^{p/(p+1)}_\alpha\right) = A^1_\alpha.$$

If X and Y are two spaces of analytic functions in $\mathbb{D}$, a function $F \in \mathrm{Hol}(\mathbb{D})$ is said to be a *coefficient multiplier* (or a convolution multiplier) from X to Y if

$$f \in X \quad \Rightarrow \quad F \star f \in Y.$$

The following result is due to Duren and Shields, it is a particular case of [15, Theorem 4].

Theorem D *Suppose that $1 < p < \infty$ and $F \in \mathrm{Hol}(\mathbb{D})$. Let m be a positive integer such that $(m+1)^{-1} \le \frac{p}{p+1} < m^{-1}$. Then F is a coefficient multiplier from $H^{p/(p+1)}$ to H^p if and only if the $(m+1)$-th derivative $F^{(m+1)}$ of F satisfies*

$$M_p\left(r, F^{(m+1)}\right) = \mathrm{O}\left((1-r)^{\frac{1}{p}-1-m}\right).$$

We can now proceed to prove Theorem 6.

Proof of the implication (i) $\Rightarrow$ (ii) in Theorem 6. Assume that μ is a Carleson measure and set

$$F(z)=\sum_{n=0}^{\infty}\mu_n z^n, \quad z\in\mathbb{D}.$$

Since μ is a Carleson measure $\mu_n = \mathrm{O}\left(\frac{1}{n}\right)$. This, the simple fact that $\{\mu_n\}$ is a deceasing sequence of nonnegative numbers, and Lemma B imply that $F\in\Lambda^p_{1/p}$ and, hence

$$M_p(r,F')=\mathrm{O}\left((1-r)^{\frac{1}{p}-1}\right).$$

Using [13, Theorem 5.5], we see that this implies

$$M_p(r,F^{(m+1)})=\mathrm{O}\left((1-r)^{\frac{1}{p}-1-m}\right), \quad m=1,2,3,\dots,$$

and then Theorem D gives that F is a coefficient multiplier from $H^{p/(p+1)}$ into H^p. Trivially, this implies that

$$F \text{ is also a coefficient multiplier from } A^{p/(p+1)}_\alpha \text{into } A^p_\alpha. \tag{16}$$

Take $f\in A^p_\alpha$. We have to prove that $\mathcal{C}_\mu(f)\in A^p_\alpha$. Set $g(z)=\frac{f(z)}{1-z}$ $(z\in\mathbb{D})$. A simple computation shows that $\frac{1}{1-z}\in A^1_\alpha$. Then, using Theorem C we deduce that $g\in A^{p/(p+1)}_\alpha$. This and (16) imply that $F\star g\in A^p_\alpha$. By Lemma 1 this is equivalent to saying that $\mathcal{C}_\mu(f)\in A^p_\alpha$. □

Proof of the implication (ii) $\Rightarrow$ (i) in Theorem 6. Suppose that $\mathcal{C}_\mu$ is a bounded operator on A^p_α. Let q be the exponent conjugate to p, that is, $\frac{1}{p}+\frac{1}{q}=1$. Let T_μ be the adjoint of $\mathcal{C}_\mu$, it is a bounded operator on A^q_α.

For $0<b<1$, set

$$f_b(z)=\frac{(1-b)^{1-\frac{1}{q}}}{(1-bz)^{1+\frac{\alpha+1}{q}}}=\sum_{k=0}^{\infty}a_{k,b}z^k, \quad z\in\mathbb{D}.$$

Using [39, Lemma 3.10], we see that

$$\|f_b\|^q_{A^q_\alpha}\asymp 1. \tag{17}$$

Also,

$$a_{k,b}\asymp(1-b)^{1-\frac{1}{q}}k^{(\alpha+1)/q}b^k.$$

For $N \in \mathbb{N}$, set

$$f_{b,N}(z) = \sum_{k=0}^{N} a_{k,b} z^k, \quad z \in \mathbb{D}.$$

Bearing in mind Proposition 2 and Proposition 3, we see that

$$T_\mu(f_{b,N})(z) = \sum_{n=0}^{N} \left(\sum_{k=n}^{N} \mu_k a_{k,b} \right) z^n.$$

Since the coefficients $a_{k,b}$ are nonnegative, it follows that the sequence of the Taylor coefficients of $T_\mu(f_{b,N})$ is a decreasing sequence of nonnegative numbers, then (see, e. g., [20, Proposition 1])

$$\begin{aligned}
\|T_\mu(f_{b,N})\|^q_{A^q_\alpha} &\gtrsim \sum_{n=1}^{N} n^{q-\alpha-3} \left(\sum_{k=n}^{N} \mu_k a_{k,b} \right)^q \\
&\gtrsim (1-b)^{q-1} \sum_{n=1}^{N} n^{q-\alpha-3} \left(\sum_{k=n}^{N} k^{\frac{\alpha+1}{q}} b^k \int_{[b,1)} t^k \, d\mu(t) \right)^q \\
&\gtrsim (1-b)^{q-1} \mu\left([b,1)\right)^q \sum_{n=1}^{N} n^{q-\alpha-3} \left(\sum_{k=n}^{N} k^{\frac{\alpha+1}{q}} b^{2k} \right)^q.
\end{aligned}$$

Since $f_{b,N} \to f_b$ in A^q_α as $N \to \infty$, using the fact that T_μ is bounded on A^q_α, (17), and simple estimations, we deduce that

$$\begin{aligned}
1 &\gtrsim (1-b)^{q-1} \mu\left([b,1)\right)^q \sum_{n=1}^{\infty} n^{q-\alpha-3} \left(\sum_{k=n}^{\infty} k^{\frac{\alpha+1}{q}} b^{2k} \right)^q \\
&\gtrsim (1-b)^{q-1} \mu\left([b,1)\right)^q \sum_{n=1}^{\infty} n^{q-\alpha-3} n^{\alpha+1} \left(\sum_{k=n}^{\infty} b^{2k} \right)^q \\
&\asymp (1-b)^{q-1} \mu\left([b,1)\right)^q \sum_{n=1}^{\infty} n^{q-2} \frac{b^{2nq}}{(1-b)^q} \\
&\asymp \left(\frac{\mu\left([b,1)\right)}{1-b} \right)^q.
\end{aligned}$$

Hence, μ is a Carleson measure. □

5 The operators $\mathcal{C}_\mu$ acting on *BMOA* and on the Bloch space

Let λ be defined by $\lambda(z) = \log \frac{1}{1-z}$ $(z \in \mathbb{D})$. Then $\lambda \in BMOA$. In fact, it is true that $\lambda \in \Lambda^p_{1/p}$ for all $p > 1$. Danikas and Siskakis [12] observed that $\mathcal{C}(\lambda) \notin BMOA$. This implies that the Cesàro operator does not map $BMOA$ into itself. Our Theorem 3 includes a characterization of those μ so that $\mathcal{C}_\mu$ maps $BMOA$ into itself.

Since $\Lambda^2_{1/2} \subset BMOA \subset \mathcal{B}$, Theorem 3 follows from the following result.

Theorem 7 *Let μ be a positive finite Borel measure on $[0, 1)$ and let X and Y be two Banach subspaces of $\mathrm{Hol}(\mathbb{D})$ with $\Lambda^2_{1/2} \subset X \subset \mathcal{B}$ and $\Lambda^2_{1/2} \subset Y \subset \mathcal{B}$. Then the following conditions are equivalent.*

(i) The measure μ is a 1-logarithmic 1-Carleson measure.
(ii) The operator $\mathcal{C}_\mu$ is bounded from X into Y.

Proof Let us start showing that (i) $\Rightarrow$ (ii). So assume that μ is a 1-logarithmic 1-Carleson measure and take $f \in X$. We recall that μ being a 1-logarithmic 1-Carleson measure is equivalent to

$$\mu_n = \mathrm{O}\left(\frac{1}{n \log(n+1)}\right). \tag{18}$$

Take $f \in X$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{D})$. Since $X \subset \mathcal{B}$, we have that $f \in \mathcal{B}$. Then, using a result of Kayumov and Wirths (see [27, Corollary 4] or [28, Corollary D]), we have

$$\left|\sum_{k=0}^{n} a_k\right| \lesssim \|f\|_{\mathcal{B}} \log(n+1). \tag{19}$$

The estimates (18) and (19) yield

$$M_2^2(r, \mathcal{C}_\mu(f)') = \sum_{n=1}^{\infty} n^2 \mu_n^2 \left|\sum_{k=0}^{n} a_k\right|^2 r^{2n-2} \lesssim \sum_{n=1}^{\infty} r^{2n-2} \lesssim \frac{1}{1-r}.$$

Hence $\mathcal{C}_\mu(f) \in \Lambda^2_{1/2} \subset Y$.

Suppose now that $\mathcal{C}_\mu(X) \subset Y$. As above, set $\lambda(z) = \log \frac{1}{1-z} = \sum_{n=1}^{\infty} \frac{z^n}{n}$ $(z \in \mathbb{D})$. We have that $\lambda \in X$ and then $\mathcal{C}_\mu(\lambda) \in Y \subset \mathcal{B}$. Now, $\mathcal{C}_\mu(\lambda)(z) = \sum_{n=1}^{\infty} \mu_n \left(\sum_{k=1}^{n} \frac{1}{k}\right) z^n$ and then it follows that

$$\sum_{n=1}^{\infty} n\mu_n \left(\sum_{k=1}^{n} \frac{1}{k}\right) r^n \lesssim \frac{1}{1-r}, \quad r \in (0, 1).$$

For $N \geq 2$ take $r_N = 1 - \frac{1}{N}$. Bearing in mind that the sequence $\{\mu_n\}$ is decreasing, simple estimations lead us to the following

$$
\begin{aligned}
N^2 \mu_N \log N &\asymp \mu_N \sum_{n=1}^{N} n \log n \\
&\lesssim \sum_{n=1}^{N} n \mu_n (\log n) r_N^n \\
&\lesssim \sum_{n=1}^{N} n \mu_n \left(\sum_{k=1}^{n} \frac{1}{k} \right) r_N^n \\
&\lesssim \sum_{n=1}^{\infty} n \mu_n \left(\sum_{k=1}^{n} \frac{1}{k} \right) r_N^n \\
&\lesssim N.
\end{aligned}
$$

Hence $\mu_N \lesssim \frac{1}{N \log N}$ which implies that μ is a 1-logarithmic 1-Carleson measure. □

We have the following result concerning compactness.

Theorem 8 *Let μ be a positive finite Borel measure on* $[0, 1)$ *and let X and Y be two Banach subspaces of* $\mathrm{Hol}(\mathbb{D})$ *with $\Lambda^2_{1/2} \subset X \subset \mathcal{B}$ and $\Lambda^2_{1/2} \subset Y \subset \mathcal{B}$. Then the following four conditions are equivalent.*

(i) μ is a vanishing 1-*logarithmic* 1-*Carleson measure.*
(ii) The operator $\mathcal{C}_\mu$ is a compact operator from X into Y.
(iii) The operator $\mathcal{C}_\mu$ is a compact operator from the Bloch space $\mathcal{B}$ into itself.
(iv) The operator $\mathcal{C}_\mu$ is a compact operator from the $BMOA$ into itself.

Proof Clearly, it suffices to prove that (i) and (ii) are equivalent. Let us prove first that (i) implies (ii). So, assume that μ is a vanishing 1-logarithmic 1-Carleson measure and $\Lambda^2_{1/2} \subset X, Y \subset \mathcal{B}$.

Take $\{f_j\} \subset X$ with $\|f_j\|_X \leq 1$, for all j, and $f_j \to 0$, as $j \to \infty$, uniformly on compact subsets of $\mathbb{D}$. Since X is continuously embedded in $\mathcal{B}$, $\{f_j\} \subset \mathcal{B}$ and there exists $K_1 > 0$ such that $\|f\|_{\mathcal{B}} \leq K_1$, for all j.

Say $f_j(z) = \sum_{k=0}^{\infty} a_k^{(j)} z^k$ $(z \in \mathbb{D})$. Using the result of Kayumov and Wirths that we have mentioned above, we see that there exists $K_2 > 0$ such that

$$
\left| \sum_{k=0}^{n} a_k^{(j)} \right| \leq K_2 \|f_j\|_{\mathcal{B}} \log(n+1) \leq K_1 K_2 \log(n+1), \quad \text{for all } n \text{ and } j.
$$

Set $K = K_1 K_2$.

Since μ is a vanishing 1-logarithmic 1-Carleson measure, $\mu_n = \mathrm{o}\left(\frac{1}{n \log(n+1)}\right)$. Say $\mu_n = \frac{\varepsilon_n}{n \log(n+1)}$, with $\{\varepsilon_n\} \to 0$. Take $\varepsilon > 0$. Take $N \in \mathbb{N}$ such that $\varepsilon_n^2 K^2 < \frac{\varepsilon}{2}$ if

$n \geq N$. We have, for all $j \in \mathbb{N}$ and $0 < r < 1$,

$$
\begin{aligned}
M_2^2\left(r, \mathcal{C}_\mu(f_j)'\right) &= \sum_{n=1}^{\infty} n^2 \mu_n^2 \left| \sum_{k=0}^{n} a_k^{(j)} \right|^2 r^{2n-2} \\
&\leq \sum_{n=1}^{N} n^2 \mu_n^2 \left| \sum_{k=0}^{n} a_k^{(j)} \right|^2 + \sum_{n=N+1}^{\infty} n^2 \mu_n^2 K^2 [\log(n+1)]^2 r^{2n-2} \\
&\leq \sum_{n=1}^{N} n^2 \mu_n^2 \left| \sum_{k=0}^{n} a_k^{(j)} \right|^2 + \frac{\varepsilon/2}{1-r}.
\end{aligned}
$$

Thus,

$$
\sup_{0 \leq r < 1} (1-r) M_2^2\left(r, \mathcal{C}_\mu(f_j)'\right) \leq \frac{\varepsilon}{2} + \sum_{n=1}^{N} n^2 \mu_n^2 \left| \sum_{k=0}^{n} a_k^{(j)} \right|^2, \quad j \in \mathbb{N}.
$$

Now, since $\sum_{n=1}^{N} n^2 \mu_n^2 \left| \sum_{k=0}^{n} a_k^{(j)} \right|^2 \to 0$ and $f_j(0) \to 0$, as $j \to \infty$, it follows that there exists $j_0 \in \mathbb{N}$ such that

$$
|f_j(0)| + \sum_{n=1}^{N} n^2 \mu_n^2 \left| \sum_{k=0}^{n} a_k^{(j)} \right|^2 < \frac{\varepsilon}{2}
$$

for all $j \geq j_0$. With this we have proved that $\mathcal{C}_\mu(f_j) \to 0$ in $\Lambda^2_{1/2}$. Since $\Lambda^2_{1/2}$ is continuously embedded in Y, it follows that $\mathcal{C}_\mu(f_j) \to 0$ in Y.

Let us prove now that (ii) implies (i). Assume that $\Lambda^2_{1/2} \subset X$, $Y \subset \mathcal{B}$ and that $\mathcal{C}_\mu$ is compact from X into Y. For $0 < a < 1$, set

$$
f_a(z) = \left(\log \frac{2}{1-a}\right)^{-1} \left(\log \frac{2}{1-az}\right)^2, \quad z \in \mathbb{D}.
$$

We have that

$$
f_a'(z) = \left(\log \frac{2}{1-a}\right)^{-1} \left(\log \frac{2}{1-az}\right) \frac{2a}{1-az}, \quad z \in \mathbb{D}, \ 0 < a < 1.
$$

Then it is clear that $f_a \in \Lambda^2_{1/2}$ for all $a \in [0, 1)$ and that there exists a constant $M_1 > 0$ such that $\|f_a\|_{2,1/2} \leq M_1$, for all $a \in (0, 1)$. Since $\Lambda^2_{1/2}$ is continuously embedded in X, it follows that $f_a \in X$ for all $a \in [0, 1)$ and that there exists $M > 0$ such that $\|f_a\|_X \leq M$, for all $a \in (0, 1)$. Also, $f_a \to 0$, as $a \to 1$, uniformly on compact subsets of $\mathbb{D}$. Since $\mathcal{C}_\mu$ is compact from X into Y, we have that $\|\mathcal{C}_\mu(f_a)\|_Y \to 0$, as $a \to 1$. This, together with the fact that Y is continuously embedded in $\mathcal{B}$, implies that

$$\|\mathcal{C}_\mu(f_a)\|_{\mathcal{B}} \to 0, \quad \text{as } a \to 1. \tag{20}$$

A simple calculation gives that for $0 < a < 1$ and $z \in \mathbb{D}$,

$$\mathcal{C}_\mu(f_a)'(z) = \int_{[0,1)} \left[\frac{tf_a'(tz)}{1-tz} + \frac{tf_a(tz)}{(1-tz)^2}\right] d\mu(t).$$

Then it follows that, for $0 < a < 1$,

$$\begin{aligned}
\left|\mathcal{C}_\mu(f_a)'(a)\right| &= \mathcal{C}_\mu(f_a)'(a) \\
&\geq \int_{[0,1)} \frac{tf_a(ta)}{(1-ta)^2} d\mu(t) \\
&= \left(\log \frac{2}{1-a}\right)^{-1} \int_{[0,1)} \frac{t\left(\log \frac{2}{1-ta}\right)^2}{(1-ta)^2} d\mu(t) \\
&\geq \left(\log \frac{2}{1-a}\right)^{-1} \int_{[a,1)} \frac{t\left(\log \frac{2}{1-ta}\right)^2}{(1-ta)^2} d\mu(t) \\
&\geq \left(\log \frac{2}{1-a}\right)^{-1} \mu([a,1)) \frac{a\left(\log \frac{2}{1-a^2}\right)^2}{(1-a^2)^2}.
\end{aligned}$$

This gives that

$$\mu([a,1)) \lesssim (1-a)\left(\log \frac{2}{1-a}\right)^{-1} \|\mathcal{C}_\mu(f_a)\|_{\mathcal{B}}.$$

This and (20) imply that μ is a vanishing 1-logarithmic 1-Carleson measure. □

Acknowledgements We wish to express our gratitude to the referee for his comments and suggestions to improve the paper.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Data Availibility All data generated or analyzed during this study are included in this article and in its bibliography

Declarations

Conflict of interest The authors declare that there is no conflict of interest.

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