



Some properties of certain close-to-convex harmonic mappings

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Abstract

In this paper, we determine the sharp estimates for Toeplitz determinants of a subclass of close-to-convex harmonic mappings. Moreover, we obtain an improved version of Bohr's inequalities for a subclass of close-to-convex harmonic mappings, whose analytic parts are Ma-Minda convex functions.

Keywords Univalent harmonic mappings · Close-to-convex harmonic mappings · Bohr radius · Toeplitz determinant · Ma-Minda convex function

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1 Introduction

A complex-valued function f in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ is called a harmonic mapping if $\Delta f = 4f_{z\bar{z}} = 0$. Let \mathcal{H} denote the class of sense-preserving harmonic

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mappings $f = h + \bar{g}$ in \mathbb{D} , where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=1}^{\infty} b_n z^n \tag{1.1}$$

are analytic functions in \mathbb{D} . Let $\mathcal{S}_{\mathcal{H}}$ be the subclass of \mathcal{H} consisting of univalent mappings. We observe that $\mathcal{S}_{\mathcal{H}}$ reduces to the class \mathcal{S} of normalized univalent analytic functions, if the co-analytic part $g \equiv 0$. Denote by $\mathcal{K}_{\mathcal{H}}$ the close-to-convex subclass of $\mathcal{S}_{\mathcal{H}}$. If $b_1 = 0$, then the class $\mathcal{K}_{\mathcal{H}}$ reduces to $\mathcal{K}_{\mathcal{H}}^0$.

Lewy [37] proved that $f = h + \bar{g}$ is locally univalent in \mathbb{D} if and only if the Jacobian $J_f = |h'|^2 - |g'|^2 \neq 0$ in \mathbb{D} . Noting that the harmonic mapping f is sense-preserving, i.e. $J_f > 0$ or $|h'| > |g'|$ in \mathbb{D} . At this point, its dilatation $\omega_f = g'/h'$ has the property $|\omega_f| < 1$ in \mathbb{D} . The reader can find much information about planar harmonic mappings from [18,22,46].

Let \mathcal{P} denote the class of analytic functions p in \mathbb{D} of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{1.2}$$

such that $\text{Re}(p(z)) > 0$ in \mathbb{D} .

Denote by \mathcal{A} the class of analytic functions in \mathbb{D} with $f(0) = f'(0) - 1 = 0$, and $\mathcal{K}(\alpha)$ denotes the class of functions $f \in \mathcal{A}$ such that

$$\text{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad \left(-\frac{1}{2} \leq \alpha < 1; z \in \mathbb{D} \right). \tag{1.3}$$

Particularly, the elements in $\mathcal{K}(-1/2)$ are close-to-convex but are not necessarily starlike in \mathbb{D} . For $0 \leq \alpha < 1$, the elements in $\mathcal{K}(\alpha)$ are known to be convex functions of order α in \mathbb{D} . For more properties of starlike and convex functions, the reader can refer to the monographs [23,53].

By making use of the subordination in analytic functions, Ma and Minda [42] introduced a more general class $\mathcal{C}(\phi)$, consisting of functions in \mathcal{S} for which

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z).$$

Here the function $\phi : \mathbb{D} \rightarrow \mathbb{C}$, called Ma-Minda function, is analytic and univalent in \mathbb{D} such that $\phi(\mathbb{D})$ has positive real part, symmetric with respect to the real axis, starlike with respect to $\phi(0) = 1$ and $\phi'(0) > 0$ (for more details, see [50,57]). A Ma-Minda function has the form

$$\phi(z) = 1 + \sum_{n=1}^{\infty} B_n z^n.$$

The extremal function K for the class $\mathcal{C}(\phi)$ is given by

$$K(z) = \int_0^z \exp\left(\int_0^\zeta \frac{\phi(t) - 1}{t} dt\right) d\zeta \quad (z \in \mathbb{D}), \tag{1.4}$$

which satisfies the condition

$$1 + \frac{zK''(z)}{K'(z)} = \phi(z).$$

We recall the following natural class of close-to-convex harmonic mappings $\mathcal{M}(\alpha, \zeta, n)$, due to Wang et al. [56] (see also [47,55]).

Definition 1.1 A harmonic mapping $f = h + \bar{g} \in \mathcal{H}$ is said to be in the class $\mathcal{M}(\alpha, \zeta, n)$ if $h \in \mathcal{K}(\alpha)$, for some $\alpha \in [-1/2, 1)$, given by (1.3) and g satisfies the condition

$$g'(z) = \zeta z^n h'(z) \quad \left(\zeta \in \mathbb{C} \text{ with } |\zeta| \leq \frac{1}{2n-1}; n \in \mathbb{N} := \{1, 2, 3, \dots\} \right). \tag{1.5}$$

For $n = 1, \alpha = -1/2$ and $|\zeta| = 1$, the class $\mathcal{M}(-1/2, \zeta, 1)$ was introduced by Bharanedhar and Ponnusamy [12]. For $n = 1$, the class $\mathcal{M}(\alpha, \zeta, 1)$ was studied in [9,52].

In 2021, Allu and Halder [9] introduced and investigated the following subclass $\mathcal{HC}(\phi)$ of close-to-convex harmonic mappings.

Definition 1.2 For $\zeta \in \mathbb{C}$ with $|\zeta| \leq 1$, let $\mathcal{HC}(\phi)$ denote the class of harmonic mappings $f = h + \bar{g}$ in \mathbb{D} of the form (1.1), whose analytic part h belongs to $\mathcal{C}(\phi)$ and $h'(0) \neq 0$, along with the condition $g'(z) = \zeta zh'(z)$.

Motivated essentially by the classes $\mathcal{M}(\alpha, \zeta, n)$ and $\mathcal{HC}(\phi)$, we define a new subclass $\mathcal{HC}_n(\phi)$ of close-to-convex harmonic mappings as follows:

Definition 1.3 A harmonic mapping $f = h + \bar{g} \in \mathcal{H}$ is said to be in the class $\mathcal{HC}_n(\phi)$ if $h \in \mathcal{C}(\phi)$ and g satisfies the condition (1.5).

In 2019, Sun et al. [51] investigated upper bounds of the third Hankel determinants for the class $\mathcal{M}(\alpha, 1, 1)$ of close-to-convex harmonic mappings. In recent years, the Toeplitz determinants and Hankel determinants of functions in the class \mathcal{S} or its subclasses have attracted many researchers' attention (see [11,17,19,20,28,29,33–36]). Among them, the symmetric Toeplitz determinant $|T_q(n)|$ for subclasses of \mathcal{S} with small values of n and q , are investigated by [2,7,10,49,54,58].

The symmetric Toeplitz determinant $T_q(n)$ for analytic functions f is defined as follows:

$$T_q(n)[f] := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n \end{vmatrix},$$

where $n, q \in \mathbb{N}$ and $a_1 = 1$. In particular, for functions in starlike and convex classes, $T_2(2)[f]$, $T_3(1)[f]$ and $T_3(2)[f]$ were studied by Ali *et al.* [7].

Let \mathcal{B} be the class of analytic functions f in \mathbb{D} such that $|f(z)| < 1$ for all $z \in \mathbb{D}$, and let $\mathcal{B}_0 = \{f \in \mathcal{B} : f(0) = 0\}$. In 1914, Bohr [16] proved that if $f \in \mathcal{B}$ is of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then the majorant series $M_f(r) = \sum_{n=0}^{\infty} |a_n| r^n$ of f satisfies

$$M_{f_0}(r) = \sum_{n=1}^{\infty} |a_n| r^n \leq 1 - |a_0| = d(f(0), \partial f(\mathbb{D})) \tag{1.6}$$

for all $z \in \mathbb{D}$ with $|z| = r \leq 1/3$, where $f_0(z) = f(z) - f(0)$. Bohr actually obtained the inequality (1.6) for $|z| \leq 1/6$. Moreover, Wiener, Riesz and Schur, independently, established the Bohr inequality (1.6) for $|z| \leq 1/3$ (known as Bohr radius for the class \mathcal{B}) and proved that $1/3$ is the best possible.

The Bohr phenomenon was reapared in the 1990s due to Dixon [21]. Furthermore, Boas and Khavinson [15] found bounds for Bohr’s radius in any complete Reinhard domains. Other works one can see [3,4,14,44,45]. In recent years, Bohr inequality and Bohr radius have become an active research field in geometric function theory (see [6,8,27,31,38,40,43]). Furthermore, initiated by the work of [32], the Bohr’s phenomenon for the complex-valued harmonic mappings have been widely studied (see [1,9,25,26,30,41]).

In this paper, we aim at determining the sharp estimates for Toeplitz determinants of the class $\mathcal{M}(\alpha, \zeta, n)$. Moreover, we will derive an improved version of Bohr’s inequalities for the class $\mathcal{HC}_n(\phi)$.

2 Preliminary results

To prove our main results, we need the following lemmas.

Lemma 2.1 ([23, p. 41]) *For a function $p \in \mathcal{P}$ of the form (1.2), the sharp inequality $|p_n| \leq 2$ holds for each $n \geq 1$. Equality holds for the function*

$$p(z) = \frac{1+z}{1-z}.$$

Lemma 2.2 ([24, Theorem 1]) *Let $p \in \mathcal{P}$ be of the form (1.2) and $\mu \in \mathbb{C}$. Then*

$$|p_n - \mu p_k p_{n-k}| \leq 2 \max\{1, |2\mu - 1|\} \quad (1 \leq k \leq n - 1).$$

If $|2\mu - 1| \geq 1$, then the inequality is sharp for the function

$$p(z) = \frac{1+z}{1-z}$$

or its rotations. If $|2\mu - 1| < 1$, then the inequality is sharp for

$$p(z) = \frac{1+z^n}{1-z^n}$$

or its rotations.

Lemma 2.3 ([56]) *Let $f = h + \bar{g} \in \mathcal{M}(\alpha, \zeta, n)$. Then the coefficients a_k ($k \in \mathbb{N} \setminus \{1\}$) of h satisfy*

$$|a_k| \leq \frac{1}{k!} \prod_{j=2}^k (j - 2\alpha) \quad (k \in \mathbb{N} \setminus \{1\}). \tag{2.1}$$

Moreover, the coefficients b_k ($k = n + 1, n + 2, \dots; n \in \mathbb{N}$) of g satisfy

$$|b_{n+1}| \leq \frac{|\zeta|}{n + 1} \quad \text{and} \quad |b_{k+n}| \leq \frac{|\zeta|}{(k + n)(k - 1)!} \prod_{j=2}^k (j - 2\alpha) \quad (k \in \mathbb{N} \setminus \{1\}; n \in \mathbb{N}). \tag{2.2}$$

The bounds are sharp for the extremal function given by

$$f(z) = \int_0^z \frac{dt}{(1 - \delta t)^{2-2\alpha}} + \overline{\int_0^z \frac{\zeta t^n}{(1 - \delta t)^{2-2\alpha}} dt} \quad (|\delta| = 1; z \in \mathbb{D}). \tag{2.3}$$

Lemma 2.4 ([56]) *Let $f \in \mathcal{M}(\alpha, \zeta, n)$ with $0 \leq \alpha < 1$ and $0 \leq \zeta < \frac{1}{2n-1}$ ($n \in \mathbb{N}$). Then*

$$\Phi(r; \alpha, \zeta, n) \leq |f(z)| \leq \Psi(r; \alpha, \zeta, n) \quad (r = |z| < 1), \tag{2.4}$$

where

$$\Phi(r; \alpha, \zeta, n) = \begin{cases} \log(1 + r) - \frac{\zeta r^{n+1} {}_2F_1(1, n + 1; n + 2; -r)}{n + 1} & (\alpha = 1/2), \\ \frac{(1 + r)^{2\alpha-1} - 1}{2\alpha - 1} - \frac{\zeta r^{n+1} {}_2F_1(n + 1, 2 - 2\alpha; n + 2; -r)}{n + 1} & (\alpha \neq 1/2), \end{cases}$$

and

$$\Psi(r; \alpha, \zeta, n) = \begin{cases} -\log(1 - r) + \frac{\zeta r^{n+1} {}_2F_1(1, n + 1; n + 2; r)}{n + 1} & (\alpha = 1/2), \\ \frac{1 - (1 - r)^{2\alpha-1}}{2\alpha - 1} + \frac{\zeta r^{n+1} {}_2F_1(n + 1, 2 - 2\alpha; n + 2; r)}{n + 1} & (\alpha \neq 1/2). \end{cases}$$

All these bounds are sharp, the extremal function is $f_{\alpha,\zeta,n} = h_{\alpha} + \overline{g_{\alpha,\zeta,n}}$ or its rotations, where

$$f_{\alpha,\zeta,n}(z) = \begin{cases} -\log(1-z) + \frac{\zeta z^{n+1} {}_2F_1(1, n+1; n+2; z)}{n+1} & (\alpha = 1/2), \\ \frac{1 - (1-z)^{2\alpha-1}}{2\alpha-1} + \frac{\zeta z^{n+1} {}_2F_1(n+1, 2-2\alpha; n+2; z)}{n+1} & (\alpha \neq 1/2). \end{cases} \tag{2.5}$$

The following two results are due to Ma and Minda [42].

Lemma 2.5 *Let $f \in \mathcal{C}(\phi)$. Then $zf''(z)/f'(z) \prec zK''(z)/K'(z)$ and $f'(z) \prec K'(z)$, where K is given by (1.4).*

Lemma 2.6 *Assume that $f \in \mathcal{C}(\phi)$ and $|z| = r < 1$. Then*

$$K'(-r) \leq |f'(z)| \leq K'(r), \tag{2.6}$$

where K is given by (1.4). Equality holds for some $z \neq 0$ if and only if f is a rotation of K .

Lemma 2.7 ([13]) *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be two analytic functions in \mathbb{D} and $g \prec f$. Then*

$$\sum_{n=0}^{\infty} |b_n| r^n \leq \sum_{n=0}^{\infty} |a_n| r^n \tag{2.7}$$

for $|z| = r \leq 1/3$.

Remark 2.1 Lemma 2.7 continues to hold for quasi-subordination (cf. [5]). Moreover, the bound $1/3$ is optimal as shown by [48, Lemma 1].

3 Toeplitz determinants for the class $\mathcal{M}(\alpha, \zeta, n)$

In this section, we will give several sharp estimates for Toeplitz determinants $|T_q(n)[\cdot]|$ of functions in the class $\mathcal{M}(\alpha, \zeta, n)$.

Theorem 3.1 *Let $f \in \mathcal{M}(\alpha, \zeta, n)$. Then*

$$|T_2(n)[h]| \leq \left(\frac{1}{n!} \prod_{j=2}^n (j - 2\alpha) \right)^2 + \left(\frac{1}{(n+1)!} \prod_{j=2}^{n+1} (j - 2\alpha) \right)^2 \quad (n \in \mathbb{N} \setminus \{1\}), \tag{3.1}$$

and

$$|T_2(n)[g]| \leq \frac{1}{[(2n-1)(n+1)]^2}. \tag{3.2}$$

The inequalities in (3.1) and (3.2) are sharp for the extremal function given by (2.3).

Proof Suppose that $f \in \mathcal{M}(\alpha, \zeta, n)$. By Lemma 2.3, we see that

$$|T_2(n)[h]| = |a_n^2 - a_{n+1}^2| \leq |a_n^2| + |a_{n+1}^2| \tag{3.3}$$

yields (3.1). Equality in (3.3) holds for the function h given by

$$\begin{aligned} h(z) &= \int_0^z \frac{dt}{(1 - \delta t)^{2-2\alpha}} \\ &= z + \frac{1}{2}(2 - 2\alpha)\delta^2 z^2 + \frac{1}{6}(2 - 2\alpha)(3 - 2\alpha)\delta^3 z^3 \\ &\quad + \frac{1}{24}(2 - 2\alpha)(3 - 2\alpha)(4 - 2\alpha)\delta^4 z^4 \\ &\quad + \frac{1}{120}(2 - 2\alpha)(3 - 2\alpha)(4 - 2\alpha)(5 - 2\alpha)\delta^5 z^5 + \dots \quad (|\delta| = 1; z \in \mathbb{D}). \end{aligned} \tag{3.4}$$

By virtue of (2.2), we get the assertion (3.2). The proof of Theorem 3.1 is thus completed. \square

Corollary 3.1 *Let $f \in \mathcal{M}(\alpha, \zeta, 2)$. Then*

$$|T_2(2)[h]| \leq \frac{2}{9}(1 - \alpha)^2 (2\alpha^2 - 6\alpha + 9), \tag{3.5}$$

and

$$|T_2(2)[g]| \leq \frac{1}{81}. \tag{3.6}$$

The inequalities in (3.5) and (3.6) are sharp for the extremal function given by (2.3) with $n = 2$.

Theorem 3.2 *Let $f \in \mathcal{M}(\alpha, \zeta, 1)$. Then*

$$|T_3(1)[h]| \leq \begin{cases} \frac{1}{9}(8\alpha^4 - 34\alpha^3 + 71\alpha^2 - 72\alpha + 36) & (-\frac{1}{2} \leq \alpha \leq \frac{1}{2}), \\ \frac{1}{9}(-2\alpha^3 + 25\alpha^2 - 44\alpha + 30) & (\frac{1}{2} \leq \alpha < 1), \end{cases} \tag{3.7}$$

and

$$|T_3(1)[g]| \leq \frac{1}{3}(1 - \alpha). \tag{3.8}$$

The inequality in (3.7) is sharp for the function h given by (3.4), and the inequality in (3.8) is sharp for the function g defined by

$$g(z) = \int_0^z \frac{\zeta t}{(1 - \delta t)^{2-2\alpha}} dt \quad (|\delta| = 1; |\zeta| \leq 1; z \in \mathbb{D}). \tag{3.9}$$

Proof For $f \in \mathcal{M}(\alpha, \zeta, 1)$, we see that

$$p(z) = \frac{1}{1 - \alpha} \left(1 + \frac{zh''(z)}{h'(z)} - \alpha \right) \in \mathcal{P} \quad \left(-\frac{1}{2} \leq \alpha < 1; z \in \mathbb{D} \right).$$

It follows that

$$n(n - 1)a_n = (1 - \alpha) \sum_{k=1}^{n-1} ka_k p_{n-k} \quad (n \geq 2). \tag{3.10}$$

From (3.10), we obtain

$$\begin{cases} a_2 = \frac{1}{2}(1 - \alpha)p_1, \\ a_3 = \frac{1}{6}(1 - \alpha) [(1 - \alpha)p_1^2 + p_2], \\ a_4 = \frac{1}{24}(1 - \alpha) [(1 - \alpha)^2 p_1^3 + 3(1 - \alpha)p_1 p_2 + 2p_3]. \end{cases} \tag{3.11}$$

By virtue of Lemma 2.2 and (3.11), we get

$$\begin{aligned} |T_3(1)[h]| &= \left| 1 - 2a_2^2 + 2a_2^2 a_3 - a_3^2 \right| \\ &\leq 1 + 2 \left| a_2^2 \right| + |a_3| \left| a_3 - 2a_2^2 \right| \\ &\leq 1 + \frac{1}{2}(1 - \alpha)^2 p_1^2 + \frac{1}{36}(1 - \alpha)^2 |(1 - \alpha)p_1^2 + p_2| |p_2 - 2(1 - \alpha)p_1^2| \\ &\leq \begin{cases} \frac{1}{9}(8\alpha^4 - 34\alpha^3 + 71\alpha^2 - 72\alpha + 36) & (-\frac{1}{2} \leq \alpha \leq \frac{1}{2}), \\ \frac{1}{9}(-2\alpha^3 + 25\alpha^2 - 44\alpha + 30) & (\frac{1}{2} \leq \alpha < 1). \end{cases} \end{aligned} \tag{3.12}$$

In what follows, we shall prove that the equality in (3.12) holds for the function h given by (3.4). It follows from (3.4) that

$$\begin{cases} |a_2| = 1 - \alpha, \\ |a_3| = \frac{1}{3}(1 - \alpha)(3 - 2\alpha). \end{cases} \tag{3.13}$$

Therefore, we obtain

$$\begin{aligned} |T_3(1)[h]| &= \left| 1 - 2a_2^2 + 2a_2^2 a_3 - a_3^2 \right| \\ &\leq 1 + 2 \left| a_2^2 \right| + |a_3| \left| a_3 - 2a_2^2 \right| \\ &= 1 + 2(1 - \alpha)^2 + \frac{1}{3}(1 - \alpha)(3 - 2\alpha) \left| \frac{1}{3}(1 - \alpha)(3 - 2\alpha) - 2(1 - \alpha)^2 \right| \\ &= \begin{cases} \frac{1}{9}(8\alpha^4 - 34\alpha^3 + 71\alpha^2 - 72\alpha + 36) & (-\frac{1}{2} \leq \alpha \leq \frac{1}{2}), \\ \frac{1}{9}(-2\alpha^3 + 25\alpha^2 - 44\alpha + 30) & (\frac{1}{2} \leq \alpha < 1). \end{cases} \end{aligned} \tag{3.14}$$

The graph of $|T_3(1)[h]|$ with $\alpha \in [-1/2, 1)$ is presented as Fig. 1.

By the power series representations of h and g for $f = h + \bar{g} \in \mathcal{M}(\alpha, \zeta, 1)$, we find that

$$(k + 1)b_{k+1} = \zeta ka_k \quad (k \in \mathbb{N}; |\zeta| \leq 1; a_1 = 1),$$

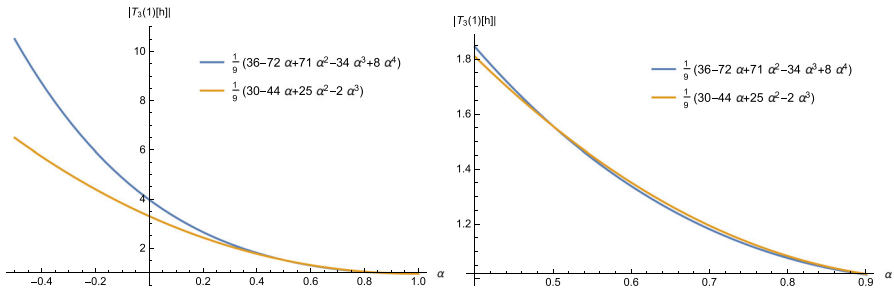


Fig. 1 The left graph is $|T_3(1)[h]|$ of $\frac{1}{9}(8\alpha^4 - 34\alpha^3 + 71\alpha^2 - 72\alpha + 36)$ and $\frac{1}{9}(-2\alpha^3 + 25\alpha^2 - 44\alpha + 30)$, respectively; The right graph is a locally enlarged version

which implies that

$$\begin{cases} b_2 = \frac{1}{2}\zeta a_1, \\ b_3 = \frac{2}{3}\zeta a_2. \end{cases} \tag{3.15}$$

Thus, by Lemma 2.1, (3.11) and (3.15), we deduce that the assertion (3.8) of Theorem 3.2 is true. The sharpness of (3.8) follows from (3.7). \square

Theorem 3.3 *Let $f \in \mathcal{M}(\alpha, \zeta, 2)$. Then*

$$|T_3(2)[h]| \leq \begin{cases} \frac{1}{108} (1 - \alpha)^3 (2\alpha^2 - 7\alpha + 12)(10\alpha^2 - 27\alpha + 36) & (-\frac{1}{2} \leq \alpha \leq \frac{1}{7}), \\ \frac{5}{108} (1 - \alpha)^3 (2\alpha^2 - 7\alpha + 12)(2\alpha^2 - 4\alpha + 7) & (\frac{1}{7} \leq \alpha < 1), \end{cases} \tag{3.16}$$

and

$$|T_3(2)[g]| = |2b_3^2 b_4| \leq \frac{1}{243} (1 - \alpha). \tag{3.17}$$

The inequality in (3.16) is sharp for the function h given by (3.4), and the inequality in (3.17) is sharp for the function g defined by

$$g(z) = \int_0^z \frac{\zeta t^2}{(1 - \delta t)^{2-2\alpha}} dt \quad \left(|\delta| = 1; |\zeta| \leq \frac{1}{3}; z \in \mathbb{D} \right). \tag{3.18}$$

Proof Suppose that $f \in \mathcal{M}(\alpha, \zeta, 2)$. It follows that

$$T_3(2)[h] = (a_2 - a_4) (a_2^2 - 2a_3^2 + a_2 a_4).$$

In view of (3.11) and Lemma 2.1, we find that

$$\begin{aligned}
 |a_2 - a_4| &\leq |a_2| + |a_4| \\
 &\leq \left| \frac{1}{2}(1 - \alpha)p_1 \right| + \left| \frac{1}{24}(1 - \alpha) \left[(1 - \alpha)^2 p_1^3 + 3(1 - \alpha)p_1 p_2 + 2p_3 \right] \right| \\
 &\leq \frac{1}{6} (1 - \alpha)(2\alpha^2 - 7\alpha + 12).
 \end{aligned}
 \tag{3.19}$$

Next, we shall maximize $|a_2^2 - 2a_3^2 + a_2 a_4|$. With the help of (3.11), Lemma 2.1 and Lemma 2.2, we get

$$\begin{aligned}
 |a_2^2 - 2a_3^2 + a_2 a_4| &= \frac{(1 - \alpha)^2}{144} | -5(1 - \alpha)^2 p_1^4 + 36p_1^2 - 7(1 - \alpha)p_1^2 p_2 - 8p_2^2 + 6p_1 p_3 | \\
 &\leq \frac{(1 - \alpha)^2}{144} \left[5(1 - \alpha)^2 |p_1|^4 + 36|p_1|^2 + 8|p_2|^2 + 6|p_1| \left| p_3 - \frac{7}{6}(1 - \alpha)p_1 p_2 \right| \right] \\
 &\leq \begin{cases} \frac{1}{18}(1 - \alpha)^2(10\alpha^2 - 27\alpha + 36) & (-\frac{1}{2} \leq \alpha \leq \frac{1}{7}), \\ \frac{5}{18}(1 - \alpha)^2(2\alpha^2 - 4\alpha + 7) & (\frac{1}{7} \leq \alpha < 1). \end{cases}
 \end{aligned}
 \tag{3.20}$$

Therefore, combining (3.19) with (3.20), we obtain (3.16). By noting that for $f \in \mathcal{M}(\alpha, \zeta, 2)$, we have

$$\begin{cases} b_3 = \frac{1}{3}\zeta a_1, \\ b_4 = \frac{1}{2}\zeta a_2. \end{cases}
 \tag{3.21}$$

By means of Lemma 2.1, we get the assertion (3.17). The sharpness of (3.16) and (3.17) are similar to that of Theorem 3.2, we choose to omit the details here. \square

Remark 3.1 By setting $\alpha = 0$ in Corollary 3.1, Theorem 3.2 and Theorem 3.3, respectively, we get $|T_2(2)[h]| \leq 2$, $|T_3(1)[h]| \leq 4$ and $|T_3(2)[h]| \leq 4$. The bounds for convex functions were recently proved by Ali et al. [7, Theorem 2.11].

4 Bohr inequality for the class $\mathcal{HC}_n(\phi)$

In this section, we firstly give the sharp growth estimate for the class $\mathcal{HC}_n(\phi)$.

Proposition 4.1 *Let $f \in \mathcal{HC}_n(\phi)$. Then*

$$L(\zeta, n, r) \leq |f(z)| \leq R(\zeta, n, r),
 \tag{4.1}$$

where

$$L(\zeta, n, r) = -K(-r) - |\zeta| \int_0^r t^n K'(-t) dt,
 \tag{4.2}$$

and

$$R(\zeta, n, r) = K(r) + |\zeta| \int_0^r t^n K'(t) dt.
 \tag{4.3}$$

The bounds are sharp for the extremal function $f_\zeta = h_\zeta + \overline{g_\zeta}$ with $h_\zeta = K$, where K satisfies (1.4) or its rotations and g_ζ satisfies $g'_\zeta = \zeta z^n h'_\zeta$.

Proof Let $f = h + \overline{g} \in \mathcal{HC}_n(\phi)$. By Lemma 2.6, we know that

$$K'(-r) \leq |h'(z)| \leq K'(r) \quad (|z| = r). \tag{4.4}$$

Let γ be the linear segment joining 0 to z in \mathbb{D} . Then, we see that

$$\begin{aligned} |f(z)| &= \left| \int_\gamma \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial \overline{\theta}} d\overline{\theta} \right| \leq \int_\gamma (|h'(\theta)| + |g'(\theta)|) |d\theta| \\ &= \int_\gamma (1 + |\zeta||\theta|^n) |h'(\theta)| |d\theta|. \end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5), we obtain

$$|f(z)| \leq \int_0^r (1 + |\zeta|t^n) K'(t) dt = K(r) + |\zeta| \int_0^r t^n K'(t) dt = R(\zeta, n, r). \tag{4.6}$$

Let Γ be the preimage of the line segment joining 0 to $f(z)$ under the function f . It follows that

$$\begin{aligned} |f(z)| &= \left| \int_\Gamma \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial \overline{\theta}} d\overline{\theta} \right| \geq \int_\Gamma (|h'(\theta)| - |g'(\theta)|) |d\theta| \\ &= \int_\Gamma (1 - |\zeta||\theta|^n) |h'(\theta)| |d\theta|. \end{aligned} \tag{4.7}$$

From (4.4) and (4.7), we have

$$|f(z)| \geq \int_0^r (1 - |\zeta|t^n) K'(-t) dt = -K(-r) - |\zeta| \int_0^r t^n K'(-t) dt = L(\zeta, n, r). \tag{4.8}$$

In view of (4.6) and (4.8), we deduce that

$$L(\zeta, n, r) \leq |f(z)| \leq R(\zeta, n, r). \tag{4.9}$$

To show the sharpness, we consider the function $f_\zeta = h_\zeta + \overline{g_\zeta}$ with $h_\zeta = K$ or its rotations. It is easy to see that $h_\zeta = K \in \mathcal{C}(\phi)$ and g_ζ satisfies $g'_\zeta(z) = \zeta z^n h'_\zeta(z)$, which shows that $f_\zeta \in \mathcal{HC}_n(\phi)$. The equality holds on both sides of (4.4) for suitable rotations of K . For $0 \leq \zeta < 1/(2n - 1)$, we see that $f_\zeta(r) = R(\zeta, n, r)$ and $f_\zeta(-r) = -L(\zeta, n, r)$. Hence $|f_\zeta(r)| = R(\zeta, n, r)$ and $|f_\zeta(-r)| = L(\zeta, n, r)$. This completes the proof of Proposition 4.1. \square

Proposition 4.2 *Let $f \in \mathcal{HC}_n(\phi)$ and S_r be the area of the image $f(\mathbb{D}_r)$ ($\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r \leq 1\}$). Then*

$$2\pi \int_0^r t \left(1 - |\zeta|^2 t^{2n}\right) (K'(-t))^2 dt \leq S_r \leq 2\pi \int_0^r t \left(1 - |\zeta|^2 t^{2n}\right) (K'(t))^2 dt. \tag{4.10}$$

Proof Let $f = h + \bar{g} \in \mathcal{HC}_n(\phi)$. Then, the area of image of \mathbb{D}_r under a harmonic mapping f is given by

$$S_r = \iint_{\mathbb{D}_r} \left(|h'(z)|^2 - |g'(z)|^2\right) dx dy = \iint_{\mathbb{D}_r} \left(1 - |\zeta|^2 |z|^{2n}\right) |h'(z)|^2 dx dy. \tag{4.11}$$

Since $h \in \mathcal{C}(\phi)$, in view of (4.4) and (4.11), we have

$$\int_0^r \int_0^{2\pi} t \left(1 - |\alpha|^2 t^2\right) (K'(-t))^2 d\theta dt \leq S_r \leq \int_0^r \int_0^{2\pi} t \left(1 - |\alpha|^2 t^2\right) (K'(t))^2 d\theta dt. \tag{4.12}$$

Therefore, the assertion (4.10) of Proposition 4.2 follows directly from (4.12). \square

In what follows, we derive the Bohr inequality for the class $\mathcal{HC}_n(\phi)$.

Theorem 4.1 *Let $f \in \mathcal{HC}_n(\phi)$. Then the majorant series of f satisfies the inequality*

$$|z| + \sum_{n=2}^{\infty} (|a_n| + |b_n|) |z|^n \leq d(f(0), \partial f(\mathbb{D})) \tag{4.13}$$

for $|z| = r \leq \min\{1/3, r_f\}$, where r_f is the smallest positive root in $(0, 1)$ of

$$L(\zeta, n, 1) = M_K(r) + |\zeta| \int_0^r t^n M_{K'}(t) dt,$$

and $L(\zeta, n, 1)$ is given by (4.2) with $r = 1$.

Proof Let $f = h + \bar{g} \in \mathcal{HC}_n(\phi)$. Since $h \in \mathcal{C}(\phi)$, from Lemma 2.5, we know that

$$h' \prec K'. \tag{4.14}$$

Let $K(z) = z + \sum_{n=2}^{\infty} k_n z^n$. In view of Lemma 2.7 and (4.14), we have

$$1 + \sum_{n=2}^{\infty} n |a_n| r^{n-1} = M_{h'}(r) \leq M_{K'}(r) = 1 + \sum_{n=2}^{\infty} n |k_n| r^{n-1} \tag{4.15}$$

for $|z| = r \leq 1/3$. By integrating (4.15) with respect to r from 0 to r , we get

$$M_h(r) = r + \sum_{n=2}^{\infty} |a_n| r^n \leq r + \sum_{n=2}^{\infty} |k_n| r^n = M_K(r) \quad (r \leq 1/3). \tag{4.16}$$

From the definition of $\mathcal{HC}_n(\phi)$, we know that

$$g'(z) = \zeta z^n h'(z).$$

This relationship along with (4.15) yields

$$\sum_{n=2}^{\infty} n|b_n|r^{n-1} = M_{g'}(r) = |\zeta|r^n M_{h'}(r) \leq |\zeta|r^n M_{K'}(r) \quad (r \leq 1/3). \quad (4.17)$$

By integrating (4.17) with respect to r from 0 to r , it follows that

$$M_g(r) = \sum_{n=2}^{\infty} |b_n|r^n \leq |\zeta| \int_0^r t^n M_{K'}(t) dt \quad (r \leq 1/3). \quad (4.18)$$

Therefore, for $|z| = r \leq 1/3$, from (4.16) and (4.18), we obtain

$$M_f(r) = |z| + \sum_{n=2}^{\infty} (|a_n| + |b_n|)r^n \leq M_K(r) + |\zeta| \int_0^r t^n M_{K'}(t) dt = R_C(n, r). \quad (4.19)$$

In view of (4.1), it is evident that the Euclidean distance between $f(0)$ and the boundary of $f(\mathbb{D})$ is given by

$$d(f(0), \partial f(\mathbb{D})) = \liminf_{|z| \rightarrow 1} |f(z) - f(0)| \geq L(\zeta, n, 1). \quad (4.20)$$

We note that $R_C(n, r) \leq L(\zeta, n, 1)$ whenever $r \leq r_f$, where r_f is the smallest positive root of $R_C(n, r) = L(\zeta, n, 1)$ in $(0, 1)$. Let

$$H_1(n, r) = R_C(n, r) - L(\zeta, n, 1).$$

Then $H_1(n, r)$ is a continuous function in $[0, 1]$. Since

$$M_K(r) \geq K(r) > -K(-r),$$

it follows that

$$\begin{aligned} H_1(n, 1) &= R_C(n, 1) - L(\zeta, n, 1) \\ &= M_K(1) + K(-1) + |\zeta| \int_0^1 t^n (M_{K'}(t) + K'(t)) dt \\ &\geq K(1) + K(-1) + |\zeta| \int_0^1 t^n (M_{K'}(t) + K'(t)) dt > 0. \end{aligned} \quad (4.21)$$

On the other hand,

$$H_1(n, 0) = -L(\zeta, n, 1) = K(-1)(1 - |\zeta|) + n|\zeta| \int_0^1 t^{n-1} K(-t) dt < 0. \quad (4.22)$$

Therefore, H_1 has a root in $(0, 1)$. Let r_f be the smallest root of H_1 in $(0, 1)$. Then $R_C(n, r) \leq L(\zeta, n, 1)$ for $r \leq r_f$. Now, in view of the inequalities (4.19) and (4.20) with the relationship $R_C(n, r) \leq L(\zeta, n, 1)$ for $r \leq r_f$, we obtain

$$|z| + \sum_{n=2}^{\infty} (|a_n| + |b_n|)r^n \leq d(f(0), \partial f(\mathbb{D}))$$

for $|z| = r \leq \min\{1/3, r_f\}$. □

For a particular choice of ϕ in Theorem 4.1, we get the following result.

Corollary 4.1 *Let $f \in \mathcal{M}(\alpha, \zeta, n)$ with $0 \leq \alpha < 1$ and $0 \leq \zeta < 1/(2n - 1)$. Then the inequality (4.13) holds for $|z| = r \leq r_f$, where r_f is the smallest root in $(0, 1)$ of*

$$F_n(r) := R(\alpha, \zeta, n, r) - L(\alpha, \zeta, n, 1) = 0.$$

The radius r_f is sharp.

Proof From Lemma 2.4, the Euclidean distance between $f(0)$ and the boundary of $f(\mathbb{D})$ shows that

$$d(f(0), \partial f(\mathbb{D})) = \liminf_{|z| \rightarrow 1} |f(z) - f(0)| \geq L(\alpha, \zeta, n, 1). \tag{4.23}$$

We note that r_f is the root of the equation $R(\alpha, \zeta, n, r) = L(\alpha, \zeta, n, 1)$ in $(0, 1)$. The existence of the root is ensured by the relation $R(\alpha, \zeta, n, 1) > L(\alpha, \zeta, n, 1)$ with (2.4). For $0 < r \leq r_f$, it is evident that $R(\alpha, \zeta, n, r) \leq L(\alpha, \zeta, n, 1)$. In view of Lemma 2.3 and (4.23), for $|z| = r \leq r_f$, we have

$$\begin{aligned} |z| + \sum_{n=2}^{\infty} (|a_n| + |b_n|)|z|^n &\leq r_f + (|a_2| + |b_2|)r_f^2 + \sum_{n=3}^{\infty} (|a_n| + |b_n|)r_f^n \\ &= R(\alpha, \zeta, n, r_f) \leq L(\alpha, \zeta, n, 1) \leq d(f(0), \partial f(\mathbb{D})). \end{aligned}$$

To show the sharpness of the radius r_f , we consider the function $f = f_{\alpha, \zeta, n}$, which is defined in Lemma 2.4. We see that $f_{\alpha, \zeta, n}$ belongs to $\mathcal{M}(\alpha, \zeta, n)$. Since the left side of the growth inequality in Lemma 2.4 holds for $f = f_{\alpha, \zeta, n}$ or its rotations, we have $d(f(0), \partial f(\mathbb{D})) = L(\alpha, \zeta, n, 1)$. Therefore, the function $f = f_{\alpha, \zeta, n}$ for $|z| = r_f$ gives

$$\begin{aligned} |z| + \sum_{n=2}^{\infty} (|a_n| + |b_n|)|z|^n &= r_f + (|a_2| + |b_2|)r_f^2 + \sum_{n=3}^{\infty} (|a_n| + |b_n|)r_f^n \\ &= R(\alpha, \zeta, n, r_f) = L(\alpha, \zeta, n, 1) = d(f(0), \partial f(\mathbb{D})), \end{aligned}$$

which reveals that the radius r_f is the best possible. □

Table 1 The roots r_f of $F_n(r) = 0$ for different values of ζ when $\alpha = 0.5$

n	1	2	3	4	5	10	100	1000
ζ	1/2	1/4	1/6	1/8	1/10	1/20	1/200	1/2000
r_f	0.386555	0.468176	0.486196	0.492459	0.495252	0.498809	0.499988	0.500000

Table 2 The roots r_f of $F_n(r) = 0$ for different values of ζ when $\alpha = 0.9$

n	1	2	3	4	5	10	100	1000
ζ	1/2	1/4	1/6	1/8	1/10	1/20	1/200	1/2000
r_f	0.567721	0.731273	0.774894	0.792253	0.800709	0.812036	0.815292	0.815323

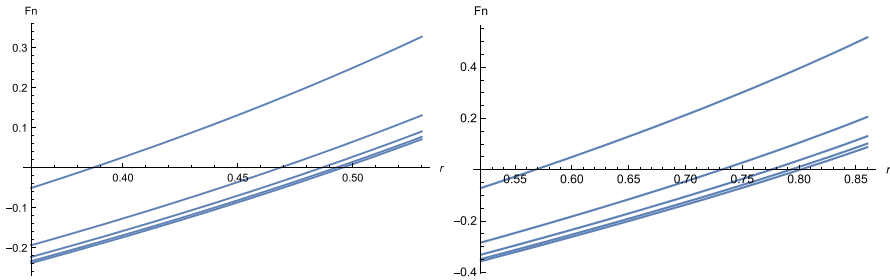


Fig. 2 The graphs of $F_n(r)$, respectively, for $\alpha = 0.5$ and $\alpha = 0.9$ when $n = 1, 2, 3, 4, 5$

The roots r_f of $F_n(r) = 0$ for different values of α, ζ and n have been shown in Tables 1, 2 and Fig. 2, respectively.

Remark 4.1 For $n = 1$ and $|\zeta| \leq 1, r_f$ can be found in [9]. For $\alpha = 0.5$, when $n \rightarrow \infty$, the sharp radius is 0.500000. For $\alpha = 0.9$, when $n \rightarrow \infty$, the sharp radius is 0.815323. For $n = 1$, when $\alpha \rightarrow 1$, the sharp radius is 0.645751.

Now, we give an improved version of Bohr inequality for the class $\mathcal{HC}_n(\phi)$. By adding area quantity $S_r/(2\pi)$ with the majorant series of $f \in \mathcal{HC}_n(\phi)$, the sum is still less than $d(f(0), \partial f(\mathbb{D}))$ for some radius $r \leq \min\{1/3, \tilde{r}_f\} < 1$.

Note that the additional term such as $S_r/(2\pi)$ to the majorant sum was first mooted by Kayumov and Ponnusamy [31] to refine and improve the Bohr inequality. This variation of Bohr inequality was proved for harmonic mappings in [25]. Subsequently, several extensions were made by many authors (cf. [39]).

Theorem 4.2 Let $f \in \mathcal{HC}_n(\phi)$ and S_r be the area of the image $f(\mathbb{D}_r)$. Then the inequality

$$M_f(r) + \frac{S_r}{2\pi} \leq d(f(0), \partial f(\mathbb{D}))$$

holds for $|z| = r \leq \min\{1/3, \tilde{r}_f\}$, where \tilde{r}_f is the smallest positive root in $(0, 1)$ of

$$L(\zeta, n, 1) = M_K(r) + |\zeta| \int_0^r t^n M_{K'}(t) dt + \int_0^r t (1 - |\zeta|^2 t^{2n}) (K'(t))^2 dt,$$

and $L(\zeta, n, 1)$ is given by (4.2) with $r = 1$.

Proof Let $f \in \mathcal{HC}_n(\phi)$ be of the form (1.1). Then, from the right hand inequality in (4.10) and (4.19), we obtain

$$\begin{aligned} M_f(r) + \frac{S_r}{2\pi} &\leq M_K(r) + |\zeta| \int_0^r t^n M_{K'}(t) dt + \int_0^r t (1 - |\zeta|^2 t^{2n}) (K'(t))^2 dt \\ &= R_C(n, r) + \int_0^r t (1 - |\zeta|^2 t^{2n}) (K'(t))^2 dt = \tilde{R}_f(n, r) \end{aligned} \tag{4.24}$$

for $r \leq 1/3$. Suppose that $H_2(n, r) = \tilde{R}_f(n, r) - L(\zeta, n, 1)$. Then $H_2(n, r)$ is a continuous function in $[0, 1]$. The inequality (4.22) yields that $H_2(n, 0) = -L(\zeta, n, 1) < 0$. By virtue of (4.21), we get

$$R_C(n, 1) - L(\zeta, n, 1) > 0. \tag{4.25}$$

For $|\zeta| \leq 1/(2n - 1)$, we observe that

$$t \left(1 - |\zeta|^2 t^{2n}\right) (K'(t))^2 \geq 0,$$

and hence

$$\int_0^r t \left(1 - |\zeta|^2 t^{2n}\right) (K'(t))^2 dt \geq 0. \tag{4.26}$$

From (4.24) and (4.25), we obtain

$$H_2(n, 1) = R_C(n, 1) - L(\zeta, n, 1) + \int_0^1 t \left(1 - |\zeta|^2 t^{2n}\right) (K'(t))^2 dt > 0.$$

Since $H_2(n, 0) < 0$ and $H_2(n, 1) > 0$, H_2 has a root in $(0, 1)$ and choose \tilde{r}_f to be the smallest root in $(0, 1)$, we know that $\tilde{R}_f(n, r) \leq L(\zeta, n, 1)$ for $r \leq \tilde{r}_f$. Therefore, by virtue of (4.20) and (4.24), we conclude that

$$M_f(r) + \frac{S_r}{2\pi} \leq d(f(0), \partial f(\mathbb{D}))$$

for $r \leq \min\{1/3, \tilde{r}_f\}$. □

Remark 4.2 By setting $n = 1$ in Theorems 4.1 and 4.2, we get the corresponding results obtained in [9].

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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