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Riesz bases of normalized reproducing kernels in Fock type spaces

K. P. Isaev[1] · R. S. Yulmukhametov[1]

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Abstract
We describe some radial Fock type spaces which possess Riesz bases of normalized reproducing kernels, the spaces $\mathcal{F}_\varphi$ of entire functions f such that $fe^{-\varphi} \in L_2(\mathbb{C})$, where $\varphi(z) = \varphi(|z|)$ is a radial subharmonic function. We prove that $\mathcal{F}_\varphi$ has Riesz basis of normalized reproducing kernels for sufficiently regular $\psi(r) = \varphi(e^r)$ such that $\psi''(r)$ is bounded above.

Keywords Hilbert spaces · Entire functions · Reproducing kernels · Unconditional bases · Riesz bases

Mathematics Subject Classification 46E22 · 30D10

1 Introduction

We consider the radial weighted Fock spaces

$$\mathcal{F}_\varphi = \left\{ f \in H(\mathbb{C}) : \ \|f\|^2 = \frac{1}{2\pi} \int\limits_{\mathbb{C}} |f(z)|^2 e^{-2\varphi(z)} dm(z) < \infty \right\},$$

where $dm(z)$ being planar Lebesgue measure, $\varphi(z)$ being a radial subharmonic function. We assume that this space is not degenerate. It has a natural Hilbert space structure,

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✉ K. P. Isaev
orbit81@list.ru

R. S. Yulmukhametov
yulmukhametov@mail.ru

1 Institute of Mathematics, Ufa Federal Research Centre, Russian Academy of Sciences (Institute of Mathematics UFRC RAS), 112, Chernyshevsky str., Ufa, Russia 450008

the evaluations $\delta_\lambda : f \to f(\lambda)$ are continuous. Since the Hilbert spaces are self-dual, it follows that each of these functionals is generated by an element $k_\lambda(z) = k(z, \lambda) \in \mathcal{F}_\varphi$ in the sense that

$$\frac{1}{2\pi} \int\limits_{\mathbb{C}} f(z)\overline{k(z, \lambda)} e^{-2\varphi(z)} dm(z) = f(\lambda), \quad \text{for any } f \in \mathcal{F}_\varphi \text{ and } \lambda \in \mathbb{C}.$$

The function $k(z, \lambda)$ is called the reproducing kernel of the space $\mathcal{F}_\varphi$. Obviously,

$$\|\delta_\lambda\|^2 = k(\lambda, \lambda) := K(\lambda), \quad \lambda \in \mathbb{C}.$$

The system $\{k(z, \lambda_j)\}_{j=1}^\infty$ will be called an unconditional basis in the space $\mathcal{F}_\varphi$ if it is complete and for some $C > 1$ we have

$$\frac{1}{C} \sum_j |a_j|^2 K(\lambda_j) \le \left\| \sum_j a_j k(z, \lambda_j) \right\|^2 \le C \sum_j |a_j|^2 K(\lambda_j),$$

for finite sequences $\{a_j\}$ of complex numbers. An unconditional basis $\{e_j, \ j = 1, 2, ...\}$ becomes Riesz basis if and only if $0 < \inf_k \|e_k\| \le \sup_k \|e_k\| < \infty$. Equivalently, Riesz basis is a linear isomorphic image of an orthonormal basis in a separable Hilbert space. We study the existence of Riesz bases of normalized reproducing kernels $\left\{ \frac{k(z,\lambda_j)}{\|k(\cdot,\lambda_j)\|} \right\}_{j=1}^\infty$ in $\mathcal{F}_\varphi$.

The issue on existence and construction of Riesz bases of normalized reproducing kernels is actively studied due to the fact, in particular, that this question is closely related to such classical problems of complex analysis as the problem of interpolation (see, for example, [1–3]) and the problem of representing by exponential series (see, for example, [4]). Summing up the studies of this issue in various aspects, we can say that Riesz bases are a rare phenomenon (see [1,3,5]). In [5], an unexpected result was obtained, which stated the existence of Riesz bases of normalized reproducing kernels in the Fock spaces $\mathcal{F}_\varphi$ with the weights $\varphi = (\ln^+ |z|)^\alpha$ as $\alpha \in (1; 2]$. Later, in paper [6], there was proved the existence of Riesz bases of normalized reproducing kernels in the Fock spaces with radial weights of essentially more general form. We prove that if φ is a radial function and the function $\psi(r) = \varphi(e^r)$ satisfies the conditions: $\lim_{r\to\infty} \psi'(r) = \infty$, ψ'' is a non-increasing positive function, and $\left|\psi'''(r)\right| = O(\psi''(r)^{\frac{5}{3}})$, $r \to \infty$, then $\mathcal{F}_\varphi$ has a Riesz basis of normalized reproducing kernels. In this paper, we prove a weaker sufficient condition for the existence of a Riesz basis of normalized reproducing kernels in Fock spaces with radial and sufficiently regular weights.

2 Notation, definitions, preliminaries, and statements of results

Definition 1 A convex function v is called regular if there exist a number $q > 1$ and a function $\gamma(x) \uparrow +\infty$ such that

$$\frac{1}{q} \le \frac{v''(x)}{v''(y)} \le q \ \text{ as } \ |x-y| \le \gamma(x)\sqrt{\frac{1}{v''(x)}}, \quad x, y \in \mathbb{R}_+.$$

Conditions of this kind are used to find the asymptotic of the Laplace integrals. In this paper we prove (see Theorem 3) that if φ is radial subharmonic function, the function $\psi(x) = \varphi(e^x)$ is regular, and

$$\sup_{x>0} \psi''(x) < \infty,$$

then the space $\mathcal{F}_\varphi$ has a Riesz basis of normalized reproducing kernels.

Definition 2 The function $\widetilde{v}(y) = \sup_x (xy - v(x)), \ y \in \mathbb{R}$, is the Young conjugate of the convex function v.

Definition 3 Let v be a continuous function, and

$$d(v, y, r) = \inf_l \left\{ \max_{t \in [y-r; y+r]} |v(t) - l(t)|, \ l \text{ is a linear function} \right\}.$$

We set

$$\rho_1(v, y, p) = \sup\{r : \ d(v, y, r) \le p\}$$

for a positive number p.

This characteristic was introduced in [7].

Definition 4 Let v be a convex function on $\mathbb{R}$, and p be a positive number. We let

$$\rho_2(v, x, p) = \sup \left\{ t > 0 : \int_{x-t}^{x+t} \left| v'_+(\tau) - v'_+(x) \right| d\tau \le p \right\},$$

where v'_+ is the right derivative of v.

This characteristic was introduced in [8]. It was proved in [7] (see Lemma 3) that

$$\rho_1(v, y, p) = \rho_2(v, y, 2p) \tag{1}$$

for convex function v.

In what follows we shall make use of the following notations. For positive functions $A, \ B$, the writing $A(x) \asymp B(x), x \in X$, means that for some constants $C, \ c > 0$ and for all $x \in X$ the estimates $cB(x) \le A(x) \le CB(x)$ hold. The symbol $A(x) \prec B(x)$, $x \in X$, ($A(x) \succ B(x)$, $x \in X$), means the existence of a constant $C > 0$ such that $A(x) \le CB(x)$ ($B(x) \le CA(x)$).

We denote $[x]$ the floor function (the integer part of x).

3 A sufficient condition for the existence of Riesz bases in general Hilbert spaces

In this section, we consider a sufficient condition for the existence of Riesz bases of normalized reproducing kernels in general Hilbert spaces of entire functions. Let H be a radial functional Hilbert space of entire functions satisfying the division property, i.e.:

1. all evaluation functionals $\delta_z : f \to f(z)$ are continuous;
2. if $F \in H$, then $\|F\| = \|F(ze^{i\varphi})\|$ for any $\varphi \in \mathbb{R}$;
3. if $F \in H$, $F(z_0) = 0$, then $F(z)(z - z_0)^{-1} \in H$.

The functional property of the space implies that it admits a reproducing kernel $k(z, \lambda)$.

It was proved in [9] (see Theorem A) that if H is a radial functional Hilbert space satisfying the division property, admitting a Riesz basis of normalized reproducing kernels, and monomials are complete in H, then there exists a convex sequence $u(n)$, $n \in \mathbb{N} \cup \{0\}$, such that $\|z\|^n \asymp e^{u(n)}$, $n \in \mathbb{N} \cup \{0\}$. The convexity of $\{u(n)\}$ means

$$u(n+1) + u(n-1) - 2u(n) \ge 0, \quad n \in \mathbb{N}.$$

If $u(t)$ be a convex piecewise linear function with integer non-negative breakpoints, and $u(t) \equiv u(0)$ as $t < 0$, then the convexity condition can be written in a more compact form

$$u'_+(n+1) - u'_+(n) \ge 0.$$

In what follows, we assume that $u(n) = \ln \|z^n\|$, $n \in \mathbb{N} \cup \{0\}$, is a convex sequence, $u(0) = 0$, and $u(t)$ is a piecewise linear function, $u(t) \equiv 0$ as $t < 0$. The following theorem was proved in [10] (see Theorem 2).

Theorem A *If the system of monomials $\{z^n,\ n \in \mathbb{N} \cup \{0\}\}$ is complete in a radial functional Hilbert space H satisfying the division property, and the function $\widetilde{u}$ satisfies the condition*

$$\sup_{x>0} (\widetilde{u}'_+(x+1) - \widetilde{u}'_+(x)) \le N < \infty, \tag{2}$$

then the space H possesses Riesz bases of normalized reproducing kernels.

Let us prove following lemmas.

Lemma 1 *For the convex piecewise linear function $u(t)$, $t \in \mathbb{R}$, condition (2) is equivalent to*

$$\inf_{x>1} \rho_2(\widetilde{u}, x, 1) > 0. \tag{3}$$

Proof Without loss of generality we can suppose that $N \geq 1$ in (2). The monotonicity of the function $\widetilde{u}'_+(x)$ implies that if (2) holds, then

$$\int\limits_{x-\frac{1}{2N}}^{x+\frac{1}{2N}} |\widetilde{u}'_+(\tau) - \widetilde{u}'_+(x)| d\tau \leq 1.$$

By definition of $\rho_2(\widetilde{u}, x, 1)$ this means that

$$\rho_2(\widetilde{u}, x, 1) \geq \frac{1}{2N}, \ x \geq 1.$$

Thus (3) holds.

Conversely, let

$$\rho_2(\widetilde{u}, x, 1) \geq 2\delta > 0, \ x \geq 1.$$

By definition of $\rho_2(\widetilde{u}, x, 1)$ we have

$$\int\limits_{\delta}^{2\delta} (\widetilde{u}'_+(x+t) - \widetilde{u}'_+(x)) dt \leq 1, \ x \geq 1,$$

and therefore,

$$\widetilde{u}'_+(x+\delta) - \widetilde{u}'_+(x) \leq \frac{1}{\delta}, \ x \geq 1.$$

Let $N = \left[\frac{1}{\delta}\right] + 1$. Taking into account that $\widetilde{u}'_+$ is an increasing function, we get

$$\widetilde{u}'_+(x+1) - \widetilde{u}'_+(x) \leq \sum_{k=0}^{N-1} (\widetilde{u}'_+(x+k\delta+\delta) - \widetilde{u}'_+(x+\delta)) \leq \frac{N}{\delta} \leq N^2, \ x > 0,$$

that is, (2) holds. □

Lemma 2 *Condition* (3) *is equivalent to the boundness of the function* $\rho_2(u, t, 1)$ *on* $\mathbb{R}_+$*:*

$$\sup_{t>0} \rho_2(u, t, 1) < \infty.$$

Proof Let $\rho_2(u, t, 1) \leq N, \ t \in \mathbb{R}_+$, for some constant $N > 0$. Without loss of generality we can suppose that N is integer. By definition of $\rho_2(u, t, 1)$ this means that

$$\int\limits_{t-N}^{t+N} |u'_+(y) - u'_+(t)| dy \geq 1, \ t \in \mathbb{R}_+.$$

Hence, since $u'_+(y)$ is a monotonic function, we have

$$u'_+(n+N) - u'_+(n-N) \geq \frac{1}{2N}, \ n \in \mathbb{N},$$

or

$$u'_+(n+2N) - u'_+(n) \geq \frac{1}{2N},\ n \in \mathbb{N}.$$

It was proved in [11] (see Lemma 2) that the the Young conjugate $\widetilde{u}$ is also piecewise linear with breakpoints $x_n = u'_+(n-1) = u(n) - u(n-1)$, and the derivative $\widetilde{u}'_+$ is the function with unite jumps at the points x_n. Thus, the last estimate can be written as

$$x_{n+2N} - x_n \geq \frac{1}{2N},\ n \in \mathbb{N}.$$

This means that the quantity of jumps of $\widetilde{u}'_+$ on an interval which length is less than $\frac{1}{2N}$ does not exceed $2N$. Since there are unit jumps, we find that for $\varepsilon < \frac{1}{2N}$

$$\widetilde{u}'_+(x+\varepsilon) - \widetilde{u}'_+(x) \leq 2N,\ x \geq 1.$$

Put $\varepsilon = \frac{1}{5N}$. Then

$$\int\limits_{t-\varepsilon}^{t+\varepsilon} |\widetilde{u}'_+(x) - \widetilde{u}'_+(t)|dx \leq 2N \cdot 2\varepsilon = \frac{4}{5} < 1,\ t \geq 1.$$

Hence,

$$\rho_2(\widetilde{u}, t, 1) \geq \frac{1}{5N},\ t \geq 1.$$

Conversely, let for some $\varepsilon > 0$

$$\rho_2(\widetilde{u}, t, 1) \geq 2\varepsilon,\ t \geq 1.$$

Then

$$\int\limits_{x+\varepsilon}^{x+2\varepsilon} |\widetilde{u}'_+(y) - \widetilde{u}'_+(x)|dy \leq \int\limits_{x-2\varepsilon}^{x+2\varepsilon} |\widetilde{u}'_+(y) - \widetilde{u}'_+(x)|dy \leq 1.$$

Hence, for any $x \geq 1$

$$\widetilde{u}'_+(x+\varepsilon) - \widetilde{u}'_+(x) \leq \frac{1}{\varepsilon}.$$

Put $N = \left[\frac{1}{\varepsilon}\right]$. Then

$$\widetilde{u}'_+(x+\varepsilon) - \widetilde{u}'_+(x) \leq N+1,$$

or

$$u'_+(n+N+1) - u'_+(n) \geq \varepsilon,\ n \in \mathbb{N} \cup \{0\}.$$

Thus,

$$\int\limits_{n+N+1}^{n+2(N+1)} |u'_+(x) - u'_+(n)|dx \geq \varepsilon(N+1) > 1.$$

Hence,

$$\rho_2(u, n, 1) \leq 2N + 2.$$

It was proved in [7] (see Lemmas 3 and 4) that the function $\rho_2(u, x, 1)$ satisfies Lipschitz condition

$$|\rho_2(u, x, 1) - \rho_2(u, y, 1)| \leq |x - y|, \; x, y \in \mathbb{R}.$$

Therefore,

$$\rho_2(u, t, 1) \leq 2N + 3, \; t \in \mathbb{R}_+.$$

□

Now we can reformulate Theorem A in the following form.

Theorem 1 *If the system of monomials $\{z^n, \; n \in \mathbb{N} \cup \{0\}\}$ is complete in a radial functional Hilbert space H satisfying the division property, and the function u satisfies the condition*

$$\sup_{x>0} \rho_2(u, t, 1) < \infty,$$

then the space H possesses Riesz bases of normalized reproducing kernels.

4 A sufficient condition for the existence of Riesz bases in radial weighted Fock spaces in terms of conjugate function

Let us turn to Fock spaces with radial weight φ. Let $\psi(x) = \varphi(e^x)$ and

$$e^{2u_1(t)} = \int_{-\infty}^{\infty} e^{2(t+1)x - 2\psi(x)} dx, \; t \in \mathbb{R}_+.$$

Then $u_1(t)$ is a convex function on $\mathbb{R}_+$, coinciding with the function $u(t)$ at the points $t \in \mathbb{N} \cup \{0\}$, in particular,

$$u_1(t) \leq u(t), \; t \in \mathbb{R}_+.$$

Let us extend u_1 to the entire axis, setting $u_1(t) \equiv 0$, $t \in \mathbb{R}_-$.

Lemma 3 *We have the relation*

$$\rho_2(u, t, 1) \leq \max_{|t-\tau| \leq \frac{1}{2}} \rho_2(u_1, \tau, 1) + 2, \; t \in \mathbb{R}_+,$$
$$\rho_2(u_1, t, 1) \leq \max_{|t-\tau| \leq \frac{1}{2}} \rho_2(u, \tau, 1) + 2, \; t \in \mathbb{R}_+.$$

Proof Let

$$\max_{|t-\tau| \leq \frac{1}{2}} \rho_2(u_1, \tau, 1) = M.$$

Let us suppose that for a natural number n, satisfying $|n - t| \le \frac{1}{2}$, the following inequality holds

$$\rho_2(u, n, 1) > [M] + 1.$$

Then, setting $k = [M] + 1$, we have

$$\int_{n-k}^{n+k} |u'_+(t) - u'_+(n)|dt < 1,$$

that is

$$u(n + k) + u(n - k) - 2u(n) < 1.$$

Since the functions u and u_1 coincide at integer points, then

$$u_1(n + k) + u_1(n - k) - 2u_1(n) < 1.$$

Hence,

$$\int_{n-k}^{n+k} |(u_1)'_+(t) - (u_1)'_+(n)|dt < 1,$$

and

$$\rho_2(u_1, n, 1) \ge k = [M] + 1 > M.$$

The resulting contradiction means that

$$\rho_2(u, n, 1) \le [M] + 1 \le M + 1, \; n \in \mathbb{N}.$$

Since the function $\rho_2(u, t, 1)$ satisfies the Lipschitz condition, we have

$$\rho_2(u, t, 1) \le \rho_2(u, [t], 1) + |t - [t]| \le M + 2, \; t \in \mathbb{R}_+.$$

The second relation is proved in a similar way. □

Lemma 4 *If the function $\widetilde{\psi}$ is regular and q is the constant in the regularity condition, then for sufficiently large numbers $t \in \mathbb{R}$ the following inequalities hold*

$$\sqrt{\frac{1}{q\widetilde{\psi}''(t)}} \le \rho_2(\widetilde{\psi}, t, 1) \le \sqrt{\frac{q}{\widetilde{\psi}''(t)}}.$$

Proof Let $\rho_0 = \frac{\gamma(t)}{\sqrt{\widetilde{\psi}''(t)}}$, then due to regularity $\widetilde{\psi}$

$$\frac{1}{q} \le \frac{\widetilde{\psi}''(t)}{\widetilde{\psi}''(x)} \le q \;\text{ for }\; |t - x| \le \rho_0.$$

Hence, by the mean value theorem for any x such that $|x-t| \le \rho_0$ we have

$$|\widetilde{\psi}'(t) - \widetilde{\psi}'(x)| = \widetilde{\psi}''(x^*)|t-x| \ge \frac{1}{q}\widetilde{\psi}''(t)|t-x|.$$

Therefore, if $\gamma(t) \ge q$, then

$$\int_{t-\rho_0}^{t+\rho_0} |\widetilde{\psi}'(x) - \widetilde{\psi}'(t)|dx \ge \frac{1}{q}\widetilde{\psi}''(t)\rho_0^2 = \frac{1}{q}\gamma^2(t) > 1.$$

Hence, $\rho_2(\widetilde{\psi}, t, 1) := \rho \le \rho_0$. By definition of the function $\rho_2(\widetilde{\psi}, t, 1)$ we have

$$\int_{t-\rho}^{t+\rho} |\widetilde{\psi}'(x) - \widetilde{\psi}'(t)|dx = 1,$$

and

$$\left(\max_{|t-x|\le\rho} \widetilde{\psi}''(x)\right)^{-1} \le \rho^2 \le \left(\min_{|t-x|\le\rho} \widetilde{\psi}''(x)\right)^{-1}.$$

From this and the regularity of the function $\widetilde{\psi}$, we obtain the assertion of the lemma for t such that $\gamma(t) \ge q$. □

Lemma 5 *If the function $\widetilde{\psi}$ is regular, then for some constant $m > 1$ we have*

$$\frac{1}{m}\rho_2(\widetilde{\psi}, t+1, 1) \le \rho_2(u_1, t, 1) \le m\rho_2(\widetilde{\psi}, t+1, 1), \ \ t \in \mathbb{R}_+.$$

The left estimate holds without the regularity condition.

Proof 1. Let us prove the left inequality. By Theorem 2(a) in [7] we have

$$e^{2u_1(y)} = \int_{-\infty}^{\infty} e^{2(y+1)x - 2\psi(x)}dx \asymp \frac{e^{2\widetilde{\psi}(y+1)}}{\rho_1(\widetilde{\psi}, y+1, 1)}, \ \ y \in \mathbb{R}_+.$$

That is,

$$e^{-2a} \le \frac{e^{2(\widetilde{\psi}(y+1) - u_1(y))}}{\rho_1(\widetilde{\psi}, y+1, 1)} \le e^{2a}, \ \ y \in \mathbb{R}_+,$$

for some $a > 0$, and

$$\left|\widetilde{\psi}(y+1) - u_1(y) - \frac{1}{2}\ln\rho_1(\widetilde{\psi}, y+1, 1)\right| \le a, \ \ y \in \mathbb{R}_+.$$

Take an arbitrary point $t \in \mathbb{R}_+$ and denote $\rho_1(\widetilde{\psi}, t+1, 1) = \rho_1$. Let $\alpha \in \left(0; \frac{1}{2}\right)$. There is a linear function $l(x)$ such that

$$\max_{x \in [t+1-\alpha\rho_1; t+1+\alpha\rho_1]} |\widetilde{\psi}(x) - l(x)| \leq 1.$$

For the linear function $l_1(x) = l(x) - \frac{1}{2} \ln \rho_1$ we have

$$\begin{aligned}
&\max_{x \in [t-\alpha\rho_1; t+\alpha\rho_1]} |u_1(x) - l_1(x+1)| \\
&\quad \leq \max_{x \in [t-\alpha\rho_1; t+\alpha\rho_1]} \left| u_1(x) - \widetilde{\psi}(x+1) + \frac{1}{2} \ln \rho_1(\widetilde{\psi}, x+1, 1) \right| \\
&\quad + \max_{x \in [t-\alpha\rho_1; t+\alpha\rho_1]} \left| \widetilde{\psi}(x+1) - l(x+1) \right| + \frac{1}{2} \max_{x \in [t-\alpha\rho_1; t+\alpha\rho_1]} \left| \ln \frac{\rho_1(\widetilde{\psi}, t+1, 1)}{\rho_1(\widetilde{\psi}, x+1, 1)} \right|
\end{aligned}$$

$$\leq a + 1 + \frac{1}{2} \max_{x \in [t-\alpha\rho_1; t+\alpha\rho_1]} \left| \ln \frac{\rho_1(\widetilde{\psi}, x+1, 1)}{\rho_1(\widetilde{\psi}, t+1, 1)} \right|. \tag{4}$$

The function $\rho_1(u, x, p)$ satisfies the Lipschitz condition too (see Lemma 4 in [7]), therefore, if $|x - t| \leq \alpha\rho_1$, then

$$|\rho_1(\widetilde{\psi}, x+1, 1) - \rho_1(\widetilde{\psi}, t+1, 1)| \leq \alpha\rho_1(\widetilde{\psi}, t+1, 1)$$

or

$$\left| \frac{\rho_1(\widetilde{\psi}, x+1, 1)}{\rho_1(\widetilde{\psi}, t+1, 1)} - 1 \right| \leq \alpha < \frac{1}{2}.$$

Hence,

$$\frac{1}{2} \max_{x \in [t-\alpha\rho_1; t+\alpha\rho_1]} \left| \ln \frac{\rho_1(\widetilde{\psi}, x+1, 1)}{\rho_1(\widetilde{\psi}, t+1, 1)} \right| \leq \frac{1}{2} \max_{|s| < \frac{1}{2}} |\ln(1+s)| \leq \frac{1}{2} \ln 2 < \frac{1}{2}.$$

Continuing estimate (4), we obtain

$$\max_{x \in [t-\alpha\rho_1; t+\alpha\rho_1]} |u_1(x) - l_1(x+1)| < a + \frac{3}{2},$$

and by that,

$$\rho_1\left(u_1, t, a + \frac{3}{2}\right) \geq \alpha\rho_1(\widetilde{\psi}, t+1, 1), \ t \in \mathbb{R}_+,$$

or taking into account the arbitrariness of $\alpha \in \left(0; \frac{1}{2}\right)$, we get

$$\rho_1\left(u_1, t, a + \frac{3}{2}\right) \geq \frac{1}{2}\rho_1(\widetilde{\psi}, t+1, 1), \ t \in \mathbb{R}_+.$$

By (1) we get

$$\rho_2(u_1, t, 2a+3) \geq \frac{1}{2}\rho_2(\widetilde{\psi}, t+1, 2),\ t \in \mathbb{R}_+.$$

Hence, by Lemma 2 in [7] we obtain the lower estimate with the constant $m = 2(2a+3)$.

2. Let $\widetilde{\psi}$ be a regular function. It is convenient to write the regularity condition in the form

$$\frac{1}{q} \leq \frac{\widetilde{\psi}''(x+1)}{\widetilde{\psi}''(y+1)} \leq q,\ |x-y| \leq \gamma_1(x)\sqrt{\frac{1}{\widetilde{\psi}''(x+1)}},$$

where $\gamma_1(x) = \gamma(x+1)$. By Theorem 2(a) in [7] we have

$$e^{2u_1(y)} = \int_{-\infty}^{\infty} e^{2(y+1)x-2\psi(x)}dx \asymp \frac{e^{2\widetilde{\psi}(y+1)}}{\rho_2(\widetilde{\psi}, y+1, 1)},\ y \in \mathbb{R}_+,$$

that is, for some $b > 0$ we have

$$e^{-2b} \leq \frac{e^{2(\widetilde{\psi}(y+1)-u_1(y))}}{\rho_2(\widetilde{\psi}, y+1, 1)} \leq e^{2b},\ y \in \mathbb{R}_+, \tag{5}$$

or

$$\left|\widetilde{\psi}(y+1) - u_1(y) - \frac{1}{2}\ln \rho_2(\widetilde{\psi}, y+1, 1)\right| \leq b,\ y \in \mathbb{R}_+.$$

By Lemma 4 we have

$$\frac{1}{q}\sqrt{\frac{\widetilde{\psi}''(y+1)}{\widetilde{\psi}''(t+1)}} \leq \frac{\rho_2(\widetilde{\psi}, t+1, 1)}{\rho_2(\widetilde{\psi}, y+1, 1)} \leq q\sqrt{\frac{\widetilde{\psi}''(y+1)}{\widetilde{\psi}''(t+1)}},$$

and by the regularity of $\widetilde{\psi}$ we get

$$q^{-\frac{3}{2}} \leq \frac{\rho_2(\widetilde{\psi}, t+1, 1)}{\rho_2(\widetilde{\psi}, y+1, 1)} \leq q^{\frac{3}{2}},\ \ |t-y| \leq \gamma_1(t)\sqrt{\frac{1}{\widetilde{\psi}''(t+1)}}.$$

Take a point $t \in \mathbb{R}_+$ so that $\gamma_1(t) > 3\sqrt{q}(b+\ln q+1)$ and denote $\rho_2(\widetilde{\psi}, t+1, 1) = \rho_2$. Let $c = \ln q$. Then by the last estimate and by (5) we obtain

$$e^{-2(b+c)} \leq e^{2(\widetilde{\psi}(y+1)-\frac{1}{2}\ln \rho_2 - u_1(y))} \leq e^{2(b+c)},\ \ |t-y| \leq \gamma_1(t)\sqrt{\frac{1}{\widetilde{\psi}''(t+1)}},$$

or

$$\left|\widetilde{\psi}(y+1) - \frac{1}{2}\ln \rho_2 - u_1(y)\right| \leq b+c,\ \ |t-y| \leq \gamma_1(t)\sqrt{\frac{1}{\widetilde{\psi}''(t+1)}}. \tag{6}$$

Suppose that

$$\rho_1(u_1, t, 1) \geq 3q(b+c+1)\rho_2(\widetilde{\psi}, t+1, 1).$$

Then there is a linear function $l(x)$ such that

$$|u_1(x) - l(x)| \leq 1, \ |x-t| \leq 3q(b+c+1)\rho_2(\widetilde{\psi}, t+1, 1).$$

By Lemma 4 we have

$$|u_1(x) - l(x)| \leq 1, \ |x-t| \leq 3\sqrt{q}(b+c+1)\sqrt{\frac{1}{\widetilde{\psi}''(t+1)}}.$$

Therefore, by (6), taking into account the choice of t, for the linear function $l_1(x) = l(x) + \frac{1}{2}\ln\rho_2$, we obtain

$$|\widetilde{\psi}(x+1) - l_1(x)| = \left|\widetilde{\psi}(x+1) - \frac{1}{2}\ln\rho_2 - l(x)\right|$$
$$\leq \left|\widetilde{\psi}(x+1) - \frac{1}{2}\ln\rho_2 - u_1(x)\right| + |u_1(x) - l(x)| \leq b+c+1$$

for $|x-t| \leq 3\sqrt{q}(b+c+1)\sqrt{\frac{1}{\widetilde{\psi}''(t+1)}}$. Hence,

$$\rho_1(\widetilde{\psi}, t+1, b+c+1) \geq 3\sqrt{q}(b+c+1)\sqrt{\frac{1}{\widetilde{\psi}''(t+1)}},$$

and by (1),

$$3\sqrt{q}(b+c+1)\sqrt{\frac{1}{\widetilde{\psi}''(t+1)}} \leq \rho_2(\widetilde{\psi}, t+1, 2(b+c+1)).$$

Then by Lemma 4 we get

$$3(b+c+1)\rho_2(\widetilde{\psi}, t+1, 1) \leq \rho_2(\widetilde{\psi}, t+1, 2(b+c+1)).$$

Hence, by Lemma 2 in [7] we obtain

$$3(b+c+1)\rho_2(\widetilde{\psi}, t+1, 1) \leq \rho_2(\widetilde{\psi}, t+1, 2(b+c+1)) \leq 2(b+c+1)\rho_2(\widetilde{\psi}, t+1, 1)$$

or

$$\frac{3}{2}\rho_2(\widetilde{\psi}, t+1, 1) \leq \rho_2(\widetilde{\psi}, t+1, 1).$$

Since $\rho_2(\widetilde{\psi}, t+1, 1) > 0$, we obtain a contradiction. Thus,

$$\rho_1(u_1, t, 1) < 3q(b+c+1)\rho_2(\widetilde{\psi}, t+1, 1).$$

Taking into account (1) again, for t such that $\gamma_1(t) \geq 3\sqrt{q}(b+c+1)$ we have

$$\rho_2(u_1, t, 1) \leq \rho_2(u_1, t, 2) = \rho_1(u_1, t, 1) < 3\sqrt{q}(b+c+1)\rho_2(\widetilde{\psi}, t+1, 1).$$

Since the functions $\rho_2(u_1, t, 1)$ and $\rho_2(\widetilde{\psi}, t, 1)$ are continuous, this implies the estimate

$$\rho_2(u_1, t, 1) \leq A\rho_2(\widetilde{\psi}, t+1, 1), \; t \in \mathbb{R}_+,$$

for some constant $A > 0$. □

Lemmas 3–5 imply the following theorem.

Theorem 2 *If $\widetilde{\psi}$ is a regular function, and $\widetilde{\psi}''(t)$ satisfies the condition*

$$\inf_{t>0} \widetilde{\psi}''(t) > 0,$$

then the Fock space with the weight $\psi(\ln|z|)$ possesses Riesz bases of normalized reproducing kernels.

5 A sufficient condition for the existence of Riesz bases in radial weighted Fock spaces in terms of weight

In this section we will prove the final theorem.

Theorem 3 *If ψ is a regular function, and*

$$\sup_{t>0} \psi''(t) < \infty,$$

then the Fock space with the weight $\psi(\ln|z|)$ possesses Riesz bases of normalized reproducing kernels.

Let us first prove a lemma.

Lemma 6 *Let $v \in C^2(\mathbb{R})$ be a convex indefinitely increasing function which is not linear on $\mathbb{R}_+$. If v is a regular function, then the conjugate function $\widetilde{v}$ is also regular on some interval $(a; +\infty)$.*

Proof By hypothesis of the lemma, v' is strictly increasing, and we have

$$v'(\widetilde{v}'(t)) \equiv t, \qquad v''(\widetilde{v}'(t))\widetilde{v}''(t) \equiv 1, \; t \in \mathbb{R}. \tag{7}$$

Let $x_\pm = x \pm \frac{1}{2}\frac{\gamma(x)}{\sqrt{v''(x)}}$ and $t = v'(x), \; t_\pm = v'(x_\pm)$. Let us note that

$$\lim_{x\to+\infty} t_+(x) = +\infty. \tag{8}$$

By regularity of v, for some $x^* \in [x_-; x_+]$ we have

$$t_+ - t_- = v''(x^*)(x_+ - x_-) \geq \frac{\gamma(x)}{q}\sqrt{v''(x)}.$$

Let $\tau = \frac{1}{2}(t_+ + t_-)$. Then $y = \widetilde{v}'(\tau) \in [x_-; x_+]$, and since $t_- \geq 0$, $\tau \geq \frac{1}{2}t_+$, then by (8) we get

$$\lim_{x \to +\infty} \tau(x) = +\infty. \tag{9}$$

If

$$|\tau - s| \leq \frac{\gamma(x)}{2q^{\frac{3}{2}}\sqrt{\widetilde{v}''(\tau)}},$$

then by (7) we get

$$|\tau - s| \leq \frac{\gamma(x)}{2q^{\frac{3}{2}}}\sqrt{v''(y)} \leq \frac{\gamma(x)}{2q}\sqrt{v''(x)},$$

that is $s \in [t_-; t_+]$ and $\widetilde{v}'(s) \in [x_-; x_+]$. Hence, by (7) we obtain

$$\frac{\widetilde{v}''(\tau)}{\widetilde{v}''(s)} = \frac{v''(\widetilde{v}'(s))}{v''(\widetilde{v}'(\tau))} \in \left[\frac{1}{q}; q\right].$$

Thus, $\widetilde{v}$ satisfies the regularity condition at the points $\tau(x)$ with the function $\frac{\gamma(x(\tau))}{2q^{\frac{3}{2}}}$. By (9), the set of such τ contains some interval $(a; +\infty)$. On this interval, the regularity condition will also hold with the increasing function

$$\gamma_0(\tau) = \frac{1}{2q^{\frac{3}{2}}} \inf_{t \geq \tau} \gamma(x(t)).$$

□

By Lemma 6 and by (7) we obtain that if the hypothesis of Theorem 3 is satisfied then the function $\widetilde{\psi}$ is regular and $\inf_{t>0} \widetilde{\psi}''(t) > 0$. Then by Theorem 2, the Fock space with the weight $\psi(\ln |z|)$ possesses Riesz bases of normalized reproducing kernels.

Corollary 1 *f* $\psi \in C^2$ *and* $0 < \psi''(t) \asymp 1,\ t \in \mathbb{R}$, *then the Fock space with the weight* $\psi(\ln |\lambda|)$ *possesses Riesz bases of normalized reproducing kernels.*

Proof In this case, the conditions of Theorem 3 are satisfied in an obvious way. □

Corollary 2 *If* $\psi \in C^3$, $\psi'(x)$ *is unlimited and*

$$0 < \psi''(t) \longrightarrow 0,\ t \longrightarrow \infty,$$
$$|\psi'''(t)| = O(\psi''(t)),\ t \longrightarrow \infty,$$

then the Fock space with the weight $\psi(\ln |\lambda|)$ *possesses Riesz bases of normalized reproducing kernels.*

Proof By (7) we obtain that

$$\widetilde{\psi}''(x) \longrightarrow \infty, \; x \longrightarrow \infty,$$

and for some $M > 0$ we have

$$|(\ln \widetilde{\psi}''(\psi'(t)))'| = \left| \frac{\psi'''(t)}{\psi''(t)} \right| \leq M, \; t \in \mathbb{R}. \tag{10}$$

Hence, by mean value theorem we have

$$\left| \ln \left(\frac{\widetilde{\psi}''(x)}{\widetilde{\psi}''(y)} \right) \right| \leq M|x - y|, \; x, y \in \mathbb{R}_+.$$

Let

$$\gamma(x) = \inf_{y \geq x} \ln \widetilde{\psi}''(y), \; x > 0.$$

Then $\gamma(x) \uparrow +\infty$ as $x \to +\infty$, and $\gamma(x) \leq \ln \widetilde{\psi}''(x)$. Put

$$C = \sup_{x>0} \frac{\gamma(x)}{\sqrt{\widetilde{\psi}''(x)}},$$

then $C < \infty$. If

$$|x - y| \leq \frac{\gamma(x)}{\sqrt{\widetilde{\psi}''(x)}}, \; x, y \in \mathbb{R}_+,$$

then $|x - y| \leq C$. Hence, by (10) we have

$$\left| \ln \left(\frac{\widetilde{\psi}''(x)}{\widetilde{\psi}''(y)} \right) \right| \leq M|x - y| \leq MC.$$

Thus, $\widetilde{\psi}(x)$ is regular with $q = e^{MC}$. By Lemma 6 $\psi(x)$ is regular too. By Theorem 3 the Fock space with the weight $\psi(\ln|\lambda|)$ possesses Riesz bases of normalized reproducing kernels. □

Note that the Corollaries 1 and 2 are close to [6, Theorem 1.2]. It proved the existence of unconditional bases provided that the nonincreasing function $\psi''(t)$ satisfies the condition

$$|\psi'''(t)| = O\left(\psi''(t)^{\frac{5}{3}}\right), \; t \longrightarrow \infty.$$

Monotonicity implies the existence of a limit

$$\lim_{t \longrightarrow \infty} \psi''(t) := \psi_0.$$

If $\psi_0 > 0$, then the condition of the Corollary 1 is satisfied and the other conditions of Theorem 1.2 are not needed. If $\psi_0 = 0$, then we get the situation of the Corollary 2 without monotonicity and with a weaker condition for ψ'''.

Data availability The authors confirm that the data supporting the findings of this study are available within the article.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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