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Selected results and open problems on Hardy–Rellich and Poincaré–Friedrichs inequalities

Farit Avkhadiev[1]

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Abstract

In this paper we give a survey of selected results and open problems on integral inequalities of Mathematical Physics connected with the papers of V. Maz'ya, S. Fillippas, A. Tertikas, R. Osserman, A. Ancona, H. Brezis, M. Marcus, Y. Pinchover, E. B. Davies, A. Laptev, J. L. Fernández, J. M. Rodríguez, P. Caldiroli, R. Musina, A. A. Balinsky, W. D. Evans, R. T. Lewis, R. G. Nasibullin, I. K. Shafigullin, the author and other mathematicians. In addition, we give some new examples and present non-linear relationships between global numerical characteristics of domains in the Euclidean space of dimension $n \geq 2$.

Keywords Hardy–Rellich and Poincaré–Friedrichs inequality · Euclidean maximum modulus · Uniformly perfect set · Exterior sphere condition

Mathematics Subject Classification 26D10 · 33C20

1 Introduction

In the theory of Sobolev spaces there are many variational integral inequalities connected with Steklov, Hardy, Rellich, Poincaré, Friedrichs, Sobolev and other mathematicians. In particular, it is very known that Hardy–Rellich type inequalities have many applications and they are closely connected with the Uncertainty Principle of Heisenberg and the Spectral theory for unbounded operators of Mathematical Physics (see [1]–[7]).

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✉ Farit Avkhadiev
avkhadiev47@mail.ru

1 Lobachevskii Institute of Mathematics and Mechanics, Kazan Federal University, Kremlevskaya st. 35, Kazan, Tatarstan 420008, Russia

The following theorem of Hardy (compare [1]) may be considered as a basic fact for the Hardy–Rellich type inequalities on domains $\Omega \subset \mathbb{R}^n$ of the Euclidean space $\mathbb{R}^n$ of dimension $n \geq 2$.

Theorem 1 (1) *Suppose that* $1 \leq p < \infty$, $1 < s < \infty$, *and* $g : [0, \infty) \to \mathbb{R}$ *is an absolutely continuous nondecreasing function such that*

$$g(0) = 0, \quad g'/t^{s/p-1} \in L^p(0, \infty).$$

Then

$$\int_0^\infty \frac{|g'(t)|^p}{t^{s-p}} dt \geq \left(\frac{s-1}{p}\right)^p \int_0^\infty \frac{|g(t)|^p}{t^s} dt. \tag{1}$$

For $p > 1$ *and* $g \not\equiv 0$ *this inequality is strict, consequently, there is no extremal function, but the constant* $((s-1)/p)^p$ *is sharp.*

If $p = 1$, *then one has the following functional identity*

$$\int_0^\infty \frac{g(t)}{t^s} dt = \frac{1}{s-1} \int_0^\infty \frac{g'(t)}{t^{s-1}} dt,$$

which is valid for all admissible functions.

(2) *Suppose that* $1 \leq p < \infty$, $-\infty < \sigma < 1$, *and that* $g : (0, \infty] \to \mathbb{R}$ *is an absolutely continuous non-increasing function such that* $g(+\infty) = 0$ *and* $g'/\tau^{\sigma/p-1} \in L^p(0, \infty)$. *Then*

$$\int_0^\infty \frac{|g'(\tau)|^p}{\tau^{\sigma-p}} d\tau \geq \left(\frac{|\sigma-1|}{p}\right)^p \int_0^\infty \frac{|g(\tau)|^p}{\tau^\sigma} d\tau. \tag{2}$$

For $p > 1$ *and* $g \not\equiv 0$ *this inequality is strict, consequently, there is no extremal function, but the constant* $(|\sigma - 1|/p)^p$ *is sharp.*

In the book [1] this theorem is presented in three steps by the cases 1) $p = s = 2$, 2) $p = s > 1$ and 3) $1 \leq p < \infty$, $1 < s < \infty$, $-\infty < \sigma < 1$.

Notice that inequality (2) may be deduced from inequality (1) by the changes of variable $\tau = 1/t$ and of parameter $\sigma = 2 - s$.

It is not difficult to show that the condition of monotonicity of the function g in the Hardy theorem is not essential and one can show that the inequalities (1) and (2) are equivalent to the following inequality:
for every $p \in [1, \infty)$ *and every* $s \in \mathbb{R}$ *one has the variational inequality*

$$\int_0^\infty \frac{|g'(t)|^p}{t^{s-p}} dt \geq \left(\frac{|s-1|}{p}\right)^p \int_0^\infty \frac{|g(t)|^p}{t^s} dt \quad \forall g \in C_0^1((0, \infty)) \tag{3}$$

with the sharp constant $(|s-1|/p)^p$.

Suppose that $n \in \mathbb{N}$ and $n \geq 2$. Taking $s = s^* - n + 1$, using spherical coordinates $x = r\omega \in \mathbb{R}^n$ ($r = |x| > 0$, $\omega \in S := \{y \in \mathbb{R}^n : |y| = 1\}$), the formula $dx = r^{n-1} dr d\omega$ and the inequality $|\nabla u(x)| \geq |\partial u(x)/\partial r|$, one can prove that inequality (3) is equivalent to the following inequality:

for every $p \in [1, \infty)$ *and every* $s^* \in \mathbb{R}$ *one has the variational inequality*

$$\int_{\mathbb{R}^n} \frac{|\nabla u(x)|^p}{|x|^{s^*-p}} dx \geq \left(\frac{|s^* - n|}{p} \right)^p \int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^{s^*}} dx \quad \forall u \in C_0^1(\mathbb{R}^n \setminus \{0\}) \tag{4}$$

with the sharp constant $(|s^* - n|/p)^p$.

Here $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, $|x| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$, $\nabla u(x)$ is the Euclidean gradient of the function $u : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$, and

$$|\nabla u(x)|^2 = \sum_{j=1}^{n} \left(\frac{\partial u(x)}{\partial x_j} \right)^2, \quad dx = dx_1 dx_2 \cdots dx_n.$$

Now, we present a short proof of equivalence of inequalities (3) and (4). Clearly, if $u \in C_0^1(\mathbb{R}^n \setminus \{0\})$ and $\omega \in S$ is fixed, then inequality (3) implies that

$$\int_0^\infty \left| \frac{\partial u(r\omega)}{\partial r} \right|^p \frac{dr}{r^{s-p}} \geq \left(\frac{|s-1|}{p} \right)^p \int_0^\infty \frac{|u(r\omega)|^p}{r^s} dr,$$

which is equivalent to the inequality

$$\int_0^\infty \left| \frac{\partial u(r\omega)}{\partial r} \right|^p \frac{r^{n-1} dr}{|x|^{s^*-p}} \geq \left(\frac{|s^* - n|}{p} \right)^p \int_0^\infty \frac{|u(r\omega)|^p}{|x|^{s^*}} r^{n-1} dr,$$

where $|x| = r$ and $s^* = s + n - 1$. Multiplying the latter inequality by $d\omega$ and integrating over the unit sphere S, one obtains that

$$\int_{\mathbb{R}^n} \left| \frac{\partial u(r\omega)}{\partial r} \right|^p \frac{dx}{|x|^{s^*-p}} \geq \left(\frac{|s^* - n|}{p} \right)^p \int_{\mathbb{R}^n} \frac{|u(r\omega)|^p}{|x|^{s^*}} dx,$$

which implies (4). On the other hand, applying (4) to radial functions, defined by $u(x) \equiv u(|x|) =: g(|x|)$, one immediately obtains (3) with $s = s^* - n + 1$ and $t = r = |x|$.

Also, it is not difficult to show that inequality (3) is equivalent to the following inequality on the half-space $\mathbb{H}_n^+ = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : x_1 > 0\}$:
for every $p \in [1, \infty)$ *and every* $s \in \mathbb{R}$ *one has the variational inequality*

$$\int_{\mathbb{H}_n^+} \frac{|\nabla u(x)|^p}{x_1^{s-p}} dx \geq \left(\frac{|s-1|}{p} \right)^p \int_{\mathbb{H}_n^+} \frac{|u(x)|^p}{x_1^s} dx \quad \forall u \in C_0^1(\mathbb{H}_n^+) \tag{5}$$

with the sharp constant $(|s-1|/p)^p$.

The Hardy inequalities (3)–(5) are widely known. In the sequel we consider several inequalities on domains Ω of the Euclidean space $\mathbb{R}^n$. In particular, Hardy type inequalities for test functions $u : \Omega \to \mathbb{R}$ are generalizations of inequalities (4) and

(5) and have the following form

$$\int_{\Omega} \frac{|\nabla u(x)|^p}{\rho^{s-p}(x,\Omega)}dx \geq c_p(s,\Omega) \int_{\Omega} \frac{|u(x)|^p}{\rho^s(x,\Omega)}dx, \quad \forall u \in C_0^1(\Omega), \tag{6}$$

where $\rho(x,\Omega) := \mathrm{dist}(x,\partial\Omega)$.

Clearly, inequality (6) is similar to the Hardy inequality (4) or (5), except three changes. Namely, one changes the set of integration by the domain $\Omega \subset \mathbb{R}^n$, instead of $|x|$ or x_1 one takes the distance $\mathrm{dist}(x,\partial\Omega)$. Finally, it is necessary to replace the constant $(|s^* - n|/p)^p$ in (4) or the constant $(|s-1|/p)^p$ in (5) by a sharp constant $c_p(s,\Omega) \in [0,\infty)$. Of course, the sharp constant $c_p(s,\Omega)$ for a given domain $\Omega \subset \mathbb{R}^n$ is well-defined as the maximum possible constant in the variational inequality (6), but it is known for certain domains, only.

As basic problems one has to describe "nice" domains such that

$$c_p(s,\Omega) > 0$$

and to estimate this constant as a function of parameters $p \in [1,\infty)$, $s \in \mathbb{R}$, and global geometric characteristics of the domain $\Omega \subset \mathbb{R}^n$.

One can find many results on Hardy–Rellich inequalities in the books by F. Rellich [2] (1969), M. Reed and B. Simon [3] (1979), O. A. Ladyzhenskaya [4] (1985), V. Maz'ya [5] (1985), A. A. Balinsky, W. D. Evans and R. T. Lewis [6] (2015), M. Ruzhansky and D. Suragan [7] (2019), the author [8] (2020).

Via domains $\Omega \subset \mathbb{R}^n$ the constants $c_p(s,\Omega)$ depend on the dimension n. For several reasons we consider the case $n = 2$ separately.

2 Integral inequalities on plane domains

Consider domains $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, of the plane $\mathbb{C}$ of the complex variable $z = x + iy$. We need the distance function defined by

$$\rho(z,\Omega) := \inf_{w \in \mathbb{C}\setminus\Omega} |z - w|, \quad z \in \Omega.$$

For functions $u : \Omega \to \mathbb{R}$ we will use the notations $u = u(z)$ and

$$\nabla u(z) = \frac{\partial u(z)}{\partial x} + i\frac{\partial u(z)}{\partial y}, \quad z = x + iy \in \Omega.$$

Consider now the following Hardy type inequality

$$\iint_{\Omega} \frac{|\nabla u(z)|^p}{\rho^{s-p}(z,\Omega)}dx\,dy \geq c_p(s,\Omega) \iint_{\Omega} \frac{|u(z)|^p}{\rho^s(z,\Omega)}dx\,dy, \quad \forall u \in C_0^1(\Omega), \tag{7}$$

where $p \in [1,\infty)$, $s \in \mathbb{R}$ are fixed numbers, the constant $c_p(s,\Omega) \in [0,\infty)$ is sharp, i. e. it is defined as the maximum possible constant at this place.

Notice that the sharp constant $c_p(s, \Omega)$ in inequality (7) is a dimensionless quantity, invariant with respect to linear conformal transformations of the domain $\Omega \subset \mathbb{C}$, i. e.

$$c_p(s, \Omega) = c_p(s, a\Omega + b) \qquad (a \in \mathbb{C} \setminus \{0\}, b \in \mathbb{C}).$$

One has the following difficult question: is it possible to describe geometrically all domains for which $c_p(s, \Omega) > 0$? For any $p \in [1, \infty)$ an explicit answer to this question is known in three following cases (see Theorems 7, 2 and 3, below):

(1) if $s > 1$, then $c_p(s, \Omega) > 0$ for any convex domain $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$;

(2) if $s > 2$, then $c_p(s, \Omega) > 0$ for any domain $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$;

(3) $c_p(2, \Omega) > 0$ if and only if the boundary of the domain $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, is a uniformly perfect set.

If $c_p(s, \Omega) > 0$, then we have a natural problem to obtain lower and upper estimates of this constant.

Since the existence of extremal functions is unknown, one has an original situation for sharp constants. For example, to prove that the constant $c_2(2, \Omega) = 1/4$ for a certain domain $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, one has to prove that $c_2(2, \Omega) \geq 1/4$ and that $c_2(2, \Omega) \leq 1/4$.

There is a remarkable result, proved independently by several mathematicians: *the constant* $c_2(2, \Omega) = 1/4$ *for every convex domain* $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$ (see [9]–[16]). In particular, the constant equals $1/4$ for any disc. It is natural to presuppose that the Hardy constant $c_2(2, \Omega) \leq 1/4$ for non-convex domains.

Problem 1 (E. B. Davies [9], [13]) *Prove that* $c_2(2, \Omega) \leq 1/4$ *for every domain* $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$.

E. B. Davies proved that $c_2(2, \Omega) \leq 1/4$ *for a domain* $\Omega \subset \mathbb{C}$*, that has a boundary point* $z_0 \in \partial\Omega$*, "regular" in a certain sense. For example, there exists a neighborhood* $U(z_0)$*, such that the intersection* $U(y_0) \cap (\partial\Omega)$ *is a smooth arc. There is a weakening of this condition of "regularity" but we have no proof of the inequality* $c_2(2, \Omega) \leq 1/4$ *for arbitrary domains. Problem 1 is not solved even for the case of simply connected domains* $\Omega \subset \mathbb{C}$*, conformally equivalent to the unit disc. Currently we can claim that* $c_2(2, \Omega) \leq 1$ *for every domain* $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$*. This assertion is a consequence of conformally invariant inequalities and the Elstrodt-Patterson-Sullivan formula (see [17], [8], p. 102).*

Problem 1 is connected with the following

Problem 2 *Describe geometrically the family of non-convex domains* $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$*, such that* $c_2(2, \Omega) = 1/4$.

Currently there are several examples of non-convex domains for which $c_2(2, \Omega) = 1/4$*. We indicate two of them.*

Consider the sectors $\Omega_\beta = \{re^{i\theta} \in \mathbb{C} : 0 < r < 1, 0 < \theta < \beta\}$*. E. B. Davies [9] proved that the constant* $c_2(2, \Omega_\beta) = 1/4$ *if and only if* $\beta \leq \beta^* \approx 4.856$*. If* $\beta \in (\beta^*, 2\pi]$*, then* $c_2(2, \Omega_\beta) < 1/4$.

The critical value β^* of the angle is defined by E. B. Davies using numerical computations. There is a formula to find its sharp value, namely (see [16]):

$$\beta^* = 3\pi - 4 \arctan \frac{\Gamma^4(1/4)}{8\pi^2}.$$

In [18] we proved that the constant $c_2(2, A_{rR}) = 1/4$ for the concentric annuli $A_{rR} = \{z \in \mathbb{C} : r < |z| < R\}$ if and only if $R/r \le c^* \approx 36.6$, the critical value $c^* \approx 36.6$ is determined by an equation for hypergeometric functions of Gauss. If $R/r \in (c^*, \infty)$, then $c_2(2, A_{rR}) < 1/4$.

In addition, in the paper [16] we described a family $\Theta_{1/4}(2)$ of non-convex domains $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, such that $c_2(2, \Omega) = 1/4$.

Problem 3 *Find the sharp segment $[A^*, A^{**}]$ of variation of constants $c_2(2, \Omega)$ for simply connected domains $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$. Is the following assertion true: for every $\beta \in (A^*, A^{**})$ there exists a simply connected domain $\Omega \subset \mathbb{C}$, such that $c_2(2, \Omega) = \beta$?*

In [19] A. Ancona proved that

$$c_2(2, \Omega) \ge 1/16$$

for every simply connected domain $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$. Consequently, one has that $A^* \ge 1/16$. If the Davies conjecture is true then $A^{**} = 1/4$. Clearly, currently we can claim that $A^{**} \le 1$, only. Thus, it is known that $[A^*, A^{**}] \subset [1/16, 1]$.

To discuss Problems 4 —7 we will need the known definitions of domains $\Omega \subset \overline{\mathbb{C}}$ with uniformly perfect boundaries and characteristics $M(\Omega)$ and $M_0(\Omega)$.

Let $\Omega \subset \overline{\mathbb{C}}$ be a domain such that its boundary contains at least two points. Here $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the extended plane (the Riemann sphere). Let $\Omega_2 \subset \overline{\mathbb{C}}$ be a doubly connected domain, conformally equivalent to the concentric annulus $A(\Omega_2) = \{z \in \mathbb{C} : r < |z| < R\}$. Then the conformal modulus of Ω_2 is defined by

$$M(\Omega_2) = \frac{1}{2\pi} \ln \frac{R}{r} \in (0, \infty]$$

with a convention that $M(\Omega_2) = \infty$ in the case, when $r = 0$ or $R = \infty$.

First we give the definition of the conformal maximum modulus $M(\Omega)$.

Definition 1 Let $\Omega \subset \overline{\mathbb{C}}$ be a domain such that its boundary contains at least two points. The conformal maximum modulus $M(\Omega)$ is defined as follows.

(1) If Ω is a simply connected domain, then $M(\Omega) = 0$.

(2) If Ω is a doubly connected domain, then $M(\Omega)$ is the conformal modulus of this domain.

(3) If Ω is a multiply connected domain, then

$$M(\Omega) := \sup_{\Omega_2} M(\Omega_2),$$

where the supremum is taken over all doubly connected domains Ω_2 such that $\Omega_2 \subset \Omega$ and Ω_2 separates the boundary of the domain Ω.

It is clear that the conformal maximum modulus $M(\Omega)$ is a conformally invariant quantity.

To present the definition of the Euclidean maximum modulus $M_0(\Omega)$ we need the set $\mathbb{A}nn(\Omega)$ of concentric annuli

$$A = A(z_0; r, R) := \{z \in \mathbb{C} : r < |z - z_0| < R\},$$

with the following properties:

(1) $0 < r < R < \infty$, $A(z_0; r, R) \subset \Omega$;

(2) centers $z_0 \in \partial\Omega$;

(3) every annulus $A(z_0; r, R)$ separates the boundary of the domain Ω.

Definition 2 Let $\Omega \subset \overline{\mathbb{C}}$ be a domain such that its boundary contains at least two points, and let $\mathbb{A}nn(\Omega)$ be the set of annuli.

(1) If $\mathbb{A}nn(\Omega) = \emptyset$, then we take $M_0(\Omega) = 0$.

(2) If $\mathbb{A}nn(\Omega)$ is a non-empty set, then we take

$$M_0(\Omega) := \sup_{A \in \mathbb{A}nn(\Omega)} \frac{1}{2\pi} \ln \frac{R}{r}, \quad (A = A(z_0; r, R)).$$

It is clear that the quantity $M_0(\Omega)$ is "visible" in Euclidean geometry, but it is not conformally invariant in the general case.

It is evident that

$$0 \le M_0(\Omega) \le M(\Omega).$$

In addition, L. Carleson and T. W. Gamelin [20] indicate the following important property of the maximum moduli $M(\Omega)$ and $M_0(\Omega)$:

$$M_0(\Omega) < \infty \Longleftrightarrow M(\Omega) < \infty. \tag{8}$$

Following Ch. Pommerenke [21] (see, also, L. Carleson and T. W. Gamelin [20], T. Sugawa [22], [23]), in the case $M_0(\Omega) < \infty$ we say that the boundary of the domain Ω is a uniformly perfect set. Because of (8) one can replace the condition $M_0(\Omega) < \infty$ by the condition $M(\Omega) < \infty$.

There are simple inequalities that imply the property (8).

Proposition 1 *Let $\Omega \subset \overline{\mathbb{C}}$ be a domain such that its boundary contains at least two points. If $\Omega \subset \mathbb{C}$, then*

$$M_0(\Omega) \le M(\Omega) \le M_0(\Omega) + \frac{1}{2}. \tag{9}$$

If $\infty \in \Omega \subset \overline{\mathbb{C}}$, then

$$M_0(\Omega) \le M(\Omega) \le 2M_0(\Omega) + 1. \tag{10}$$

The inequality $M(\Omega) \leq M_0(\Omega) + 1/2$ in (9) is proved by F. G. Avkhadiev and K.-J. Wirths [24], inequality $M(\Omega) \leq 2M_0(\Omega) + 1$ in (9) is obtained by F. G. Avkhadiev [25] (see also the recent paper [26] by A. Golberg, T. Sugawa, M. Vuorinen for generalizations of (9) to higher dimensions).

One has that $M_0(\mathbb{D}') = M(\mathbb{D}') = \infty$ for the punctured disc

$$\mathbb{D}' := \{z : 0 < |z| < 1\}.$$

It is clear that $M(\Omega) = 0$ if and only if Ω is a simply connected domain conformally equivalent to the unit disc. The following example shows that there exist multiply connected domains for which $M_0(\Omega) = 0$.

Example 1 Let $\mathbb{K}$ be the classical Cantor set on the segment $[0, 1]$, and let $\Omega_0 := \{x + iy \in \mathbb{C} : |x| < \infty, |y| < 1\}$. Consider a domain defined by

$$\Omega(\mathbb{K}) = \Omega_0 \setminus \{x + iy \in \mathbb{C} : x \in \mathbb{K}, |y| \leq 3/4\}.$$

One has that $M_0(\Omega(\mathbb{K})) = 0$ since $\mathbb{A}nn(\Omega(\mathbb{K})) = \emptyset$.

In the paper [27] we proved

Theorem 2 *Suppose that $1 \leq p < \infty$ and that $2 < s < \infty$. Let Ω be an open proper subset of $\mathbb{C}$. Then for any real-valued function $u \in C_0^1(\Omega)$*

$$\iint_\Omega \frac{|\nabla u(z)|^p}{\rho^{s-p}(z,\Omega)} dx\, dy \geq \left(\frac{s-2}{p}\right)^p \iint_\Omega \frac{|u(z)|^p}{\rho^s(z,\Omega)} dx\, dy. \tag{11}$$

There exist domains Ω' such that $c_p(s, \Omega') = ((s-2)/p)^p$.

In addition, in the paper [28] we proved the following assertion:

if $M_0(\Omega') = \infty$, then the constant $c_p(s, \Omega') = ((s-2)/p)^p$ for all admissible values of parameters $p \in [1, \infty)$ and $s \in (2, \infty)$.

Problem 4 *Suppose that $1 \leq p < \infty$ and that $2 < s < \infty$. In geometrical terms describe all extremal domains in Theorem 2.*

Conjecture*: the constant $c_p(s, \Omega) = ((s-2)/p)^p$ in inequality (11) if and only if the boundary of the domain Ω is not a uniformly perfect set, i. e. the Euclidean maximum modulus $M_0(\Omega) = \infty$.*

Since the condition $M_0(\Omega) = \infty$ for the Euclidean maximum modulus implies equality $c_p(s, \Omega) = ((s-2)/p)^p$, one has to prove that the condition $M_0(\Omega) < \infty$ implies the strict inequality $c_p(s, \Omega) > ((s-2)/p)^p$.

In the limit case, when $s = 2$, we have

Theorem 3 *Let $1 \leq p < \infty$, and let $\Omega \subset \mathbb{C}$ be a domain such that $\Omega \neq \mathbb{C}$. Then the constant $c_p(2, \Omega) > 0$ if and only if the boundary of the domain Ω is a uniformly perfect set.*

For $p = 2$ Theorem 3 is proved by J. L. Fernández [29]. For $p \in [1, \infty) \setminus \{2\}$ Theorem 3 is proved in our paper [27]. In addition, in the paper [27] we proved the following estimates

$$c_p(2, \Omega) \geq \frac{c_1^p(2, \Omega)}{p^p} \quad (\forall p > 1),$$

$$c_1(2, \Omega) \geq \frac{1}{2\,(\pi M_0(\Omega) + \gamma_0)^2} \quad \left(\gamma_0 = \frac{\Gamma^4(1/4)}{4\pi^2} \approx 4.38\right).$$

By the way, a question on possible improvement of the indicated lower estimate for the quantity $c_1(2, \Omega)$ is still open.

Next, we consider the following Rellich type inequality:

$$\iint_\Omega \frac{|\Delta u(z)|^2}{\rho^{s-4}(z, \Omega)} dx\, dy \geq C_2(s, \Omega) \iint_\Omega \frac{|u(z)|^2}{\rho^s(z, \Omega)} dx\, dy, \quad \forall u \in C_0^2(\Omega), \tag{12}$$

where $s \in \mathbb{R}$ is a fixed number, the constant $C_2(s, \Omega) \in [0, \infty)$ is the sharp constant, i. e. it is defined as the maximum possible constant at this place.

In [2] F. Rellich proved that $C_2(4, \mathbb{C} \setminus \{0\}) = 0$. There are several generalizations of the Rellich result about inequality (12) on the domain $\Omega = \mathbb{C} \setminus \{0\}$. Finally, for this domain P. Caldiroli, R. Musina [30] proved the following remarkable theorem.

Theorem 4 *For every $s \in \mathbb{R}$ one has that*

$$C_2\,(s, \mathbb{C} \setminus \{0\}) = \min_{k \in \mathbb{N} \cup \{0\}} \left| k^2 - (s/2 - 1)^2 \right|.$$

Thus, the constant $C_2(2m, \mathbb{C} \setminus \{0\}) = 0$ for any $m \in \mathbb{Z}$. In addition, it is evident that $M_0(\mathbb{C} \setminus \{0\}) = \infty$.

In the paper [31] we proved

Theorem 5 *Let $\Omega \subset \mathbb{C}$ be a domain such that $\Omega \neq \mathbb{C}$. Then*

$$C_2(2, \Omega) > 0 \Longleftrightarrow M_0(\Omega) < \infty \Longleftrightarrow C_2(4, \Omega) > 0.$$

In addition, in the paper [32] we proved the following assertion: *if $m \in \mathbb{Z}$ then*

$$M_0(\Omega) = \infty \Longrightarrow C_2(2m, \Omega) = 0$$

for domains $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$.

Comparing this assertion, Theorems 4 and 5, it is natural to consider the following problem.

Problem 5 *Prove or disprove the following assertion:*
for every value of $m \in \mathbb{Z} \setminus \{1, 2\}$

$$M_0(\Omega) < \infty \Longleftrightarrow C_2(2m, \Omega) > 0$$

over the set of all domains $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$.

Now, we consider a new version of inequality (7) using the hyperbolic radius instead of the distance to the boundary.

Let $\Omega \subset \overline{\mathbb{C}}$ be a domain of hyperbolic type, i. e. its boundary contains at least three points (see L. V. Ahlfors [33], A. Yu. Solynin and M. Vuorinen [34]). In such a domain the hyperbolic radius is defined by

$$R(z, \Omega) := 1/\lambda(z, \Omega), \quad z \in \Omega,$$

where $\lambda(z, \Omega)$ is the coefficient of the Poincaré metric with Gaussian curvature $\kappa = -4$. If $\infty \in \Omega \subset \overline{\mathbb{C}}$, then $R(\infty, \Omega) = \rho(\infty, \Omega) = \infty$. More precisely, there exist finite limits

$$\lim_{z\to\infty} \frac{R(z, \Omega)}{|z|^2} = \lim_{z\to\infty} \frac{R(z, \Omega)}{\rho^2(z, \Omega)} > 0.$$

It is well known that $R(z, \Omega) \geq \rho(z, \Omega)$ at any point $z \in \Omega$. If $\infty \in \Omega \subset \overline{\mathbb{C}}$, then $\inf_{z\in\Omega} \rho(z, \Omega)/R(z, \Omega) = 0$. On the other hand, according to the Beardon-Pommerenke theorem [35]

$$\alpha(\Omega) := \inf_{z\in\Omega} \rho(z, \Omega)/R(z, \Omega) > 0 \Longleftrightarrow M_0(\Omega) < \infty$$

for every domain $\Omega \subset \mathbb{C}$ of hyperbolic type.

In the papers [21]–[24] one can find other relationship between the Euclidean characteristic $M_0(\Omega)$ and conformal characteristics of a domain Ω. In particular, it is proved that for every domain $\Omega \subset \mathbb{C}$ of hyperbolic type

$$\sup_{z\in\Omega} |\nabla R(z, \Omega)| < \infty \Longleftrightarrow M_0(\Omega) < \infty,$$

and that

$$|M_0(\Omega') - M_0(\Omega'')| \leq \frac{1}{2}$$

for conformally equivalent hyperbolic type domains $\Omega' \subset \mathbb{C}$ and $\Omega'' \subset \mathbb{C}$.

Let $\Omega \subset \overline{\mathbb{C}}$ be a domain of hyperbolic type. Consider the following conformally invariant inequality

$$\iint_\Omega \frac{|\nabla u(z)|^p}{R^{2-p}(z, \Omega)} dx\, dy \geq c_p^*(2, \Omega) \iint_\Omega \frac{|u(z)|^p}{R^2(z, \Omega)} dx\, dy, \quad \forall u \in C_0^1(\Omega), \tag{13}$$

where $p \in [1, \infty)$ is a fixed number, the constant $c_p^*(2, \Omega) \in [0, \infty)$ is sharp, i. e. it is defined as the maximum possible constant at this place.

Problem 6 *(see J. L. Fernández, J. M. Rodríguez [36]). In terms of the Euclidean geometry describe the set of all hyperbolic type domains $\Omega \subset \mathbb{C}$ such that $c_2^*(2, \Omega) > 0$.*

In [29] J. L. Fernández proved that the condition $M_0(\Omega) < \infty$ guarantees positivity of the constant $c_2^*(2, \Omega)$. In [36] J. L. Fernández, J. M. Rodríguez proved two theorems that give the existence of a family of hyperbolic type domains such that $M_0(\Omega) = \infty$,

$c_2^*(2, \Omega) > 0$, as well as the existence of a family of hyperbolic type domains such that $M_0(\Omega) = \infty$, $c_2^*(2, \Omega) = 0$.

For $p \in [1, \infty) \setminus \{2\}$ the properties of the constant $c_p^*(2, \Omega)$ and its generalizations are studied in the papers [37], [25] and [38]. Clearly, for $p \in [1, \infty) \setminus \{2\}$ one has a natural generalization of Problem 6.

Now, we will attract reader's attention to an optimistic problem.

If $s > 2$, then the constant $c_p(s, \Omega) \geq ((s-2)/p)^p > 0$ for all domains $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$ (see Theorem 2). In the case $s = 2$, according to Theorem 3, the constant $c_p(2, \Omega) > 0$ if and only if the boundary of the domain Ω is a uniformly perfect set. In addition, in the paper [25] we proved the following assertion that presents a universal inequality.

Theorem 6 *Let $\Omega \subset \overline{\mathbb{C}}$ be a hyperbolic type domain. Then*

$$\iint_\Omega \frac{|\nabla u(z)|}{\rho(z, \Omega)}\, dxdy \geq 2 \iint_\Omega \frac{|u(z)|}{R^2(z, \Omega)}\, dxdy \quad \forall u \in C_0^1(\Omega).$$

Problem 7 *Using the radius $R(z, \Omega)$ and the distance $\rho(z, \Omega)$ construct new integral inequalities that are universal in the sense to be valid with a positive constant on every hyperbolic type domain $\Omega \subset \mathbb{C}$.*

3 Integral inequalities on domains of $\mathbb{R}^n$, $n \geq 2$

We will consider domain $\Omega \subset \mathbb{R}^n$ for fixed $n \geq 2$. Again, we need the distance function defined by

$$\rho(x, \Omega) := \inf_{y \in \mathbb{R}^n \setminus \Omega} |x - y|, \quad x = (x_1, x_2, ..., x_n) \in \Omega,$$

and the Hardy type inequality

$$\int_\Omega \frac{|\nabla u(x)|^p}{\rho^{s-p}(x, \Omega)} dx \geq c_p(s, \Omega) \int_\Omega \frac{|u(x)|^p}{\rho^s(x, \Omega)} dx, \quad \forall u \in C_0^1(\Omega), \tag{14}$$

where $dx = dx_1 dx_2 ... dx_n$, $p \in [1, \infty)$ and $s \in \mathbb{R}$ are fixed numbers, the constant $c_p(s, \Omega) \in [0, \infty)$ is sharp, i. e. it is defined to be the maximum possible constant in inequality (14).

Inequality (14) is invariant with respect to linear conformal and anticonformal transformations of the domain Ω. In particular, the constant $c_p(s, \Omega)$ is a dimensionless quantity such that

$$c_p(s, \Omega) = c_p(s, k\,\Omega + x_0) \quad (\forall k \in \mathbb{R} \setminus \{0\}, \ \forall x_0 \in \mathbb{R}^n). \tag{15}$$

We begin by an assertion that is basic for convex domains $\Omega \subset \mathbb{R}^n$ and test functions $u : \Omega \to \mathbb{R}$, $u \in C_0^1(\Omega)$.

Theorem 7 *Suppose that*

$$n \geq 2, \quad 1 \leq p < \infty, \quad 1 < s < \infty,$$

and that $\Omega \subset \mathbb{R}^n$ *is a convex domain such that* $\Omega \neq \mathbb{R}^n$. *Then for every real-valued function* $u \in C_0^1(\Omega)$

$$\int_\Omega \frac{|\nabla u(x)|^p}{\rho^{s-p}(x,\Omega)} dx \geq \left(\frac{s-1}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{\rho^s(x,\Omega)} dx. \tag{16}$$

The constant $((s-1)/p)^p$ *is sharp, more precisely,* $c_p(s,\Omega) = ((s-1)/p)^p$ *for every convex domain* $\Omega \neq \mathbb{R}^n$ *and for all admissible values of parameters* $p \in [1,\infty)$ *and* $s \in (1,\infty)$.

Theorem 7 is proved in several papers (for the case $p = s > 1$ see T. Matskewich, P. E. Sobolevskii [10], M. Marcus, V. J. Mitzel, Y. Pinchover [11], A. A. Balinsky, W. D. Evans, R. T. Lewis [6], and for the general case, when $p \in [1,\infty)$ and $s \in (1,\infty)$, see F. G. Avkhadiev [27], F. G. Avkhadiev and I. K. Shafigullin [39]).

Notice that a convex domain satisfies automatically the condition of the next theorem, proved in [39].

Theorem 8 *Suppose that* $n \geq 2$, $1 \leq p < \infty$, $1 < s < \infty$, *and* $\Omega \subset \mathbb{R}^n$ *is a domain satisfying the property: there exist a boundary point* y_0, *two* n*-dimensional balls* $B^+ \subset \Omega$ *and* $B^- \subset \mathbb{R}^n \setminus \Omega$ *such that*

$$y_0 \in (\partial B^+) \cap (\partial B^-) \cap (\partial \Omega).$$

Then $c_p(s,\Omega) \leq ((s-1)/p)^p$.

The restriction $1 < s < \infty$ in Theorem 7 is natural: if $-\infty < s \leq 1$, then $c_p(s,B) = 0$ for any n-dimensional ball $B \subset \mathbb{R}^n$. Because of formula (15) it is sufficient to consider the case of the unit ball.

Proposition 2 *Suppose that* $n \geq 2$, $1 \leq p < \infty$, *but* $-\infty < s \leq 1$. *Then* $c_p(s,B) = 0$ *for the unit ball* $B = \{x = r\omega \in \mathbb{R}^n : 0 \leq r < 1\}$.

Proof of Proposition 2. Suppose the contrary, namely, suppose that $c_p(s,B) > 0$ for some fixed $p \in [1,\infty)$ and $s \in (-\infty,1]$. Then there exists a number $\delta \in (0,1)$ such that $c_p(s,B) \geq \delta$. Therefore for every real-valued function $u \in C_0^1(B)$

$$\int_B \frac{|\nabla u(r\omega)|^p}{(1-r)^{s-p}} r^{n-1} dr d\omega \geq \delta \int_B \frac{|u(r\omega)|^p}{(1-r)^s} r^{n-1} dr d\omega.$$

Taking radial functions $u(r\omega) \equiv g(r)$ one obtains

$$\int_0^1 \frac{|g'(r)|^p}{(1-r)^{s-p}} r^{n-1} dr \geq \delta \int_0^1 \frac{|g(r)|^p}{(1-r)^s} r^{n-1} dr. \tag{17}$$

It is clear that inequality (17) must be valid for any absolutely continuous function $g : [0, 1] \to \mathbb{R}$ satisfying the boundary condition $g(1) = 0$.

Let ε be a parameter such that $\varepsilon \in (0, 1)$. For functions $g = g_\varepsilon$ defined by

$$g_\varepsilon(r) = \varepsilon, \quad 0 \le r \le 1 - \varepsilon; \qquad g_\varepsilon(r) = 1 - r, \quad 1 - \varepsilon < r \le 1,$$

inequality (17) gives that

$$\int_{1-\varepsilon}^{1} \frac{r^{n-1}}{(1-r)^{s-p}} dr \ge \delta \int_{1-\varepsilon}^{1} \frac{r^{n-1}}{(1-r)^{s-p}} dr + \delta \varepsilon^p \int_0^{1-\varepsilon} \frac{r^{n-1}}{(1-r)^s} dr.$$

Therefore, for any $\varepsilon \in (0, 1)$

$$Y_s(\varepsilon) := \frac{1-\delta}{\varepsilon^p \delta} \int_{1-\varepsilon}^{1} \frac{r^{n-1}}{(1-r)^{s-p}} dr \ge X_s(\varepsilon) := \int_0^{1-\varepsilon} \frac{r^{n-1}}{(1-r)^s} dr.$$

Letting $\varepsilon \to 0$ we obtain a contradiction. Indeed, one has that

$$\lim_{\varepsilon \to 0} Y_1(\varepsilon) = \frac{1-\delta}{p\delta}, \qquad \lim_{\varepsilon \to 0} X_1(\varepsilon) = \infty$$

in the case $s = 1$ and that

$$\lim_{\varepsilon \to 0} Y_s(\varepsilon) = 0, \qquad \lim_{\varepsilon \to 0} X(\varepsilon) = \int_0^1 \frac{r^{n-1}}{(1-r)^s} dr \in (0, \infty)$$

in the case $s < 1$. Thus, the proof of Proposition 2 is complete.

Problem 8 *(a generalization of Problem 1, see I. K. Shafigullin [40]). Suppose that $n \ge 3$, $p \in [1, \infty)$. Prove that the constant $c_p(2, \Omega) \le (n-2)^p / p^p$ for all domains $\Omega \subset \mathbb{R}^n$, $\Omega \ne \mathbb{R}^n$.*

The dimension n plays an essential role in the case of non-convex domains. In particular, the constant $c_2(2, \mathbb{R}^n \setminus \{0\}) = (n-2)^2/4 > 1/4$ for $n \ge 4$. Thus, in the case $n \ge 4$ there exists "exotic" domain Ω with $c_2(2, \Omega) > 1/4$.

In the paper [40] I. K. Shafigullin examined Problem 8. In particular, he proved certain estimates of the form

$$c_2(2, \Omega) \le c\, n^2 \quad (c = \text{const} > 0)$$

for arbitrary domains $\Omega \subset \mathbb{R}^n$, $\Omega \ne \mathbb{R}^n$.

If $n \ge 2$, then $c_2(2, \Omega) = 1/4$ for all convex domains $\Omega \subset \mathbb{R}^n$, $\Omega \ne \mathbb{R}^n$ and for some non-convex domains. Therefore, one can formulate a generalization of Problem 2.

Problem 9 *Suppose that $n \ge 3$. Describe geometrically the family of non-convex domains $\Omega \subset \mathbb{R}^n$, $\Omega \ne \mathbb{R}^n$, such that $c_2(2, \Omega) = 1/4$.*

In the paper [16] we described geometrically a family $\Theta_{1/4}(n)$ of non-convex domains $\Omega \subset \mathbb{R}^n$, $\Omega \ne \mathbb{R}^n$ with the property $c_2(2, \Omega) = 1/4$.

In the next Theorem we shall present a simple subfamily of $\Theta_{1/4}(3)$ *from the paper [16], using the inradius*

$$\rho(\Omega) := \sup_{x \in \Omega} \rho(x, \Omega)$$

and the following definition of a family of non-convex domains.

Definition 3 Suppose that $n \geq 2$ and $\lambda \in (0, \infty)$.

A domain $\Omega \subset \mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$, is called λ-close-to-convex, if for every point $y \in (\partial\Omega) \setminus \{\infty\}$ there exists a point x_y such that

$$|y - x_y| = \lambda \quad \text{and} \quad B_y = \{x \in \mathbb{R}^n : |x - x_y| < \lambda\} \subset \mathbb{R}^n \setminus \overline{\Omega}.$$

In other words, a domain $\Omega \neq \mathbb{R}^n$ is λ-close-to-convex, if every finite boundary point of this domain satisfies the exterior sphere condition with prescribed radius $\lambda \in (0, \infty)$.

Remark 1 Suppose that $n \geq 2$ and $\lambda \in (0, \infty)$. If $\Omega' \subset \mathbb{R}^n$ is a domain λ-close-to-convex, then the domain $\Omega := \Omega' \times \mathbb{R} \subset \mathbb{R}^{n+1}$ is λ-close-to-convex, too.

Theorem 9 (see [16]) *Let* $\Omega \subset \mathbb{R}^3$ *be a non-convex domain with finite inradius* $\rho(\Omega)$. *If the domain* Ω *is* λ*-close-to-convex with a radius* $\lambda = \lambda(\Omega)$ *such that*

$$\lambda(\Omega) \geq \rho(\Omega),$$

then $c_2(2, \Omega) = 1/4$.

Example 2 Consider two domains $\Omega_2 \subset \mathbb{R}^2$ and $\Omega_3 \subset \mathbb{R}^3$ defined by

$$\Omega_2 = \{(x, y) \in \mathbb{R}^2 : 0 < x < \infty,\ 0 < y < 1/x\},$$

$$\Omega_3 = \Omega_2 \times \mathbb{R} = \{(x, y, z) \in \mathbb{R}^3 : 0 < x < \infty,\ 0 < y < 1/x,\ -\infty < z < \infty\}.$$

It is clear that the domain Ω_2 is λ-close-to-convex with the radius $\lambda = \min R(x)$, where $R(x)$ is the radius of curvature of the hyperbola at the point $(x, 1/x)$. We have that

$$R(x) = \frac{(1 + y'^2(x))^{3/2}}{|y''(x)|} = \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right)^{3/2} \geq R(1) = \sqrt{2},$$

where $y = y(x) = 1/x$, $0 < x < \infty$. Therefore, Ω_2 and Ω_3 are λ-close-to-convex with $\lambda = \lambda(\Omega_2) = \lambda(\Omega_3) = \sqrt{2}$.

On the other hand, it is clear that

$$\rho(\Omega_2) := \sup_{(x,y) \in \Omega_2} \rho((x, y), \Omega_2) \leq 1.$$

Consequently, $\rho(\Omega_3) = \rho(\Omega_2) \leq 1$ (in fact, $\rho(\Omega_2) = \rho(\Omega_3) = 2 - \sqrt{2}$). We see that $\lambda(\Omega_3) > \rho(\Omega_3)$. Thus, the Hardy constant $c_2(2, \Omega_3) = 1/4$ by Theorem 9.

In [16] for the case $n \geq 2$ we proved a general version of Theorem 9 connected with the condition $\Lambda_n \lambda(\Omega) \geq \rho(\Omega)$, where Λ_n is a constant defined as a root of an equation for hypergeometric functions. In particular, $\Lambda_2 \approx 2.49$, $\Lambda_3 = 1$, $\Lambda_4 \approx 0.61$, and $1/(n-2) < \Lambda_n < 1$ for $n > 3$.

Next, we will need a definition (see, for instance, [26,27]) about the Euclidean maximum moduli $M_0(\Omega)$ for spatial domains $\Omega \subset \mathbb{R}^n$, having at least two boundary points. Let $\mathbb{A}nn(\Omega)$ be the set of domains

$$A = A(x_0; r,\ R) := \{x \in \mathbb{R}^n : r < |x - x_0| < R\},$$

with the following properties: $0 < r < R < \infty$, $A(x_0; r,\ R) \subset \Omega$; $x_0 \in \partial\Omega$.

Definition 4 Let $n \geq 3$, and let $\Omega \subset \mathbb{R}^n$ be a domain, having at least two boundary points.

1) If $\mathbb{A}nn(\Omega) = \emptyset$, then we take $M_0(\Omega) = 0$.

2) If $\mathbb{A}nn(\Omega)$ is a non-empty set, then we take

$$M_0(\Omega) := \sup_{A \in \mathbb{A}nn(\Omega)} \frac{1}{2\pi} \ln \frac{R}{r}, \quad (A = A(x_0; r,\ R)).$$

The following assertion is a not difficult geometrical exercise.

Proposition 3 *Suppose that $\Omega \subset \mathbb{R}^n$ is a domain, λ-close-to-convex with a radius $\lambda = \lambda(\Omega) \in (0, \infty)$. If the inradius $\rho(\Omega) < \infty$, then*

$$e^{2\pi\ M_0(\Omega)} \leq 1 + \frac{\rho(\Omega)}{\lambda(\Omega)}.$$

In the paper [27] we proved the following generalization of Theorem 2 (also, see [28] concerning the case $M_0(\Omega') = \infty$).

Theorem 10 *Suppose that $n \geq 3$, $1 \leq p < \infty$, $n < s < \infty$, and that Ω is a proper open subset of $\mathbb{R}^n$. Then for every real-valued function $u \in C_0^1(\Omega)$*

$$\int_\Omega \frac{|\nabla u(x)|^p}{\rho^{s-p}(x, \Omega)} dx \geq \left(\frac{s-n}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{\rho^s(x, \Omega)} dx.$$

There exist domains Ω', for which the constant $((s-n)/p)^p$ is sharp, moreover, if $M_0(\Omega') = \infty$, then $c_p(s, \Omega') = ((s-n)/p)^p$.

Problem 10 *Suppose that $n \geq 3$, $1 \leq p < \infty$, $n < s < \infty$, $\Omega \subset \mathbb{R}^n$ is a domain having at least two boundary points. Is it true the following assertion: the condition $M_0(\Omega) < \infty$ implies the strict inequality $c_p(s, \Omega) > ((s-n)/p)^p$.*

It is sufficient to prove the following assertion: the condition $M_0(\Omega) < \infty$ implies the strict inequality $c_1(s, \Omega) > s - n$.

Remark 2 Suppose that $n \geq 2$, $1 \leq p < \infty$, but $-\infty < s \leq n$. Then there exist domains $\Omega' \subset \mathbb{R}^n$ and $\Omega'' \subset \mathbb{R}^n$, such that $c_p(s, \Omega') > 0$ and $c_p(s, \Omega'') = 0$.

Remark 3 If $s > n$ and $\Omega \subset \mathbb{R}^n$ is a domain such that $\mathbb{R}^n \setminus \Omega$ is a non-empty compact set, then $M_0(\Omega) = \infty$. Therefore, $c_p(s, \Omega') = ((s-n)/p)^p$ according to Theorem 10.

Recently, in the paper [41] F. G. Avkhadiev and R. V. Makarov proved the following theorem.

Theorem 11 *Suppose that $n \geq 2$, $1 \leq p < \infty$, $-\infty < s < n$, and that $\Omega \subset \mathbb{R}^n$ is a domain such that $\mathbb{R}^n \setminus \Omega$ is a non-empty convex compact set. Then*

$$c_p(s, \Omega) \geq c_{psn} := \min_{k=1,2,\ldots,n} \frac{|s-k|^p}{p^p},$$

i. e. for every real-valued function $u \in C_0^1(\Omega)$

$$\int_\Omega \frac{|\nabla u(x)|^p}{\rho^{s-p}(x, \Omega)} dx \geq c_{psn} \int_\Omega \frac{|u(x)|^p}{\rho^s(x, \Omega)} dx.$$

There exist admissible domains $\Omega' \subset \mathbb{R}^n$ for which the constant c_{psn} is sharp.

It is useful to compare Theorem 11 and the case $-\infty < \sigma < 1$ of the Hardy Theorem 1 with the boundary condition $g(+\infty) = 0$.

Taking into account formulas (4) and (5), Proposition 2 and Theorem 11 one can formulate the following problem.

Problem 11 *Suppose that $n \geq 2$, $1 \leq p < \infty$,but $-\infty < s \leq 1$. Prove that $c_p(s, \Omega) = 0$ for any bounded domain $\Omega \subset \mathbb{R}^n$.*

Because of the absence of extremal functions one can improve several Hardy and Rellich type inequalities with sharp constants using some positive remainders. In this direction there are several interesting results due to V. Maz'ya, H. Brezis, M. Marcus and other mathematicians (see [5]–[8], [12]–[15], [42,43]).

We describe some examples on inequalities with remainders. First, consider the classical Poincaré-Friedrichs inequality

$$\int_\Omega |\nabla u(x)|^2 \, dx \;\geq\; \lambda_1(\Omega) \int_\Omega |u(x)|^2 \, dx, \quad \forall u \in C_0^1(\Omega), \tag{18}$$

where $\lambda_1(\Omega)$ is the first eigenvalue of the Dirichlet problem for the Laplace equation. In the paper [12] H. Brezis, M. Marcus proved the following assertion.

Theorem 12 *Let $n \geq 2$. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded convex domain and $\lambda = (1/4)/\,(diam\,(\Omega))^2$, then*

$$\int_\Omega |\nabla u(x)|^2 \, dx \;\geq\; \frac{1}{4} \int_\Omega \frac{|u(x)|^2}{\rho^2(x, \Omega)} \, dx + \lambda \int_\Omega |u(x)|^2 \, dx, \quad \forall u \in C_0^1(\Omega). \tag{19}$$

In [12] there is a question: is it possible that one takes $\lambda = c_n/(vol(\Omega))^{2/n}$ in inequality (19) with a positive constant c_n? In [14], M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and A. Laptev proved that the answer to the question is positive and $c_n \geq (n/4)\omega_n^{2/n}$, where $\omega_n = 2\pi^{2/n}/(n\Gamma(n/2))$ is the volume of the unit ball in $\mathbb{R}^n$. But the sharp value of c_n is unknown.

Clearly, the choice of λ by H. Brezis, M. Marcus in inequality (19) is connected with the Poincaré–Friedrichs inequality (18), the Poincaré estimate

$$\lambda_1(\Omega) \geq \pi^2/(diam\ \Omega)^2$$

and the isoperimetric inequality of Rayleigh-Faber-Krahn

$$\lambda_1(\Omega)\ \geq\ \omega_n^{2/n}\, j_{n/2-1}^2/(vol\,(\Omega))^{2/n},$$

where j_ν is the first zero of the Bessel function J_ν of order ν.

There are several open problems about Hardy and Rellich type inequalities with sharp constants and some positive remainders. We indicate one of them formulated explicitly as a conjecture in the paper [42] by F. G. Avkhadiev and K.-J. Wirths.

Problem 12 *Prove that among all n-dimensional domains with given inradius* $\rho(\Omega) := \sup_{x\in\Omega} \rho(x,\Omega)$ *the maximum of the best Brezis-Marcus constants* λ *in (19) is presented by* B_n*, where* B_n *is an n-dimensional ball of radius* $\rho(\Omega)$.

Next, we consider a natural parametric generalization of inequality (14) on domains $\Omega \subset \mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$ $(n \geq 2)$*:*

$$\left(\int_\Omega \frac{|\nabla u(x)|^p dx}{\rho^\alpha(x,\Omega)}\right)^{1/p} \geq c_{pq}(s,\alpha,\Omega)\left(\int_\Omega \frac{|u(x)|^q dx}{\rho^s(x,\Omega)}\right)^{1/q},\ \ \forall u \in C_0^1(\Omega), \quad (20)$$

where $p \in [1,\infty)$, $q \in [1,\infty)$, $\alpha \in \mathbb{R}$ *and* $s \in \mathbb{R}$ *are fixed numbers, the constant* $c_{pq}(s,\alpha,\Omega) \in [0,\infty)$ *is sharp, i. e. the maximum possible at this place.*

There are a few non-trivial results on inequality (20) in the non-standard case, when $s \in [1,\infty)$*, but* $\alpha \neq s - p$ *(see, for instance, [27,44]).*

Problem 13 *Suppose that* $n \geq 2$ *and that parameters* $p \in [1,\infty)$, $q \in [1,\infty)$, $\alpha \in \mathbb{R}$, $s \in \mathbb{R}$ *are fixed numbers. In terms of the Euclidean geometry describe non-trivial families of domains* $\Omega \subset \mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$*, such that* $c_{pq}(s,\alpha,\Omega) > 0$.

It is evident that Theorem 3 gives a solution to this problem in the case when $n = s = 2$, $p = q \in [1,\infty)$, $\alpha = 2 - p$.

Now, we will describe a problem, connected with Problem 13 and the classical Poincaré–Friedrichs inequality (18).

Observe that $c_{22}(0,0,\Omega) = \lambda_1(\Omega)$*, where* $\lambda_1(\Omega)$ *is the first eigenvalue of the Dirichlet problem for the Laplace equation.*

Problem 14 *Let* $n \geq 2$*. In terms of the Euclidean geometry describe all domains* $\Omega \subset \mathbb{R}^n$*, such that*

$$\rho(\Omega) := \sup_{x\in\Omega} \rho(x,\Omega) < \infty \Longrightarrow \lambda_1(\Omega) > 0.$$

On implications $\rho(\Omega) < \infty \Longrightarrow \lambda_1(\Omega) > 0$ *there are several interesting results (see, for instance, R. Osserman [45]), but Problem 14 is still open even in the case of dimension* $n = 2$*. We have to note that Problem 14 is one of many interesting and* *difficult problems connected with Dirichlet and Neumann eigenvalues for the Laplacian (see, for instance, the recent paper [46] by V. Gol'dshtein, R. Hurri-Syrjänen, V. Pchelintsev, A. Ukhlov).*

Let $m \geq 2$ be a fixed natural number. For smooth functions $u \in C^m(\Omega)$ consider the polyharmonic operators defined by

$$\Delta^{m/2}u := \begin{cases} \Delta^j u, & \text{if } m = 2j \text{ is an even number}, \\ \nabla\Delta^j u, & \text{if } m = 2j+1 \text{ is an odd number}, \end{cases}$$

with a formal convention $\Delta^{1/2}u := \nabla u$. Thus, the function $\Delta^{m/2}u$ is well-defined for every natural number m (see the book [47] by F. Gazzola, H. Ch. Grunau, G. Sweers on polyharmonic boundary value problems).

Consider the following generalization of Hardy–Rellich inequalities:

$$\int_\Omega \frac{|\Delta^{m/2}u(x)|^2}{\rho^{s-2m}(x,\Omega)}dx \geq A_2^{(m)}(s,\Omega)\int_\Omega \frac{|u(x)|^2}{\rho^s(x,\Omega)}dx, \quad \forall u \in C_0^m(\Omega), \tag{21}$$

where the constant $A_2^{(m)}(s,\Omega) \in [0,\infty)$ is chosen to be maximum possible.

For a real-valued function $u \in C_0^m(\Omega)$ one has the generalized Ladyghenskaya identity (see [4], ch. 2, (6.26) for m=2 and [47], ch. 2, (2.12) for the general case):

$$\int_\Omega \left|\Delta^{m/2}u(x)\right|^2 dx = \int_\Omega \sum_{k_1=1}^n \sum_{k_2=1}^n \cdots \sum_{k_m=1}^n \left(\frac{\partial^m u(x)}{\partial x_{k_1}\partial x_{k_2}\cdots\partial x_{k_m}}\right)^2 dx. \tag{22}$$

If $s = 2m$, then the left hand part in (21) has the form $\int_\Omega \left|\Delta^{m/2}u(x)\right|^2 dx$ and one can use formula (22). In several papers this fact is used to examine inequality (21) in the case $s = 2m$ (see, for instance, the papers [48]–[51]).

In the paper [48] M. P. Owen proved that $A_2^{(m)}(2m,\Omega) \geq ((2m-1)!!)^2/4^m$ for every convex domain $\Omega \neq \mathbb{R}^n$ and that this estimate is optimal since it is sharp for the half-space $x_1 > 0$.

For any convex domain $\Omega \neq \mathbb{R}^n$ in the papers [31] and [50] we proved that the opposite estimate $A_2^{(m)}(2m,\Omega) \leq ((2m-1)!!)^2/4^m$ is valid. Consequently, one has the following assertion.

Theorem 13 *Suppose that $n \geq 2$ and $m \geq 2$. Then for every convex domain $\Omega \subset \mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$*

$$A_2^{(m)}(2m,\Omega) = \frac{((2m-1)!!)^2}{4^m}.$$

We exclude the case $m = 1$ since $A_2^{(1)}(2,\Omega) \equiv c_2(2,\Omega) = 1/4$.

For non-convex plane domains in [51] we proved

Theorem 14 *Suppose that $m \geq 2$, $\Omega \subset \mathbb{C}$ is a domain such that $\Omega \neq \mathbb{C}$. Then*

$$A_2^{(m)}(2m,\Omega) \geq ((m-1)!)^2 c_2(2,\Omega),$$

$$A_2^{(m)}(2m, \Omega) > 0 \Longleftrightarrow M_0(\Omega) < \infty.$$

In the case $s \neq 2m$, we have

Problem 15 *Suppose that $n \geq 2$, $m \geq 2$, $s \in \mathbb{R}$, $s \neq 2m$. In terms of the Euclidean geometry describe non-trivial families of domains $\Omega \subset \mathbb{R}^n$, such that $A_2^{(m)}(s, \Omega) > 0$.*

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Declarations

Conflict of interest The author declares that he has no conflict of interest.

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