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The dispersionless Veselov–Novikov equation: symmetries, exact solutions, and conservation laws

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Abstract
We study symmetries, invariant solutions, and conservation laws for the dispersionless Veselov–Novikov equation. The emphasis is placed on cases when the ODEs involved in description of the invariant solutions are integrable by quadratures. Then we find some non-invariant solutions, in particular, solutions that are polynomials of an arbitrary degree $N \geq 3$ with respect to the spatial variables. Finally we compute all conservation laws that are associated to cosymmetries of second order.

Keywords Dispersionless Veselov–Novikov equation · Symmetry · Exact solution · Conservation law

Mathematics Subject Classification 35G20 · 35Q60 · 17B50 · 22E70

1 Introduction

We consider the dispersionless Veselov-Novikov equation (dVN) [11] written in the form

$$u_{txy} = (u_{xx}u_{xy})_x + (u_{xy}u_{yy})_y. \tag{1}$$

This equation describes the propagation of the high frequency electromagnetic waves in certain nonlinear media, see [12] and references therein. Nontrivial t-independent

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solutions of dVN are related to the problem of existence of the first integrals for the geodesic or magnetic geodesic flows on a two-torus, [23]. Equation (1) is the dispersionless reduction of the Nizhnik–Veselov–Novikov equation [17,24]. The Lax representation [21]

$$\begin{cases} q_t = \dfrac{1}{3}\left(q_x^3 - \dfrac{u_{xy}^3}{q_x^3}\right) + u_{xx}\,q_x - \dfrac{u_{xy}\,u_{yy}}{q_x}, \\ q_y = -\dfrac{u_{xy}}{q_x}. \end{cases} \tag{2}$$

of dVN is the dispersionless reduction of the Lax representation of the Nizhnik–Veselov–Novikov equation. In [2] the Lax representation (2) was used to construct two-dimensional reductions of dVN.

In the present paper we study exact solutions and conservation laws of dVN. We find the contact symmetry algebra and the explicit form for the transformations from the contact symmetry pseudogroup of dVN. Then we employ the pseudogroup to find the optimal system of one-dimensional subalgebras of the symmetry algebra. We factorize dVN with respect to the symmetries from the optimal system and obtain two-dimensional partial differential equations (PDEs) (8) and (50) for the invariant solutions as well as their Lax representations. Then we find the symmetry algebras and their optimal systems of one-dimensional subalgebras for equations (8) and (50). The factorization with respect to the subalgebras provide the collection of ordinary differential equations (ODEs) that describe invariant solutions to (8) and (50). We find some cases when the obtained ODEs are integrable by quadratures, thus providing exact solutions for dVN. Further, we study solutions that are not invariant with respect to contact symmetries. In particular, we find a class of solutions to dVN that are polynomials in x and y of arbitrary degree. Finally we find the whole set of conservation laws that are associated to cosymmetries defined on the second order jets.

2 Preliminaries

The presentation in this section closely follows [13,15,25]. Let $\pi\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $\pi\colon (x^1, \dots, x^n, u) \mapsto (x^1, \dots, x^n)$, be a trivial bundle, and $J^\infty(\pi)$ be the bundle of its jets of infinite order. The local coordinates on $J^\infty(\pi)$ are (x^i, u, u_I), where $I = (i_1, \dots, i_n)$ are multi-indices, and for every local section $f\colon \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}$ of π the corresponding infinite jet $j_\infty(f)$ is a section $j_\infty(f)\colon \mathbb{R}^n \to J^\infty(\pi)$ such that $u_I(j_\infty(f)) = \dfrac{\partial^{\#I} f}{\partial x^I} = \dfrac{\partial^{i_1+\cdots+i_n} f}{(\partial x^1)^{i_1} \dots (\partial x^n)^{i_n}}$. We put $u = u_{(0,\dots,0)}$. Also, we will simplify notation in the following way, e.g., in the case of $n = 3$: we denote $x^1 = t$, $x^2 = x$, $x^3 = y$, and $u_{(i,j,k)} = u_{t\dots tx\dots xy\dots y}$ with i times t, j times x, and k times y.

The vector fields

$$D_{x^k} = \frac{\partial}{\partial x^k} + \sum_{\#I \ge 0} u_{I+1_k}\,\frac{\partial}{\partial u_I}, \qquad k \in \{1, \dots, n\},$$

$(i_1, \dots, i_k, \dots, i_n) + 1_k = (i_1, \dots, i_k + 1, \dots, i_n)$, are called *total derivatives*. They commute everywhere on $J^\infty(\pi)$ and are annihilated by the ideal of contact forms $\langle du_I - \sum_{k=1}^n u_{I+1_k}\, dx^k \mid \#I \geq 0\rangle$.

The *evolutionary vector field* associated to an arbitrary smooth function $\varphi\colon J^\infty(\pi) \to \mathbb{R}$ is defined as

$$\mathbf{E}_\varphi = \sum_{\#I \geq 0} D_I(\varphi)\, \frac{\partial}{\partial u_I}$$

with $D_I = D_{(i_1,\dots i_n)} = D_{x^1}^{i_1} \circ \dots \circ D_{x^n}^{i_n}$.

A PDE $F(x^i, u_I) = 0$ of order $s \geq 1$ with $\#I \leq s$ defines the submanifold $\mathcal{E} = \{(x^i, u_I) \in J^\infty(\pi) \mid D_K(F(x^i, u_I)) = 0,\ \#K \geq 0\}$ in $J^\infty(\pi)$.

A function $\varphi\colon J^\infty(\pi) \to \mathbb{R}$ is called a *(generator of an infinitesimal) symmetry* of equation $\mathcal{E}$ when $\mathbf{E}_\varphi(F) = 0$ on $\mathcal{E}$. The symmetry φ is a solution to the *defining system*

$$\ell_{\mathcal{E}}(\varphi) = 0, \tag{3}$$

where $\ell_{\mathcal{E}} = \ell_F|_{\mathcal{E}}$ with the differential operator

$$\ell_F = \sum_{\#I \geq 0} \frac{\partial F}{\partial u_I}\, D_I.$$

The *symmetry algebra* $\mathrm{Sym}(\mathcal{E})$ of equation $\mathcal{E}$ is the linear space of solutions to (3) endowed with the structure of a Lie algebra over $\mathbb{R}$ by the *Jacobi bracket* $\{\varphi, \psi\} = \mathbf{E}_\varphi(\psi) - \mathbf{E}_\psi(\varphi)$. The *algebra of contact symmetries* $\mathrm{Sym}_0(\mathcal{E})$ is the Lie subalgebra of $\mathrm{Sym}(\mathcal{E})$ defined as $\mathrm{Sym}(\mathcal{E}) \cap C^\infty(J^1(\pi))$. The *point symmetries* are the contact symmetries whose generators are polynomials in u_{x^i} of degree 1.

The contact symmetry pseudogroup of a PDE $\mathcal{E} \subset J^\infty(\pi)$ is the collection of all the local diffeomorphisms $\varGamma_\infty\colon J^\infty(\pi) \to J^\infty(\pi)$ that preserve the submanifold $\mathcal{E}$ as well as the ideal of contact forms.

Let ϕ be a symmetry of $\mathcal{E}$, then the *ϕ–invariant solution* of $\mathcal{E}$ is the solution of the compatible system $F = 0, \phi = 0$.

For a PDE $\mathcal{E}$ in three independent variables t, x, y a *conservation law*, [18, § 4.3], [25, Ch. 5], is a horizontal two-form

$$\varOmega = P_1\, dx \wedge dy + P_2\, dy \wedge dt + P_3\, dt \wedge dx,$$

closed with respect to the *horizontal differential* d_h, which means that

$$d_h \varOmega = \big(D_t(P_1) + D_x(P_2) + D_y(P_3)\big)\, dt \wedge dx \wedge dy = 0$$

on $\mathcal{E}$. Functions P_i are smooth functions on $\mathcal{E}$. The conservation law is referred to as a *trivial conservation law* when there exists a horizontal one-form ω such that $\varOmega = d_h\omega$. Nontrivial conservation laws are associated with cosymmetries of equation

$\mathcal{E}$, see discussion in [14, Ch. 1]. Let $\tilde{P}_i$ be arbitrary extensions of P_i on $J^\infty(\pi)$, then for $\tilde{\Omega} = \tilde{P}_1\, dx \wedge dy + \tilde{P}_2\, dy \wedge dt + \tilde{P}_3\, dt \wedge dx$ there holds $d_h \tilde{\Omega} = \psi \cdot F\, dt \wedge dx \wedge dy$ for some function ψ on $J^\infty(\pi)$. The restriction $\psi|_{\mathcal{E}}$ depends on Ω only and is called the *generating function* or *characteristic* of the conservation law Ω. The conservation law is trivial if and only if its generating function vanishes. Generating functions are solutions to equation

$$\ell^*_{\mathcal{E}}(\psi) = 0 \tag{4}$$

with $\ell^*_{\mathcal{E}} = \ell^*_F|_{\mathcal{E}}$, where the adjoint operator to ℓ_F is

$$\ell^*_F = \sum_{\#I \geq 0} (-1)^{|I|}\, D_I \circ \frac{\partial F}{\partial u_I}.$$

A solution of (4) is referred to as a *cosymmerty* of equation $\mathcal{E}$.

3 Symmetries of dVN

3.1 The symmetry algebra

Direct computations[1] show that the contact symmetry algebra $\mathrm{Sym}_0(\mathrm{dVN})$ is generated by functions

$$\begin{aligned}
\phi_0(A) &= -A\, u_t - \tfrac{1}{3}\, A'\, (x\, u_x + y\, u_y) - \tfrac{1}{18}\, A''\, (x^3 + y^3),\\
\phi_{1,1}(A) &= -A\, u_x - \tfrac{1}{2}\, A'\, x^2,\\
\phi_{2,1}(A) &= -A\, u_y - \tfrac{1}{2}\, A'\, y^2,\\
\phi_{1,2}(A) &= -A\, x,\\
\phi_{2,2}(A) &= -A\, y,\\
\phi_3(A) &= A,\\
\psi &= 3\, u - x\, u_x - y\, u_y,
\end{aligned}$$

where $A = A(t)$ and $B = B(t)$ below are arbitrary smooth functions of t. Actually, all the contact symmetries of (1) turn out to be point symmetries. The structure of the Lie algebra $\mathrm{Sym}_0(\mathrm{dVN})$ is defined by the commutator table

$$\begin{aligned}
\{\phi_0(A), \phi_0(B)\} &= \phi_0(A\, B' - B\, A'),\\
\{\phi_0(A)\, \phi_{i,j}(B)\} &= \phi_{i,j}\left(A\, B' - \left(1 - \tfrac{2}{3}\, j\right)\, B\, A'\right), \qquad i, j \in \{1, 2\},\\
\{\phi_0(A), \phi_3(B)\} &= \phi_3(A\, B'),\\
\{\phi_{i,1}(A), \phi_{j,1}(B)\} &= \delta_{ij}\, \phi_{i,2}(A\, B' - B\, A'),
\end{aligned}$$

[1] Computations of symmetries, their commutators, cosymmetries, and conservation laws were supported by the *Jets* software [1].

$$
\begin{aligned}
\{\phi_{i,1}(A), \phi_{j,2}(B)\} &= \delta_{ij}\, \phi_3(A\, B'),\\
\{\phi_{i,1}(A), \phi_3(B)\} &= 0,\\
\{\phi_{i,2}(A), \phi_{j,2}(B)\} &= 0,\\
\{\phi_{i,2}(A), \phi_3(B)\} &= 0,\\
\{\phi_3(A), \phi_3(B)\} &= 0,\\
\{\psi, \phi_0(A)\} &= 0,\\
\{\psi, \phi_{i,j}(A)\} &= -j\, \phi_{i,j}(A),\\
\{\psi, \phi_3(A)\} &= -3\, \phi_3(A).
\end{aligned}
$$

Direct computations show that the contact symmetry pseudogroup of equation (1) is generated by the infinite prolongations of the (local) diffeomorphisms $\Gamma_0\colon (t, x, y, u) \mapsto (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u})$ of the form

$$
\begin{cases}
\tilde{t} = B_0, \qquad \tilde{x} = \varepsilon\left(B_0'\, x + B_1\right), \qquad \tilde{y} = \varepsilon\left(B_0'\, y + B_2\right),\\
\tilde{u} = \varepsilon^3 \Big(u - \dfrac{B_0''}{18 B_0'}\,(x^3 + y^3) - \dfrac{1}{2(B_0')^{1/3}}\,(B_1'\, x^2 + B_2'\, y^2)\\
\qquad + B_3\, x + B_4\, y + B_5\Big),
\end{cases}
\tag{5}
$$

where $\varepsilon \neq 0$, $B_i = B_i(t)$ are arbitrary functions, $B_i' = \dfrac{dB_i}{dt}$, and $B_0'(t) \neq 0$. In other words, substitution for (5) into dVN written in the tilded variables yields (1).

3.2 The optimal system of one-dimensional subalgebras

Since the symmetry algebra of dVN is infinite-dimensional and depends on 6 arbitrary functions of one variable, the problem of examining all invariant solutions is very complicated. To overcome the difficulty, we use the following observation: transformations from the symmetry pseudogroup (5) preserve equation (1), while changing the symmetry generators. Therefore we can classify the orbits of the action of (5) on the $\mathrm{Sym}_0(\mathrm{dVN})$. In order to use symmetries for computing invariant solutions we consider symmetries whose generators depend explicitly on at least one of the variables u, u_t, u_x, or u_y.

Proposition 1 *Each symmetry*

$$
\Phi = \phi_0(A_0) + \phi_{1,1}(A_{1,1}) + \phi_{2,1}(A_{2,1}) + \phi_{2,1}(A_{2,1}) + \phi_{2,2}(A_{2,2}) + \phi_3(A_3) + \mu\,\psi
$$

from $\mathrm{Sym}_0(\mathrm{dVN})$ *with* $A_0^2 + \mu^2 + A_{1,1}^2 + A_{2,1}^2 \not\equiv 0$ *is equivalent with respect to the action of the pseudogroup (5) to one of symmetries*

$$
\begin{aligned}
\chi_1 &= \phi_0(1) + \mu\,\psi = -u_t + \mu\,(3\,u - x\,u_x - y\,u_y),\\
\chi_2 &= \psi = 3\,u - x\,u_x - y\,u_y,\\
\chi_3 &= \phi_{1,1}(1) + \phi_{2,1}(A) = -u_x - A\,u_y - \tfrac{1}{2}\,A'\,y^2,
\end{aligned}
$$

$$\chi_4 = \phi_{2,1}(1) + \phi_{1,2}(A) = -u_y - A\,x.$$

Proof Let $A_0(t) \neq 0$. Put $\varepsilon = 1$ and consider solutions $B_0(t)$, ... , $B_5(t)$ to the system of ODEs

$$\begin{cases} B_0' = A_0^{-1}, \\ B_1' = \mu\, A_0^{-1}\, B_1 - A_0^{-4/3}\, A_{1,1}, \\ B_2' = \mu\, A_0^{-1}\, B_2 - A_0^{-4/3}\, A_{2,1}, \\ B_3' = \frac{1}{3}\, A_0^{-1}\, (6\,\mu - A_0')\, B_3 + \mu\, A_0^{-5/3}\, A_{1,1}\, B_1 - A_0^{-2}\, (A_0\, A_{1,2} + A_{1,1}^2), \\ B_4' = \frac{1}{3}\, A_0^{-1}\, (6\,\mu - A_0')\, B_4 + \mu\, A_0^{-5/3}\, A_{2,1}\, B_2 - A_0^{-2}\, (A_0\, A_{2,2} + A_{2,1}^2), \\ B_5' = A_0^{-1}\, (3\,\mu\, B_5 - A_{1,1}\, B_3 - A_{2,1}\, B_4 - A_3). \end{cases}$$

Direct computations show that for $\tilde{\chi}_1 = -\tilde{u}_{\tilde{t}} + \mu\,(3\,\tilde{u} - \tilde{x}\,\tilde{u}_{\tilde{x}} - \tilde{y}\,\tilde{u}_{\tilde{y}})$ there holds $\Gamma_1^*(\tilde{\chi}_1) = \Phi$, where Γ_1 is the first prolongation of (5). Therefore the inverse diffeomorphism maps Φ to $\tilde{\chi}_1$.

When $A_0(t) \equiv 0$ and $\mu \neq 0$, put $\varepsilon = \mu^{1/3}$ and define (5) by $B_0 = t$, $B_1 = \mu^{-1}\, A_{1,1}$, $B_2 = \mu^{-1}\, A_{2,1}$, $B_3 = \frac{1}{2}\,\mu^{-2}\,(\mu\, A_{1,2} - A_{1,1}\, A_{1,1}')$, $B_4 = \frac{1}{2}\,\mu^{-2}\,(\mu\, A_{2,2} - A_{2,1}\, A_{2,1}')$, $B_5 = \frac{1}{6}\,\mu^{-3}\,(2\,\mu^2\, A_3 + \mu\,(A_{1,1}\, A_{1,2} + A_{2,1}\, A_{2,2}) - A_{1,1}^2\, A_{1,1}' - A_{2,1}^2\, A_{2,1}')$. Then we have $\Gamma_1^*(\tilde{\chi}_2) = \Phi|_{A_0=0}$.

Suppose now that $A_0(t) \equiv 0$, $\mu = 0$, $A_{1,1}(t) \neq 0$, and $A_{2,1}(t) \neq 0$. Put $\varepsilon = 1$ and define functions B_i, $A(\tilde{t})$ by equations $B_0' = A_{1,1}^{-3}$, $B_1' = A_{1,2}A_{1,1}^{-2}$, $B_2' = (A_{2,2} + (A_{1,1}A_{2,1}' - A_{2,1}A_{1,1}')\, B_2)\, A_{1,1}^{-1}A_{2,1}^{-1}$, $B_3 = \frac{1}{2}\,(A_{1,1}A_{2,1}' - A_{2,1}A_{1,1}')\, B_2^2 - A_3\, A_{1,1}^{-1}$, $B_4 = 0$, and $A(B_0(t)) = A_{2,1}(t)/A_{1,1}(t)$. Then $\Gamma_1^*(\tilde{\chi}_3) = \Phi|_{A_0=0,\mu=0}$.

Finally, when $A_0(t) \equiv 0$, $\mu = 0$, $A_{1,1}(t) \equiv 0$, and $A_{2,1}(t) \neq 0$, put $\varepsilon = 1$ and define functions B_i, $A(\tilde{t})$ by equations $B_0' = A_{2,1}^{-3}$, $B_1 \equiv 0$, $B_2 = A_{2,2}A_{2,1}^{-2}$, $B_3 \equiv 0$, $B_4 = -A_3\, A_{2,1}^{-1}$, and $A(B_0(t)) = A_{1,2}(t)\, A_{2,1}(t)$. Then $\Gamma_1^*(\tilde{\chi}_4) = \Phi|_{A_0=0,\mu=0,A_{1,1}=0}$. □

4 Invariant solutions

In this section we analyze reductions of dVN with respect to the symmetries χ_1, ... , χ_4.

4.1 Reduction w.r.t. χ_1

The χ_1–invariant solutions of dVN satisfy equation (1) and

$$\chi_1 = -u_t + \mu\,(3\,u - x\,u_x - y\,u_y) = 0,$$

so they are of the form

$$u = \mathrm{e}^{3\mu t}\,\hat{U}(s, w), \qquad s = x\,\mathrm{e}^{-\mu t}, \qquad w = y\,\mathrm{e}^{-\mu t}. \tag{6}$$

Table 1 Commutator table of the Lie algebra $\mathrm{Sym}_0(\mathcal{E}_\mu)$

	η_2	η_3	η_4	η_5	η_6
η_1	$-\eta_2$	$-\eta_3$	$-2\,\eta_4$	$-2\,\eta_5$	$-3\,\eta_6$
η_2		0	η_6	0	0
η_3			0	η_6	0
η_4				0	0
η_5					0

We introduce function

$$\hat{U} = U - \frac{\mu}{6}\,(s^3 + w^3) \tag{7}$$

for convenience of the further computations. Substitution for (6), (7) into (1) shows that $U(z, w)$ is a solution to equation $\mathcal{E}_\mu$ defined by

$$U_{www} = -U_{sss} - \frac{U_{ss}U_{ssw} + U_{ww}U_{sww}}{U_{sw}} + 3\,\mu. \tag{8}$$

This equation admits a Lax representation. Indeed, symmetry χ_1 has the lift $(\chi_1, \hat{\chi}_1)$ with $\hat{\chi}_1 = -q_t + \mu\,\left(\frac{3}{2}\,q - x\,q_x - y\,q_y\right)$ to the Lax representation (2). Solutions to equation $\hat{\chi}_1 = 0$ are of the form $q = \mathrm{e}^{\frac{3}{2}\,\mu\,t}\,Q(s, w)$. Substituting this into (2) yields the Lax representation

$$\begin{cases} Q_s^6 &= -3\,U_{ss}\,Q_s^4 + \dfrac{9}{2}\,\mu\,Q\,Q_s^3 + 3\,U_{sw}\,U_{ww}\,Q_s^2 + U_{sw}^3, \\ Q_w &= -\dfrac{U_{sw}}{Q_s} \end{cases}$$

for equation (8). We put $Q_s = S$ and obtain another Lax representation for (8):

$$\begin{cases} S_s &= -\dfrac{S}{2}\,\dfrac{(3\,\mu + 2\,U_{sss})\,S^4 - 2\,(U_{ww}U_{ssw} + U_{sw}U_{sww})\,S^2 - 2\,U_{sw}^2U_{ssw}}{S^6 + U_{ss}\,S^4 + U_{sw}U_{ww}\,S^2 + U_{sw}^3}, \\ S_w &= -\dfrac{S}{2}\,\dfrac{2\,U_{ssw}S^4 + (2\,U_{ss}U_{ssw} + U_{sw}\,(2\,U_{sss} + 3\,\mu)\,S^2 - 2\,U_{sw}U_{sww}}{S^6 + U_{ss}\,S^4 + U_{sw}U_{ww}\,S^2 + U_{sw}^3}. \end{cases}$$

Now we analyze invariant solutions to equation (8). We consider cases $\mu \neq 0$ and $\mu = 0$ separately.

4.1.1 Case $\mu \neq 0$.

When $\mu \neq 0$, the symmetry algebra $\mathrm{Sym}_0(\mathcal{E}_\mu)$ of equation (8) is generated by functions $\eta_1 = 3\,U - s\,U_s - w\,U_w$, $\eta_2 = -U_s$, $\eta_3 = -U_w$, $\eta_4 = s$, $\eta_5 = w$, $\eta_6 = 1$. The structure of this Lie algebra is defined by the commutators in Table 1.

Table 2 The adjoint reperesentation of the Lie group G_μ

	η_1	η_2	η_3	η_4	η_5	η_6
η_1	η_1	$e^{-\tau}\eta_2$	$e^{-\tau}\eta_3$	$e^{-2\tau}\eta_4$	$e^{-2\tau}\eta_5$	$e^{-3\tau}\eta_6$
η_2	$\eta_1+\tau\,\eta_2$	η_2	η_3	$\eta_4+\tau\,\eta_6$	η_5	η_6
η_3	$\eta_1+\tau\,\eta_3$	η_2	η_3	η_4	$\eta_5+\tau\,\eta_6$	η_6
η_4	$\eta_1+2\,\tau\,\eta_4$	$\eta_2-\tau\,\eta_6$	η_3	η_4	η_5	η_6
η_5	$\eta_1+2\,\tau\,\eta_5$	η_2	$\eta_3-\tau\,\eta_6$	η_4	η_5	η_6
η_6	$\eta_1+3\,\tau\,\eta_6$	η_2	η_3	η_4	η_5	η_6

The adjoint representation of the Lie group G_μ associated with the finite-dimensional Lie algebra $\mathrm{Sym}_0(\mathcal{E}_\mu)$ is defined by the Lie series

$$\mathrm{Ad}_{\tau\,\eta_i}(\eta_j) = \exp(\tau\,\mathrm{ad}\,\eta_i)(\eta_j) = \sum_{k\ge 0}\frac{\tau^k}{k!}(\mathrm{ad}\,\eta_i)^k(\eta_j),$$

where $\mathrm{ad}\,\eta_i(\eta_j) = \{\eta_i, \eta_j\}$. This representation is given by Table 2. In this table the (i, j)-th entry is the expression for $\mathrm{Ad}_{\tau\eta_i}(\eta_j)$. Using this table one can classify the orbits of action of the adjoint representation of the Lie group G_μ on its Lie algebra $\mathrm{Sym}_0(\mathcal{E}_\mu)$:

Proposition 2 *Each symmetry of equation $\mathcal{E}_\mu$ with $\mu \neq 0$ is equivalent under the action of the adjoint representation of G_μ to one of the symmetries*

$$\begin{aligned}
\zeta_1 &= \eta_1 = 3\,U - s\,U_s - w\,U_w,\\
\zeta_2 &= \eta_3 + \alpha\,\eta_2 + \beta\,\eta_4 + \gamma\,\eta_5 = -U_w - \alpha\,U_s + \beta\,s + \gamma\,w,\\
\zeta_3 &= \eta_2 + \beta\,\eta_4 + \gamma\,\eta_5 = -U_s + \beta\,s + \gamma\,w,
\end{aligned}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$.

Proof is obtained by the standard computation, see, e.g., [18, § 3.3]. □

Thus each invariant solution to equation (8) can be obtained by an action of appropriate superposition of transformations $\mathrm{Ad}_{\tau\,\eta_i}(\eta_j)$ from ζ_k–invariant solutions. Below we examine such solutions.

4.1.1.1. Solutions invariant w.r.t. ζ_1.
The ζ_1–invariant solutions satisfy (8) and $\zeta_1 = 0$, therefore they are of the form

$$U = s^3\,W(z), \qquad z = w\,s^{-1}, \tag{9}$$

where W is a solution to the ODE

$$W_{zzz} = \frac{(5\,z^3-1)\,W_{zz}^2 - z\,(22\,z\,W_z - 18\,W + 3\,\mu)\,W_{zz}}{2\,\big((z^3-1)\,W_{zz} - (3\,z^3-1)\,W_z + 3\,z^2\,W\big)}$$

$$+\frac{20\,z\,W_z^2-6\,(4\,W-\mu)\,W_z}{2\,\left((z^3-1)\,W_{zz}-(3\,z^3-1)\,W_z+3\,z^2\,W\right)}. \tag{10}$$

We could not obtain the general solution to this ODE. Instead, we find a family of particular solutions given by

$$W=\frac{1}{18\,a_1\,a_2}\left((3\,\mu\,a_1a_2-4\,a_1^3+2\,a_2^3)\,z^3+3\,\mu\,a_1a_2-4\,a_2^3+2\,a_1^3\right)+a_2\,z^2+a_1\,z,$$

where $a_1a_2\neq 0$. Then (6), (7), (9) give the t-independent solution

$$u=x\,y\,(a_1\,x+a_2\,y)+\frac{1}{9\,a_1a_2}\left((a_1^3-2\,a_2^3)\,x^3+(a_2^3-2\,a_1^3)\,y^3\right) \tag{11}$$

to dVN.

4.1.1.2. Solutions invariant w.r.t. ζ_2.
For ζ_2–invariant solutions there holds $\zeta_2=0$, or

$$U_w+\alpha\,U_s-\beta\,s-\gamma\,w=0.$$

When $\alpha\neq 0$, this equation gives

$$U=V(z)+\frac{\beta}{2\,\alpha}\,s^2+\frac{\gamma}{2}\,w^2,\qquad z=s-\alpha\,w. \tag{12}$$

Substituting for (12) into (8) yields

$$V_{zzz}=-\frac{3\,\alpha\,\mu\,V_{zz}}{2\,\alpha\,(\alpha^3-1)\,V_{zz}-\beta+\alpha^2\,\gamma}. \tag{13}$$

This equation is integrable by quadratures. When $\beta=\alpha^2\,\gamma$, $\alpha\neq -1$, the general solution to (13) is

$$V=-\frac{\mu}{4\,(\alpha^3-1)}\,z^3+c_2\,z^2+c_1\,z+c_0,$$

where c_0, c_1, c_2 are arbitrary constants. This function produces solution

$$\begin{aligned}u=&\frac{\mu}{12\,(\alpha^3-1)}\left((\alpha^3+2)\,y^3-(2\,\alpha^3+1)\,x^3\right)+\frac{3\,\alpha\,\mu}{4\,(\alpha^3-1)}\,x\,y\,(x-\alpha\,y)\\&+\frac{1}{2}\,\mathrm{e}^{\mu\,t}\left((\alpha\,\gamma+2\,c_2)\,x^2+(\gamma+2\,c_2\,\alpha^2)\,y^2-4\,c_2\,\alpha\,x\,y\right)\\&+c_1\,\mathrm{e}^{2\,\mu\,t}\,(x-\alpha\,y)+c_0\,\mathrm{e}^{3\,\mu\,t}\end{aligned} \tag{14}$$

of dVN.

Remark 1 If u is a solution to (1), then for arbitrary functions $b_0(t)$, $b_1(t)$, $b_2(t)$ the linear combination $u + b_0(t) + b_1(t)\, x + b_2(t)\, y$ is a solution to dVN as well. So we can put $c_0 = c_1 = 0$ in (14) without loss of generality. In what follows we will ignore the linear in x and y terms in solutions of dVN. □

When $\beta \neq \alpha^2\,\gamma$, $\alpha \neq -1$, equation (13) has the first integral

$$2\,\alpha\,(\alpha^3 - 1)\,V_{zz} + (\alpha^2\,\gamma - \beta)\,\ln V_{zz} = -3\,\alpha\,\mu\,z + c_0, \qquad c_0 \in \mathbb{R}.$$

hence the general solution in this case is of form

$$V = \int \left(\int H(-3\,\alpha\,\mu\,z + c_0)\,dz \right) dz, \tag{15}$$

where function $H(\tau)$ is defined by formula

$$2\,\alpha\,(\alpha^3 - 1)\,H(\tau) + (\alpha^2\,\gamma - \beta)\,\ln H(\tau) \equiv \tau. \tag{16}$$

This function can be expressed in terms of the Lambert W function, [6], while the expression is too complicated to be written here.

When $\alpha = -1$, $\beta \neq \gamma$, the general solution of (13) reads

$$V = c_2\,\exp\left(\frac{3\,\mu\,z}{\beta - \gamma}\right) + c_1\,z + c_0, \qquad c_i \in \mathbb{R},$$

to dVN of the form

$$u = c_2\,\exp\left(3\,\mu\,\left(t + \mathrm{e}^{-\mu\,t}\,\frac{x - y}{\beta - \gamma}\right)\right) - \frac{\mu}{6}\,(x^3 + y^3) + \frac{1}{2}\,\mathrm{e}^{\mu\,t}(\beta\,x^2 + \gamma\,y^2), \tag{17}$$

cf. Remark 1.

Finally, when $\alpha = 0$ we have solution

$$U = \frac{\mu}{2}\,s^3 + \frac{\gamma}{2}\,w^2 + \beta\,s\,w + c_2\,z^2 + c_1\,z + c_0$$

of equation (8). This produces the following solution to dVN:

$$u = \frac{\mu}{6}\,(2\,x^3 - y^3) + \mathrm{e}^{\mu\,t}\,(\beta\,x\,y + c_2\,x^2 + \gamma\,y^2). \tag{18}$$

4.1.1.3. Solutions invariant w.r.t. ζ_3

The ζ_3–invariant solution of equation (8) has the form

$$U = \frac{\beta}{2}\,s^2 + \gamma\,s\,w + c_0 + c_1\,w + c_2\,w^3 + \frac{\mu}{2}\,w^3, \qquad c_i \in \mathbb{R},$$

the corresponding solution of dVN is obtained from (18) by renaming $x \leftrightarrow y$, $\beta \leftrightarrow \gamma$.

Table 3 Commutator table of the Lie algebra $\mathrm{Sym}_0(\mathcal{E}_0)$

	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7
ξ_1	$-\xi_2$	$-\xi_3$	ξ_4	ξ_5	0	0
ξ_2		0	ξ_6	0	0	0
ξ_3			0	ξ_6	0	0
ξ_4				0	0	ξ_4
ξ_5					0	ξ_5
ξ_6						ξ_6

Table 4 The adjoint action of the Lie group G_0

	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7
ξ_1	ξ_1	$\mathrm{e}^{-\tau}\xi_2$	$\mathrm{e}^{-\tau}\xi_3$	$\mathrm{e}^{\tau}\xi_4$	$\mathrm{e}^{\tau}\xi_5$	ξ_6	ξ_7
ξ_2	$\xi_1+\tau\,\xi_2$	ξ_2	ξ_3	$\xi_4+\tau\,\xi_6$	ξ_5	ξ_6	ξ_7
ξ_3	$\xi_1+\tau\,\xi_3$	ξ_2	ξ_3	ξ_4	$\xi_5+\tau\,\xi_6$	ξ_6	ξ_7
ξ_4	$\xi_1-\tau\,\xi_4$	$\xi_2-\tau\,\xi_6$	ξ_3	ξ_4	ξ_5	ξ_6	$\xi_7-\tau\,\xi_4$
ξ_5	$\xi_1-\tau\,\xi_5$	ξ_2	$\xi_3-\tau\,\xi_6$	ξ_4	ξ_5	ξ_6	$\xi_7-\tau\,\xi_5$
ξ_6	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	$\xi_7-\tau\,\xi_6$
ξ_6	ξ_1	ξ_2	ξ_3	$\mathrm{e}^{-\tau}\,\xi_4$	$\mathrm{e}^{-\tau}\,\xi_5$	$\mathrm{e}^{-\tau}\,\xi_6$	ξ_7

Remark 2 For each solution $u = f(t, x, y)$ of dVN expression $u = f(t, y, x)$ defines a solution as well. □

4.1.2 Case $\mu = 0$

Now consider equation $\mathcal{E}_0$ obtained by putting $\mu = 0$ in (8). Notice that solutions of this equation produce t-independent solutions to dVN, and t-independent solutions are defined up to a nonzero constant multiple. The symmetry algebra $\mathrm{Sym}_0(\mathcal{E}_0)$ of this equation has generators $\xi_1 = -s\,U_s - w\,U_w$, $\xi_2 = -U_s$, $\xi_3 = -U_w$, $\xi_4 = s$, $\xi_5 = w$, $\xi_6 = 1$, and $\xi_7 = U$, with the commutators given in Table 3.

The adjoint action of the symmetry group G_0 of equation $\mathcal{E}_0$ on $\mathrm{Sym}_0(\mathcal{E}_0)$ is defined by the Table 4. Then direct computations give the optimal system of one-dimensional subalgebras of $\mathrm{Sym}_0(\mathcal{E}_0)$:

Proposition 3 *Each symmetry of equation $\mathcal{E}_0$ is equivalent under the action of the adjoint representation of G_0 to one of the following symmetries:*

$$\begin{aligned}
\sigma_1 &= \xi_1 + \alpha\,\xi_7 = -s\,U_s - w\,U_w + \alpha\,U,\\
\sigma_2 &= \xi_1 + \alpha\,\xi_6 = -s\,U_s - w\,U_w + \alpha,\\
\sigma_3 &= \xi_1 + \xi_7 + \xi_4 + \alpha\,\xi_5 = -s\,U_s - w\,U_w + U + s + \alpha\,w,\\
\sigma_4 &= \xi_1 + \xi_7 + \xi_5 = -s\,U_s - w\,U_w + U + w,\\
\sigma_5 &= \xi_2 + \alpha\,\xi_3 + \beta\,\xi_7 = -U_s - \alpha\,U_w + \beta\,U, \qquad \alpha \neq 0\\
\sigma_6 &= \xi_2 + \alpha\,\xi_3 + \xi_4 + \beta\,\xi_5 = -U_s - \alpha\,U_w + s + \beta\,w,
\end{aligned}$$

$$\sigma_7 = \xi_2 + \alpha\, \xi_3 + \xi_5 = -U_s - \alpha\, U_w + w,$$
$$\sigma_8 = \xi_2 + \alpha\, \xi_3 = -U_s - \alpha\, U_w,$$
$$\sigma_9 = \xi_3 + \alpha\, \xi_7 = -U_w + \alpha\, U,$$
$$\sigma_{10} = \xi_3 + \alpha\, \xi_4 + \beta\, \xi_5 = -U_w + \alpha\, s + \beta\, w.$$

4.1.2.1. Solutions invariant w.r.t. σ_1.
The σ_1–invariant solutions to $\mathcal{E}_0$ have the form

$$U = s^{\alpha}\, W(z), \qquad z = w\, s^{-1}.$$

Substituting this into (8) gives the ODE

$$W_{zzz} = (\alpha - 2)\, \frac{F(z, W, W_z, W_{zz})}{G(z, W, W_z, W_{zz})}, \tag{19}$$

where

$$F(z, W, W_z, W_{zz}) = (5\, z^3 - 1)\, W_{zz}^2 - (\alpha - 1)\, z\, (11\, z\, W_z - 3\, \alpha\, W)\, W_{zz} + (\alpha - 1)^2\, (5\, z\, W_z - 2\, \alpha\, W)\, W_z$$

and

$$G(z, W, W_z, W_{zz}) = 2\, z\, (z^3 - 1)\, W_{zz} - (\alpha - 1)\, (3\, z^3 - 1)\, W_z + \alpha\, (\alpha - 1)\, z^2\, W.$$

We did not find the general solution to this equation for any α. For each α equation (19) admits solutions of the form

$$W = (z + z_0)^{\alpha}, \tag{20}$$

where z_0 is a root of equation $z_0\, (z_0^3 + 1) = 0$. Thus we get solution

$$u = (y + z_0\, x)^{\alpha} \tag{21}$$

of equation (1). We found some other solutions of (19) for $\alpha \in \{1, 2, 3, 4\}$.

Equation (19) with $\alpha = 1$

$$W_{zzz} = -\frac{1}{2}\, \frac{5\, z^3 - 1}{z\, (z^3 - 1)}\, W_{zz}.$$

is integrable by quadratures, its general solution

$$W(z) = c_0 + c_1\, z + \int \left(\int z^{-1/2}\, (z^3 - 1)^{-2/3}\, dz \right) dz \tag{22}$$

produces solution to dVN of the form

$$u = x^4\, W(y\, x^{-1}). \tag{23}$$

Notice that integral in (22) can not be expressed in elementary functions, [3, § VIII], [22, Ch. 3, § 14].

When $\alpha = 2$, equation (19) reduces to $W_{zzz} = 0$, its general solution $W = c_0 + c_1\, z + c_2\, z^2$, $c_i \in \mathbb{R}$, gives solution of dVN

$$u = c_0\, x^2 + c_1\, x\, y + c_2\, y^2.$$

Remark 3 Expression $u = b_1(t)\, x^2 + c\, x\, y + b_2(t)\, y^2$ with arbitrary functions $b_1(t)$, $b_2(t)$ and constant c defines a trivial solution to dVN. We will ignore such solutions below. □

For $\alpha = 3$ we obtain two algebraic solutions

$$W = z^{\frac{3}{2}} \tag{24}$$

and

$$W = (c_2^3 - 2\, c_1^3)\, z^3 + 9\, c_1\, c_2\, z\, (c_2\, z + c_1) + c_1^3 - 2\, c_2^3. \tag{25}$$

of (19). They give solutions

$$u = x^{\frac{3}{2}}\, y^{\frac{3}{2}} \tag{26}$$

and

$$u = (c_2^3 - 2\, c_1^3)\, y^3 + 9\, c_1\, c_2^2\, x\, y\, (c_2\, y + c_1\, x) + (c_1^3 - 2\, c_2^3)\, x^3 \tag{27}$$

of dVN.

When $\alpha = 4$, equation (19) admits the polynomial solution $W = 17\, z^4 - 36\, z^3 - 90\, z^2 - 36\, z + 17$. This corresponds to solution of dVN of the form

$$u = 17\, x^4 - 36\, x^3 y - 90\, x^2 y^2 - 36\, x\, y^3 + 17\, y^4. \tag{28}$$

4.1.2.2. Solutions invariant w.r.t. σ_2.
For σ_2–invariant solutions of $\mathcal{E}_0$ there holds $\sigma_2 = -U_w + \alpha\, U = 0$, therefore these solutions have the form

$$U = \mathrm{e}^{\alpha\, w}\, V(s). \tag{29}$$

Substituting this into $\mathcal{E}_0$ gives the reduced equation $V_s\, V_{sss} + V_{ss}^2 + 2\, \alpha^3\, V\, V_s = 0$. Integrating this twice, we obtain the first order ODE

$$V_s^3 + \alpha^3\, (V^3 + \gamma\, V + \delta) = 0 \tag{30}$$

with arbitrary constants γ and δ. This equation is integrable by quadratures, and the general solution can be expressed in elliptic functions, [10, Ch. V, § 21]. To prove this claim, consider transformation

$$\begin{cases} P = \dfrac{\gamma^{1/2}\,(3\,p^2 + r - 1)}{6\,p}, \\ R = -\dfrac{\gamma^{1/2}\,(3\,p^2 - r + 1)}{6\,p}, \end{cases}$$

that maps the curve $R^3 = P^3 + \gamma\,P + \delta$ onto the curve

$$r^2 = 1 - 2\,\beta\,p - 6\,p^2 - 3\,p^4 \tag{31}$$

with $\beta = 6\,\delta\,\gamma^{-3/2}$. We have $2\,r\,dr = -2\,(\beta + 6\,p + 6\,p^3)\,dp$, hence

$$\frac{dP}{R} = \frac{1}{2p}\left(\frac{3\,p^2+1}{r} - 1\right)dp.$$

Then (29), (6), (7), and $\mu = 0$ give solution

$$u = \mathrm{e}^{\alpha\,y}\,V(x), \tag{32}$$

where function V is defined implicitly by equations

$$V = \frac{\sqrt{\gamma}}{6\,p}\left(3\,p^2 - 1 + \sqrt{1 - 2\,\beta\,p - 6\,p^2 - 3\,p^4}\right), \tag{33}$$

$$\int \frac{1}{2\,p}\left(1 - \frac{3\,p^2+1}{\sqrt{1 - 2\,\beta\,p - 6\,p^2 - 3\,p^4}}\right)dp = \alpha^3\,x + C. \tag{34}$$

The integral in (34) includes Legendre's integrals, therefore x can be expressed as an elliptic function of p.

When polynomial $V^3 + \gamma\,V + \delta$ has multiple roots, the general solution to equation (30) can be expressed in elementary functions. Indeed, the presence of the triple root implies $\gamma = \delta = 0$, and then solution to (30) is $V = c\,\mathrm{e}^{-\alpha\,s}$, $c \in \mathbb{R}$. This corresponds to solution

$$u = \mathrm{e}^{\alpha\,(y-x)} \tag{35}$$

of dVN. For a double root we have $V^3 + \gamma\,V + \delta = (V + 2\,\varepsilon)\,(V - \varepsilon)^2$ for some $\varepsilon \neq 0$. Then we obtain solution to dVN of the form (32), where function V is defined in the implicit form by equations

$$V = \frac{\varepsilon\,(\tau^3 + 2)}{t^3 - 1}, \tag{36}$$

$$\frac{1}{2}\,\ln\frac{\tau^2+\tau+1}{\tau-1}+\sqrt{3}\arctan\left(\frac{\sqrt{3}}{3}\,(2\,\tau+1)\right)=-\alpha^3\,x+c \tag{37}$$

with $c \in \mathbb{R}$.

There are some other values of parameters γ and δ when integral $\int (V^3+\gamma\,V+\delta)^{-\frac{1}{3}}\,dV$ can be expressed by elementary functions. From the results of [4] (see also [9]) we obtain three following cases when function $V(x)$ in (32) admits elementary expressions in implicit form:
(i) for $\gamma=0$, $\delta \neq 0$ there holds

$$\frac{3+i\,\sqrt{3}}{6}\left(\ln(V-(V^3+\delta)^{\frac{1}{3}})+\mathrm{e}^{\frac{\pi i}{3}}\,\ln(V-\mathrm{e}^{\frac{2\pi i}{3}}(V^3+\delta)^{\frac{1}{3}})\right)=\alpha^3\,x+c, \tag{38}$$

(ii) when $\gamma \neq 0$, $\delta=0$, we get

$$\frac{3+i\,\sqrt{3}}{4}\left(\ln(V^{\frac{2}{3}}-(V^2+\gamma)^{\frac{1}{3}})+\mathrm{e}^{\frac{\pi i}{3}}\,\ln(V^{\frac{2}{3}}-\mathrm{e}^{\frac{2\pi i}{3}}(V^2+\gamma)^{\frac{1}{3}})\right)=\alpha^3\,x+c, \tag{39}$$

(iii) conditions $\gamma^3=-6\,\delta^2 \neq 0$ imply

$$\ln H(V,1)+\mathrm{e}^{\frac{2\pi i}{3}}\,\ln H(V,\mathrm{e}^{\frac{2\pi i}{3}})+\mathrm{e}^{\frac{4\pi i}{3}}\,\ln H(V,\mathrm{e}^{\frac{4\pi i}{3}})=6\,\alpha^3\,x+c, \tag{40}$$

where

$$H(\tau,\omega)=\frac{3\,\delta\,\tau-\gamma\,(\tau+\gamma)-\omega\,(\gamma\,\tau+3\,\delta)\,T+2\,\gamma\,\omega^2\,T^2}{2\,\gamma}\cdot(\tau-\omega\,T) \tag{41}$$

with $T=(\tau^2+\gamma\,\tau+\delta)^{\frac{1}{3}}$.

4.1.2.3. Solutions invariant w.r.t. σ_3, σ_4, σ_5, *and* σ_6.

For symmetries σ_3, σ_4, σ_5, and σ_6 the reduced ODEs are of the form $W_{zzz}=0$. This gives only trivial solutions to dVN, cf. Remark 3.

4.1.2.4. Solutions invariant w.r.t. σ_7, σ_8, σ_9, *and* σ_{10}.

For symmetries σ_7, ..., σ_{10} we found neither general nor particular solutions of the reduced equations. Below we write the forms of the invariant solutions of dVN that include functions W and reduced ODEs that define these functions.

For σ_7 we have

$$u=\alpha\,\ln x+W(y\,x^{-1}), \tag{42}$$

where $W(z)$ satisfies

$$W_{zzz}=-2\,\frac{(5\,z^3-1)\,W_{zz}^2+z\,(11\,z\,W_z-3\,\alpha)\,W_{zz}+(5\,z\,W_z-2\,\alpha)\,W_z}{2\,z\,(z^3-1)\,W_{zz}+(3\,z^3-1)\,W_z-\alpha\,z^2}. \tag{43}$$

The σ_8–invariant solutions are

$$u = (x + \alpha\, y)\, \ln x + x\, W(y\, x^{-1}), \tag{44}$$

where for $W(z)$ there holds

$$W_{zzz} = -\frac{(5\, z^3 - 1)\, W_{zz}^2 - z\, (8\, z - 3\, \alpha)\, W_{zz} + 3\, z - 2\, \alpha}{2\, z\, (z^3 - 1)\, W_{zz} - 2\, z^3 + \alpha\, z^2 + 1}. \tag{45}$$

Symmetry σ_9 produces the invariant solution

$$u = y\, (\ln y + W(y\, x^{-1})) \tag{46}$$

with the reduced ODE

$$W_{zzz} = -\frac{z^2 (11 z^3 - 7)\, W_{zz}^2 + (16\, z\, (2 z^3 - 1)\, W_z - 3)\, W_{zz} + 4\, (5 z^3 - 1)\, W_z}{2\, z^3\, (z^3 - 1)\, W_{zz} + 4\, z^2\, (z^3 - 1)\, W_z - z}. \tag{47}$$

Finally, for σ_{10} we get

$$u = \mathrm{e}^{\beta\, x}\, W(y - \alpha\, x). \tag{48}$$

with the defining equation for $W(z)$ of the form

$$\begin{aligned} W_{zzz} = \beta\, &\frac{(5\, \alpha^3 - 1)\, W_{zz}^2 - \alpha\, \beta\, (11\, \alpha\, W_z - 3\, \beta\, W)\, W_{zz}}{2\, \alpha\, (\alpha^3 - 1)\, W_{zz} - \beta\, (3\, \alpha^3 - 1)\, W_z + \alpha^2\, \beta^2\, W} \\ &+ \beta^3\, \frac{5\, \alpha\, W_z^2 - 4\, \beta\, W\, W_z}{2\, \alpha\, (\alpha^3 - 1)\, W_{zz} - \beta\, (3\, \alpha^3 - 1)\, W_z + \alpha^2\, \beta^2\, W}. \end{aligned} \tag{49}$$

4.2 Reduction of dVN w.r.t. χ_2

The χ_2–invariant solutions of dVN satisfy $\chi_2 = 3\, u - x\, u_x - y\, u_y = 0$, therefore they have the form $u = x^3\, U(t, z)$, $z = y\, x^{-1}$, where U is a solution to

$$\begin{aligned} U_{tzz} = \frac{1}{z}\, &(2\, U_{tz} - 2\, (z\, (z^3 - 1)\, U_{zz} - (3\, z^3 - 1)\, U_z + 3\, z^2\, U)\, U_{zzz} \\ &+ (5\, z^3 - 1)\, U_{zz}^2 - 2\, z\, (11\, z\, U_z - 9\, U)\, U_{zz} + 4\, U_z\, (5\, z\, U_z - 6\, U)). \end{aligned} \tag{50}$$

This equation admits a Lax representation. To show this, consider the lift $(\chi_2, \hat{\chi}_2)$ of χ_2 to the Lax representation (2) with $\hat{\chi}_2 = \frac{3}{2}\, q - x\, q_x - y\, q_y$. Solutions to $\hat{\chi}_2 = 0$

have the form $q = x^{\frac{3}{2}}\, Q(t,z)$. Substituting this into (2) yields

$$\begin{cases} Q_t = \dfrac{2}{3}\,(z^3-1)\,Q_z^3 - \dfrac{3\,(z^3-1)}{2\,z}\,Q\,Q_z^2 + \dfrac{2}{z}\,((z^3+1)\,U_z - 3\,z^2\,U)\,Q_z \\ \qquad + \dfrac{9}{8}\,Q^3 - 3\,(3\,z\,U_z - 3\,U)\,Q, \\ Q_z^2 = \dfrac{1}{2\,z}\,(3\,Q\,Q_z - 2\,(z\,U_{zz} - 2\,U_z)). \end{cases}$$

For $S = Q_z$ we obtain another Lax representation

$$\begin{cases} S_t = \dfrac{S^6 + 3\,U_{zz}\,S^4 + F(z,U,U_z,U_{zz},U_{zzz})\,S^2 + G(z,U,U_z,U_{zz})}{S\,(z\,S^2 + 2\,U_z - U_{zz})}, \\ S_z = \dfrac{S\,(S^2 + 2\,U_{zz} - 2\,z\,U_{zzz})}{2\,(z\,S^2 + 2\,U_z - U_{zz})}, \end{cases}$$

where

$$\begin{aligned} F(z,U,U_z,U_{zz},U_{zzz}) &= 4\,(z\,(z^3-1)\,U_{zz} - (3\,z^3-1)\,U_z + 3\,z^2\,U)\,U_{zzz} \\ &\quad -(7\,z^2-2)\,U_{zz}^2 + 2\,z\,(13\,U_z - 9\,U)\,U_{zz} - 16\,z\,U_z^2 + 12\,U\,U_z, \\ G(z,U,U_z,U_{zz}) &= z^3\,U_{zz}^3 - 6\,z\,U_z\,U_{zz}\,(z\,U_{zz} - 2\,U_z) - 8\,U_z^3. \end{aligned}$$

The symmetry algebra of equation (50) is generated by the family $A(t)\,U_t + A'(t)\,U + \frac{1}{18}\,A''(t)\,(z^3+1)$ with arbitrary function $A(t)$. This symmetry provides invariant solutions to (50) of the form $U = \frac{1}{18}\,(W(z) - A'(t)\,(z^3+1))\,A(t)^{-1}$, where W is a solution to ODE (19) with $\alpha = 3$. Hence solutions (20), (24), (25) of (19) generate the following solutions of dVN:

$$u = \frac{1}{A(t)}\left((y + z_0\,x)^3 - \frac{A'(t)}{18}\,(x^3 + y^3)\right), \qquad z_0\,(z_0^3 + 1) = 0, \tag{51}$$

$$u = \frac{1}{A(t)}\left(x^{\frac{3}{2}}\,y^{\frac{3}{2}} - \frac{A'(t)}{18}\,(x^3 + y^3)\right), \tag{52}$$

and

$$u = \frac{(c_2^3 - 2\,c_1^3)\,y^3 + 9\,c_1 c_2^2\,x\,y\,(c_2 y + c_1 x) + (c_1^3 - 2c_2^3)\,x^3 - A'(t)(x^3 + y^3)}{A(t)} \tag{53}$$

with $c_1\,c_2 \neq 0$.

4.3 Reduction of dVN w.r.t. χ_3

For χ_3–invariant solutions we have $u_x + A\, u_y + \frac{1}{2}\, A'\, y^2 = 0$, thus such solutions have the form

$$u = W(t, z) - \frac{A'(t)\, y^3}{6\, A(t)}, \qquad z = y - A(t)\, x. \tag{54}$$

Substituting this into (1) and denoting $V = W_{zz}$, we get the nonlinear PDE of first order

$$V_t + \left(2\,(A^3 - 1)\, V + \frac{A'}{A}\, z\right) V_z + \frac{2\, A'}{A}\, V = 0. \tag{55}$$

We did not find the general solution to (55), instead we obtain three families of particular solutions for this PDE.

First, we note that substitution $V = -\frac{1}{2}\, A' A^{-1}\, (A^3 - 1)^{-1}\, z$ reduces (55) to the ODE $A'' = (2\, A^3 + 1)\, (A')^2\, A^{-1}\, (A^3 - 1)^{-1}$ for function A. Integrating this when $A \neq \text{const}$, we get solution of dVN of the form

$$u = -\frac{1}{12}\, \frac{A'}{A\,(A^3 - 1)}\, (y - A\, x)^3 - \frac{1}{6}\, \frac{A'}{A}\, y^3, \tag{56}$$

where function $A(t)$ is defined implicitly by equation

$$\ln \frac{(A - 1)^2}{A^2 + A + 1} + 2\sqrt{3} \arctan \left(\frac{\sqrt{3}}{3}\, (2\, A + 1)\right) = c_1\, t + c_0, \tag{57}$$

and c_0, $c_1 \neq$ are constants.

When $A(t) \equiv \varepsilon \in \mathbb{R}\backslash\{1\}$, equation (55) gets the form $V_t + 2\,(\varepsilon^3 - 1)\, V\, V_z = 0$. The (multi-valued) solutions of this equation have the form $2\,(\varepsilon^3 - 1)\, t\, V + G(V) = z$, where G is an arbitrary function of one variable. They produce the family of solutions

$$u = W(t, z), \qquad z = y - \varepsilon\, x, \tag{58}$$

where function W is defined implicitly by ODE

$$2\,(\varepsilon^3 - 1)\, t\, W_{zz} + G(W_{zz}) = z. \tag{59}$$

When $A(t) \equiv 1$, we have $W_{tzz} = 0$, and hence

$$u = H(y - x) + p_1(t)(y - x) + p_0(t) \tag{60}$$

with arbitrary functions H, p_0, p_1 of one variable. We can put $p_0 \equiv p_1 \equiv 0$ without loss generality in accordance with Remark 3.

4.4 Reduction of dVN w.r.t. χ_4

Solutions of dVN that are invariant with respect to χ_4 satisfy equation $u_y = A(t)\,x$ and therefore they have the form $u = A(t)\,x\,y + W(t,x)$. When $A \neq 0$, substituting this into (1) gives $W_{xxx} = A' A^{-1}$, so we have solutions of dVN in the form

$$u = \frac{1}{6}\,\frac{A'(t)}{A(t)}\,x^3 + A(t)\,x\,y + p_2(t)\,x^2 \tag{61}$$

with arbitrary function p_2.

Finally, for $A \equiv 0$, equation dVN is satisfied identically, so

$$u = W(t,x) \tag{62}$$

is a solution to dVN for arbitrary function W of two variables.

5 Non-invariant solutions

Some of the solutions obtained in Sect. 4 admit natural generalizations that are not invariant with respect to symmetries from $\mathrm{Sym}_0(\mathrm{dVN})$.

For example, solution (62) is a particular case of the family of solutions $u = f(t,x) + g(t,y)$ with arbitrary functions f and g. The σ_2– invariant solutions from subsection 4.1.2.2 are included is the set of separable solutions of the form $u = X_\alpha(x)\,Y_{-\alpha}(y)$, where X_α and Y_α are independently defined by either one of systems (33)–(34), (36)–(37), (40)–(41), or equation (38), or (39).

Nontrivial polynomial solutions (11), (14), (18), (27), (28), (51), (53), (56)–(57), (61) lead to idea to consider solutions of the form

$$u = \sum_{1 \le i+j \le N} T_{ij}(t)\,x^i\,y^j. \tag{63}$$

Below we present such solutions for $N \geq 3$.

5.1 $N = 3$

Substituting (63) into (1) yields system of ODEs

$$\begin{cases} T'_{21} = 6\,(2\,T_{30} + T_{03})\,T_{21} + 2\,T_{12}^2, \\ T'_{12} = 6\,(T_{30} + 2\,T_{03})\,T_{12} + 2\,T_{21}^2, \\ T'_{11} = 6\,(T_{30} + T_{03})\,T_{11} + 4\,(T_{20}T_{21} + T_{02}T_{12}). \end{cases} \tag{64}$$

for unknown functions T_{21}, T_{12}, T_{11} and arbitrary functions T_{30}, T_{03}, T_{20}, T_{02} of t.

Solutions (11), (14), (18), (27), (51), (53), (56)–(57), (61) are particular cases of solutions given by system (63)–(64) for appropriate choices of functions T_{30}, T_{03}, T_{20}, T_{02}.

Analysis of system (64) gives two families of solutions to dVN. The first one reads

$$u = -\frac{1}{18}\left(\frac{T_{21}'}{T_{21}}\left(y^3 - 2\,x^3 + \frac{3}{2}\,\frac{T_{11}}{T_{21}}\,x^2\right) + \frac{T_{12}'}{T_{12}}\left(x^3 - 2\,y^3 + \frac{3}{2}\,\frac{T_{11}}{T_{21}}\,x^2\right)\right)$$
$$+\frac{1}{4}\,\frac{T_{11}'}{T_{11}}\,x^2 + T_{02}\,y^2 + \frac{1}{9\,T_{21}T_{12}}\left((T_{21}^3 - 2\,T_{12}^3)\,x^3 + (T_{12}^3 - 2\,T_{21}^3)\,y^3\right)$$
$$+x\,y\,(T_{21}\,x + T_{12}\,y + T_{11}), \tag{65}$$

where T_{21}, T_{12}, T_{11}, and T_{02} are arbitrary functions of t such that $T_{21} \neq 0$ and $T_{12} \neq 0$. When $T_{21} \equiv T_{12} \equiv 0$, we obtain solution

$$u = T_{30}\,(x^3 - y^3) + \frac{1}{6}\,\frac{T_{11}'}{T_{11}}\,y^3 + T_{20}\,x^2 + T_{11}\,x\,y + T_{02}\,y^2 \tag{66}$$

with arbitrary functions T_{11}, T_{30}, T_{20}, T_{02} of t such that $T_{11} \neq 0$.

5.2 $N = 4$

Substituting (63) with $N = 4$ into (1) and analyzing the resulting system we get four families of solutions of dVN:

$$u = T_1\left(4\,x^3\,y - 3\,y^4\right) + \frac{T_1'}{6\,T_1}\,y^3 + T_2\,y^2, \quad T_1 \neq 0, \tag{67}$$

$$u = T\left(17\,x^4 - 36\,x^3\,y - 90\,x^2\,y^2 - 36\,x\,y^3 + 17\,y^4\right) + \frac{1}{6}\,\frac{T'}{T}\,(x^3 + y^3),$$
$$T \neq 0, \tag{68}$$

$$u = T\left(17\,x^4 - 36\,x^3\,y - 90\,x^2\,y^2 - 36\,x\,y^3 + 17\,y^4\right) + \frac{1}{6}\,\frac{T'}{T}\,(x^3 + y^3),$$
$$T \neq 0, \quad c \in \mathbb{R}, \tag{69}$$

$$u = \frac{4\,T^2\,T'\,T''' - 8\,T^2\,(T'')^2 + 3\,(T')^4}{1728\,T^2\,(T')^{16/3}}\left(2\,T\,T''\,(x^2 + y^2) - 3\,(T')^2\,y^2\right)$$
$$+(T')^{4/3}\,(x - y)^4 - \frac{1}{18}\,\frac{T''}{T'}\,y\,(3\,x^2 - 3\,x\,y + 2\,y^2)$$
$$+\frac{1}{12}\,\frac{T'}{T}\,y\,(3\,x^2 + y^2), \qquad T \neq 0,\ T' \neq 0. \tag{70}$$

Solution (68) with $T \equiv 1$ coincides with solution (28).

5.3 $N = 5$

For $N = 5$ we obtain the family of solutions

$$u = T_{50}\,(x - y)^5 + T_{30}\,x\,(x^2 + 3\,y^2) + T_{03}\,y\,(y^2 + 3\,x^2)$$

$$+\frac{1}{30}\,\frac{T_{50}'}{T_{50}}\,y^2\,(y-3\,x)+T_{11}\,x\,y, \tag{71}$$

where T_{50} is an arbitrary nonzero function of t, while T_{30}, T_{03}, and T_{11} satisfy the following system of ODEs:

$$\begin{cases} T_{30}' = \dfrac{1}{150}\left(\dfrac{5\,T_{50}''}{T_{50}}-\dfrac{7\,(T_{50}')^2}{T_{50}^2}+\dfrac{30\,T_{50}'}{T_{50}}\right)(T_{30}-2\,T_{03}) \\ \qquad +6\,(T_{30}+T_{03})^2, \\ T_{03}' = \dfrac{(T_{50}')^2}{150\,T_{50}}+\dfrac{T_{50}'}{5\,T_{50}}\,(T_{03}-2\,T_{30})+6\,(T_{30}+T_{03})^2, \\ T_{11}' = \left(\dfrac{T_{50}'}{5\,T_{50}}+6\,(T_{30}+T_{03})\right)T_{11}. \end{cases} \tag{72}$$

5.4 $N \geq 6$

For each $N \geq 6$ we find the family of solutions

$$\begin{aligned} u = {} & \sum_{k=4}^{N} T_k\,(x-y)^k + T_{30}\,(x^3+3\,x^2\,y)+T_{03}\,(y^3+3\,x\,y^2) \\ & +\frac{1}{2\,N^2}\,\frac{T_N'}{T_N}\,y\,\left((N-1)\,T_{N-1}\,y-N\,T_N\,x\,(x+y)\right) \\ & -\frac{1}{2\,N}\,\frac{T_{N-1}'}{T_N}\,y^2+T_{20}\,(x^2+y^2)+T_{11}\,x\,y, \end{aligned} \tag{73}$$

where T_N, T_{N-1}, and T_{20} are arbitrary functions of t such that $T_N \neq 0$, $T_{N-1} \neq 0$, while functions T_k with $k \in \{4, \ldots N-2\}$, T_{30}, T_{03}, and T_{11} obey the following system of ODEs:

$$\begin{cases} T_k' = \dfrac{1}{N^2\,T_N^2}\left((k\,N\,T_k\,T_N-(k+1)\,(N-1)\,T_{k+1}\,T_{N-1})\,T_N'\right. \\ \qquad \left.+(k+1)\,N\,T_{k+1}T_N T_{N-1}'\right), \qquad k \in \{4,\ldots,N-2\}, \\ T_{30}' = \dfrac{1}{6\,N^2\,T_N^2}\left(N\,T_N\,T_N''-(N-1)\,(T_N')^2\right. \\ \qquad -24\,T_4\,((N-1)\,T_{N-1}T_N'-N\,T_N\,T_{N-1}') \\ \qquad \left.+36\,N^2\,T_N^2\,(T_{30}+T_{03})^2-6\,N\,(T_{30}+4\,T_{03})\,T_N\,T_N'\right), \\ T_{03}' = \dfrac{1}{6\,N^2\,T_N^2}\left(N\,T_N\,T_N''-(N-1)\,(T_N')^2\right. \\ \qquad -24\,T_4\,((N-1)\,T_{N-1}T_N'-N\,T_N\,T_{N-1}') \\ \qquad \left.+36\,N^2\,T_N^2\,(T_{30}+T_{03})^2-6\,N\,(4\,T_{30}+T_{03})\,T_N\,T_N'\right), \\ T_{11}' = \dfrac{1}{N^3\,T_N^3}\left((N\,T_N\,T_{N-1}'-(N-1)\,T_{N-1}\,T_N')\,T_N'\right. \\ \qquad +6\,N^3\,T_N^3\,(T_{30}+T_{03})\,(T_{11}+2\,T_{20})-6\,N^2\,T_N^2 T_{30}T_{N-1}' \\ \qquad \left.+6\,N\left((N-1)\,T_N T_{30}-\dfrac{2}{3}\,N\,T_N\,T_{20}\right)T_N T_N'\right). \end{cases} \tag{74}$$

6 Cosymmetries and conservation laws

Cosymmetries of equation (1) are solutions ψ to equation (4), which is of the form

$$\begin{aligned} D_t D_x D_y(\psi) = u_{xy}\,(D_x^3(\psi) + D_y^3(\psi)) + u_{xx}\, D_x^2 D_y(\psi) + u_{yy} D_x D_y^2(\psi) \\ +3\, u_{xxy}\, D_x^2(\psi) + (u_{xxx} + u_{yyy})\, D_x D_y(\psi) + 3\, u_{xyy}\, D_y^2(\psi) \\ +2\, u_{xxxy}\, D_x(\psi) + 2\, u_{xyyy}\, D_y(\psi). \end{aligned} \tag{75}$$

Direct computations show that each solution $\psi \in C^\infty(J^2(\pi))$ of equation (75) is a linear combination of the following cosymmetries:

$$\begin{aligned} \psi_1 &= \ln u_{xy}, \\ \psi_2 &= A, \\ \psi_3 &= 2\,A\,u_{xx} + A'\,x, \\ \psi_4 &= 2\,A\,u_{yy} + A'\,y, \\ \psi_5 &= 4\,A\,(2\,(u_{tx} - u_{xy}\,u_{yy}) + u_{xx}^2) + 2\,A'\,(2\,x\,u_{xx} + u_x) + A''\,x^2, \\ \psi_6 &= 4\,A\,(2\,(u_{ty} - u_{xy}\,u_{xx}) + u_{yy}^2) + 2\,A'\,(2\,y\,u_{yy} + u_y) + A''\,y^2, \\ \psi_7 &= 12\,A\,(3\,u_{tt} + 6\,(u_{tx}\,u_{xx} + u_{ty}\,u_{yy}) - 18\,u_{xx}\,u_{xy}\,u_{yy} - 2\,u_{xy}^3) \\ &\quad -6\,A'\,(3\,u_t + 4\,(x\,(u_{tx} - u_{xy}\,u_{yy}) + y\,(u_{ty} - u_{xy}\,u_{xx})) \\ &\quad +2\,(u_{xx}\,(x\,u_{xx} + u_x) + u_{yy}\,(y\,u_{yy} + u_y))) - A'''\,(x^3 + y^3) \\ &\quad -6\,A''\,((x\,u_x + y\,u_y + y^2\,u_{yy} + x^2\,u_{xx}). \end{aligned}$$

Here $A = A(t)$ are arbitrary (smooth) functions.

The conservation laws $\Omega_1, \dots, \Omega_7$ associated to cosymmetries $\psi_1, \dots, \psi_7$ are given by the formulas

$$\begin{aligned} \Omega_1 &= u_{xy}\,(\ln u_{xy} - 1)\,dx \wedge dy + \left(\frac{1}{2}\,u_{yy}^2 - u_{xx}\,u_{xy}\,\ln u_{xy}\right) dy \wedge dt \\ &\quad + \left(\frac{1}{2}\,u_{xx}^2 - u_{xy}\,u_{yy}\,\ln u_{xy}\right) dt \wedge dx, \\ \Omega_2 &= 2\,A\,u_{xy}\,(dx \wedge dy - u_{xx}\,dy \wedge dt - u_{yy}\,dx \wedge dt) - A'(u_y\,dy - u_x\,dx) \wedge dt, \\ \Omega_3 &= A\,(3\,u_{xx}\,u_{xy}\,dx \wedge dy - 3\,u_{xy}\,(u_{tx} + u_{xx}^2 + u_{xy}\,u_{yy})\,dy \wedge dt \\ &\quad +(3\,u_{xx}\,(u_{tx} - 2\,u_{xy}\,u_{yy}) + u_{xx}^3 + u_{xy}^3)\,dt \wedge dx) \\ &\quad -3\,A'\,x\,(u_{xx}\,u_{xy}\,dy + (u_{tx} - u_{xy}\,u_{yy})\,dx) \wedge dt, \\ \Omega_4 &= A\,(3\,u_{yy}\,u_{xy}\,dx \wedge dy - 3\,u_{xy}\,(u_{ty} + u_{yy}^2 + u_{xy}\,u_{xx})\,dx \wedge dy \\ &\quad +(3\,u_{yy}\,(u_{ty} - 2\,u_{xy}\,u_{xx}) + u_{yy}^3 + u_{xy}^3)\,dy \wedge dt) \\ &\quad +3\,A'\,y\,(u_{yy}\,u_{xy}\,dx + (u_{ty} - u_{xy}\,u_{xx})\,dy) \wedge dt, \\ \Omega_5 &= u_{xy}\,(4\,A\,u_{xx}^2 + 2\,A'\,x\,u_{xx} + A''\,x^2)\,dx \wedge dy \end{aligned}$$

$$
\begin{aligned}
&+\Big(A\,(8\,u_{xy}\,u_{xx}\,(u_{yy}\,u_{xy}-u_{tx})-\tfrac{2}{3}\,u_{xy}\,(u_{xx}^3-u_{xy}^3))\\
&+2\,A'\,(x\,u_{xy}(u_{xy}u_{yy}-u_{tx}-u_{xx}^2)-u_x u_{xx} u_{xy})-A''\,x^2\,u_{xx}u_{xy}\Big)\,dy\wedge dt\\
&+\Big(A\,(4\,((u_{tx}-u_{yy}\,u_{xy})^2-u_{xy}\,u_{yy}\,u_{xx}^2)-\tfrac{2}{3}\,u_{xx}\,(4\,u_{xy}^3+u_{xx}^3)\\
&+A'\,(2\,u_{tx}\,(x\,u_{xx}+u_x)-2\,u_{yy}\,u_{xy}\,(2\,x\,u_{xx}+u_x)-\tfrac{2}{3}\,x\,(u_{xy}^3+u_{xx}^3))\\
&-A''\,x^2\,u_{xy}\,u_{yy}-A'''\,x^2\,u_x\Big)\,dt\wedge dx,\\
\Omega_6&=u_{xy}\,(4\,A\,u_{yy}^2+2\,A'\,y\,u_{yy}+A''\,y^2)\,dx\wedge dy\\
&+\Big(A\,(4\,((u_{ty}-u_{xx}\,u_{xy})^2-u_{xx}\,u_{xy}\,u_{yy}^2)-\tfrac{2}{3}\,u_{yy}\,(4\,u_{xy}^3+u_{yy}^3)\\
&+A'\,(2\,u_{ty}\,(y\,u_{yy}+u_y)-2\,u_{xx}\,u_{xy}\,(2\,y\,u_{yy}+u_y)-\tfrac{2}{3}\,y\,(u_{xy}^3+u_{yy}^3))\\
&-A''\,y^2\,u_{xx}\,u_{xy}-A'''\,y^2\,u_y\Big)\,dy\wedge dt\\
&+\Big(A\,(8\,u_{xy}\,u_{yy}\,(u_{xx}\,u_{xy}-u_{ty})-\tfrac{2}{3}\,u_{xy}\,(u_{yy}^3-u_{xy}^3))\\
&+2\,A'\,(y\,u_{xy}\,(u_{xx}\,u_{xy}-u_{ty}-u_{yy}^2)-u_y\,u_{xy}\,u_{yy})-A''\,y^2\,u_{xy}\,u_{yy}\Big)\,dt\wedge dx,\\
\Omega_7&=\sum_{k=0}^{3}\frac{d^kA}{dt^k}\,(P_k\,dx\wedge dy+Q_k\,dy\wedge dt+R_k\,dt\wedge dx),
\end{aligned}
$$

where

$$
\begin{aligned}
P_0&=\frac{1}{12}\,(2\,u_{xy}\,(6\,(u_{tx}\,u_{xx}+u_{ty}\,u_{yy})+u_{xx}^3-3\,u_{xx}\,u_{xy}\,u_{yy}+u_{yy}^3)-3\,u_{tx}\,u_{ty}),\\
P_1&=\frac{1}{12}\,u_{xy}\,(u_x\,u_{xx}+u_y\,u_{yy}+3\,(y\,u_{ty}+u_t)+2\,(x\,u_{xx}^2+y\,u_{yy}^2)-3\,y\,u_{tx}\,u_{yy}),\\
P_2&=\frac{1}{24}\,(6\,t\,((x\,u_{xy}+y\,u_{yy})\,u_{tx}+u_{xy}\,(x^2\,u_{xx}+y^2\,u_{yy}+2\,(x\,u_x+y\,u_y)))\\
&\quad-(x\,u_{xx}+y\,u_{xy})\,u_{ty}),\\
Q_0&=\frac{1}{12}\,(3\,u_{ty}^2\,u_{yy}+u_{xy}\,(6\,u_{xy}\,u_{yy}\,(6\,u_{tx}+3\,u_{xx}^2-u_{xy}\,u_{yy})-u_{tx}\,(u_{tx}+6\,u_{xx}^2))\\
&\quad+(3\,u_{tt}-12\,u_{xx}\,u_{xy}\,u_{yy}-2\,u_{yy}^3-2\,u_{xy}^3)\,u_{ty}+2\,u_{xy}\,(2\,u_{xy}^3-3\,u_{tt})\,u_{xx}),\\
Q_1&=\frac{1}{36}\,(u_{xy}^3\,(x\,u_{xy}-u_y)-3\,u_{xy}\,(4\,x\,u_{xx}+3\,u_x)\,u_{tx}-3\,y\,u_{ty}^2-2\,x\,u_{xx}^3\,u_{xy}\\
&\quad-3\,(4\,y\,u_{xx}\,u_{xy}-u_y\,u_{yy})\,u_{ty}+3\,u_{xy}\,(2\,y\,u_{xy}-u_x)\,u_{xx}^2-u_{yy}^3\,(y\,u_{yy}+u_y)\\
&\quad-3\,u_{xy}\,(3\,u_t+2\,y\,u_{yy}^2-2\,(2\,x\,u_{xy}-u_y)\,u_{yy})\,u_{xx}\\
&\quad+(3\,u_x\,u_{xy}^2+9\,y\,u_{tt}-4\,y\,u_{xy}^3)\,u_{yy}),\\
Q_2&=\frac{1}{72}\,(3\,(y^2\,u_{ty}\,u_{yy}+6\,t\,(y\,u_{ty}^2+x\,u_{tt}\,u_{xy})-(x^2\,u_{xy}+6\,t\,x\,u_{ty})\,u_{tx})\\
&\quad-3\,(x^2\,u_{xy}u_{xx}^2+2\,u_{xy}(x\,u_x+y\,u_y+y^2\,u_{yy})\,u_{xx}+(6\,t\,y\,u_{tt}-x^2\,u_{xy}^2)\,u_{yy})\\
&\quad-y^2\,(u_{xy}^3+u_{yy}^3)-18\,x\,u_t\,u_{xy}),
\end{aligned}
$$

$$Q_3 = \frac{1}{72}\,((18\,t\,(x\,u_x + y\,u_y - u) + y^3)\,u_{ty} - (x^3 + y^3)\,u_{xx}\,u_{xx} - 3\,y\,u_y^2),$$

$$R_0 = \frac{1}{12}\,(3\,(u_{tx}^2\,u_{xx} - u_{xy}\,u_{ty}^2) + (3\,u_{tt} - 12\,u_{xx}\,u_{xy}\,u_{yy} - 2\,(u_{xx}^3 + u_{xy}^3))\,u_{tx}$$
$$-6\,(u_{xx}^2\,u_{xy}^3 - u_{xy}\,(u_{xx}\,u_{xy} - u_{yy}^2)\,u_{ty} - 3\,u_{xx}\,u_{xy}^2\,u_{yy}^2)$$
$$+2\,u_{xy}\,(4\,u_{xy}^4 - 3\,u_{tt})\,u_{yy}),$$

$$R_1 = \frac{1}{36}\,(3\,(2\,x\,u_{tx}^2 + (3\,y\,u_{ty} - 4\,x\,u_{xy}\,u_{yy} + 2\,u_x\,u_{xx})\,u_{tx}) - u_{xx}^3\,(x\,u_{xx} + u_x)$$
$$-3\,u_{xy}\,((4\,y\,u_{yy} + 3\,u_y)\,u_{ty} - (2\,x\,u_{xy} - u_y)\,u_{yy}^2 + 3\,(y\,u_{tt} - u_t\,u_{yy}))$$
$$+2\,u_{xy}\,u_{yy}\,((2\,y\,u_{xy} - u_x) - 3\,x\,u_{xx}^2\,u_{yy} - y\,u_{yy}^2) + u_{xy}^3\,(y\,u_{xy} - u_x)$$
$$+u_{xx}\,u_{xy}^2\,(3\,u_y - 4\,x\,u_{xy})),$$

$$R_2 = \frac{1}{72}\,(18\,t\,(y\,u_{tt}\,u_{xy} - x\,u_{tx}^2) + (3\,x^2\,u_{xx} - 18\,t\,y\,u_{ty})\,u_{tx} - x^2\,(u_{xx}^3 + u_{xy}^3)$$
$$-3\,(y^2\,u_{xy}\,(u_{ty} + u_{yy}^2) - (6\,x\,(t\,u_{tt} + u_t) - u_{xy}\,(2\,x^2\,u_{yy} + y^2\,u_{xy}))\,u_{xx})$$
$$-6\,u_{xy}\,(x\,u_x + y\,u_y)\,u_{yy}),$$

$$R_3 = \frac{1}{72}\,(18\,t\,(u - x\,u_x - y\,u_y) + x^3)\,u_{tx} - (x^3 + y^3)\,u_{xy}\,u_{yy} - 3\,x\,u_x^2).$$

7 Concluding remarks

The results of the paper can be summarized as follows. Employing the methods of the Lie symmetry group analysis we have found a number of exact solutions for the dispersionless Veselov–Novikov equation, including solutions in elementary or elliptic functions (17), (33)–(34), (36)–(37), (38), (39), (40)–(41), (52), or functions represented by quadratures (15)–(16), (22)–(23), (58)–(59). We have indicated ordinary differential equations (10), (19), (43), (45), (47), (49) that describe all other invariant solutions. We have studied some non-invariant solutions and found a broad set of polynomial solutions (65)—(74). Furthermore, we have presented all the local conservation laws of order up to two.

While ODEs (10), (19), (43), (45), (47), (49) are not integrable by quadratures in general, their origin as reductions of Lax-integrable PDEs (8) and (50) allows one to hope that the method of prolongation structures in the version implemented in [16] could be applicable to examine these ODEs. Likewise, the methods of weak symmetries [19,20], nonclassical symmetry reductions, see [5] and references therein, conditional symmetries [7,8], or stable-range approach [26], can be useful to generate new non-invariant solutions of the dispersionless Veselov–Novikov equation. We intend to address these issues in our future work.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval The authors declare that they have adhered to the ethical standards of research execution.

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