



Realizations of Lie algebras on the line and the new group classification of (1+1)-dimensional generalized nonlinear Klein–Gordon equations

Vyacheslav M. Boyko¹ · Oleksandra V. Lokaziuk¹ · Roman O. Popovych^{1,2}

Received: 6 January 2021 / Revised: 14 May 2021 / Accepted: 15 May 2021 / Published online: 19 June 2021
© The Author(s), under exclusive licence to Springer Nature Switzerland AG 2021

Abstract

Essentially generalizing Lie's results, we prove that the contact equivalence groupoid of a class of (1+1)-dimensional generalized nonlinear Klein–Gordon equations is the first-order prolongation of its point equivalence groupoid, and then we carry out the complete group classification of this class. Since it is normalized, the algebraic method of group classification is naturally applied here. Using the specific structure of the equivalence group of the class, we essentially employ the classical Lie theorem on realizations of Lie algebras by vector fields on the line. This approach allows us to enhance previous results on Lie symmetries of equations from the class and substantially simplify the proof. After finding a number of integer characteristics of cases of Lie-symmetry extensions that are invariant under action of the equivalence group of the class under study, we exhaustively describe successive Lie-symmetry extensions within this class.

1 Introduction

Quasilinear second-order hyperbolic equations model many phenomena and processes in physics and mathematics, especially, various kinds of wave propagation, see [14, 16, 17, 47] and references therein. Such equations even with two independent variables are

✉ Vyacheslav M. Boyko
boyko@imath.kiev.ua

Oleksandra V. Lokaziuk
sasha.lokaziuk@gmail.com

Roman O. Popovych
rop@imath.kiev.ua

¹ Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivska Str, Kyiv 01024, Ukraine

² Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria

important in a number of areas, including differential geometry, quantum field theory, cosmology, hydro- and gas dynamics, superconductivity, crystal dislocation, waves in ferromagnetic materials, nonlinear optics, low temperature physics, to name a few. This is why these equations have been and are intensively studied in many branches of mathematics, in particular, within the framework of integrability theory and symmetry analysis of differential equations.

In the present paper, we carry out the exhaustive group classification of the class of $(1+1)$ -dimensional generalized nonlinear Klein–Gordon equations of the form

$$u_{tx} = f(t, x, u) \quad \text{with} \quad f_{uu} \neq 0 \quad (1)$$

in the light-cone (characteristic) coordinates, which we denote by \mathcal{K} and simultaneously refer as the class (1) and the class \mathcal{K} . Here $u = u(t, x)$ is the unknown function of the independent variables (t, x) , and subscripts of functions denote derivatives with respect to the corresponding variables, e.g., $u_{tx} := \partial^2 u / \partial t \partial x$ and $f_{uu} := \partial^2 f / \partial u^2$. The arbitrary element f of the class \mathcal{K} runs through the set of smooth functions of (t, x, u) that are not affine in u . The last constraint is imposed on f for excluding linear equations from the class \mathcal{K} , which is natural in view of several arguments, see Remark 6. The problem of studying the class \mathcal{K} within the framework of group analysis of differential equations was posed by Sophus Lie [35].

We discover four important properties of the class \mathcal{K} , which allows us to obtain stronger results than the ordinary group classification of this class and to simplify all computations. We prove that, firstly, the class \mathcal{K} is normalized with respect to its point equivalence group G^\sim and, secondly, the first prolongations of its point admissible transformations exhaust its contact admissible transformations. In view of these two properties, the classification of Lie symmetries of equations from the class \mathcal{K} up to the G^\sim -equivalence, roughly speaking, coincides with the similar classification up to the general point equivalence and with the classifications of continuous contact symmetries of these equations modulo the equivalences generated by the contact equivalence group and groupoid of \mathcal{K} . Moreover, this classification problem can be effectively solved by the algebraic method. Thirdly, the dimensions of the maximal Lie invariance algebras of equations from the class \mathcal{K} are not greater than four, except the equations that are G^\sim -equivalent to the Liouville equation and whose maximal Lie invariance algebras are infinite-dimensional. The fourth property is that each point transformation between any two equations from the class \mathcal{K} is projectable both on the space with coordinate t and on the space with coordinate x . As a result, all the point-transformation structures associated with the class \mathcal{K} , including its equivalence group, its equivalence algebra and the maximal Lie invariance algebras of equations from this class, are also projectable on the same two spaces. This twofold projectability leads to the possibility of extended application of the following classical Lie theorem on realizations of finite-dimensional Lie algebras by vector fields on the line [34, Satz 6, p. 455] in the course of the group classification, see also [43, Theorem 2.70], [51] and references therein.

Theorem 1 (Lie theorem) *Inequivalent (up to the local diffeomorphisms of the line) realizations of finite-dimensional Lie algebras by vector fields on the t -line are exhausted by the algebras*

$$\{0\}, \quad \langle \partial_t \rangle, \quad \langle \partial_t, t\partial_t \rangle, \quad \langle \partial_t, t\partial_t, t^2\partial_t \rangle.$$

Previously, the Lie theorem was applied to group classification of classes of evolution and Schrödinger equations [5,28,29,44] and the class of linear ordinary differential equations of an arbitrary fixed order $r \geq 2$ [10, Section 3], where there is the similar projectability to the single space coordinatized by the time variable or the independent variable, respectively.

The most important and well-studied subclass \mathcal{K}_9 of the class \mathcal{K} is singled out from \mathcal{K} by the additional auxiliary equations $f_t = f_x = 0$, i.e., this subclass consists of the nonlinear Klein–Gordon equations, which are of the form $u_{tx} = f(u)$ with $f_{uu} \neq 0$; cf. Remark 14 for justifying the notation. The subclass \mathcal{K}_9 contains a number of famous equations, which we present in the canonical forms, where the constant parameters are removed by equivalence transformations:

- the Liouville equation $u_{tx} = e^u$,
- the Tzitzeica equation $u_{tx} = e^u \pm e^{-2u}$ called also the Dodd–Bullough–Mikhailov equation,
- the sine-Gordon (or Bonnet) equation $u_{tx} = \sin u$,
- the sinh-Gordon equation $u_{tx} = \sinh u$,
- the double sine-Gordon equation $u_{tx} = \sin u + C \sin 2u$ with $C \neq 0$,

see Section 7.5.1 in [47] and references therein. Contact symmetry transformations of equations from the subclass \mathcal{K}_9 were described by Sophus Lie himself [35]. In particular, any such transformation was proved to be the first prolongation of a point transformation. Then Lie singled out the Liouville equations, where $f(u) = Ce^{\kappa u}$ with nonzero constants κ and C , as the only equations in \mathcal{K}_9 admitting infinite-dimensional point symmetry groups. The other equations were shown to possess only point symmetry transformations, where each of the t -, x - and u -components depends only on the respective variable, and this dependence is affine. In the introduction of [35], Lie remarked that results obtained therein can be extended to the class \mathcal{K} . In fact, the present paper essentially generalizes several extensions of Lie’s results to the class \mathcal{K} .

Specific equations from the subclass \mathcal{K}_9 and the subclass \mathcal{K}_9 itself were intensively studied within the framework of symmetry analysis of differential equations. In particular, generalized symmetries of equations from \mathcal{K}_9 with characteristics not depending on (t, x) were classified over the complex field in [60]. The Liouville equation, the sine-Gordon equation and the Tzitzeica equation were singled out in the subclass \mathcal{K}_9 as the only equations with infinite-dimensional algebras of such symmetries, see also [23, Section 21.2]. The same equations were also singled out in [13] as the only equations in \mathcal{K}_9 admitting infinite-dimensional spaces of conservation laws with so-called “polynomial densities” and in [12,39] as the only ones with the Painlevé property among the equations $u_{tx} = f(u)$, where the function f is a linear combination of exponential functions $e^{j\alpha u}$ for some fixed nonzero complex constant α and $j \in \mathbb{Z}$. Equations from the subclass \mathcal{K}_9 admitting nonlinear separation of variables in the standard spacetime

coordinates to two first-order ordinary differential equations were classified in [58]; see also references therein and [20] for other kinds of nonlinear separation of variables for these equations. The classification of local conservation laws of equations from the subclass \mathcal{K}_9 over the complex field was begun in [16]. Singular reduction operators [9,27], i.e., singular nonclassical (or conditional, or Q -conditional) symmetries, of all the equations of the form $u_{tx} = f(u)$ were exhaustively studied in [27]. At the same time, there are still no complete classifications of generalized symmetries, local conservation laws and regular reduction operators of equations from the subclass \mathcal{K}_9 as well as no exhaustive classifications of such equations admitting nonlinear separation of variables or the Painlevé property, not to mention the entire class \mathcal{K} . An exception is the general description of regular reduction operators for equations from the class \mathcal{K} that was given in [57].

The framework of the algebraic method of group classification originated in Lie's classification of second-order ordinary differential equations [37] but it became a common tool of group analysis of differential equations considerably later, only since the 1990s, although its applications to problems of complete group classification still involved the normalization property implicitly [2,18,19,22,32,33,38,59]. When straightforwardly applied to non-normalized classes of differential equations, the algebraic method results in the so-called preliminary group classification of such classes [1,3,15,24]. The algebraic method is usually used to solve group classification problems for classes of differential equations with arbitrary elements depending on several arguments, for which the direct method of group classification, including the method of furcate splitting [4,41,45] as its most advanced version, is unproductive. To carry out the group classification of the class \mathcal{K} , we use the advanced version of the algebraic method, which is based on the normalization of the class of differential equations to be classified [48,52,54] and involves the classification of appropriate subalgebras [3,15] of the corresponding equivalence algebra. This version of the algebraic method was suggested in [52,54] and was effectively applied to solving group classification problems for various classes of differential equations [3,5,10,28,29,44,50,52,54,55].

The papers [3,32,33,55] are especially relevant in the context of the present paper since they are devoted to group classification of classes of quasilinear hyperbolic second-order equations with two independent variables using the algebraic method. See also references therein for group classifications of other classes of such equations. In particular, the group classification problem for the superclass $\check{\mathcal{K}}$ of \mathcal{K} that is constituted by the equations of the form $u_{\check{t}\check{x}} - u_{\check{x}\check{x}} = f(\check{t}, \check{x}, u, u_{\check{x}})$ in the standard spacetime coordinates $(\check{t}, \check{x}) = (x + t, x - t)$ was studied in the seminal papers [32,33]. The superclass $\check{\mathcal{K}}$ was partitioned into four subclasses, which are in fact normalized and are not related by point transformations to each other, and \mathcal{K} is one of these subclasses (under using the light-cone coordinates). As a result, the group classification problem for the entire superclass $\check{\mathcal{K}}$ was split into four group classification problems for the subclasses that each was separately studied within the framework of the algebraic method of group classification. Unfortunately, the consideration of the class \mathcal{K} had several drawbacks (see the second paragraph of Sect. 7 below), and hence Lie symmetries of equations from this class have needed a more accurate and comprehensive classification, which is done in the present paper. The group classification problems for the non-normalized class of quasilinear hyperbolic and elliptic equations

of the form $u_{\check{t}\check{t}} - h(\check{x}, u, u_{\check{x}})u_{\check{x}\check{x}} = f(\check{x}, u, u_{\check{x}})$ up to the equivalences generated by the corresponding equivalence group and groupoid, respectively, were exhaustively solved in [55] using an original version of the algebraic method of group classification for non-normalized classes of differential equations. When written in the light-cone coordinates, this class nontrivially intersects the class \mathcal{K} .

The further organization of the present paper is as follows. In Sect. 2, we prove that any contact-transformation structure associated with equations from the class (1) is the first-order prolongation of its point-transformation counterpart, thus justifying the restriction of the further consideration to point-transformation structures. Then we show the normalization of the class (1) (with respect to point transformations) and construct its (point) equivalence group G^{\sim} and its (point) equivalence algebra \mathfrak{g}^{\sim} . Therein, we also single out the equations in the class (1) with infinite-dimensional Lie invariance algebras and derive properties of finite-dimensional appropriate subalgebras of the projection $\varpi_*\mathfrak{g}^{\sim}$ of \mathfrak{g}^{\sim} onto the space with coordinates (t, x, u) , which are important in the course of the group classification of this class. The main result of the group classification, which is a complete list of G^{\sim} -inequivalent Lie-symmetry extensions within the class (1), is given by Theorem 13 in Sect. 3. The structure of the partially ordered set of such extensions is represented as a Hasse diagram in Fig. 1. We also discuss relations between Lie-symmetry extensions via limit processes. Section 4 is devoted to the proof of Theorem 13 and its analysis. We find a number of G^{\sim} -invariant integer characteristics of subalgebras of \mathfrak{g}^{\sim} or, equivalently, of $\varpi_*\mathfrak{g}^{\sim}$, which allow us to completely identify G^{\sim} -inequivalent cases of Lie-symmetry extensions within the class (1). In Sect. 5, we use these characteristics to distinguish, modulo the G^{\sim} -equivalence, successive Lie-symmetry extensions among the found ones, thus exhaustively describing the structure of partially ordered set of G^{\sim} -inequivalent Lie-symmetry extensions within the class (1). Possible ways for the group classifications of subclasses of the class (1) are analyzed in Sect. 6. As an example, we use results of [55] to carry out the group classification of the important subclass \mathcal{K}_2 associated with the constraint $f_x + f_t = 0$ up to the equivalence generated by the equivalence group G_2^{\sim} of this subclass. In Sect. 7, we discuss the obtained results and overview related problems for the further study.

2 Preliminary analysis

Consider the superclass \mathcal{K}_{gen} of all the equation of the general form $u_{tx} = f(t, x, u)$, $\mathcal{K} \subset \mathcal{K}_{\text{gen}}$. For a fixed value of the arbitrary element f , let K_f denote the equation from the class \mathcal{K}_{gen} with this value of f . We begin with the study of contact admissible transformations within the subclass $\tilde{\mathcal{K}}$ of \mathcal{K}_{gen} singled out by the constraint $f_u \neq 0$, i.e., we attach to \mathcal{K} the linear equations of the form $u_{tx} = f(t, x, u)$ with $f_{uu} = 0$ and $f_u \neq 0$. We essentially generalize Lie's consideration in [35].

Lemma 2 *Any contact admissible transformation within the class $\tilde{\mathcal{K}}$ is the first-order prolongation of a point admissible transformation within this class.*

Lemma 2 can be proved by the direct method. It is also a consequence of later results of [49] for the more general class of equations of the form $u_{tx} = f(t, x, u, u_t, u_x)$.

In view of Lemma 2, all structures related to contact transformations of equations from the superclass $\tilde{\mathcal{K}}$ and all its subclasses, including the class \mathcal{K} , are the first-order prolongations of analogous structures related to point transformations of equations from the same classes. These structures include equivalence groupoids, equivalence groups of these classes and the symmetry groups of equations from them. This is why we restrict the further consideration to point transformations within the class \mathcal{K} .

The following lemma is an obvious corollary of Lemma 2 and [26, Theorem 4.3c].

Lemma 3 *The class (1) is normalized in the usual sense with respect to both point and contact transformations, i.e., its point and contact equivalence groupoids coincide with the action groupoids of its point equivalence group G^\sim and of the first prolongation of this group, respectively. The group G^\sim is generated by the transformations of the form*

$$\tilde{t} = T(t), \quad \tilde{x} = X(x), \quad \tilde{u} = Cu + U^0(t, x), \quad \tilde{f} = \frac{Cf + U_{tx}^0}{T_t X_x} \quad (2)$$

and the discrete equivalence transformation $\mathcal{J}^0: \tilde{t} = x, \tilde{x} = t, \tilde{u} = u, \tilde{f} = f$. Here T, X and U^0 are arbitrary smooth functions of their arguments with $T_t X_x \neq 0$, and C is an arbitrary nonzero constant.

Corollary 4 *A complete list of discrete equivalence transformations of the class (1) that are independent up to combining with each other and with continuous equivalence transformations of this class is exhausted by the (t, x) -permutation \mathcal{J}^0 and three transformations alternating signs of variables, $\mathcal{J}^1: (t, x, u, f) \mapsto (-t, x, u, -f)$, $\mathcal{J}^2: (t, x, u, f) \mapsto (t, -x, u, -f)$, $\mathcal{J}^3: (t, x, u, f) \mapsto (t, x, -u, -f)$. The quotient group of the equivalence group G^\sim of the class (1) with respect to its identity component is isomorphic to the group $D_4 \times \mathbb{Z}_2$, where the dihedral group D_4 is the symmetry group of a square.*

Corollary 5 *There is no contact transformation that maps an equation of the form $u_{tx} = f(t, x, u)$ with $f_u \neq 0$ to equations of the same form with $f_u = 0$.*

Proof Suppose that there exists such a contact transformation Φ . Repeating the proof of Lemma 2 for $f_u \neq 0$ and $\tilde{f}_u = 0$, we derive that the transformation Φ is the first prolongation of a point transformation in the space with the coordinates (t, x, u) . Theorem 4.3c from [26] implies that the (point) equivalence group of the superclass \mathcal{K}_{gen} of all the equation of the form $u_{tx} = f(t, x, u)$ coincides with the group G^\sim , and any point admissible transformation within \mathcal{K}_{gen} is generated by an element of G^\sim . The conditions $f_u \neq 0$ and $f_u = 0$ are G^\sim -invariant, which contradicts the supposition. \square

Analyzing results of [35,36], one can deduce that assertions like Lemmas 2 and 3 and Corollary 5 may have been known to Sophus Lie. In general, similar assertions are typical for the theory of contact equivalence of Monge–Ampère equations, see, e.g., Lemma 1 in [31, p. 205] and references therein. In particular, the above Corollary 5 follows from Corollary 1 in [30, p. 238].

Remark 6 The above assertions imply that the class \mathcal{K}_{gen} and its subclasses $\mathcal{K}_{\text{gen}} \setminus \bar{\mathcal{K}}$, $\bar{\mathcal{K}}$, $\bar{\mathcal{K}} \setminus \mathcal{K}$ and \mathcal{K} , which are singled out by the auxiliary constraints $f_u = 0$, $f_u \neq 0$, $f_u \neq 0 \wedge f_{uu} = 0$ and $f_{uu} \neq 0$, respectively, have the same point equivalence group G^\sim and are normalized in the point sense. The subclasses $\bar{\mathcal{K}}$, $\bar{\mathcal{K}} \setminus \mathcal{K}$ and \mathcal{K} are also normalized in the contact sense. Equations from the class \mathcal{K} are not mapped by contact transformations to equations from the class $\mathcal{K}_{\text{gen}} \setminus \mathcal{K}$. Although the group classification of the class $\mathcal{K}_{\text{gen}} \setminus \mathcal{K}$ up to the G^\sim -equivalence has not been carried out in the literature, Lie's solution of the group classification problem for the wider class of all linear hyperbolic equations with two independent variables is well known [36]. Moreover, symmetry properties of the linear and the nonlinear equations from \mathcal{K}_{gen} , which constitute the classes $\mathcal{K}_{\text{gen}} \setminus \mathcal{K}$ and \mathcal{K} , respectively, are quite different. The last three facts justify the exclusion of linear equations from the further consideration.

Lemma 3 implies that the transformations of the form (2) constitute a subgroup H of G^\sim . Any such transformation \mathcal{T} can be represented as a composition

$$\mathcal{T} = \mathcal{D}^t(T) \circ \mathcal{D}^x(X) \circ \mathcal{Z}(U^0) \circ \mathcal{D}^u(C)$$

of the elementary equivalence transformations

$$\begin{aligned} \mathcal{D}^t(T): \quad & \tilde{t} = T(t), \quad \tilde{x} = x, \quad \tilde{u} = u, & \tilde{f} &= f/T_t, \\ \mathcal{D}^x(X): \quad & \tilde{t} = t, \quad \tilde{x} = X(x), \quad \tilde{u} = u, & \tilde{f} &= f/X_x, \\ \mathcal{D}^u(C): \quad & \tilde{t} = t, \quad \tilde{x} = x, \quad \tilde{u} = Cu, & \tilde{f} &= Cf, \\ \mathcal{Z}(U^0): \quad & \tilde{t} = t, \quad \tilde{x} = x, \quad \tilde{u} = u + U^0(t, x), & \tilde{f} &= f + U_{tx}^0, \end{aligned}$$

which are an arbitrary transformation in t , an arbitrary transformation in x , a scaling of u and a shift of u with arbitrary functions of (t, x) , respectively. The transformation parameters are described in Lemma 3, and their values are the same as in the form (2). The families of elementary transformations $\{\mathcal{D}^t(T)\}$, $\{\mathcal{D}^x(X)\}$, $\{\mathcal{D}^u(C)\}$ and $\{\mathcal{Z}(U^0)\}$, where the corresponding constant or functional parameter varies, are subgroups of G^\sim . One more elementary equivalence transformation of the class (1) is \mathcal{J}^0 . Each transformation \mathcal{T} from $G^\sim \setminus H$ can be decomposed as

$$\mathcal{T} = \mathcal{J}^0 \circ \mathcal{D}^t(T) \circ \mathcal{D}^x(X) \circ \mathcal{Z}(U^0) \circ \mathcal{D}^u(C).$$

Corollary 7 *The equivalence algebra of the class (1) is $\mathfrak{g}^\sim := \langle \hat{D}^t(\tau), \hat{D}^x(\xi), \hat{I}, \hat{Z}(\eta^0) \rangle$, where the parameter functions $\tau = \tau(t)$, $\xi = \xi(x)$ and $\eta^0 = \eta^0(t, x)$ run through the sets of smooth functions of their arguments, and*

$$\begin{aligned} \hat{D}^t(\tau) &:= \tau(t)\partial_t - \tau_t(t) f \partial_f, \quad \hat{D}^x(\xi) := \xi(x)\partial_x - \xi_x(x) f \partial_f, \\ \hat{I} &:= u\partial_u + f\partial_f, \quad \hat{Z}(\eta^0) := \eta^0(t, x)\partial_u + \eta_{tx}^0(t, x)\partial_f. \end{aligned}$$

Using the infinitesimal invariance criterion [6,7,42], we prove the following assertion.

Proposition 8 *The maximal Lie invariance algebra \mathfrak{g}_f of an equation K_f from the class (1) consists of the vector fields of the form $\tau(t)\partial_t + \xi(x)\partial_x + (\eta^1 u + \eta^0(t, x))\partial_u$, where the parameter functions $\tau = \tau(t)$, $\xi = \xi(x)$ and $\eta^0 = \eta^0(t, x)$ and the constant η^1 satisfy the classifying equation*

$$\tau f_t + \xi f_x + (\eta^1 u + \eta^0) f_u = (\eta^1 - \tau_t - \xi_x) f + \eta_{tx}^0. \tag{3}$$

Consider the linear span $\mathfrak{g}_{\langle \rangle}$ of all the maximal Lie invariance algebras of equations from the class (1),

$$\begin{aligned} \mathfrak{g}_{\langle \rangle} &:= \sum_f \mathfrak{g}_f = \{ Q = \tau(t)\partial_t + \xi(x)\partial_x + (\eta^1 u + \eta^0(t, x))\partial_u \} \\ &= \langle D^t(\tau), D^x(\xi), I, Z(\eta^0) \rangle \neq \bigcup_f \mathfrak{g}_f, \end{aligned} \tag{4}$$

where the parameters τ , ξ and η^0 run through the sets of smooth functions of their arguments, η^1 is an arbitrary constant, and

$$D^t(\tau) := \tau(t)\partial_t, \quad D^x(\xi) := \xi(x)\partial_x, \quad I := u\partial_u, \quad Z(\eta^0) := \eta^0(t, x)\partial_u.$$

It is obvious that any vector field Q of the above form with $(\tau, \xi) \neq (0, 0)$ belongs to \mathfrak{g}_f for any f satisfying the classifying equation with the components of Q , and such a value of the arbitrary element f necessarily exists. The last inequality in (4) holds since the vector fields from $\langle I, Z(\eta^0) \rangle$ do not belong to \mathfrak{g}_f for any f in view of the auxiliary inequality $f_{uu} \neq 0$ for the arbitrary element f within the class (1). At the same time, we can represent such vector fields as linear combinations of vector fields from $\mathfrak{g}_{\langle \rangle}$ with $(\tau, \xi) \neq (0, 0)$. This is why the second equality in (4) holds as well, and thus the algebra $\mathfrak{g}_{\langle \rangle}$ coincides with the projection $\varpi_* \mathfrak{g}^{\sim}$ of \mathfrak{g}^{\sim} . Moreover, the span $\mathfrak{g}_{\langle \rangle}$ is a Lie algebra, since it is closed with respect to the Lie bracket of vector fields.

Here and in what follows ϖ denotes the projection of the space with coordinates (t, x, u, f) onto the space with coordinates (t, x, u) . We also use the notation $\pi^{t,x}$, π^t and π^x for the projections of the space with coordinates (t, x, u) onto the spaces with coordinates (t, x) , t and x , respectively.

The nonidentity actions of elementary equivalence transformations on the above vector fields spanning $\mathfrak{g}_{\langle \rangle}$ are the following:

$$\begin{aligned} (\varpi_* \mathcal{D}^t(T))_* D^t(\tau) &= D^t(\tau(\hat{T})/\hat{T}_t), & (\varpi_* \mathcal{D}^t(T))_* Z(\eta^0) &= Z(\eta^0(\hat{T}, x)), \\ (\varpi_* \mathcal{D}^x(X))_* D^x(\xi) &= D^x(\xi(\hat{X})/\hat{X}_x), & (\varpi_* \mathcal{D}^x(X))_* Z(\eta^0) &= Z(\eta^0(t, \hat{X})), \\ (\varpi_* \mathcal{Z}(U^0))_* D^t(\tau) &= D^t(\tau) + Z(\tau U_t^0), & (\varpi_* \mathcal{D}^u(C))_* Z(\eta^0) &= Z(C\eta^0), \\ (\varpi_* \mathcal{Z}(U^0))_* D^x(\xi) &= D^x(\xi) + Z(\xi U_x^0), & (\varpi_* \mathcal{Z}(U^0))_* I &= I - Z(U^0), \end{aligned}$$

where $\hat{T} = \hat{T}(t)$ and $\hat{X} = \hat{X}(x)$ are the inverses of the functions T and X , respectively.

Definition 9 A subalgebra \mathfrak{s} of $\mathfrak{g}_{(\cdot)}$ is said to be appropriate if there exists a value of the arbitrary element f such that $\mathfrak{s} = \mathfrak{g}_f$.

In view of Lemma 3, the group classification of the class (1) reduces to the classification of appropriate subalgebras of $\mathfrak{g}_{(\cdot)} = \varpi_*\mathfrak{g}^\sim$ up to the ϖ_*G^\sim -equivalence.

Splitting the classifying equation (3) with respect to the arbitrary element f and its derivations, we obtain the trivial system $\tau = 0, \xi = 0, \eta^1 = 0$ and $\eta^0 = 0$. This means that the following assertion holds.

Lemma 10 *The kernel Lie invariance algebra of the equations from the class (1) is $\mathfrak{g}^\cap = \{0\}$.*

Since the kernel Lie invariance algebra \mathfrak{g}^\cap is zero, the condition of the necessary inclusion of it into each appropriate algebra makes no constraint for such algebras. Analyzing the classifying equation (3) deeper, we derive really essential constraints for such algebras.

Lemma 11 (i) $\mathfrak{g}_f \cap \langle I, Z(\eta^0) \rangle = \{0\}$ for any $f = f(t, x, u)$ with $f_{uu} \neq 0$, and therefore $\dim \mathfrak{g}_f = \dim \pi_*^{t,x} \mathfrak{g}_f$. Here η^0 runs through the set of smooth functions depending on (t, x) .

(ii) $\dim \mathfrak{g}_f = \infty$ if and only if $f = e^u \pmod{G^\sim}$.

(iii) If $f \neq e^u \pmod{G^\sim}$, then $\dim \mathfrak{g}_f \leq 4$.

Proof Suppose that for a value of the arbitrary element f , the algebra \mathfrak{g}_f contains a vector field $\eta^1 I + Z(\eta^0)$ with $(\eta^1, \eta^0) \neq (0, 0)$. The classifying equation (3) implies that the function f satisfies the equation $(\eta^1 u + \eta^0) f_u = \eta^1 f + \eta_{tx}^0$. Considering the cases $\eta^1 \neq 0$ and $\eta^1 = 0$ separately, we easily show that in both cases f is affine in u , which contradicts the auxiliary inequality $f_{uu} \neq 0$ for the class (1). This proves item (i) of the lemma.

We differentiate the classifying equation (3) with respect to u and, in view of the auxiliary inequality $f_{uu} \neq 0$, divide the result of differentiation by f_{uu} . Then we differentiate the obtained equation once more with respect to u , which gives

$$\tau \left(\frac{f_{ut}}{f_{uu}} \right)_u + \xi \left(\frac{f_{ux}}{f_{uu}} \right)_u + \eta^1 = -(\tau_t + \xi_x) \left(\frac{f_u}{f_{uu}} \right)_u. \tag{5}$$

We need to consider two cases depending on whether or not the expression $(f_u/f_{uu})_u$ vanishes.

Upon the condition $(f_u/f_{uu})_u \neq 0$, we can rewrite the equation (5) as

$$\tau_t + \xi_x = -\tau \frac{(f_{ut}/f_{uu})_u}{(f_u/f_{uu})_u} - \xi \frac{(f_{ux}/f_{uu})_u}{(f_u/f_{uu})_u} - \eta^1 \frac{1}{(f_u/f_{uu})_u}.$$

After fixing a value $u = u_0$, the last equation takes the form $\tau_t + \xi_x = A(t, x)\tau + B(t, x)\xi + C(t, x)$, where the coefficients A, B and C are obviously expressed via derivatives of f . After additionally fixing a value $t = t_0$, we derive the first-order inhomogeneous linear ordinary differential equation

$$\xi_x = B(t_0, x)\xi - \tau_t(t_0) + A(t_0, x)\tau(t_0) + \eta^1 C(t_0, x)$$

with respect to ξ , and its inhomogeneity involves three constant parameters $\tau(t_0)$, $\tau_t(t_0)$ and η^1 . The general solution of this equation can be represented in the form

$$\xi = C_1 \xi^1(x) + \tau(t_0) \xi^2(x) + \tau_t(t_0) \xi^3(x) + \eta^1 \xi^4(x),$$

where $\xi^k(x)$, $k = 1, \dots, 4$, are fixed smooth functions of x , and hence it is linearly parameterized at most four independent arbitrary constants. In other words, $\dim \pi_*^x \mathfrak{g}_f \leq 4$. Similarly, after taking into account the derived expression for ξ , we fix a value x_0 for x instead of t and obtain the first-order inhomogeneous linear ordinary differential equation

$$\begin{aligned} \tau_t = A(t, x_0)\tau + B(t, x_0) & (C_1 \xi^1(x_0) + \tau(t_0) \xi^2(x_0) + \tau_t(t_0) \xi^3(x_0) + \eta^1 \xi^4(x_0)) \\ & - C^1 \xi_x^1(x_0) - \tau(t_0) \xi_x^2(x_0) - \tau_t(t_0) \xi_x^3(x_0) - \eta^1 \xi_x^4(x_0) + C(t, x_0) \end{aligned}$$

with respect to τ , where the inhomogeneity involves the four constant parameters $\tau(t_0)$, $\tau_t(t_0)$, η^1 and C_1 . Since the value of τ in the fixed point $t = t_0$ is among these parameters, the general solution of this equation merely involves these very parameters, i.e., $\dim \pi_*^t \mathfrak{g}_f \leq 4$. Therefore, $\dim \pi_*^{t,x} \mathfrak{g}_f \leq 4$ and thus, in view of item (i) of the lemma, $\dim \mathfrak{g}_f \leq 4$.

Now we study the second case $(f_u/f_{uu})_u = 0$, i.e., $f = \gamma(t, x)e^{\alpha(t,x)u} + \beta(t, x)$, where $\alpha\gamma \neq 0$ in view of the auxiliary inequality $f_{uu} \neq 0$ for the class (1). Hence $\gamma = 1 \pmod{G^\sim}$. Substituting the expression for f into the classifying equation (3), we collect the coefficients of the linearly independent functions $ue^{\alpha u}$, $e^{\alpha u}$ and 1, treated as functions of u , which leads to the system

$$\begin{aligned} \tau\alpha_t + \xi\alpha_x + \eta^1\alpha &= 0, \quad \alpha\eta^0 = \eta^1 - \tau_t - \xi_x, \\ \tau\beta_t + \xi\beta_x &= (\eta^1 - \tau_t - \xi_x)\beta + \eta_{tx}^0. \end{aligned} \tag{6}$$

If $\alpha_x \neq 0$, then $\xi = -((\tau\alpha_t + \eta^1\alpha)/\alpha_x)|_{t=t_0}$, i.e., the component ξ involves at most two varying constants, $\tau(t_0)$ and η^1 . Similarly, if $\alpha_t \neq 0$, then $\tau = ((\xi\alpha_x + \eta^1\alpha)/\alpha_t)|_{x=x_0}$, i.e., the component τ also involves at most two varying constants, $\xi(x_0)$ and η^1 . Therefore, in the case $\alpha_t\alpha_x \neq 0$, the components τ and ξ in total involve at most three different varying constants, and hence $\dim \mathfrak{g}_f \leq 3$. If exactly one of the derivatives α_t and α_x is nonzero, then up to the equivalence transformation \mathcal{J}^0 , we can assume that $\alpha_x \neq 0$ and $\alpha_t = 0$. Then the substitution of the expressions for ξ and η^0 implied by the first two equations of (6), $\xi = -\eta^1\alpha/\alpha_x$ and $\eta^0 = (\eta^1 - \tau_t - \xi_x)/\alpha$, into the last equation of (6) leads to the second-order inhomogeneous linear ordinary differential equation $(1/\alpha)_x \tau_{tt} - (\beta\tau)_t = \eta^1(\alpha\beta/\alpha_x)_x - \eta^1\beta$ with respect to τ , where the leading coefficient $(1/\alpha)_x$ does not vanish, and the inhomogeneity involves the single constant parameter η^1 . Analogously to the above consideration, we obtain $\dim \mathfrak{g}_f \leq 3$. Otherwise, $\alpha_t = \alpha_x = 0$, and hence $\alpha = \text{const}$. Since $\alpha \neq 0$, we can set $\alpha = 1$ by scalings of u and \mathcal{J}^u . The above system reduces to $\eta^1 = 0$, $\eta^0 = -\tau_t - \xi_x$, $\tau\beta_t + \xi\beta_x = -(\tau_t + \xi_x)\beta$. For $\beta \neq 0$, we treat the last equation in the same way as the equation (5) and conclude that $\dim \mathfrak{g}_f \leq 3$ in this case. If $\beta = 0$, then the equation K_f

coincides with the Liouville equation $u_{tx} = e^u$, whose maximal Lie invariance algebra is $\mathfrak{g}_f = \langle \tau(t)\partial_t + \xi(x)\partial_x - (\tau_t(t) + \xi_x(x))\partial_u \rangle$, where the components τ and ξ run through the sets of smooth functions of t or x , respectively. Therefore, $\dim \mathfrak{g}_f = \infty$, and up to the G^\sim -inequivalence, this is the only case with the infinite dimension of $\dim \mathfrak{g}_f$, which proves item (ii) of the lemma.

For all the other equations from the class (1), we derive that the dimensions of the corresponding maximal Lie invariance algebras do not exceed four, which implies item (iii) of the lemma. \square

Corollary 12 *If $f \neq e^u \pmod{G^\sim}$, then $\dim \pi_*^t \mathfrak{g}_f \leq 3$ and $\dim \pi_*^x \mathfrak{g}_f \leq 3$.*

Proof In view of item (iii) of Lemma 11, we have $\dim \mathfrak{g}_f \leq 4$ if $f \neq e^u \pmod{G^\sim}$. Then $\dim \pi_*^t \mathfrak{g}_f \leq \dim \mathfrak{g}_f \leq 4$ and $\dim \pi_*^x \mathfrak{g}_f \leq \dim \mathfrak{g}_f \leq 4$, i.e., $\pi_*^t \mathfrak{g}_f$ and $\pi_*^x \mathfrak{g}_f$ are finite-dimensional Lie algebras of vector fields on the t - and the x -lines, respectively. Then the required inequalities directly follow from the Lie theorem on such algebras. \square

3 Result of group classification

The main result of the paper is the following theorem.

Theorem 13 *A complete list of G^\sim -inequivalent (maximal) Lie-symmetry extensions in the class (1) is exhausted by the following cases:*

0. General case $f = f(t, x, u)$: $\{0\}$;
1. $f = \hat{f}(x, u)$: $\langle \partial_t \rangle$;
2. $f = \hat{f}(x - t, u)$: $\langle \partial_t + \partial_x \rangle$;
3. $f = e^t \hat{f}(x, e^{-t}u)$: $\langle \partial_t + u\partial_u \rangle$;
4. $f = e^{x+t} \hat{f}(x - t, e^{-x-t}u)$: $\langle \partial_t + \partial_x + 2u\partial_u \rangle$;
5. $f = e^t \hat{f}(e^{-t}u)$: $\langle \partial_t + u\partial_u, \partial_x \rangle$;
6. $f = e^{x+t} \hat{f}(e^{-x-t}u)$: $\langle \partial_t + u\partial_u, \partial_x + u\partial_u \rangle$;
7. $f = |x - t|^{-q-2} \hat{f}(|x - t|^q u)$, $q \neq 0$: $\langle \partial_t + \partial_x, t\partial_t + x\partial_x - qu\partial_u \rangle$;
8. $f = |x|^{-q-2} \hat{f}(|x|^q u)$, $q \neq 0$: $\langle \partial_t, t\partial_t + x\partial_x - qu\partial_u \rangle$;
9. $f = \hat{f}(u)$: $\langle \partial_t, \partial_x, t\partial_t - x\partial_x \rangle$;
10. $f = (x - t)^{-2} \hat{f}(u)$: $\langle \partial_t + \partial_x, t\partial_t + x\partial_x, t^2\partial_t + x^2\partial_x \rangle$;
11. $f = e^{u/x}$: $\langle \partial_t, t\partial_t - x\partial_u, x\partial_x + u\partial_u \rangle$;
12. $f = |u|^p u$, $p \neq -1, 0$: $\langle \partial_t, \partial_x, t\partial_t - x\partial_x, -pt\partial_t + u\partial_u \rangle$;
13. $f = e^u$: $\langle \tau(t)\partial_t + \xi(x)\partial_x - (\tau_t(t) + \xi_x(x))\partial_u \rangle$.

Here \hat{f} is an arbitrary smooth function of its arguments whose second derivative with respect to the argument involving u is nonzero, and q and p are arbitrary constants that satisfy the conditions indicated in the corresponding cases. In Case 13, the components τ and ξ run through the sets of smooth functions of t or x , respectively.

The proof of Theorem 13 is presented in the next Section 4.

Remark 14 There are two ways of interpreting the classification cases listed in Theorem 13, in a *weak* sense and in a *strong* sense. Within the framework of the weak group classification, we consider the entire subclass \mathcal{K}_N of the equations from the class \mathcal{K} with the form of f presented in Case N ,

$$N \in \Gamma := \{0, \dots, 6, 7_q, 8_q, 9, 10, 11, 12_p, 13 \mid q \neq 0, p \neq -1, 0\},$$

and then the corresponding algebra is the kernel Lie invariance algebra \mathfrak{g}_N^\cap of the equations from the subclass \mathcal{K}_N . Here we use the notation $7_q := (7, q)$, $8_q := (8, q)$ and $12_p := (12, p)$. We refer to Cases 7, 8 and 12 as to collections of Cases 7_q , 8_q and 12_p with fixed values of q or p , respectively. It is obvious that $\mathcal{K}_0 = \mathcal{K}$. Under the strong group classification, Case N includes only the equations from the subclass \mathcal{K}_N for which $\mathfrak{g}_f = \mathfrak{g}_N^\cap$. Thus, the discussion after the equation (4) implies that $\mathfrak{g}_f = \{0\}$ and hence K_f belongs to strong Case 0 if and only if f does not satisfy the classifying equation (3) for any constant η^1 and any smooth functions $\tau = \tau(t)$, $\xi = \xi(x)$ and $\eta^0 = \eta^0(t, x)$ with $(\tau, \xi) \neq (0, 0)$. For Cases 1–9 to merely collect maximal Lie-symmetry extensions, the parameter function \hat{f} should take only values for which the associated values of the arbitrary element f are not G^\sim -equivalent to ones from the other listed cases with maximal Lie invariance algebras of greater dimensions. In other words, a value of the parameter function \hat{f} leads to a maximal Lie-symmetry extension if and only if it satisfies no equation among those associated with the corresponding case in Proposition 25 or 26 below. Case 10 is special since $\mathfrak{g}_f = \mathfrak{g}_{10}^\cap$ for any $K_f \in \mathcal{K}_{10}$, and thus it does not depend on interpreting the group classification. We will mostly omit the attributes “weak” and “strong”, explicitly indicating all places where the weak interpretation is used.

Remark 15 Cases 3–6 and 8 can be replaced by G^\sim -equivalent cases, for each of which the arbitrary element f just runs through the set of arbitrary smooth functions of either one or two arguments without an additional multiplier:

- 3'. $f = \hat{f}(x, t^{-1}u)$: $\mathfrak{g}_f = \langle t\partial_t + u\partial_u \rangle$;
- 4'. $f = \hat{f}(t^{-1}x, (tx)^{-1}u)$: $\mathfrak{g}_f = \langle t\partial_t + x\partial_x + 2u\partial_u \rangle$;
- 5'. $f = \hat{f}(t^{-1}u)$: $\mathfrak{g}_f = \langle t\partial_t + u\partial_u, \partial_x \rangle$;
- 6'. $f = \hat{f}((tx)^{-1}u)$: $\mathfrak{g}_f = \langle t\partial_t + u\partial_u, x\partial_x + u\partial_u \rangle$;
- 8'a. $f = \hat{f}(|x|^{q'}u)$, $q' \neq 0, -1$: $\mathfrak{g}_f = \langle \partial_t, (q' + 1)t\partial_t - x\partial_x + q'u\partial_u \rangle$;
- 8'b. $f = \hat{f}(e^{-x}u)$: $\mathfrak{g}_f = \langle \partial_t, t\partial_t + \partial_x + u\partial_u \rangle$.

Here Case 8 splits into two subcases, 8'a and 8'b, respectively associated with $q \neq 0, -1$ and $q = -1$.

Remark 16 The group classification of the class (1) can be easily mapped via the point transformation $\check{t} = x + t$, $\check{x} = x - t$, $\check{u} = u$, $\check{f} = f$ to the group classification of

nonlinear Klein–Gordon equations in the standard spacetime variables, $\check{u}_{\check{t}\check{t}} - \check{u}_{\check{x}\check{x}} = \check{f}(\check{t}, \check{x}, \check{u})$.

Remark 17 We found eight triples of G^\sim -invariant integer characteristics of subalgebras \mathfrak{s} of $\mathfrak{g}_{(\cdot)}$, which suffice for distinguishing the G^\sim -inequivalent cases of Lie-symmetry extensions from each other, see Remark 24 below. The most significant among these eight triples is (r_3, j_1, r_2) , where

$$\begin{aligned} r_3 = r_3(\mathfrak{s}) &:= 3 - \min \{ \dim \langle D^t(\tau), D^x(\xi) \rangle \mid \\ &\quad \exists \eta^0 : D^t(\tau) + D^x(\xi) + I + Z(\eta^0) \in \mathfrak{s} \}, \\ j_1 = j_1(\mathfrak{s}) &:= \max(\dim \mathfrak{s}^1, \dim \mathfrak{s}^2), \\ r_2 = r_2(\mathfrak{s}) &:= \min(\dim \pi_*^t \mathfrak{s}^{12}, \dim \pi_*^x \mathfrak{s}^{12}) \end{aligned}$$

with $\mathfrak{s}^1 := \mathfrak{s} \cap \langle D^t(\tau), Z(\eta^0) \rangle$, $\mathfrak{s}^2 := \mathfrak{s} \cap \langle D^x(\xi), Z(\eta^0) \rangle$ and $\mathfrak{s}^{12} := \mathfrak{s} \cap \langle D^t(\tau), D^x(\xi), Z(\eta^0) \rangle$.

In the next four remarks, we discuss weak Lie-symmetry extensions, omitting the attribute “weak”.

Remark 18 We can partially order the collection of Lie-symmetry extensions within the class (1). Here Case $N < \text{Case } \bar{N}$ with $N, \bar{N} \in \Gamma$ means that Case \bar{N} is a further Lie-symmetry extension of Case N modulo the G^\sim -equivalence, i.e., there exists $\mathcal{T} \in G^\sim$ such that $\mathfrak{s} \subsetneq (\varpi_* \mathcal{T})\bar{\mathfrak{s}}$, where \mathfrak{s} and $\bar{\mathfrak{s}}$ are subalgebras of $\mathfrak{g}_{(\cdot)}$ associated with Cases N and \bar{N} , respectively. For the corresponding subclasses \mathcal{K}_N and $\mathcal{K}_{\bar{N}}$, we have the inverse inclusion, $\mathcal{K}_N \supsetneq (\varpi_* \mathcal{T})\mathcal{K}_{\bar{N}}$ with the same $\mathcal{T} \in G^\sim$.

Remark 19 We assign two more G^\sim -invariant integer characteristics of subalgebras \mathfrak{s} of $\mathfrak{g}_{(\cdot)}$ to those from Remark 17,

$$n = n(\mathfrak{s}) := \dim \mathfrak{s}, \quad k = k(\mathfrak{s}) := \min(\dim \pi_*^t \mathfrak{s}, \dim \pi_*^x \mathfrak{s}),$$

see Remark 24 below. Thus, we consider tuples of the form (n, r_3, r_2, j_1, k) , where the entries are ordered according to their importance, see the beginning of Sect. 5. We partially order the set of these tuples, assuming $(n, r_3, r_2, j_1, k) < (\bar{n}, \bar{r}_3, \bar{r}_2, \bar{j}_1, \bar{k})$ if $n < \bar{n}$, $r_3 \leq \bar{r}_3$, $r_2 \leq \bar{r}_2$, $j_1 \leq \bar{j}_1$ and $k \leq \bar{k}$. Let (n, r_3, r_2, j_1, k) and $(\bar{n}, \bar{r}_3, \bar{r}_2, \bar{j}_1, \bar{k})$ correspond to the algebras \mathfrak{s} and $\bar{\mathfrak{s}}$ of Cases N and \bar{N} , respectively. Then the relation Case $N < \text{Case } \bar{N}$ is equivalent to the relation $(n, r_3, r_2, j_1, k) < (\bar{n}, \bar{r}_3, \bar{r}_2, \bar{j}_1, \bar{k})$.

Remark 20 The Hasse diagram for the partially ordered set of G^\sim -inequivalent Lie-symmetry extensions within the class (1) is presented in Fig. 1. According to the rule of constructing Hasse diagrams, the arrows in Fig. 1 depict only the direct Lie-symmetry extensions. We say that the pair (Case N , Case \bar{N}) of classification cases with Case $N < \text{Case } \bar{N}$ presents a *direct Lie-symmetry extension* if there does not exist $\check{N} \in \Gamma$ such that Case $N < \text{Case } \check{N} < \text{Case } \bar{N}$.

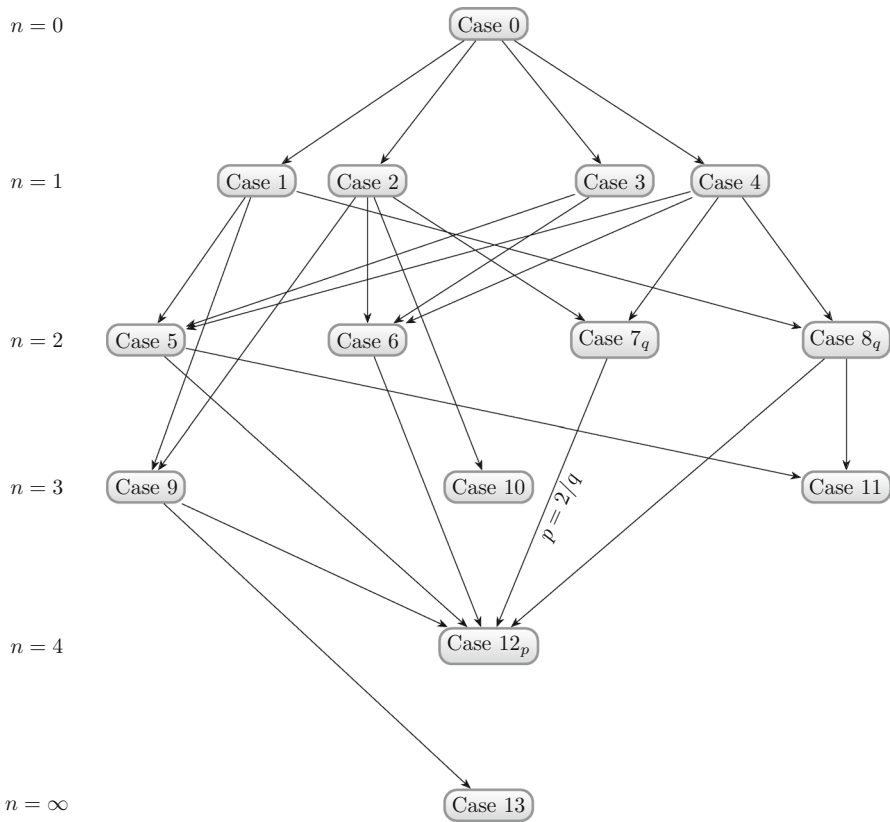


Fig. 1 The Hasse diagram of Lie-symmetry extensions within the class (1)

Remark 21 There are many pairs among $\{(Case\ N, Case\ \bar{N}), N, \bar{N} \in \Gamma\}$, where the pair components are related to each other via limit processes supplemented, if necessary, with preliminary or subsequent equivalence transformations.¹ All these limits lead to contractions $\mathfrak{g}_N^\cap \rightarrow \mathfrak{h} \subseteq \mathfrak{g}_{\bar{N}}^\cap$ as contractions of realizations of abstract Lie algebras by vector fields, where \mathfrak{h} is a subalgebra of $\mathfrak{g}_{\bar{N}}^\cap$, which often coincides with the entire $\mathfrak{g}_{\bar{N}}^\cap$. The most obvious limit process is

$$Case\ 7_q \rightarrow Case\ 10\ \text{ as } q \rightarrow 0\ \text{ with } \mathfrak{g}_{7,1}^\cap \simeq \mathfrak{g}_{7,q}^\cap \rightarrow \langle \partial_t + \partial_x, t\partial_t + x\partial_x \rangle \subsetneq \mathfrak{g}_{10}^\cap.$$

An example with a necessary subsequent equivalence transformation is

$$\begin{aligned} Case\ 8_q &\rightarrow \mathcal{D}^x(-x^{-1})_*(Case\ 9)\ \text{ as } q \rightarrow 0 \\ &\text{ with } \mathfrak{g}_{8,1}^\cap \simeq \mathfrak{g}_{8,q}^\cap \rightarrow \langle \partial_t, t\partial_t + x\partial_x \rangle \subsetneq (\varpi_* \mathcal{D}^x(-x^{-1}))_* \mathfrak{g}_9^\cap. \end{aligned}$$

¹ Such a limit process was considered for the class of nonlinear diffusion equations in [8] and [7, p. 181], where the exponential nonlinearity was excluded from the classification list as a limit of power nonlinearities. A theory of such limit processes and a number of their examples were presented in [25,53,56].

The limit processes

Case 2 → Case 1, Case 4 → Case 2, Case 6 → Case 5, Case 7 → Case 8,
 Case 10 → $\mathcal{D}^x(-x^{-1})_*(\text{Case 9})$

as $q \rightarrow 0$ with contractions between the entire corresponding Lie algebras are realized via preliminarily introducing the parameter q using a scaling equivalence transformation. For instance, for each value of the parameter function \hat{f} in Case 2, we take the family $f^q = q^{-1}\hat{f}(x - t, u)$ with $q \neq 0$ and act on each equation K_{f^q} by the equivalence transformation $\mathcal{D}^t(q^{-1}t)$. For each $q \neq 0$, we obtain the equation $K_{\tilde{f}^q}$ with $\tilde{f}^q = \hat{f}(\tilde{x} - q\tilde{t}, \tilde{u})$, which is invariant with respect to the algebra $\langle \partial_{\tilde{t}} + q\partial_{\tilde{x}} \rangle$. The limit as $q \rightarrow 0$ then gives Case 1. One more set of limit processes

Case 7 → Case 6, Case 8 → $(\mathcal{J}^t \circ \mathcal{J}^0)_*(\text{Case 5})$, Case 12 → Case 13

is based on the remarkable limit $(1 + q^{-1})^q \rightarrow e$ as $q \rightarrow +\infty$. The first two limit processes again give the contractions between the entire corresponding Lie algebras, whereas the last limit process results in a contraction to a proper subalgebra. We describe in detail the second limit process. For each value of the parameter function \hat{f} in Case 8, we act on the equation K_{f^q} with $f^q = q|x|^{-q-2}\hat{f}(|x|^q u)$, $q \in \mathbb{R}_{\neq 0}$, by the equivalence transformation $\mathcal{D}^x(q(x - 1))$, which leads to the equation $K_{\tilde{f}^q}$ with $\tilde{f}^q = |1 + \tilde{x}/q|^{-q-2}\hat{f}(|1 + \tilde{x}/q|^q \tilde{u})$. The Lie invariance algebra of $K_{\tilde{f}^q}$ is $\langle q^{-1}\tilde{t}\partial_{\tilde{t}} + (1 + q^{-1}\tilde{x})\partial_{\tilde{x}} - \tilde{u}\partial_{\tilde{u}} \rangle$. Proceeding to the limit as $q \rightarrow +\infty$, we get $(\mathcal{J}^t \circ \mathcal{J}^0)_*(\text{Case 5})$. A less standard limit process is Case 11 → Case 13, where we introduce the parameter q via the transformation $\mathcal{D}^t(qt) \circ \mathcal{D}^x(q^{-1}(x - 1))$ and take the limit as $q \rightarrow 0$. The contracted algebra $\langle \partial_t, t\partial_t - \partial_u, \partial_x \rangle$ is a (proper) subalgebra of the infinite-dimensional algebra \mathfrak{g}_{13}^\cap .

4 Proof of group classification

In view of Lemma 11(ii), we can exclude Case 13, which corresponds to the Liouville equation, from the further consideration. For convenience, denote by \mathcal{L} the subclass of \mathcal{K} consisting of the equations that are G^\sim -equivalent to the Liouville equation. The equations from the class \mathcal{K} with finite-dimensional Lie invariance algebras, for which $f \neq e^u \pmod{G^\sim}$, constitute the complement $\mathcal{K} \setminus \mathcal{L}$. In this notation, we need to classify only Lie symmetries of equations from the class $\mathcal{K} \setminus \mathcal{L}$. The classification of such equations splits into different cases depending on the following three G^\sim -invariant integer values

$$m := \max(\dim \pi_*^t \mathfrak{g}_f, \dim \pi_*^x \mathfrak{g}_f), \quad n := \dim \mathfrak{g}_f, \quad k := \min(\dim \pi_*^t \mathfrak{g}_f, \dim \pi_*^x \mathfrak{g}_f),$$

which are defined by f and are listed in accordance with their influence on the classification splitting. In view of their definition, they satisfy the inequality $0 \leq n - m \leq$

$k \leq m \leq n$. According to Lemma 11(iii), we can suppose from the very beginning that $n \leq 4$, whereas Corollary 12 implies that $m \leq 3$. Moreover, it follows from Lemma 11(i) that $m > 0$ for any equation in the class (1) with Lie-symmetry extension. In other words, looking for G^\sim -inequivalent equations K_f from $\mathcal{K} \setminus \mathcal{L}$ with nonzero maximal Lie invariance algebra \mathfrak{g}_f , we select the appropriate triples (m, n, k) among those that satisfy the condition

$$m \in \{1, 2, 3\}, \quad n \in \{1, 2, 3, 4\}, \quad 0 \leq n - m \leq k \leq m \leq n.$$

Since the class (1) possesses the discrete equivalence transformation \mathcal{J}^0 , which permutes the variables t and x , without loss of generality we can assume that $m = \dim \pi_*^t \mathfrak{g}_f$. We choose an initial basis of \mathfrak{g}_f consisting of vector fields

$$Q^i = \tau^i(t)\partial_t + \xi^i(x)\partial_x + (\eta^{1i}u + \eta^{0i}(t, x))\partial_u, \quad i = 1, \dots, n \quad \text{with } n \leq 4,$$

see Proposition 8. In addition to Corollary 12, the application of the Lie theorem to the group classification of the class (1) is based on the fact that $\pi_*^t G^\sim$ and $\pi_*^x G^\sim$ coincide with the diffeomorphism groups on the t - and the x -lines, respectively.

We separately consider the cases with different values of $m \in \{1, 2, 3\}$ in descending order.

$m = 3$. Therefore, $n \in \{3, 4\}$. We change the basis (Q^1, \dots, Q^n) in such a way that τ^1, τ^2 and τ^3 are linearly independent and, if $n = 4$, $\tau^4 = 0$. Then $\pi_*^t \mathfrak{g}_f = \langle \tau^1\partial_t, \tau^2\partial_t, \tau^3\partial_t \rangle$ is a faithful realization of the algebra $\mathfrak{sl}(2, \mathbb{R})$ on the line. This fact has several implications. In particular, the algebra $\mathfrak{f} := \langle Q^1, Q^2, Q^3 \rangle$ is, up to combining Q^1, Q^2 and Q^3 with Q^4 if $n = 4$, a faithful realization of $\mathfrak{sl}(2, \mathbb{R})$ as well, and $\pi_*^x \mathfrak{f} = \langle \xi^1\partial_x, \xi^2\partial_x, \xi^3\partial_x \rangle$ is also a (not necessarily faithful) realization of $\mathfrak{sl}(2, \mathbb{R})$.² The last realization should be either faithful or zero since the kernel of any homomorphism of a Lie algebra \mathfrak{g} to a Lie algebra is an ideal of \mathfrak{g} , and the algebra $\mathfrak{sl}(2, \mathbb{R})$ has no proper ideals. In other words, either $\xi^1 = \xi^2 = \xi^3 = 0$ or $\xi^1\xi^2\xi^3 \neq 0$. In view of the Lie theorem, up to linearly combining Q^1, Q^2 and Q^3 with each other, we can set $(\tau^1, \tau^2, \tau^3) = (1, t, t^2) \pmod{\pi_*^t G^\sim}$. Moreover, the derived algebra $[\mathfrak{f}, \mathfrak{f}]$ of the algebra \mathfrak{f} coincides with the algebra \mathfrak{f} itself, and thus $\eta^{1i} = 0, i = 1, 2, 3$.

In the case $\xi^1 = \xi^2 = \xi^3 = 0$, we have $Q^i = t^{i-1}\partial_t + \eta^{0i}\partial_u, i = 1, 2, 3$, with $\eta^{01} = 0 \pmod{G^\sim}$ and, if $n = 4$, $\tau^4 = 0$. The commutator relations of \mathfrak{f} ,

$$\begin{aligned} [Q^1, Q^2] &= \partial_t + \eta_t^{02}\partial_u = Q^1, \\ [Q^1, Q^3] &= 2t\partial_t + \eta_t^{03}\partial_u = 2Q^2, \\ [Q^2, Q^3] &= t^2\partial_t + (t\eta_t^{03} - t^2\eta_t^{02})\partial_u = Q^3, \end{aligned}$$

² Indeed, this claim is obvious for $n = 3$. Suppose that $n = 4$. The algebra \mathfrak{g}_f is not solvable since otherwise both the algebra \mathfrak{f} and the projection $\pi_*^t \mathfrak{g}_f = \pi_*^t \mathfrak{f}$ are solvable, which is not the case. There are only two four-dimensional unsolvable real Lie algebras, $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{a}$ and $\mathfrak{so}(3) \oplus \mathfrak{a}$, where \mathfrak{a} is the one-dimensional abelian Lie algebra. The algebra \mathfrak{g}_f cannot be isomorphic to $\mathfrak{so}(3) \oplus \mathfrak{a}$ in view of $\pi_*^t \mathfrak{g}_f \simeq \mathfrak{sl}(2, \mathbb{R})$. Thus, $\mathfrak{g}_f \simeq \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{a}$, and the existence of required basis is obvious.

imply the system $\eta_t^{02} = 0, \eta_t^{03} = 2\eta^{02}, t\eta_t^{03} = \eta^{03}$, and thus $Q^1 = \partial_t, Q^2 = t\partial_t - \mu(x)\partial_u, Q^3 = t^2\partial_t - 2t\mu(x)\partial_u$, where μ is a smooth function of x . Successively substituting the obtained vector fields Q^1, Q^2 and Q^3 into the classifying equation (3), we derive the following system with respect to the arbitrary element f : $f_t = 0, \mu f_u = f, 2\mu t f_u = 2tf + \mu_x$. An obvious consequence of this system is $\mu_x = 0$ and, since $f \neq 0$, we also get $\mu \neq 0$ and $f = v(x)e^{u/\mu}$. Therefore, the equation K_f is G^\sim -equivalent to the Liouville equation, cf. the proof of Lemma 11, i.e., $\dim \mathfrak{g}_f = \infty$, which contradicts the supposition $\dim \mathfrak{g}_f \leq 4$.

Now we consider the case $\xi^1 \xi^2 \xi^3 \neq 0$ and, up to the G^\sim -equivalence, set $\xi^1 = 1$ and $\eta^{01} = 0$. After expanding the commutator relations of \mathfrak{f} and collecting the components of vector fields,

$$\begin{aligned} [Q^1, Q^2] &= \partial_t + \xi_x^2 \partial_x + (\eta_t^{02} + \eta_x^{02}) \partial_u = Q^1, \\ [Q^1, Q^3] &= 2t\partial_t + \xi_x^3 \partial_x + (\eta_t^{03} + \eta_x^{03}) \partial_u = 2Q^2, \\ [Q^2, Q^3] &= t^2\partial_t + (\xi^2 \xi_x^3 - \xi^3 \xi_x^2) \partial_x + (t\eta_t^{03} + \xi^2 \eta_x^{03} - t^2 \eta_t^{02} - \xi^3 \eta_x^{02}) \partial_u = Q^3, \end{aligned}$$

we obtain the system

$$\begin{aligned} \xi_x^2 = 1, \quad \eta_t^{02} + \eta_x^{02} = 0, \quad \xi_x^3 = 2\xi^2, \quad \eta_t^{03} + \eta_x^{03} = 2\eta^{02}, \\ \xi^2 \xi_x^3 - \xi^3 \xi_x^2 = \xi^3, \quad t\eta_t^{03} + \xi^2 \eta_x^{03} - t^2 \eta_t^{02} - \xi^3 \eta_x^{02} = \eta^{03}. \end{aligned}$$

From the first, the third and the fifth equations of this system, up to the equivalence transformations of shifts with respect to x , we derive $\xi^2 = x$ and $\xi^3 = x^2$. From the second and the fourth equations, we find $\eta^{02} = \rho(\omega^-)$ and $\eta^{03} = \rho(\omega^-)\omega^+ + \theta(\omega^-)$ with $\omega^- := x - t$ and $\omega^+ := x + t$. Then the sixth equation of the system reduces to $\omega^- \theta_{\omega^-} = \theta$, which integrates to $\theta = \lambda \omega^-$, where λ is an arbitrary constant. Thus, $Q^1 = \partial_t + \partial_x, Q^2 = t\partial_t + x\partial_x + \rho(\omega^-)\partial_u, Q^3 = t^2\partial_t + x^2\partial_x + ((x + t)\rho(\omega^-) + \lambda(x - t))\partial_u$, where we can set the parameter function $\rho = \rho(\omega^-)$ to zero by the transformation $\tilde{t} = t, \tilde{x} = x, \tilde{u} = u - \int (\omega^-)^{-1} \rho(\omega^-) d\omega^-$, which is the projection of an equivalence transformation. We successively substitute the components of the vector fields Q^1, Q^2 and Q^3 into the classifying equation (3). The derived system with respect to the arbitrary element f ,

$$f_t + f_x = 0, \quad t f_t + x f_x = -2f, \quad t^2 f_t + x^2 f_x + \lambda(x - t) f_u = -2(x + t) f,$$

is consistent under the inequality $f_u \neq 0$ if and only if $\lambda = 0$.

Suppose that $n = 4, \tau^4 = 0$ and $\xi^4 \neq 0$. Since $\dim \pi_*^x \mathfrak{g}_f \leq 3$, the component ξ^4 should belong to $\langle \xi^1, \xi^2, \xi^3 \rangle = \langle 1, x, x^2 \rangle$. Moreover, $[\mathfrak{f}, \langle Q^4 \rangle] \subseteq \langle Q^4 \rangle$, which implies $[\pi_*^x \mathfrak{f}, \langle \xi^4 \partial_x \rangle] \subseteq \langle \xi^4 \partial_x \rangle$, i.e., $\langle \xi^4 \partial_x \rangle$ is an ideal of $\pi_*^x \mathfrak{f} \simeq \mathfrak{sl}(2, \mathbb{R})$. The algebra $\mathfrak{sl}(2, \mathbb{R})$ has no proper ideals. Hence $\xi^4 = 0$, which is a contradiction. This gives Case 10 of Theorem 13. Moreover, Case 10 admits no further Lie-symmetry extensions via specifying \hat{f} ; cf. the beginning of Sect. 5.

$m = 2$. We change the basis (Q^1, \dots, Q^n) in such a way that τ^1, τ^2 are linearly independent and $\tau^i = 0, i = 3, \dots, n$. Then $\pi_*^t \mathfrak{g}_f = \langle \tau^1 \partial_t, \tau^2 \partial_t \rangle$ is a faithful

realization of a two-dimensional Lie algebra on the t -line, and according to the Lie theorem, up to linearly combining Q^1 and Q^2 , we can set $(\tau^1, \tau^2) = (1, t) \pmod{\pi_*^t G^\sim}$. The possible values of $n = \dim \mathfrak{g}_f$ are 2, 3 and 4.

Suppose that $n = 4$. Then $\dim \pi_*^x \mathfrak{g}_f = \dim \langle \xi^3 \partial_x, \xi^4 \partial_x \rangle = 2$ in view of Lemma 11(i). Up to linearly combining Q^3 and Q^4 , we can set $(\xi^3, \xi^4) = (1, x) \pmod{\pi_*^x G^\sim}$. Since $\xi^1, \xi^2 \in \langle \xi^3, \xi^4 \rangle$, we can further linearly combine Q^1 and Q^2 with Q^3 and Q^4 to annihilate ξ^1 and ξ^2 . We have

$$\begin{aligned} Q^1 &= \partial_t + (\eta^{11}u + \eta^{01})\partial_u, & Q^2 &= t\partial_t + (\eta^{12}u + \eta^{02})\partial_u, \\ Q^3 &= \partial_x + (\eta^{13}u + \eta^{03})\partial_u, & Q^4 &= x\partial_x + (\eta^{14}u + \eta^{04})\partial_u, \end{aligned}$$

where $\eta^{11}, \dots, \eta^{14}$ are constants and $\eta^{01}, \dots, \eta^{04}$ are smooth functions of (t, x) . The commutation relations $[Q^1, Q^2] = Q^1$ and $[Q^3, Q^4] = Q^3$ imply $\eta^{11} = \eta^{13} = 0$. Acting by a transformation $\varpi_* \mathcal{Z}(U^0)$, we can set $\eta^{01} = 0$. From the commutation relation $[Q^1, Q^3] = \eta_t^{03} \partial_u = 0$, we derive $\eta_t^{03} = 0$. All transformations $\varpi_* \mathcal{Z}(U^0)$ with $U^0 = U^0(x)$ preserve the reduced form of $Q^1 = \partial_t$, and among them there is a transformation allowing us to set $\eta^{03} = 0$. We still need to take into account the following commutation relations:

$$\begin{aligned} [Q^1, Q^2] &= \partial_t + \eta_t^{02} \partial_u = Q^1, & [Q^3, Q^2] &= \eta_x^{02} \partial_u = 0, \\ [Q^3, Q^4] &= \partial_x + \eta_x^{04} \partial_u = Q^3, & [Q^1, Q^4] &= \eta_t^{04} \partial_u = 0, \end{aligned}$$

which leads to $\eta_t^{02} = \eta_x^{02} = \eta_t^{04} = \eta_x^{04} = 0$, i.e., η^{02} and η^{04} are constants. Substituting the components of the vector fields Q^1, Q^2, Q^3 and Q^4 into the classifying equation (3), we obtain a system with respect to the arbitrary element f ,

$$f_t = f_x = 0, \quad (\eta^{12}u + \eta^{02})f_u = f, \quad (\eta^{14}u + \eta^{04})f_u = f.$$

This system is consistent if and only if $\eta^{12} = \eta^{14}$ and $\eta^{02} = \eta^{04}$. Moreover, $\eta^{12} = \eta^{14} \neq 0$ since otherwise we have the Liouville equation. In view of the derived conditions, we can set $\eta^{02} = \eta^{04} = 0$ using a shift of u by a constant, which preserves all the posed constraints of the components of Q^1, \dots, Q^4 . Denoting $p := -1/\eta^{12}$, we derive Case 12, where the constant multiplier \hat{C} of f can be removed by the transformation $\mathcal{D}^t(\hat{C}t)$.

Let $n = 3$. Hence $\xi^3 \neq 0$ and $k := \dim \pi_*^x \mathfrak{g}_f = \dim \langle \xi^1, \xi^2, \xi^3 \rangle \in \{1, 2\}$.

If $k = 2$, then the Lie theorem implies that $\xi^i = a_i x + b_i$ with some constants a_i and $b_i, i = 1, 2, 3$. Therefore, we need to examine the cases with respect to coefficients of ξ^i . For $a_3 \neq 0$, we can set $a_3 = 1$ and $b_3 = 0$ by rescaling the vector field Q^3 and by shifting x , respectively. We can further linearly combine Q^1 and Q^2 with Q^3 to make $a_1 = a_2 = 0$. Thus, the basis elements Q^1, Q^2 and Q^3 satisfy the commutation relations $[Q^1, Q^2] = Q^1, [Q^1, Q^3] = 0$ and $[Q^2, Q^3] = 0$. The last two commutation relations imply $b_1 = b_2 = 0$, which means $k = 1$, contradicting the supposed condition $k = 2$. For $a_3 = 0$, the condition $\xi^3 \neq 0$ is equivalent to $b_3 \neq 0$. Rescaling Q^3 and linearly combining Q^1 and Q^2 with Q^3 , we make $b_3 = 1$ and $b_1 = b_2 = 0$. The commutation relation $[Q^1, Q^2] = Q^1$ leads to $a_1 = 0$ and

$\eta^{11} = 0$, and thus $a_2 \neq 0$ since $k = 2$. Acting by a transformation $\varpi_*\mathcal{Z}(U^0)$, we can set $\eta^{01} = 0$. The basis elements take the form

$$Q^1 = \partial_t, \quad Q^2 = t\partial_t + a_2x\partial_x + (\eta^{12}u + \eta^{02})\partial_u, \quad Q^3 = \partial_x + (\eta^{13}u + \eta^{03})\partial_u,$$

where η^{1i} is a constant and η^{0i} is a smooth function of (t, x) , $i = 2, 3$. From the commutation relation $[Q^1, Q^3] = \eta_t^{03}\partial_u = 0$, we derive $\eta_t^{03} = 0$. Since all the pushforwards $\varpi_*\mathcal{Z}(U^0)$ with $U^0 = U^0(x)$ preserve the reduced form of $Q^1 = \partial_t$, in view of the equation $\eta_t^{03} = 0$ we can assume that $\eta^{03} = 0$ up to these pushforwards. More conditions on the basis elements follow from the commutation relations

$$[Q^1, Q^2] = \partial_t + \eta_t^{02}\partial_u = Q^1, \quad [Q^3, Q^2] = a_2\partial_x + (\eta_x^{02} - \eta^{13}\eta^{02})\partial_u = a_2Q^3,$$

which are $\eta^{13} = 0, \eta_t^{02} = 0, \eta_x^{02} = 0$, i.e., $\eta^{02} = \text{const}$. Substituting the components of the vector fields Q^1, Q^2 and Q^3 into the classifying equation (3), we obtain a system with respect to the arbitrary element f ,

$$f_t = f_x = 0, \quad (\eta^{12}u + \eta^{02})f_u = (\eta^{12} - a_2 - 1)f.$$

If $\eta^{12} = \eta^{02} = 0$, then $a_2 = -1$ since $f \neq 0$, and we get Case 9. For $\eta^{12} = 0$ and $\eta^{02} \neq 0$, the inequality $f_u \neq 0$ implies $a_2 \neq -1$, which leads to the Liouville equation. If $\eta^{12} \neq 0$, then the corresponding values of the arbitrary element f are, up to the shifts of u , of the form as in Case 12, where $n = 4$.

Consider the case $k = 1$. Then $\xi^3 \neq 0$, and we can assume $\xi^3 = 1, \xi^1 = \xi^2 = 0$. In view of the commutation relations $[Q^1, Q^2] = Q^1$ and $[Q^1, Q^3] = 0$, analogously to previous cases, we have $\eta^{11} = 0$ and we can set $\eta^{01} = \eta^{03} = 0$ acting by the transformation $\mathcal{Z}(U^0)$. Further expanding commutation relations, $[Q^1, Q^2] = \partial_t + \eta_t^{02}\partial_u = Q^1, [Q^3, Q^2] = (\eta_x^{02} - \eta^{13}\eta^{02})\partial_u = 0$, we derive $\eta_t^{02} = 0, \eta_x^{02} = \eta^{13}\eta^{02}$. From the classifying equation (3), we obtain the system on the arbitrary element f ,

$$f_t = 0, \quad f_x + \eta^{13}uf_u = \eta^{13}f, \quad (\eta^{12}u + \eta^{02})f_u = (\eta^{12} - 1)f.$$

If $\eta^{13} = 0$, then $f_t = f_x = 0$, and thus $\langle t\partial_t - x\partial_x \rangle \in \mathfrak{g}_f$, which contradicts the condition $\xi^2 = 0$. Hence $\eta^{13} \neq 0$, and we can set $\eta^{13} = 1$ by scaling of x , and thus $\eta^{02} = Ce^x$ for some constant C . If additionally $\eta^{12} \neq 0$, then we can make $\eta^{02} = 0$ acting by the transformation $\mathcal{Z}(-pCe^x)$ with $p := -1/\eta^{12} \neq 0$, which preserves Q^1 and Q^3 . The general solution of the system for f with the auxiliary inequality $f_{uu} \neq 0$ is $f = \hat{C}e^{-px}|u|^p u$, where \hat{C} is an arbitrary nonzero constant. The family of equivalence transformations $\tilde{t} = \hat{C}t, \tilde{x} = -e^{-px}/p, \tilde{u} = u, \tilde{f} = e^{px}f/\hat{C}$ maps such values of f to that from Case 12, where $n = 4$. Otherwise, $\eta^{12} = 0$, and thus $C \neq 0$ since $f \neq 0$. Scaling u , we can set $C = 1$. From the above system for f , we derive $f = \hat{C}e^{x-e^{-x}u}$. The equivalence transformation $\tilde{t} = -\hat{C}t, \tilde{x} = e^x, \tilde{u} = -u, \tilde{f} = e^{-x}f/\hat{C}$ leads to Case 11.

If $n = 2$, then $k \in \{0, 1, 2\}$.

Suppose that $k = 2$. From the commutation relation $[Q^1, Q^2] = Q^1$, we derive $\eta^{11} = 0$ and, modulo the transformations $\varpi_*\mathcal{D}^x(X)$ and $\varpi_*\mathcal{Z}(U^0)$, $\xi^1 = 1, \xi^2 = x$

and $\eta^{01} = 0$, and then $\eta_t^{02} + \eta_x^{02} = 0$. Hence η^{02} is a function of $x - t$, and additionally acting by a transformation $\varpi_*\mathcal{Z}(\theta)$, where θ is also a function of $x - t$, we can set $\eta^{02} = 0$. The basis vector fields take the form $Q^1 = \partial_t + \partial_x$, $Q^2 = t\partial_t + x\partial_x + \eta^{12}u\partial_u$. Substituting the components of Q^1 and Q^2 into the classifying equation (3), we obtain the system

$$f_t + f_x = 0, \quad tf_t + xf_x + \eta^{12}uf_u = (\eta^{12} - 2)f.$$

This system implies that $\eta^{12} \neq 0$ since otherwise we get Case 10 with $n = 3$. The condition $\eta^{12} \neq 0$ singles out Case 7, where $q := -\eta^{12}$.

In the case $k = 1$, the commutation relation $[Q^1, Q^2] = Q^1$ implies $\xi^1 = 0$ and $\eta^{11} = 0$. It is possible and convenient to make $\xi^2 = x$ using a transformation $\varpi_*\mathcal{D}^x(X)$. Similarly to the previous cases, we can simultaneously set $\eta^{01} = 0$ and $\eta^{02} = 0$ via acting by a transformation $\varpi_*\mathcal{Z}(U^0)$. The system $f_t = 0$, $xf_x + \eta^{12}uf_u = (\eta^{12} - 2)f$ following from the classifying equation (3) implies that $\eta^{12} \neq 0$ since otherwise we again get Case 9 with $n = 3$. As a result, we have Case 8, where $q := -\eta^{12}$.

Remark 22 The (nonzero) parameter q in Cases 7 and 8 cannot be further gauged by equivalence transformations. We show this only for Case 8 since the argumentation for Case 7 is similar. For each value of f from Case 8, the derived algebra of the corresponding algebra \mathfrak{g}_f is spanned by the vector field $Q^1 = \partial_t$. Hence the projection $\varpi_*\mathcal{T}$ of any element \mathcal{T} of G^\sim that maps the equation K_f to an equation from the same Case 8 should preserve the span of Q^1 , i.e., $(\varpi_*\mathcal{T})_*\partial_t \in \langle \partial_t \rangle$. This implies that the t -component of \mathcal{T} is affine in t . Such a transformation cannot change the ratio of the coefficients of $t\partial_t$ and of $u\partial_u$ in the vector field Q^2 .

For $k = 0$, $\xi^1 = \xi^2 = 0$, and the further consideration is similar to the case $k = 1$. Modulo the G^\sim -equivalence, from the commutation relation $[Q^1, Q^2] = Q^1$ we derive that $\eta^{11} = 0$ and $\eta^{01} = \eta^{02} = 0$. The associated system for f is $f_t = 0$, $\eta^{12}uf_u = (\eta^{12} - 1)f$, where $\eta^{12} \neq 0, 1$, and hence such a value of f can be reduced by a transformation $\mathcal{D}^x(X)$ to the form from Case 12, where $n = 4$.

$m = 1$. Then $n \leq 2$.

If $n = 2$, then linearly combining basis elements and pushing forward the algebra \mathfrak{g}_f by $\varpi_*\mathcal{D}^t(T)$ and $\varpi_*\mathcal{Z}(U^0)$, we make $\tau^1 = 1$, $\tau^2 = 0$ and $\eta^{01} = 0$. Hence $\xi^2 \neq 0$, i.e., $\xi^2 = 1 \pmod{G^\sim}$, and we additionally set $\xi^1 = 0$ by linearly combining the basis elements and repairing the gauge $\eta^{01} = 0$ using a transformation $\varpi_*\mathcal{Z}(U^0)$. In view of the commutation relation $[Q^1, Q^2] = (\eta_t^{02} - \eta^{11}\eta^{02})\partial_u = 0$, we can conclude that $\eta^{02} = e^{\eta^{11}t}\zeta(x)$ for some smooth function ζ of x . Therefore, we can set $\eta^{02} = 0$ by the transformation $\varpi_*\mathcal{Z}(e^{\eta^{11}t}\theta(x))$, where θ is a solution of the first-order linear ordinary differential equation $\theta_x - \eta^{12}\theta + \zeta = 0$; the vector field Q^1 is preserved by this transformation. The basis elements take the form $Q^1 = \partial_t + \eta^{11}u\partial_u$, $Q^2 = \partial_x + \eta^{12}u\partial_u$. Here $(\eta^{11}, \eta^{12}) \neq (0, 0)$ since otherwise we have Case 9 with $n = 3$. If both the coefficients η^{11} and η^{12} are nonzero, then by scaling t and x , we can set $\eta^{11} = \eta^{12} = 1$, and the associated system for the arbitrary element f is $f_t + uf_u = f$, $f_x + uf_u = f$, which leads to Case 6. If one of the parameters η^{11} or

η^{12} is nonzero, then up to the discrete equivalence transformation \mathcal{J}^0 , which permutes the variables t and x , we can assume that $\eta^{11} \neq 0$, and we can set $\eta^{11} = 1$ by scaling t . From the classifying equation (3), we obtain the system $f_t + uf_u = f$, $f_x = 0$, whose integration gives Case 5.

In the case $n = 1$, since $\tau^1 \neq 0$, we can set $\tau^1 = 1$, $\xi^1 = \delta$, $\eta^{01} = 0$ and $\eta^{11} = (1 + \delta) \delta'$ with $\delta, \delta' \in \{0, 1\}$ modulo the transformations $\varpi_* \mathcal{D}^t(T)$, $\varpi_* \mathcal{D}^x(X)$ and $\varpi_* \mathcal{Z}(U^0)$ and of a simultaneous scaling of (t, x) , respectively. As a result, we obtain the following G^\sim -inequivalent cases for Q^1 : ∂_t , $\partial_t + \partial_x$, $\partial_t + u\partial_u$, and $\partial_t + \partial_x + 2u\partial_u$, which correspond to Cases 1–4.

Remark 23 We have split the group classification of equations from the class (1) with finite-dimensional maximal Lie invariance algebras into different cases depending on values of the triple (m, n, k) of G^\sim -invariant integers, which are defined as

$$\begin{aligned} m &= m(f) := \max(\dim \pi_*^t \mathfrak{g}_f, \dim \pi_*^x \mathfrak{g}_f), \\ n &= n(f) := \dim \mathfrak{g}_f, \\ k &= k(f) := \min(\dim \pi_*^t \mathfrak{g}_f, \dim \pi_*^x \mathfrak{g}_f). \end{aligned}$$

It is clear that most of the values in \mathbb{Z}^3 are inappropriate for (m, n, k) . As a preliminary step of the classification, we have significantly narrowed down the set of candidates to be looked through for such triples. According to Lemma 11, Corollary 12 and the definition of (m, n, k) , we have the constraints $m \in \{1, 2, 3\}$, $n \in \{1, 2, 3, 4\}$, and $0 \leq n - m \leq k \leq m \leq n$ but even the (quite restricted) set S of triples satisfying these constraints contains many elements that are not realized for equations from the class (1). The appropriate values for (m, n, k) are exhausted by $(1, 1, 0)$, $(1, 1, 1)$, $(1, 2, 1)$, $(2, 2, 2)$, $(2, 2, 1)$, $(2, 3, 2)$, $(3, 3, 3)$, $(2, 3, 1)$, $(2, 4, 2)$, which are associated with pairs of Cases 1 and 3, 2 and 4, 5 and 6, and single Cases 7, 8, 9, 10, 11, 12 of Theorem 13, respectively. Therefore, the inappropriate triples in S are $(3, 4, k)$ with $k = 1, 2, 3$, $(3, 3, k)$ with $k = 0, 1, 2$ and $(2, 2, 0)$, and their number is significant but less than the number of appropriate triples. The only appropriate triple with $m = 3$ is $(3, 3, 3)$, i.e., the value $m = 3$ uniquely defines the possible values for n and k . This interesting observation may be related to the fact that for $m = 3$ both the projections $\pi_*^t \mathfrak{g}_f$ and $\pi_*^x \mathfrak{g}_f$ as well as the algebra \mathfrak{g}_f itself are necessarily isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ in view of the Lie theorem and the simplicity of $\mathfrak{sl}(2, \mathbb{R})$. The separation of the appropriate values from the inappropriate ones in the set S cannot be implemented in the course of preliminary analysis of Lie symmetries of equations from the class (1) since it is an integral part of the group classification procedure for this class.

Remark 24 To check algebraically that the cases given in Theorem 13 are G^\sim -inequivalent to each other, we need more values associated with the maximal Lie invariance algebras of equations from the class (1), since there are pairs of G^\sim -inequivalent cases with the same triples (m, n, k) . These are Cases 1 and 3 with $(m, n, k) = (1, 1, 0)$, Cases 2 and 4 with $(m, n, k) = (1, 1, 1)$ and Cases 5 and 6 with $(m, n, k) = (1, 2, 1)$. To introduce additional G^\sim -invariant values for complete identification of classification cases, we represent the span $\mathfrak{g}_{(\cdot)}$ as a direct sum of its

subspaces,

$$\mathfrak{g}_{\langle \rangle} = \langle D^t(\tau) \rangle \dot{+} \langle D^x(\xi) \rangle \dot{+} \langle I \rangle \dot{+} \langle Z(\eta^0) \rangle,$$

where the parameter functions $\tau = \tau(t)$, $\xi = \xi(x)$ and $\eta^0 = \eta^0(t, x)$ run through the sets of smooth functions of their arguments. By \mathfrak{P}_i we denote the projection from $\mathfrak{g}_{\langle \rangle}$ onto the i th summand of the above representation for $\mathfrak{g}_{\langle \rangle}$, $i = 1, \dots, 4$. Although $\dim \mathfrak{P}_1 \mathfrak{g}_f = \dim \pi_*^t \mathfrak{g}_f$ and $\dim \mathfrak{P}_2 \mathfrak{g}_f = \dim \pi_*^x \mathfrak{g}_f$, we can define the new G^\sim -invariant integer values

$$\begin{aligned} l &= l(f) := \dim \mathfrak{P}_3 \mathfrak{g}_f, \\ j_1 &= j_1(f) := \max(\dim \mathfrak{g}_f^1, \dim \mathfrak{g}_f^2), \\ j_2 &= j_2(f) := \min(\dim \mathfrak{g}_f^1, \dim \mathfrak{g}_f^2), \\ j_{12} &= j_{12}(f) := \dim \mathfrak{g}_f^{12}, \\ j_{13} &= j_{13}(f) := \max(\dim \mathfrak{g}_f^{13}, \dim \mathfrak{g}_f^{23}), \\ j_{23} &= j_{23}(f) := \min(\dim \mathfrak{g}_f^{13}, \dim \mathfrak{g}_f^{23}), \\ r_1 &= r_1(f) := \max(\dim \pi_*^t \mathfrak{g}_f^{12}, \dim \pi_*^x \mathfrak{g}_f^{12}), \\ r_2 &= r_2(f) := \min(\dim \pi_*^t \mathfrak{g}_f^{12}, \dim \pi_*^x \mathfrak{g}_f^{12}), \\ r_3 &= r_3(f) := 3 - \min \{ \dim \langle D^t(\tau), D^x(\xi) \rangle \mid \\ &\quad \exists \eta^0 : D^t(\tau) + D^x(\xi) + I + Z(\eta^0) \in \mathfrak{g}_f \}, \end{aligned}$$

where $r_3 := 0$ if the set in its definition is empty,

$$\begin{aligned} \mathfrak{g}_f^1 &:= \mathfrak{g}_f \cap \langle D^t(\tau), Z(\eta^0) \rangle, & \mathfrak{g}_f^2 &:= \mathfrak{g}_f \cap \langle D^x(\xi), Z(\eta^0) \rangle, \\ \mathfrak{g}_f^{12} &:= \mathfrak{g}_f \cap \langle D^t(\tau), D^x(\xi), Z(\eta^0) \rangle, \\ \mathfrak{g}_f^{13} &:= \mathfrak{g}_f \cap \langle D^t(\tau), I, Z(\eta^0) \rangle, & \mathfrak{g}_f^{23} &:= \mathfrak{g}_f \cap \langle D^x(\xi), I, Z(\eta^0) \rangle. \end{aligned}$$

The values of $(m, n, k, l, j_1, j_2, j_{12}, j_{13}, j_{23}, r_1, r_2, r_3)$ differ from each other for different cases of Theorem 13,

- | | |
|---|---|
| 1. (1, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0); | 2. (1, 1, 1, 0, 0, 0, 1, 0, 0, 1, 1, 0); |
| 3. (1, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 2); | 4. (1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1); |
| 5. (1, 2, 1, 1, 1, 0, 1, 1, 1, 1, 0, 2); | 6. (1, 2, 1, 1, 0, 0, 1, 1, 1, 1, 1, 2); |
| 7. (2, 2, 2, 1, 0, 0, 1, 0, 0, 1, 1, 1); | 8. (2, 2, 1, 1, 1, 0, 1, 1, 0, 1, 0, 1); |
| 9. (2, 3, 2, 0, 1, 1, 3, 1, 1, 2, 2, 0); | 10. (3, 3, 3, 0, 0, 0, 3, 0, 0, 3, 3, 0); |
| 11. (2, 3, 1, 1, 2, 0, 2, 2, 1, 2, 0, 2); | 12. (2, 4, 2, 1, 1, 1, 3, 2, 1, 2, 2, 2); |
| 13. ($\infty, \infty, \infty, 0, \infty, \infty, \infty, \infty, \infty, \infty, \infty, 0$). | |

This is why these cases are G^\sim -inequivalent. At the same time, this integer tuple is redundant for distinguishing the classification cases from each other. The most

remarkable minimal sufficient tuple is the triple (r_3, j_1, r_2) , where we order the characteristics according to their importance. The other sufficient triples are (r_3, j_1, n) , (r_3, j_1, k) , (r_3, j_1, j_{12}) , (r_3, j_1, r_1) , (r_3, r_2, n) , (r_3, r_2, j_{12}) , (r_3, r_2, r_1) . Nevertheless, in the course of the study of successive extension in the next section, we need to extend triples with other values among the above ones, although the values r_3, j_1 and r_2 jointly with n are still of primary importance.

5 Successive Lie-symmetry extensions

Throughout this section, the classification cases listed in Theorem 13 are interpreted in the weak sense. We intend to identify all the pairs (Case N , Case \bar{N}) of G^\sim -equivalent Lie-symmetry extensions with Case $N < \text{Case } \bar{N}$, i.e., where Case \bar{N} is an additional Lie-symmetry extension of Case N modulo the G^\sim -equivalence, see Remark 18. For this purpose, we use a technique similar to that for the classification of contractions of low-dimensional Lie algebras, see, e.g., [11,21,40] and references therein. Let \mathfrak{s} and $\bar{\mathfrak{s}}$ be subalgebras of $\mathfrak{g}_{(\cdot)}$ associated with Cases N and \bar{N} , and

$$(m, n, k, l, j_1, j_2, j_{12}, j_{13}, j_{23}, r_1, r_2, r_3)$$

and

$$(\bar{m}, \bar{n}, \bar{k}, \bar{l}, \bar{j}_1, \bar{j}_2, \bar{j}_{12}, \bar{j}_{13}, \bar{j}_{23}, \bar{r}_1, \bar{r}_2, \bar{r}_3)$$

are the tuples of their G^\sim -invariant characteristics that are defined in Remarks 23 and 24. It is obvious that the relation Case $N < \text{Case } \bar{N}$ implies

$$n < \bar{n}, \quad m \leq \bar{m}, \quad k \leq \bar{k}, \quad l \leq \bar{l}, \quad j_1 \leq \bar{j}_1, \quad j_2 \leq \bar{j}_2, \\ j_{12} \leq \bar{j}_{12}, \quad j_{13} \leq \bar{j}_{13}, \quad j_{23} \leq \bar{j}_{23}, \quad r_1 \leq \bar{r}_1, \quad r_2 \leq \bar{r}_2, \quad r_3 \leq \bar{r}_3.$$

In other words, if at least one of the above inequalities does not hold, then Case $N \not< \text{Case } \bar{N}$. Examining all the pairs of the cases listed in Theorem 13, we exclude the pairs (Case N , Case \bar{N}) with Case $N \not< \text{Case } \bar{N}$. It turns out that for this exclusion it suffices to use only a tuple of five of the above G^\sim -invariant integer values, e.g., (n, r_3, r_2, j_1, k) , which is minimally sufficient. The other minimally sufficient tuples of five characteristics are obtained by replacing k by m or r_1 . We order the characteristics according to their importance in the elimination procedure. The principal characteristic is the dimension n of the entire general Lie invariance algebra of the corresponding case, and the inequality between n and \bar{n} should only be strict for Case N and Case \bar{N} to be ordered. The characteristics r_3, r_2, j_1, k, m and r_1 detect the following cases of disordering with $n < \bar{n}$:

- r_3 : Case 3 $\not<$ Cases 7, 8, 9, 10, 13, Cases 4, 5, 6, 7, 8 $\not<$ Cases 9, 10, 13,
- Cases 11, 12 $\not<$ Case 13,

- r_2 : Case 2 $\not\prec$ Cases 5, 8, 11, Cases 6, 7 $\not\prec$ Case 11,
- j_1 : Case 1 $\not\prec$ Cases 6, 7, 10, Cases 5, 8 $\not\prec$ Case 10, Case 11 $\not\prec$ Case 12,
- k : Case 7 $\not\prec$ Case 11, Case 10 $\not\prec$ Case 12, or m, r_1 : Case 10 $\not\prec$ Case 12.

The direct inspection shows that the remaining pairs (Case N , Case \bar{N}) with $n < \bar{n}$ are necessarily ordered, except the pairs (Case 7, Case 10) and (Case 8, Case 9), which are related to limit processes for Cases 7 and 8 as $q \rightarrow 0$. Therefore, the Hasse diagram in Fig. 1 represents the structure of the partially ordered set of Lie-symmetry extensions within the class (1), cf. Remark 20. Note that there are two characteristics, j_2 and j_{12} , that detect no cases of disordering with $n < \bar{n}$. Each of the characteristics l, j_{13} and j_{23} detects only cases of disordering with $n < \bar{n}$, that are detected by other characteristics. For example, the characteristic l detects Cases 3, 4, 5, 6, 7, 8 $\not\prec$ Cases 9, 10, 13 and Cases 11, 12 $\not\prec$ Case 13, which is completely covered by the characteristic r_3 .

We derive the necessary and sufficient conditions for the parameter function \hat{f} under which equations from Cases 1–9 have wider Lie invariance algebras than equations with general values of \hat{f} . Here we omit Case 10 since we have shown in Sect. 4 that this case admits no further Lie-symmetry extension. We first consider Cases 1–4, for each of which the parameter function \hat{f} depends on two arguments and the corresponding common Lie invariance algebra is one-dimensional. By default, we assume that the second derivative of \hat{f} with respect to the argument involving u is nonzero.

Case 1 possesses, modulo the G^\sim -equivalence, three families of further Lie-symmetry extensions, which are given by Cases 5, 8 and 9. Analyzing them, we conclude that for any further Lie-symmetry extension of Case 1, the corresponding invariance algebra contains $Q^2 \in \mathfrak{g}_{\langle \cdot \rangle}$ with $\pi_*^x Q^2 \neq 0$ and $[Q^1, Q^2] \in \langle Q^1, Q^2 \rangle$. Therefore, $[Q^1, Q^2] \in \langle Q^1 \rangle$. Up to rescaling of Q^2 , we can assume $[Q^1, Q^2] = \delta Q^1$ with $\delta \in \{0, 1\}$. We split the last commutation relation componentwise and integrate the obtained equations for the components of Q^2 . Linearly combining Q^2 with Q^1 if necessary, we derive the representation $Q^2 = \delta t \partial_t + \xi(x) \partial_x + (\eta^1 u + \eta^0(x)) \partial_u$, where ξ and η^0 are arbitrary smooth functions of x with $\xi \neq 0$, and η^1 is an arbitrary constant. The substitution of this representation into the classifying equation (3) leads to the equation

$$\xi \hat{f}_x + (\eta^1 u + \eta^0) \hat{f}_u = (\eta^1 - \delta - \xi_x) \hat{f}. \tag{7}$$

For any value of the parameter function \hat{f} satisfying the last equation, we indeed have a further Lie-symmetry extension of Case 1, which belongs, up to the G^\sim -equivalence, to Case 5 if $\eta^1 \neq 0$ and $\delta = 0$, to Case 8 if $\eta^1 \neq 0$ and $\delta = 1$, or to Case 9 if $\eta^1 = 0$.

Case 3 is considered similarly to Case 1. The further Lie-symmetry extensions of Case 3 are exhausted, modulo the G^\sim -equivalence, by Cases 5 and 6. The additional Lie-symmetry vector field $Q^2 \in \mathfrak{g}_{\langle \cdot \rangle}$ satisfies the conditions $\pi_*^x Q^2 \neq 0$ and $[Q^1, Q^2] = 0$. This is why we can assume without loss of generality that, up to linearly combining Q^2 with Q^1 and rescaling Q^2 , $Q^2 = \xi(x) \partial_x + (\delta u + \eta^0(x)) \partial_u$, where ξ and η^0 are arbitrary smooth functions of x with $\xi \neq 0$, and $\delta \in \{0, 1\}$. Substituting such Q^2 into the classifying equation (3) and successively splitting with respect to t under assuming x and $\omega := e^{-t} u$ as the other independent variables, we derive one

more constraint $\eta^0 = 0$ for components of Q^2 and the equation

$$\xi \hat{f}_x + \delta \omega \hat{f}_\omega = (\delta - \xi_x) \hat{f}. \tag{8}$$

The last equation defines, up to the G^\sim -equivalence, further Lie-symmetry extensions to Case 5 or Case 6 if $\delta = 0$ or $\delta = 1$, respectively.

Up to the G^\sim -equivalence, Case 2 has further Lie-symmetry extensions to Cases 6, 7, 9 and 10. For the additional Lie-symmetry vector field $Q^2 \in \mathfrak{g}(\cdot)$, we have $[Q^1, Q^2] = \delta Q^1 + \kappa Q^2$ for some constants δ and κ . If $\kappa = 0$, then, up to rescaling of Q^2 and linearly recombining Q^2 with Q^1 , we can set $Q^2 = (\delta t + \kappa') \partial_t + \delta x \partial_x + (\eta^1 u + \hat{\eta}^0(\omega)) \partial_u$, where $\hat{\eta}^0$ is an arbitrary smooth function of $\omega := x - t$, η^1 is an arbitrary constant, $\delta \in \{0, 1\}$, κ' is an arbitrary constant if $\delta = 1$, and $\kappa' = 1$ if $\delta = 0$. Analogously to the previous cases, we substitute Q^2 into the classifying equation (3) and obtain the equation

$$(\delta \omega - \kappa') \hat{f}_\omega + (\eta^1 u + \hat{\eta}^0(\omega)) \hat{f}_u = (\eta^1 - 2\delta) \hat{f} - \hat{\eta}^0_{\omega\omega}. \tag{9}$$

In view of this equation, up to the G^\sim -equivalence, we have further Lie-symmetry extensions to Case 6 if $\delta = 0$ and $\eta^1 \neq 0$, to Case 7 if $\delta = 1$ and $\eta^1 \neq 0$, to Case 9 if $\delta = \eta^1 = 0$, and to Case 10 if $\delta = 1$ and $\eta^1 = 0$. If $\kappa \neq 0$, then $\eta^1 = 0$. Linearly recombining Q^2 with Q^1 , we can set $\delta = 0$. Hence $Q^2 = C_1 e^{\kappa t} \partial_t + C_2 e^{\kappa x} \partial_x + e^{\kappa t} \hat{\eta}^0(\omega) \partial_u$, where $\hat{\eta}^0$ is again an arbitrary smooth function of $\omega := x - t$, and C_1 and C_2 are arbitrary constants with $(C_1, C_2) \neq (0, 0)$. The classifying equation (3) with such Q^2 results in the equation

$$(C_2 e^{\kappa \omega} - C_1) \hat{f}_\omega + \hat{\eta}^0 \hat{f}_u = -\kappa (C_1 + C_2 e^{\kappa \omega}) \hat{f} + \kappa \hat{\eta}^0_\omega - \hat{\eta}^0_{\omega\omega}. \tag{10}$$

Here the conditions $C_1 C_2 = 0$ and $C_1 C_2 \neq 0$ are associated with further Lie-symmetry extensions to Cases 9 and 10, respectively.

All the classification cases with $n > 1$ and $l > 0$ are, up to the G^\sim -equivalence, further Lie-symmetry extensions of Case 4. Its direct Lie-symmetry extensions are exhausted, modulo the G^\sim -equivalence, by Cases 5, 6, 7 and 8. In view of the form of Q^1 , we have the following commutation relation of Q^1 with the additional Lie-symmetry vector field $Q^2 \in \mathfrak{g}(\cdot)$: $[Q^1, Q^2] = \kappa Q^2$ for some constant κ . The commutation relation implies the representation $Q^2 = C_1 e^{\kappa t} \partial_t + C_2 e^{\kappa x} \partial_x + e^{(\kappa+2)t} \hat{\eta}^0(\omega_1) \partial_u$, where $\hat{\eta}^0$ is again an arbitrary smooth function of $\omega_1 := x - t$, and C_1 and C_2 are arbitrary constants with $(C_1, C_2) \neq (0, 0)$ and, if $\kappa = 0$, additionally $C_1 \neq C_2$. Substituting this representation into the classifying equation (3), we obtain the equation

$$\begin{aligned} & (C_2 e^{\kappa \omega_1} - C_1) \hat{f}_{\omega_1} + (e^{-\omega_1} \hat{\eta}^0(\omega_1) - C_2 \omega_2 e^{\kappa \omega_1} - C_1 \omega_2) \hat{f}_{\omega_2} \\ & = -(\kappa + 1)(C_1 + C_2 e^{\kappa \omega_1}) \hat{f} + (\kappa + 2) e^{-\omega_1} \hat{\eta}^0_{\omega_1} - e^{-\omega_1} \hat{\eta}^0_{\omega_1 \omega_1}, \end{aligned} \tag{11}$$

where $\omega_2 := e^{-x-t}u$. Modulo the G^\sim -equivalence, we obtain extensions to Case 5 if $\kappa = 0$ and $C_1C_2 = 0$, to Case 6 if $\kappa = 0$ and $C_1C_2 \neq 0$, to Case 7 if $\kappa \neq 0$ and $C_1C_2 \neq 0$, and to Case 8 if $\kappa \neq 0$ and $C_1C_2 = 0$.

We summarize the above consideration in the following proposition.

Proposition 25 *A generalized nonlinear Klein–Gordon equation from Cases 1–4 admits an additional Lie-symmetry extension if and only if the corresponding value of the parameter function \hat{f} satisfies an equation (7) in Case 1, an equation (8) in Case 3, an equation (9) or (10) in Case 2, or an equation (11) in Case 4.*

Now we derive the conditions on the parameter function \hat{f} for the Lie invariance algebras presented in Cases 5–9 of Theorem 13 to be maximal for the corresponding equations from the class (1). In each of these cases, the arbitrary element f takes a value of the form $f = \alpha(t, x)\hat{f}(\omega)$, where $\omega := \beta(t, x)u$, α and β are nonzero known functions of (t, x) , and $\hat{f}_{\omega\omega} \neq 0$ since $f_{uu} \neq 0$. Substituting this form for f into the classifying equation (3), we obtain the classifying equation in terms of \hat{f} ,

$$\begin{aligned} & \left(\left(\frac{\beta_t}{\beta} \tau + \frac{\beta_x}{\beta} \xi + \eta^1 \right) \omega + \beta \eta^0 \right) \hat{f}_\omega \\ & + \left(\tau_t + \xi_x + \frac{\alpha_t}{\alpha} \tau + \frac{\alpha_x}{\alpha} \xi - \eta^1 \right) \hat{f} - \frac{\eta^0_{tx}}{\alpha} = 0. \end{aligned} \tag{12}$$

We apply the method of furcate splitting, see [4,41,45] and references therein. Fixing values of the variables t and x gives the template form of equations for values of \hat{f} , for which the equation K_f possesses an additional Lie-symmetry extension,

$$(a\omega + b)\hat{f}_\omega + c\hat{f} - d = 0, \tag{13}$$

where a, b, c and d are constants with $(a, b) \neq (0, 0)$. Additionally, in view of $\hat{f}_{\omega\omega} \neq 0$ we have $c \neq -a$ if $a \neq 0$ and $c \neq 0$ if $a = 0$. Moreover, the number of equations of the form (13) with linearly independent tuples (a, b, c, d) cannot exceed one since otherwise $\hat{f}_{\omega\omega} = 0$. In other words, we have exactly one independent equation of the form (13) if the equation K_f possesses an additional Lie-symmetry extension. This means that the left-hand side of (12) is proportional to that of (13) with nonvanishing multiplier λ depending on (t, x) ,

$$\begin{aligned} & \left(\left(\frac{\beta_t}{\beta} \tau + \frac{\beta_x}{\beta} \xi + \eta^1 \right) \omega + \beta \eta^0 \right) \hat{f}_\omega + \left(\tau_t + \xi_x + \frac{\alpha_t}{\alpha} \tau + \frac{\alpha_x}{\alpha} \xi - \eta^1 \right) \hat{f} - \frac{\eta^0_{tx}}{\alpha} \\ & = \lambda((a\omega + b)\hat{f}_\omega + c\hat{f} - d). \end{aligned}$$

The last equation can be split with respect to \hat{f} and \hat{f}_ω into the system

$$\begin{aligned} \frac{\beta_t}{\beta} \tau + \frac{\beta_x}{\beta} \xi + \eta^1 &= a\lambda, & \tau_t + \xi_x + \frac{\alpha_t}{\alpha} \tau + \frac{\alpha_x}{\alpha} \xi - \eta^1 &= c\lambda, \\ \beta \eta^0 &= b\lambda, & \eta^0_{tx} &= d\alpha\lambda. \end{aligned} \tag{14}$$

If $a \neq 0$, then we can make $a = 1$ by rescaling the template-form equation (13), and thus $c \neq -1$. The first two equations of the system (14) are combined to

$$\begin{aligned} \lambda &= \frac{\beta_t}{\beta} \tau + \frac{\beta_x}{\beta} \xi + \eta^1, \\ \tau_t + \left(\frac{\alpha_t}{\alpha} - c \frac{\beta_t}{\beta} \right) \tau + \xi_x + \left(\frac{\alpha_x}{\alpha} - c \frac{\beta_x}{\beta} \right) \xi &= (c + 1)\eta^1, \end{aligned} \tag{15}$$

and only the last equation plays the role of a classifying condition. The third and fourth equations of the system (14) merely establish a relation between the constant parameters b and d . Indeed, in each of Cases 5–9, we have $(1/\beta)_{tx}$ is proportional to α , $(1/\beta)_{tx} = C\alpha$ for some constant C . Hence we can always set $b = 0$ by the equivalence transformation $\mathcal{Z}(b/\beta)$, which adds bC to \hat{f} and also makes $\eta^0 = 0$ and $d = 0$. In the general case $a \neq 0$, the relation between b and d is $ad = bcC$. For specific values of the parameter functions α and β in Cases 5–9, we obtain the following values of the constant parameter C and the following forms of the reduced classifying equation (15):

- Case 5: $C = 0, \quad \tau_t + c\tau + \xi_x = (c + 1)\eta^1;$
- Case 6: $C = 1, \quad \tau_t + c\tau + \xi_x + c\xi = (c + 1)\eta^1;$
- Case 7: $C = q(q + 1),$
 $(x - t)(\tau_t + \xi_x) + (q + 2 + cq)(\tau - \xi) = (c + 1)(x - t)\eta^1;$
- Case 8: $C = 0, \quad \tau_t + \xi_x - (q + 2 + cq)x^{-1}\xi = (c + 1)\eta^1;$
- Case 9: $C = 0, \quad \tau_t + \xi_x = (c + 1)\eta^1.$

In each of Cases 5, 6, 8 and 9, the corresponding classifying equation implies that there is a further Lie-symmetry extension if and only if the associated value of the parameter function \hat{f} satisfies the equation (13) with arbitrary $a \neq 0$, c and b and with $d = bcC/a$. Here the extension is given by Case 12 with an arbitrary nonzero constant p . Case 7 is similar but the constant c is related to a according to $c = -(1 + 2/q)a$, $d = -(q + 1)(q + 2)b$, and the extension for a fixed q is given by Case 12 _{p} with $p = 2/q$.

If $a = 0$, then $bc \neq 0$, and we rescale the template-form equation (13) for making $c = 1$. Then

$$\begin{aligned} \frac{\beta_t}{\beta} \tau + \frac{\beta_x}{\beta} \xi + \eta^1 &= 0, \quad \lambda = \tau_t + \xi_x + \frac{\alpha_t}{\alpha} \tau + \frac{\alpha_x}{\alpha} \xi - \eta^1, \quad \eta^0 = b \frac{\lambda}{\beta}, \\ \left(\frac{\lambda}{\beta} \right)_{tx} &= \frac{d}{b} \alpha \lambda. \end{aligned} \tag{16}$$

In Cases 5–9, the first two equations of (16) respectively reduce to

- Case 5: $\tau = \eta^1, \quad \lambda = \xi_x;$
- Case 6: $\tau + \xi - \eta^1 = 0, \quad \lambda = \tau_t + \xi_x;$

Case 7: $q \frac{\tau - \xi}{x - t} - \eta^1 = 0, \lambda = \tau_t + \xi_x + 2 \frac{\tau - \xi}{x - t};$

Case 8: $q\xi + \eta^1 x = 0, \lambda = \tau_t - 2 \frac{\eta^1}{q};$ Case 9: $\eta^1 = 0, \lambda = \tau_t + \xi_x.$

In Case 5, we have no additional constraints on the parameters b, c and d . The last two equations of (16) in view of the first ones just define η^0 and $\xi, \eta^0 = b\xi_x e^t, b\xi_{xx} = d\xi_x.$

In Cases 6 and 7, the above equations imply $\lambda = 0,$ which contradicts the inequality $\lambda \neq 0.$ This means that these cases possess no further Lie-symmetry extensions with $a = 0.$

The system (16) implies $\tau_{tt} = 0$ and $d = 0$ in Case 8 and $d = 0$ in Case 9, which correspond to the Lie-symmetry extensions to Cases 11 and 13, respectively.

Merging the conditions derived separately for $a \neq 0$ and $a = 0,$ we obtain the following proposition.

Proposition 26 *A generalized nonlinear Klein–Gordon equation from Cases 5–9 admits an additional Lie-symmetry extension if and only if the corresponding value of the parameter function \hat{f} satisfies an equation (13) with $(a, b) \neq (0, 0),$ where $ad = 0$ in Case 5, $ad = bc$ in Case 6, $c = -(1 + 2/q)a, d = -(q + 1)(q + 2)b$ in Case $7_q,$ and $d = 0$ in Cases 8 and 9.*

Remark 27 In view of the infinitesimal counterpart of [54, Proposition 10], all the subclasses $\mathcal{K}'_N, N \in \Gamma,$ of \mathcal{K} that are associated with strong Cases 0–13 are normalized, see Remark 14 for notation and definitions. At the same time, this is not the case for most of the subclasses $\mathcal{K}_N, N \in \Gamma.$ More specifically, the subclasses $\mathcal{K}_1, \dots, \mathcal{K}_6, \mathcal{K}_{7_q}, \mathcal{K}_{8_q}$ and \mathcal{K}_9 are not normalized in view of the following arguments:

- $\mathcal{K}_1 \supset \mathcal{K}_9$ but $G_1 \not\supseteq G_9$ since $\mathcal{J}^0 \in G_9 \setminus G_1;$
- $\mathcal{K}_2 \supset \mathcal{K}_9$ but $G_2 \not\supseteq G_9$ since $\mathcal{J}^t \in G_9 \setminus G_2;$
- $\mathcal{K}_3 \ni K_f$ with $f = e^{-t}u^2$ but $\varpi_* G_3 \not\supseteq G_f$ since $\varpi_*(\mathcal{D}^t(-\ln t) \circ \mathcal{J}^0 \circ \mathcal{D}^t(e^{-t})) \in G_f \setminus \varpi_* G_3;$
- $\mathcal{K}_4 \ni K_f$ with $f = e^{-p(t+x)}|u|^p u$ for any $p \neq 0$ but $\varpi_* \mathfrak{g}_4 \not\supseteq \mathfrak{g}_f$ since $\mathcal{D}^t(e^{pt}) \in \mathfrak{g}_f \setminus \varpi_* \mathfrak{g}_4;$
- $\mathcal{K}_5 \ni K_f$ with $f = e^{-t}u^2$ but $\varpi_* G_5 \not\supseteq G_f$ since $\varpi_*(\mathcal{D}^t(-\ln t) \circ \mathcal{J}^0 \circ \mathcal{D}^t(e^{-t})) \in G_f \setminus \varpi_* G_5;$
- $\mathcal{K}_6 \ni K_f$ with $f = e^{-p(t+x)}|u|^p u$ for any $p \neq 0$ but $\varpi_* \mathfrak{g}_6 \not\supseteq \mathfrak{g}_f$ since $\mathcal{D}^t(e^{pt}) \in \mathfrak{g}_f \setminus \varpi_* \mathfrak{g}_6;$
- $\mathcal{K}_{7,q} \supset \mathcal{K}_{12,p}$ with $p = 2/q$ but $G_{7,q} \not\supseteq G_{12,p}$ since $\mathcal{D}^t(2t) \circ \mathcal{D}^x(x/2) \in G_{12,p} \setminus G_{7,q};$
- $\mathcal{K}_{8,q} \supset \mathcal{K}_{12,p}$ with $p = 2/q$ but $G_{8,q} \not\supseteq G_{12,p}$ since $\mathcal{J}^0 \in G_{12,p} \setminus G_{8,q};$
- $\mathcal{K}_9 \supset \mathcal{K}_{13} = \{K_{e^t}\}$ but $G_9 \not\supseteq G_{13}$ since $\mathcal{D}^t(-t^{-1}) \circ \mathcal{Z}(2\ln|t|) \in G_{13} \setminus G_9.$

Here G_f denotes the point symmetry group of the equation $K_f.$ The subclasses $\mathcal{K}_0, \mathcal{K}_{10}, \mathcal{K}_{11}, \mathcal{K}_{12,p}$ and \mathcal{K}_{13} are normalized since they coincide with $\mathcal{K}, \mathcal{K}'_{10}$ and the sin-

gletons $\{K_{e^{u/x}}\}$, $\{K_{|u|^p u}\}$ and $\{K_{e^u}\}$, respectively. (It is obvious that a class consisting of a single system of differential equations is normalized.)

6 On group classification of subclasses

Although we have exhaustively solved the group classification problem for the class \mathcal{K} , which consists of the equations of the form (1), this does not directly lead to the solution of the group classification problem for each of the subclasses of \mathcal{K} . Given a subclass $\hat{\mathcal{K}}$ of \mathcal{K} , Theorem 13 is used for the group classification of the subclass $\hat{\mathcal{K}}$ with respect to its equivalence group $G_{\hat{\mathcal{K}}}^{\sim}$ as follows.

- Recalling the normalization of \mathcal{K} , construct the equivalence group $G_{\hat{\mathcal{K}}}^{\sim}$ as the subgroup of G^{\sim} that consists of the elements of G^{\sim} preserving the subclass $\hat{\mathcal{K}}$.
- For each $N \in \Gamma$, intersect the subclass $\hat{\mathcal{K}}$ with the G^{\sim} -orbit $G_*^{\sim}\mathcal{K}'_N$ the subclass \mathcal{K}'_N (resp. with the G^{\sim} -orbit $G_*^{\sim}\mathcal{K}_N$ the subclass \mathcal{K}_N). This is realized via selecting those values of the arbitrary element f for equations from the orbit that satisfy the additional auxiliary constraint singling out the subclass $\hat{\mathcal{K}}$ from the class \mathcal{K} . The collection of the intersections presents a complete list of Lie symmetry extensions within the subclass $\hat{\mathcal{K}}$.
- In the selected values of f , gauge parameters by transformations from $G_{\hat{\mathcal{K}}}^{\sim}$.

Since the subclass $\hat{\mathcal{K}}$ is in general not normalized, the above procedure looks easier than directly solving the group classification problem for the subclass $\hat{\mathcal{K}}$ although its computational complexity is quite high.

The subclasses \mathcal{K}_N , $N \in \Gamma$, of \mathcal{K} that are associated with weak Cases 1–9, cf. Remark 14, are specific in regard to the above procedure. The group classification of each of these subclasses up to the equivalence generated by the corresponding equivalence groupoid can be easily derived via analyzing the Hasse diagram in Fig. 1, which depicts the structure of the partially ordered set of these cases. Nevertheless, this is not the case for the group classification up to the equivalence generated by the corresponding equivalence group since most of the subclasses \mathcal{K}_N , $N \in \Gamma$, are not normalized, see Remark 27. Note that both the group classifications of the subclasses \mathcal{K}_{10} , \mathcal{K}_{11} , $\mathcal{K}_{12,p}$ and \mathcal{K}_{13} are trivial since the kernel Lie invariance algebra of equations from each of these subclasses is the maximal Lie invariance algebra for every such equation.

Consider in detail the subclass \mathcal{K}_2 of the class (1), which is related to weak Case 2 of Theorem 13 and thus consists of the equations of the form

$$u_{tx} = f(\omega, u), \quad \text{where } \omega := x - t, \quad f_{uu} \neq 0,$$

or, in the variables $(\check{t}, \check{x}, \check{u}) = (x + t, x - t, u)$ with $\check{f}(\check{x}, \check{u}) = f(\omega, u)$,

$$\check{u}_{\check{t}\check{t}} - \check{u}_{\check{x}\check{x}} = \check{f}(\check{x}, \check{u}), \quad \check{f}_{\check{u}\check{u}} \neq 0.$$

Lemma 28 *The equivalence group G_2^\sim of the class \mathcal{K}_2 consists of the transformations of the form*

$$\tilde{t} = c_1 t + c_2, \quad \tilde{x} = c_1 x + c_3, \quad \tilde{u} = c_4 u + U^0(\omega), \quad \tilde{f} = c_1^{-2}(c_4 f - U_{\omega\omega}^0) \quad (17)$$

and the discrete equivalence transformation $\mathcal{J}^0: \tilde{t} = x, \tilde{x} = t, \tilde{u} = u, \tilde{f} = f$. Here c_1, \dots, c_4 are arbitrary constants with $c_1 c_4 \neq 0$, and U^0 is an arbitrary smooth function of $\omega := x - t$.

Proof Since the class (1) is normalized, the equivalence group of any subclass of (1) is the subgroup of G^\sim that consists of the elements of G^\sim preserving this subclass. It is obvious that the transformation \mathcal{J}^0 belongs to G_2^\sim . Any transformation of the form (2) that is contained by the group G_2^\sim satisfies the equation

$$T_t X_x \tilde{f}(X - T, U) = C f(x - t, u) + U_{tx}^0. \quad (18)$$

We act on (18) by the operator $\partial_t + \partial_x$, obtaining

$$\begin{aligned} (X_x - T_t) \tilde{f}_{\tilde{\omega}}(X - T, U) + (U_t^0 + U_x^0) \tilde{f}_{\tilde{u}}(X - T, U) \\ + \frac{T_{tt} X_x + T_t X_{xx}}{T_t X_x} \tilde{f}(X - T, U) = \frac{U_{ttx}^0 + U_{txx}^0}{T_t X_x}. \end{aligned}$$

Since \tilde{f} is an unconstrained value of the arbitrary element of the class \mathcal{K}_2 , for computing G_2^\sim , we can split the last equation with respect to \tilde{f} and its derivatives. As a result, we derive the equations $T_t = X_x$ and $U_t^0 + U_x^0 = 0$ on the parameters involved in the form (2). The integration of these equations implies the form (17). \square

The appropriate subalgebra $\mathfrak{s} = \langle \partial_t + \partial_x \rangle$ of $\mathfrak{g}_{(1)}$ is the kernel Lie invariance algebra of the equations from the class \mathcal{K}_2 . In other words, Case 2 is the general case with no Lie-symmetry extensions within this class. It is not normalized since the action groupoid of G_2^\sim is properly contained in the equivalence groupoid \mathcal{G}_2^\sim . Indeed, many equations in \mathcal{K}_2 , e.g., the Liouville equation, possess points symmetries that are not related to equivalence transformations of \mathcal{K}_2 . Moreover, Theorem 30 below implies that the class \mathcal{K}_2 is not semi-normalized as well. This is why it is natural that the group classifications of the class \mathcal{K}_2 up to the \mathcal{G}_2^\sim - and the G_2^\sim -equivalences are different. We easily see from the Hasse diagram in Fig. 1 that \mathcal{G}_2^\sim -inequivalent cases of Lie-symmetry extensions within the class \mathcal{K}_2 are exhausted by Cases 6, 7, 9, 10, 12 and 13, which gives the complete group classification of this class up to the \mathcal{G}_2^\sim -equivalence. (In Case 6, we should additionally alternate the sign of x .)

The complete group classification of the class \mathcal{K}_2 up to the G_2^\sim -equivalence is more delicate. It can be derived from the group classification of the superclass \mathcal{W} of \mathcal{K}_2 , which consists of the equations of the following form in the variables $(\check{t}, \check{x}, \check{u})$:

$$\check{u}_{\check{t}\check{t}} - \check{g}(\check{x}, \check{u})\check{u}_{\check{x}\check{x}} = \check{f}(\check{x}, \check{u}), \quad (\check{g}_{\check{u}}, \check{f}_{\check{u}\check{u}}) \neq (0, 0).$$

It is obvious that the class \mathcal{K}_2 is singled out from the superclass \mathcal{W} by the constraint $\check{g} = 1$. The comprehensive group analysis of the class \mathcal{W} was carried out in [55], where a different notation of the arbitrary elements \check{g} and \check{f} was used, $\check{g} \rightsquigarrow f$ and $\check{f} \rightsquigarrow g$. The equivalence group $G_{\mathcal{W}}^{\sim}$ and the equivalence groupoid $\mathcal{G}_{\mathcal{W}}^{\sim}$ of \mathcal{W} were described in [55, Theorem 6] and in [55, Theorem 9], respectively. The action groupoid of $G_{\mathcal{W}}^{\sim}$ is a proper subgroupoid of the groupoid $\mathcal{G}_{\mathcal{W}}^{\sim}$, i.e., the superclass \mathcal{W} is not normalized. The restriction of the action groupoid of $G_{\mathcal{W}}^{\sim}$ to the subclass \mathcal{K}_2 of \mathcal{W} coincides with the action groupoid of G_2^{\sim} . This is why the complete group classification of the class \mathcal{K}_2 up to the G_2^{\sim} -equivalence can be singled out from the complete group classification of the superclass \mathcal{W} up to the $G_{\mathcal{W}}^{\sim}$ -equivalence, which was presented in [55, Theorem 8]. Since the gauge $\check{g} = 1$ modulo the $G_{\mathcal{W}}^{\sim}$ -equivalence was used for representatives of Lie-symmetry extensions whenever it was possible, to classify Lie symmetries of equations from \mathcal{K}_2 up to the G_2^{\sim} -equivalence it suffices to select all the cases of [55, Table 1] with $\check{g} = 1$, i.e., $f = 1$ in the notation of [55], write them in the variables (t, x, u) , and supplement the result with Cases 6 and 7 of Theorem 13 of the present paper, which are the counterparts of the appropriate portions of Cases 1 and 2 of [55, Table 1]. As a result, we prove the following theorem.

Theorem 29 *A complete list of G_2^{\sim} -inequivalent cases of Lie-symmetry extensions of the kernel Lie invariance algebra $\mathfrak{g}^{\cap} = \langle \partial_t + \partial_x \rangle$ in the class \mathcal{K}_2 are exhausted by the following cases:*

- 2. *General case $f = \hat{f}(x - t, u)$: $\mathfrak{g}_f = \langle \partial_t + \partial_x \rangle$;*
- 6. *$f = e^{-x+t} \hat{f}(e^{x-t}u)$: $\mathfrak{g}_f = \langle \partial_t + u\partial_u, \partial_x - u\partial_u \rangle$;*
- 7. *$f = |x - t|^{-q-2} \hat{f}(|x - t|^q u)$, $q \neq 0$:
 $\mathfrak{g}_f = \langle \partial_t + \partial_x, t\partial_t + x\partial_x - qu\partial_u \rangle$;*
- 9a. *$f = \hat{f}(u)$: $\mathfrak{g}_f = \langle \partial_t, \partial_x, t\partial_t - x\partial_x \rangle$;*
- 9b. *$f = \hat{f}(u)e^{x-t}$: $\mathfrak{g}_f = \langle e^t\partial_t, e^{-x}\partial_x, \partial_t + \partial_x \rangle$;*
- 10a. *$f = \hat{f}(u)(x - t)^{-2}$: $\mathfrak{g}_f = \langle \partial_t + \partial_x, t\partial_t + x\partial_x, t^2\partial_t + x^2\partial_x \rangle$;*
- 10b. *$f = \hat{f}(u) \cos^{-2}(x - t)$:
 $\mathfrak{g}_f = \langle \partial_t + \partial_x, \cos 2t \partial_t - \cos 2x \partial_x, \sin 2t \partial_t - \sin 2x \partial_x \rangle$;*
- 10c. *$f = \hat{f}(u) \cosh^{-2}(x - t)$:
 $\mathfrak{g}_f = \langle \partial_t + \partial_x, e^{2t} \partial_t - e^{2x} \partial_x, e^{-2t} \partial_t - e^{-2x} \partial_x \rangle$;*
- 10d. *$f = \hat{f}(u) \sinh^{-2}(x - t)$:
 $\mathfrak{g}_f = \langle \partial_t + \partial_x, e^{2t} \partial_t + e^{2x} \partial_x, e^{-2t} \partial_t + e^{-2x} \partial_x \rangle$;*
- 12a. *$f = |u|^p u$, $p \neq -1, 0$: $\mathfrak{g}_f = \langle \partial_t, \partial_x, t\partial_t - x\partial_x, -pt\partial_t + u\partial_u \rangle$;*
- 12b. *$f = |u|^p u e^{x-t}$, $p \neq -1, 0$: $\mathfrak{g}_f = \langle e^t \partial_t, e^{-x} \partial_x, \partial_t + \partial_x, p\partial_t + u\partial_u \rangle$;*
- 13. *$f = e^u$: $\mathfrak{g}_f = \langle \tau(t)\partial_t + \xi(x)\partial_x - (\tau_t(t) + \xi_x(x))\partial_u \rangle$.*

Here \hat{f} is an arbitrary smooth function of its arguments whose second derivative with respect to the argument involving u is nonzero, q and p are arbitrary constants that sat-

isfy the conditions indicated in the corresponding cases. In Case 13, the components τ and ξ run through the sets of smooth functions of t or x , respectively.

We use the two-level numeration for the classification cases listed in Theorem 29 for indicating the presence of additional equivalences between these cases. Namely, numbers with the same Arabic numerals and different Roman letters correspond to cases that are G_2^\sim -inequivalent but \mathcal{G}_2^\sim -equivalent and hence G^\sim -equivalent as Lie-symmetry extensions within the class \mathcal{K} . Related cases in Theorems 13 and 29 have numbers with the same Arabic numerals.

To find all additional equivalence transformations among G_2^\sim -inequivalent classification cases for \mathcal{K}_2 and thus to relate the group classification of \mathcal{K}_2 modulo the G_2^\sim -equivalence to that modulo the \mathcal{G}_2^\sim -equivalence, we need to classify admissible transformations within the class \mathcal{K}_2 up to the G_2^\sim -equivalence. The description of the equivalence groupoid \mathcal{G}_2^\sim of the class \mathcal{K}_2 can be derived from [55, Theorem 9] analogously to the above derivation of the group classification of \mathcal{K}_2 up to the G_2^\sim -equivalence. See necessary notions in [55, Section 2].

Theorem 30 *A generating (up to the G_2^\sim -equivalence) set of admissible transformations for the class \mathcal{K}_2 , which is minimal and self-consistent with respect to the G_2^\sim -equivalence, is the union of the following families of admissible transformations (f, Φ, \tilde{f}) :*

- T1. $f = \hat{f}(u), \tilde{f} = -f, \Phi: \tilde{t} = -t, \tilde{x} = x, \tilde{u} = u;$
- T2. $f = \hat{f}(u), \tilde{f} = f, \Phi: \tilde{t} = te^\gamma, \tilde{x} = xe^{-\gamma}, \tilde{u} = u, \gamma \in \mathbb{R}_{\neq 0};$
- T3. $f = \hat{f}(u)e^{x-t}, \tilde{f} = \hat{f}(\tilde{u}), \Phi: \tilde{t} = -e^{-t}, \tilde{x} = e^x, \tilde{u} = u;$
- T4a. $f = \hat{f}(u)(x-t)^{-2}, \tilde{f} = \hat{f}(\tilde{u})(\tilde{x}-\tilde{t})^{-2},$
 $\Phi: \tilde{t} = t^{-1}, \tilde{x} = x^{-1}, \tilde{u} = u;$
- T4b. $f = -\hat{f}(u) \cos^{-2}(x-t), \tilde{f} = \hat{f}(\tilde{u})(\tilde{x}-\tilde{t})^{-2},$
 $\Phi: \tilde{t} = \tan t, \tilde{x} = \cot x, \tilde{u} = u;$
- T4c. $f = -\hat{f}(u) \cosh^{-2}(x-t), \tilde{f} = \hat{f}(\tilde{u})(\tilde{x}-\tilde{t})^{-2},$
 $\Phi: \tilde{t} = -\frac{1}{2}e^{2t}, \tilde{x} = \frac{1}{2}e^{2x}, \tilde{u} = u;$
- T4d. $f = \hat{f}(u) \sinh^{-2}(x-t), \tilde{f} = \hat{f}(\tilde{u})(\tilde{x}-\tilde{t})^{-2},$
 $\Phi: \tilde{t} = \frac{1}{2}e^{2t}, \tilde{x} = \frac{1}{2}e^{2x}, \tilde{u} = u;$
- T5. $f = e^u, \tilde{f} = e^{\tilde{u}}, \Phi: \tilde{t} = T(t), \tilde{x} = X(x), \tilde{u} = u - \ln(T_t X_x),$

where (T, X) runs through a complete set of representatives of cosets of (T, X) with $T_t X_x > 0$ and $(T_{tt}, X_{xx}) \neq (0, 0)$ with respect to the action of the group constituted by the transformations of the form $\hat{t} = c_1 t + c_2, \hat{x} = c_1 x + c_3, \hat{T} = \tilde{c}_1 T + \tilde{c}_2, \hat{X} = \tilde{c}_1 X + \tilde{c}_3$, where $c_1, c_2, c_3, \tilde{c}_1, \tilde{c}_2$ and \tilde{c}_3 are arbitrary constants with $c_1 \tilde{c}_1 \neq 0$.

Having the classification of admissible transformations within the class \mathcal{K}_2 up to the G_{\sim}^2 -equivalence, we can directly find all independent additional equivalence transformations among classification cases listed in Theorem 29. These transformations are

- T1: Case 9a $_{\hat{f}}$ \rightarrow Case 9a $_{-\hat{f}}$,
 T3: Case 9b \rightarrow Case 9a, Case 12b \rightarrow Case 12a,
 T4b: Case 10b \rightarrow Case 10a,
 T4c: Case 10c \rightarrow Case 10a,
 T4d: Case 10d \rightarrow Case 10a.

7 Conclusion

In the present paper, we have carried out the complete (contact) group classification of the class (1) of (1+1)-dimensional generalized nonlinear Klein–Gordon equations up to the G_{\sim} -equivalence. This has substantially enhanced the results on Lie symmetries of such equations that were obtained in the seminal papers [32,33]. At first, extending results of Lie's paper [35], we have shown in Lemma 2 that any contact admissible transformation within the class (1) is the first-order prolongation of a point admissible transformation within this class. In other words, the study of contract-transformation structures related to equations from the class (1) reduces to the study of their point-transformation counterparts. We have proved in Lemma 3 that the class (1) is normalized. Therefore, applying the algebraic method, we have reduced the group classification of (1) to classifying the appropriate subalgebras of the projection $\varpi_*\mathfrak{g}_{\sim} = \mathfrak{g}_{(\cdot)}$ of the equivalence algebra \mathfrak{g}_{\sim} . In addition to this, we have employed the specific structure of \mathfrak{g}_{\sim} for twofold involving the classical Lie theorem on realizations of Lie algebras by vector fields on the line [34] into the classification procedure. Moreover, the normalization of the class (1) means that the action groupoid [55] of the equivalence group G_{\sim} coincides with the entire equivalence groupoid \mathcal{G}_{\sim} of the class (1). Hence the complete group classification of the class (1) up to the G_{\sim} -equivalence coincides with its complete group classification up to the \mathcal{G}_{\sim} -equivalence, which is just the general point equivalence within this class. In other words, we have no additional point equivalences between G_{\sim} -inequivalent classification cases.

Lie symmetries of equations from the class (1) were considered in Section 6 of [33], and cases with two-, three- and four-dimensional Lie invariance algebras were listed in Table 1 therein, see also Section V and Table I in [32]. Cases 1–6, 8 and 9 of Table 1 and the equation (5.4) in [33] correspond to Cases 5, 6, $7_{q=1}$, $8_{q=-1}$, 10, 11, 9, 12 and 13 of Theorem 13 in the present paper. Case 7 of Table 1 in [33] should be excluded from the classification since it is equivalent to Case 9 therein, see the discussion of the case $(m, n, k) = (2, 3, 1)$ in Sect. 4. Cases $7_{q \neq 1}$ and $8_{q \neq -1}$ were missed in [33] owing to superfluously constraining the parameter q , see Remark 22.

We have additionally enhanced the results of [32,33] by explicitly singling out the equations from the class (1) with infinite-dimensional maximal Lie invariance algebras in Lemma 11. It turns out that any such equation is G_{\sim} -equivalent to the Liouville equation. For the other equations from the class (1), whose maximal Lie

invariance algebras are finite-dimensional, we have found the least upper bound of dimensions of these algebras, which is equal to four. One more tool for arranging the classification is to assign a value of the triple (m, n, k) of G^\sim -invariant integers to each case of Lie-symmetry extension in the class (1). We have strongly restricted the set of candidates for appropriate values of the triple at the stage of preliminary analysis using Lemma 11 and the Lie theorem. The final selection of the appropriate values has been done in the course of the group classification. It was important for simplifying the computations on all classification stages that for equations from the class (1), in contrast to evolution equations, the Lie theorem can be applied to both the t - and the x -projections of Lie-symmetry vector fields.

Although the characteristic triple (m, n, k) has a simple interpretation and is principal for the proof of Theorem 13, it does not suffice for completely distinguishing G^\sim -inequivalent classification cases. This is why we attempted to find as many G^\sim -invariant integer characteristics of the classification cases as possible, and have found even twelve of them in total, $m, n, k, l, j_1, j_2, j_{12}, j_{13}, j_{23}, r_1, r_2$ and r_3 . The comprehensive analysis has shown that the complete tuple of these twelve characteristics is redundant. As found out in Remark 24, these characteristics can constitute no pairs and exactly eight triples that suffice for distinguishing G^\sim -inequivalent classification cases, and the most remarkable triple is (r_3, j_1, r_2) . The same triple is, simultaneously with n , of primary importance for identifying the pairs of G^\sim -inequivalent weak classification cases that do not represent successive Lie-symmetry extensions, i.e., the pairs (Case N , Case \bar{N}) with Case $N \not\sim$ Case \bar{N} , see Sect. 5. To be sufficient for this task, the tuple (n, r_3, r_2, j_1) should be extended with one of the characteristics k, m or r_1 , and the thus obtained three tuples exhaust the set of such sufficient tuples of minimum size, which is equal to five. In the same section, we have directly checked that all the other pairs of G^\sim -inequivalent classification cases are indeed associated with successive Lie-symmetry extensions. The consideration has been summarized in the Hasse diagram in Fig. 1, which represents the structure of the partially ordered set of G^\sim -inequivalent Lie-symmetry extensions within the class (1). Analyzing the Hasse diagram allows one to easily solve the group classification problems up to the general point equivalence for the subclasses \mathcal{K}_N , $N \in \Gamma$, of \mathcal{K} , which correspond, under the interpretation in the weak sense, to the classification cases that have been listed in Theorem 13; see Remark 14. Since the subclasses $\mathcal{K}_1, \dots, \mathcal{K}_6, \mathcal{K}_{7,q}, \mathcal{K}_{8,q}$ and \mathcal{K}_9 are not normalized, the group classification of any such \mathcal{K}_N up to the G^\sim -equivalence is not so easy. The last claim has been illustrated in Sect. 6 by carrying out the group classification of the subclass \mathcal{K}_2 up to the G_2^\sim -equivalence. Therein, we have also discussed a procedure of using Theorem 13 for the group classification of any subclass of the class \mathcal{K} with respect to the equivalence group of this subclass.

The classification of Lie symmetries is the first necessary step for extended symmetry analysis of equations from the class (1). It can be used for the classification of Lie reductions and further finding exact invariant solutions of these equations. Since the general solution of the Liouville equation is well known, Lie reductions should be carried out only for equations from the class (1) with finite-dimensional maximal Lie invariance algebras. In view of Lemma 11(iii), which states upper bound four for the dimensions of such algebras, the classification of subalgebras of three- and four-dimensional Lie algebras in [46] is extremely relevant here. As an example, con-

sider Case 10. It is the only case among those with finite-dimensional maximal Lie invariance algebras, where the algebra \mathfrak{g}_f is not solvable. More precisely, it is isomorphic to the algebra $\mathfrak{sl}(2, \mathbb{R})$, and its inequivalent one-dimensional subalgebras and the associated Lie reductions to ordinary differential equations are the following:

1. $\langle \partial_t + \partial_x \rangle$: $u = \varphi(\omega)$, $\omega = x - t$, $\varphi_{\omega\omega} = -\hat{f}(\varphi)\omega^{-2}$;
2. $\langle t\partial_t + x\partial_x \rangle$: $u = \varphi(\omega)$, $\omega = \frac{1}{2} \ln |x| - \frac{1}{2} \ln |t|$, $\varphi_{\omega\omega} = -\hat{f}(\varphi) \sinh^{-2} \omega$;
3. $\langle (1 + t^2)\partial_t + (1 + x^2)\partial_x \rangle$: $u = \varphi(\omega)$, $\omega = \arctan x - \arctan t$,
 $\varphi_{\omega\omega} = -\hat{f}(\varphi) \sin^{-2} \omega$.

The knowledge of Lie symmetries of equations from the class (1) is also needed for classification of reduction operators of these equations. In the course of classifying reduction operators, it is natural to exclude those of them that are induced by Lie symmetries, i.e., to look only for non-Lie reduction operators. Unfortunately, the general description of regular reduction operators for equations from the class \mathcal{K} as a whole from [57] cannot be used in a reasonable way for describing reduction operators of particular equations from this class or for classifying reduction operators of equations constituting its proper subclasses. Recall that regarding singular reduction operators of equations from the class (1), a systematic study has been done in the literature only for equations with $f = \hat{f}(u)$, which constitute the subclass \mathcal{K}_9 associated with Case 9 of Theorem 13, see [27, Section 6].

The consideration of the presented paper can be extended to the much wider superclass of generalized nonlinear Klein–Gordon equations of the form

$$u_{tx} = f(t, x, u, u_t, u_x). \quad (19)$$

A preliminary study has shown that Lemma 2 may be generalized to this superclass, and the Lie theorem should be relevant for its group classification in the same way as for that of the class (1). The principal precondition for applying the Lie theorem to the group classification of the superclass (19) is to prove an analogue of Lemma 11 for this superclass, which seems a much more difficult problem than proving Lemma 11. In particular, one needs to single out, within the superclass (19), the equations with infinite-dimensional maximal Lie invariance algebras.

Acknowledgements The authors are sincerely grateful to the anonymous referees for a number of valuable remarks and suggestions. The authors thank Michael Kunzinger, Dmytro Popovych, Galyna Popovych and Olena Vaneeva for helpful discussions and acknowledge the partial financial support provided by the NAS of Ukraine under the Project 0116U003059. The research of ROP was supported by the Austrian Science Fund (FWF), Projects P25064 and P28770.

Data Availability Statement Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The author(s) declare that they have no competing interests.

References

1. Akhatov, I.S., Gazizov, R.K., Ibragimov, N.K.: Nonlocal symmetries. A heuristic approach. *J. Soviet Math.* **55**, 1401–1450 (1991)
2. Basarab-Horwath, P., Lahno, V., Zhdanov, R.: The structure of Lie algebras and the classification problem for partial differential equations. *Acta Appl. Math.* **69**, 43–94 (2001). [arXiv:math-ph/0005013](https://arxiv.org/abs/math-ph/0005013)
3. Bihlo, A., Dos Santos Cardoso-Bihlo E., Popovych, R.O.: Complete group classification of a class of nonlinear wave equations. *J. Math. Phys.* **53**, 123515 (2012). [arXiv:1106.4801](https://arxiv.org/abs/1106.4801)
4. Bihlo, A., Poltavets, N., Popovych, R.O.: Lie symmetries of two-dimensional shallow water equations with variable bottom topography. *Chaos* **30**, 073132 (2020). [arXiv:1911.02097](https://arxiv.org/abs/1911.02097)
5. Bihlo, A., Popovych, R.O.: Group classification of linear evolution equations. *J. Math. Anal. Appl.* **448**, 982–2015 (2017). [arXiv:1605.09251](https://arxiv.org/abs/1605.09251)
6. Bluman, G.W., Cheviakov, A.F., Anco, S.C.: *Application of Symmetry Methods to Partial Differential Equations*. Springer, New York (2010)
7. Bluman, G., Kumei, S.: *Symmetries and Differential Equations*. Springer, New York (1989)
8. Bluman, G.W., Reid, G.J., Kumei, S.: New classes of symmetries for partial differential equations. *J. Math. Phys.* **29**, 806–811 (1988)
9. Boyko, V.M., Kunzinger, M., Popovych, R.O.: Singular reduction modules of differential equations. *J. Math. Phys.* **57**, 101503 (2016). [arXiv:1201.3223](https://arxiv.org/abs/1201.3223)
10. Boyko, V.M., Popovych, R.O., Shapoval, N.M.: Equivalence groupoids of classes of linear ordinary differential equations and their group classification. *J. Phys. Conf. Ser.* **621**, 012001 (2015). [arXiv:1403.6062](https://arxiv.org/abs/1403.6062)
11. Burde, D., Steinhoff, C.: Classification of orbit closures of 4-dimensional complex Lie algebras. *J. Algebra* **214**, 729–739 (1999)
12. Clarkson, P.A., McLeod, J.B., Olver, P.J., Ramani, R.: Integrability of Klein–Gordon equations. *SIAM J. Math. Anal.* **17**, 798–802 (1986)
13. Dodd, R.K., Bullough, R.K.: Polynomial conserved densities for the sine-Gordon equations. *Proc. R. Soc. Lond. Ser. A* **352**, 481–503 (1977)
14. Dodd, R.K., Eilbeck, J.C., Gibbon, J.D., Morris, H.C.: *Solitons and Nonlinear Wave Equations*. Academic Press, London (1982)
15. Dos Santos Cardoso-Bihlo, E., Bihlo, A., Popovych, R.O.: Enhanced preliminary group classification of a class of generalized diffusion equations. *Commun. Nonlinear Sci. Numer. Simulat.* **16**, 3622–3638 (2011). [arXiv:1012.0297](https://arxiv.org/abs/1012.0297)
16. Fox, D., Goertsches, O.: Higher-order conservation laws for the nonlinear Poisson equation via characteristic cohomology. *Selecta Math. (N.S.)* **17**, 795–831 (2011). [arXiv:0906.3143](https://arxiv.org/abs/0906.3143)
17. Fushchich, W.I., Shtelen, W.M., Serov, N.I.: *Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics*. Kluwer Academic Publisher, Dordrecht (1993)
18. Gagnon, L., Winternitz, P.: Symmetry classes of variable coefficient nonlinear Schrödinger equations. *J. Phys. A* **26**, 7061–7076 (1993)
19. Gazeau, J.P., Winternitz, P.: Symmetries of variable coefficient Korteweg–de Vries equations. *J. Math. Phys.* **33**, 4087–4102 (1992)
20. Grundland, A.M., Infeld, E.: A family of nonlinear Klein–Gordon equations and their solutions. *J. Math. Phys.* **33**, 2498–2503 (1992)
21. Grunewald, F., O’Halloran, J.: Varieties of nilpotent Lie algebras of dimension less than six. *J. Algebra* **112**, 315–325 (1988)
22. Güngör, F., Lahno, V.I., Zhdanov, R.Z.: Symmetry classification of KdV-type nonlinear evolution equations. *J. Math. Phys.* **45**, 2280–2313 (2004). [arXiv:nlin/0201063](https://arxiv.org/abs/nlin/0201063)
23. Ibragimov, N.H.: *Transformation Groups Applied to Mathematical Physics, Mathematics and its Applications (Soviet Series)*. D. Reidel Publishing Co., Dordrecht (1985)
24. Ibragimov, N.H., Torrisi, M., Valenti, A.: Preliminary group classification of equations $v_{tt} = f(x, v_x)v_{xx} + g(x, v_x)$. *J. Math. Phys.* **32**, 2988–2995 (1991)
25. Ivanova, N.M., Popovych, R.O., Sophocleous C.: Group analysis of variable coefficient diffusion–convection equations. II. Contractions and exact solutions. [arXiv:0710.3049](https://arxiv.org/abs/0710.3049)
26. Kingston, J.G., Sophocleous, C.: On form-preserving point transformations of partial differential equations. *J. Phys. A* **31**, 1597–1619 (1998)
27. Kunzinger, M., Popovych, R.O.: Singular reduction operators in two dimensions. *J. Phys. A* **41**, 505201 (2008). [arXiv:0808.3577](https://arxiv.org/abs/0808.3577)

28. Kurujyibwami, C., Basarab-Horwath, P., Popovych, R.O.: Algebraic method for group classification of (1+1)-dimensional linear Schrödinger equations. *Acta Appl. Math.* **157**, 171–203 (2018). [arXiv:1607.04118](#)
29. Kurujyibwami, C., Popovych, R.O.: Equivalence groupoids and group classification of multidimensional nonlinear Schrödinger equations. *J. Math. Anal. Appl.* **491**, 124271 (2020). [arXiv:2003.02781](#)
30. Kushner A.G.: Classification of Monge–Ampère equations. In: *Differential Equations: Geometry, Symmetries and Integrability*. Abel Symp. Vol. 5, Springer, Berlin, 2009, pp. 223–256
31. Kushner, A.G.: On contact equivalence of Monge–Ampère equations to linear equations with constant coefficients. *Acta Appl. Math.* **109**, 197–210 (2010)
32. Lahno, V., Zhdanov, R.: Group classification of nonlinear wave equations. *J. Math. Phys.* **46**, 053301 (2005)
33. Lahno, V., Zhdanov, R., Magda, O.: Group classification and exact solutions of nonlinear wave equations. *Acta Appl. Math.* **91**, 253–313 (2006). [arXiv:nlin.SI/0405069](#)
34. Lie, S.: Theorie der Transformationsgruppen I. *Math. Ann.* **16**, 441–528 (1880)
35. Lie S.: Diskussion der Differentialgleichung $d^2z/dx dy = F(z)$, *Arch. for Math.* **6**, 112–124, (1881). (Reprinted in: Lie S., *Gesammelte Abhandlungen*, Vol. 3, B.G. Teubner, Leipzig and H. Aschehoug & Co, Kristiania, 469–478.)
36. Lie S.: Über die Integration durch bestimmte Integrale von einer Klasse linearer partieller Differentialgleichungen, *Arch. for Math.* **6**, 328–368, (1881). (Translation by N.H. Ibragimov: S. Lie: On integration of a class of linear partial differential equations by means of definite integrals, *CRC Handbook of Lie Group Analysis of Differential equations*. Vol. 2. Applications in Engineering and Physical Sciences, CRC Press, Boca Raton, FL, 1995, pp. 473–508.)
37. Lie, S.: Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen. B.G. Teubner, Leipzig (1891)
38. Magadeev B.A.: Group classification of nonlinear evolution equations, *Algebra i Analiz* **5**, 141–156, (1993) (in Russian); English translation in *St. Petersburg Math. J.* **5**, 345–359, (1994)
39. McLeod, J.B., Olver, P.J.: The connection between partial differential equations soluble by inverse scattering and ordinary differential equations of Painlevé type. *SIAM J. Math. Anal.* **14**, 488–506 (1983)
40. Nesterenko, M., Popovych, R.O.: Contractions of low-dimensional Lie algebras. *J. Math. Phys.* **47**, 123515 (2006). [arXiv:math-ph/0608018](#)
41. Nikitin, A.G., Popovych, R.O.: Group classification of nonlinear Schrödinger equations. *Ukrainian Math. J.* **53**, 1255–1265 (2001). [arXiv:math-ph/0301009](#)
42. Olver, P.J.: *Application of Lie Groups to Differential Equations*. Springer, New York (1993)
43. Olver, P.J.: *Equivalence, Invariants, and Symmetry*. Cambridge University Press, Cambridge (1995)
44. Opanasenko, S., Bihlo, A., Popovych, R.O.: Group analysis of general Burgers–Korteweg–de Vries equations. *J. Math. Phys.* **58**, 081511 (2017). [arXiv:1703.06932](#)
45. Opanasenko, S., Boyko, V., Popovych, R.O.: Enhanced group classification of nonlinear diffusion–reaction equations with gradient-dependent diffusion. *J. Math. Anal. Appl.* **484**, 123739 (2020). [arXiv:1804.08776](#)
46. Patera, J., Winternitz, P.: Subalgebras of real three- and four-dimensional Lie algebras. *J. Math. Phys.* **18**, 1449–1455 (1977)
47. Polyanin, A.D., Zaitsev, V.F.: *Handbook of Nonlinear Partial Differential Equations*, 2nd edn. Chapman & Hall, Boca Raton (2012)
48. Popovych, R.O.: Classification of admissible transformations of differential equations. In: *Collection of Works of Institute of Mathematics*, vol. 3, pp. 239–254. Institute of Mathematics, Kyiv (2006)
49. Popovych, R.O.: Point and contact equivalence groupoids of two-dimensional quasilinear hyperbolic equations. *Appl. Math. Lett.* **116**, 107068 (2021). [arXiv:2009.07383](#)
50. Popovych, R.O., Bihlo, A.: Symmetry preserving parameterization schemes, *J. Math. Phys.* **53**, 073102, (2012), [arXiv:1010.3010](#)
51. Popovych, R.O., Boyko, V.M., Nesterenko, M.O., Lutfullin, M.W.: Realizations of real low-dimensional Lie algebras. *J. Phys. A* **36**, 7337–7360 (2003). [arXiv:math-ph/0301029](#)
52. Popovych, R.O., Eshraghi, H.: Admissible point transformations of nonlinear Schrödinger equations. In: *Proceedings of 10th International Conference in MODern GROup ANalysis (MOGRAN X)* (Larnaca, Cyprus, 2004), University of Cyprus, Nicosia, pp. 167–174 (2005)
53. Popovych, R.O., Ivanova, N.M.: Potential equivalence transformations for nonlinear diffusion–convection equations. *J. Phys. A* **38**, 3145–3155 (2005). [arXiv:math-ph/0402066](#)

54. Popovych, R.O., Kunzinger, M., Eshraghi, H.: Admissible transformations and normalized classes of nonlinear Schrödinger equations. *Acta Appl. Math.* **109**, 315–359 (2010). [arXiv:math-ph/0611061](#)
55. Vaneeva, O.O., Bihlo, A., Popovych, R.O.: Generalization of the algebraic method of group classification with application to nonlinear wave and elliptic equations. *Commun. Nonlinear Sci. Numer. Simul.* **91**, 105419 (2020). [arXiv:2002.08939](#)
56. Vaneeva, O.O., Popovych, R.O., Sophocleous, C.: Extended group analysis of variable coefficient reaction–diffusion equations with exponential nonlinearities. *J. Math. Anal. Appl.* **396**, 225–242 (2012). [arXiv:1111.5198](#)
57. Yehorchenko, I.: Conditional symmetry and reductions for the two-dimensional nonlinear wave equation. I. General case. [arXiv:1010.4913](#)
58. Zhdanov, R.Z.: Separation of variables in the non-linear wave equation. *J. Phys. A* **27**, L291–L297 (1994)
59. Zhdanov, R.Z., Lahno, V.I.: Group classification of heat conductivity equations with a nonlinear source. *J. Phys. A* **32**, 7405–7418 (1999). [arXiv:math-ph/9906003](#)
60. Zhiber, A.V., Shabat, A.B.: Klein–Gordon equations with a nontrivial group. *Soviet Phys. Dokl.* **24**, 607–609 (1979)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.