New subclasses of analytic and bi-univalent functions endowed with coefficient estimate problems

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Abstract

Inspired by the recent works of Srivastava et al. (Appl Math Lett 23(10):1188–1192, 2010), Frasin and Aouf (Appl Math Lett 24(9):1569–1573, 2011), and Çağlar et al. (Filomat 27(7):1165–1171, 2013), we introduce and investigate in the present paper two new general subclasses of the class consisting of normalized analytic and bi-univalent functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For functions belonging to these general subclasses introduced here, we obtain estimates on the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$. Several connections to some of the earlier known results are also pointed out. The results presented in this paper would generalize and improve those in related works of several earlier authors.

1 Introduction and definitions

Let $\mathcal{A}$ denote the class of all analytic functions f defined in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Thus

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each $f \in \mathcal{A}$ has a Taylor–Maclaurin series expansion of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathbb{U}). \tag{1.1}$$

Further, let $\mathcal{S}$ denote the class of all functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$ (for details, see [8]; see also some of the recent investigations [1–3,10,19]).

Two of the important and well-investigated subclasses of the analytic and univalent function class $\mathcal{S}$ are the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in $\mathbb{U}$ and the class $\mathcal{K}(\alpha)$ of convex functions of order α in $\mathbb{U}$. By definition, we have

$$\mathcal{S}^*(\alpha) := \left\{ f : f \in \mathcal{S} \text{ and } \operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \quad (z \in \mathbb{U}; 0 \le \alpha < 1) \right\}, \tag{1.2}$$

and

$$\mathcal{K}(\alpha) := \left\{ f : f \in \mathcal{S} \text{ and } \operatorname{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, \quad (z \in \mathbb{U}; 0 \le \alpha < 1) \right\}. \tag{1.3}$$

It is clear from the definitions (1.2) and (1.3) that $\mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha)$. Also we have

$$f(z) \in \mathcal{K}(\alpha) \text{ iff } zf'(z) \in \mathcal{S}^*(\alpha), \tag{1.4}$$

and

$$f(z) \in \mathcal{S}^*(\alpha) \text{ iff } \int_0^z \frac{f(t)}{t} dt = F(z) \in \mathcal{K}(\alpha). \tag{1.5}$$

It is well-known that, if $f(z)$ is an univalent analytic function from a domain $\mathbb{D}_1$ onto a domain $\mathbb{D}_2$, then the inverse function $g(z)$ defined by

$$g(f(z)) = z, \quad (z \in \mathbb{D}_1),$$

is an analytic and univalent mapping from $\mathbb{D}_2$ to $\mathbb{D}_1$. Moreover, by the familiar Koebe one-quarter theorem (for details, see [8]), we know that the image of $\mathbb{U}$ under every function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$.

According to this, every function $f \in \mathcal{S}$ has an inverse map f^{-1} that satisfies the following conditions:

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U}),$$

and

$$f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f); r_0(f) \ge \tfrac{1}{4}\right).$$

In fact, the inverse function is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{1.6}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). Examples of functions in the class Σ are

$$\frac{z}{1-z}, \ -\log(1-z), \ \frac{1}{2}\log\left(\frac{1+z}{1-z}\right), \cdots.$$

It is worth noting that the familiar Koebe function is not a member of Σ, since it maps the unit disk $\mathbb{U}$ univalently onto the entire complex plane except the part of the negative real axis from $-1/4$ to $-\infty$. Thus, clearly, the image of the domain does not contain the unit disk $\mathbb{U}$. For a brief history and some intriguing examples of functions and characterization of the class Σ, see Srivastava et al. [15], Frasin and Aouf [9], and Yousef et al. [20].

In 1967, Lewin [11] investigated the bi-univalent function class Σ and showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [5] conjectured that $|a_2| \leq \sqrt{2}$. The best known estimate for functions in Σ has been obtained in 1984 by Tan [16], that is, $|a_2| < 1.485$. The coefficient estimate problem for each of the following Taylor–Maclaurin coefficients $|a_n|$ ($n \in \mathbb{N}\setminus\{1, 2\}$) for each $f \in \Sigma$ given by (1.1) is presumably still an open problem.

Brannan and Taha [6] introduced certain subclasses of a bi-univalent function class Σ similar to the familiar subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of starlike and convex functions of order α ($0 \leq \alpha < 1$), respectively (see [4]). Thus, following the works of Brannan and Taha [6], for $0 \leq \alpha < 1$, a function $f \in \Sigma$ is in the class $\mathcal{S}^*_\Sigma(\alpha)$ of bi-starlike functions of order α; or $\mathcal{K}_\Sigma(\alpha)$ of bi-convex functions of order α if both f and f^{-1} are respectively starlike or convex functions of order α. Also, a function $f \in \mathcal{A}$ is in the class $\mathcal{S}^*_\Sigma[\alpha]$ of strongly bi-starlike functions of order α ($0 \leq \alpha < 1$) if each of the following conditions is satisfied:

$$f \in \Sigma \ \text{ and } \ \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 \leq \alpha < 1, z \in \mathbb{U})$$

and

$$\left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 \leq \alpha < 1, w \in \mathbb{U}),$$

where g is the extension of f^{-1} to $\mathbb{U}$.

Recently, many researchers have introduced and investigated several interesting subclasses of the bi-univalent function class Σ and they have found non-sharp estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$. In fact, the aforecited work of Srivastava et al. [15] essentially revived the investigation of various subclasses of the bi-univalent function class Σ in recent years; it was followed by such works as those by Frasin and Aouf [9], Xu et al. [17], Çağlar et al. [7], and others (see, for example, [12,14] and [18]). Motivated by the aforementioned works, the main object of the present investigation is to introduce two new subclasses of the function class Σ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses

of the function class Σ employing the techniques used earlier by Srivastava et al. [15]. Our work will extend the aforementioned results of Srivastava et al. [15], Frasin and Aouf [9], and Çağlar et al. [7]. Moreover, our estimates for the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ will improve the results of Srivastava et al. ([15], Theorem 2) and Frasin and Aouf ([9], Theorem 3.2). Various known or new special cases of our results are also pointed out.

The following lemma will be required in order to derive our main results.

Lemma 1.1 [13] *If $p \in \mathcal{P}$, then $|c_k| \leq 2$ for each k , where $\mathcal{P}$ is the family of all functions p analytic in $\mathbb{U}$ for which $Re\,(p(z)) > 0$, $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ for $z \in \mathbb{U}$.*

2 Coefficient bounds for the function class $\mathscr{B}_{\Sigma}^{\mu}(\alpha, \lambda, \delta)$

We begin this section by introducing the following subclass of the function class Σ.

Definition 2.1 For $\lambda \geq 1, \mu \geq 0, \delta \geq 0$ and $0 < \alpha \leq 1$, a function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathscr{B}_{\Sigma}^{\mu}(\alpha, \lambda, \delta)$ if the following conditions hold for all $z, w \in \mathbb{U}$:

$$\left| \arg \left((1 - \lambda) \left(\frac{f(z)}{z} \right)^{\mu} + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} + \xi \delta z f''(z) \right) \right| < \frac{\alpha \pi}{2} \quad (2.1)$$

and

$$\left| \arg \left((1 - \lambda) \left(\frac{g(w)}{w} \right)^{\mu} + \lambda g'(w) \left(\frac{g(w)}{w} \right)^{\mu-1} + \xi \delta w g''(w) \right) \right| < \frac{\alpha \pi}{2}, \quad (2.2)$$

where the function $g(w) = f^{-1}(w)$ is defined by (1.6) and $\xi = \frac{2\lambda + \mu}{2\lambda + 1}$.

Remark 1 Note that for $\lambda = 1, \mu = 1$ and $\delta = 0$, the class of functions $\mathscr{B}_{\Sigma}^{1}(\alpha, 1, 0) := \mathscr{B}_{\Sigma}(\alpha)$ have been introduced and studied by Srivastava et al. [15], for $\mu = 1$ and $\delta = 0$, the class of functions $\mathscr{B}_{\Sigma}^{1}(\alpha, \lambda, 0) := \mathscr{B}_{\Sigma}(\alpha, \lambda)$ have been introduced and studied by Frasin and Aouf [9], for $\delta = 0$, the class of functions $\mathscr{B}_{\Sigma}^{\mu}(\alpha, \lambda, 0) := \mathscr{B}_{\Sigma}^{\mu}(\alpha, \lambda)$ have been introduced and studied by Çağlar et al. [7], and for $\lambda = 1, \mu = 0$ and $\delta = 0$, we obtain the well-known class $\mathscr{B}_{\Sigma}^{0}(\alpha, 1, 0) := \mathcal{S}_{\Sigma}^{*}[\alpha]$ of strongly bi-starlike functions of order α.

We first state and prove the following result.

Theorem 2.2 *Let the function $f(z)$ given by (1.1) be in the class $\mathscr{B}_{\Sigma}^{\mu}(\alpha, \lambda, \delta)$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{(\lambda + \mu + 2\xi\delta)^2 + \alpha \left[2\lambda + \mu - (\lambda + 2\xi\delta)^2 + (12 - 4\mu)\xi\delta \right]}} \quad (2.3)$$

and

$$|a_3| \leq \frac{4\alpha^2}{(\lambda + \mu + 2\xi\delta)^2} + \frac{2\alpha}{2\lambda + \mu + 6\xi\delta}.$$
(2.4)

Proof Let $f \in \mathscr{B}_{\Sigma}^{\mu}(\alpha, \lambda, \delta)$. From (2.1) and (2.2), we have

$$(1 - \lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} + \xi\delta z f''(z) = [p(z)]^{\alpha}$$
(2.5)

and

$$(1 - \lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} + \xi\delta w g''(w) = [q(w)]^{\alpha},$$
(2.6)

where $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ and $q(w) = 1 + q_1 w + q_2 w^2 + \cdots$ in $\mathcal{P}$.

Now, equating the coefficients in (2.5) and (2.6), we get

$$(\lambda + \mu + 2\xi\delta)\, a_2 = \alpha p_1, \quad (2.7)$$

$$(2\lambda + \mu)\left[\left(\frac{\mu - 1}{2}\right)a_2^2 + \left(1 + \frac{6\delta}{2\lambda + 1}\right)a_3\right] = \alpha p_2 + \frac{\alpha(\alpha - 1)}{2}p_1^2, \quad (2.8)$$

and

$$-(\lambda + \mu + 2\xi\delta)\, a_2 = \alpha q_1, \quad (2.9)$$

$$(2\lambda + \mu)\left[\left(\frac{\mu + 3}{2} + \frac{12\delta}{2\lambda + 1}\right)a_2^2 - \left(1 + \frac{6\delta}{2\lambda + 1}\right)a_3\right] = \alpha q_2 + \frac{\alpha(\alpha - 1)}{2}q_1^2. \quad (2.10)$$

From (2.7) and (2.9), we obtain

$$p_1 = -q_1,$$
(2.11)

and

$$2\,(\lambda + \mu + 2\xi\delta)^2\, a_2^2 = \alpha^2(p_1^2 + q_1^2).$$
(2.12)

By adding (2.8) to (2.10), we get

$$(2\lambda + \mu)\left[1 + \mu + \frac{12\delta}{2\lambda + 1}\right]a_2^2 = \alpha(p_2 + q_2) + \frac{\alpha(\alpha - 1)}{2}(p_1^2 + q_1^2). \quad (2.13)$$

By using (2.12) in (2.13), we obtain

$$\left[(2\lambda + \mu)(\mu + 1) + 12\xi\delta - \frac{(\alpha - 1)}{\alpha}(\lambda + \mu + 2\xi\delta)^2\right] a_2^2 = \alpha\,(p_2 + q_2).$$
(2.14)

By considering Lemma 1.1 we get from (2.14) the desired inequality (2.3).
Next, by subtracting (2.10) from (2.8), we have

$$2(2\lambda + \mu)\left(1 + \frac{6\delta}{2\lambda + 1}\right) a_3 - 2(2\lambda + \mu)\left(1 + \frac{6\delta}{2\lambda + 1}\right) a_2^2$$

$$= \alpha\,(p_2 - q_2) + \frac{\alpha(\alpha - 1)}{2}\left(p_1^2 - q_1^2\right).$$
(2.15)

Further, in view of (2.11), it follows from (2.15) that

$$a_3 = a_2^2 + \frac{\alpha}{2(2\lambda + \mu + 6\xi\delta)}\,(p_2 - q_2).$$
(2.16)

By considering (2.12) and Lemma 1.1, we get from (2.16) the desired inequality (2.4).

This complete the proof of Theorem 2.2.   □

Now, we would like to draw attention to some remarkable results which are obtained for some values of λ, μ and δ in Theorem 2.2.

If we choose $\lambda = 1$, $\mu = 1$ and $\delta = 0$ in Theorem 2.2, we get the following consequence.

Corollary 2.3 [15] *Let the function $f(z)$ given by (1.1) be in the class $\mathscr{B}_\Sigma(\alpha)$. Then*

$$|a_2| \leq \alpha\sqrt{\frac{2}{\alpha + 2}}$$

and

$$|a_3| \leq \frac{\alpha(3\alpha + 2)}{3}.$$

If we choose $\mu = 1$ and $\delta = 0$ in Theorem 2.2, we get the following consequence.

Corollary 2.4 [9] *Let the function $f(z)$ given by (1.1) be in the class $\mathscr{B}_\Sigma(\alpha, \lambda)$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{(\lambda + 1)^2 + \alpha(1 + 2\lambda - \lambda^2)}}$$

and

$$|a_3| \leq \frac{4\alpha^2}{(\lambda + 1)^2} + \frac{2\alpha}{(2\lambda + 1)}.$$

If we choose $\delta = 0$ in Theorem 2.2, we get the following consequence.

Corollary 2.5 [7] *Let the function* $f(z)$ *given by* (1.1) *be in the class* $\mathscr{B}_{\Sigma}^{\mu}(\alpha, \lambda)$. *Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{(\lambda + \mu)^2 + \alpha(2\lambda + \mu - \lambda^2)}}$$

and

$$|a_3| \leq \frac{4\alpha^2}{(\lambda + \mu)^2} + \frac{2\alpha}{(2\lambda + \mu)}.$$

If we choose $\lambda = 1, \mu = 0$ and $\delta = 0$ in Theorem 2.2, we get the following consequence.

Corollary 2.6 [7] *Let the function* $f(z)$ *given by* (1.1) *be in the class* $\mathcal{S}_{\Sigma}^{*}[\alpha]$. *Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{1 + \alpha}}$$

and

$$|a_3| \leq \alpha(4\alpha + 1).$$

3 Coefficient bounds for the function class $\mathscr{B}_{\Sigma}^{\mu}(\beta, \lambda, \delta)$

This section is concerned with the coefficient bounds for the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ of the function $f \in \mathscr{B}_{\Sigma}^{\mu}(\beta, \lambda, \delta)$. Various known spacial cases of the main result are pointed out.

Definition 3.1 For $\lambda \geq 1, \mu \geq 0, \delta \geq 0$ and $0 \leq \beta < 1$, a function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathscr{B}_{\Sigma}^{\mu}(\beta, \lambda, \delta)$ if the following conditions hold for all $z, w \in \mathbb{U}$:

$$\mathrm{Re}\left((1 - \lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu - 1} + \xi\delta z f''(z)\right) > \beta \qquad (3.1)$$

and

$$\mathrm{Re}\left((1 - \lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu - 1} + \xi\delta w g''(w)\right) > \beta, \qquad (3.2)$$

where the function $g(w) = f^{-1}(w)$ is defined by (1.6) and $\xi = \frac{2\lambda + \mu}{2\lambda + 1}$.

Remark 2 Note that for $\lambda = 1$, $\mu = 1$ and $\delta = 0$, the class of functions $\mathscr{B}^1_\Sigma(\beta, 1, 0) := \mathscr{B}_\Sigma(\beta)$ have been introduced and studied by Srivastava et al. [15], for $\mu = 1$ and $\delta = 0$, the class of functions $\mathscr{B}^1_\Sigma(\beta, \lambda, 0) := \mathscr{B}_\Sigma(\beta, \lambda)$ have been introduced and studied by Frasin and Aouf [9], for $\delta = 0$, the class of functions $\mathscr{B}^\mu_\Sigma(\beta, \lambda, 0) := \mathscr{B}^\mu_\Sigma(\beta, \lambda)$ have been introduced and studied by Çağlar et al. [7], and for $\lambda = 1$, $\mu = 0$ and $\delta = 0$, we obtain the well-known class $\mathscr{B}^0_\Sigma(\beta, 1, 0) := \mathcal{S}^*_\Sigma(\beta)$ of bi-starlike functions of order β.

Theorem 3.2 *r Let the function $f(z)$ given by* (1.1) *be in the class* $\mathscr{B}^\mu_\Sigma(\beta, \lambda, \delta)$. *Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{4(1-\beta)}{(2\lambda+\mu)(1+\mu+\frac{12\delta}{2\lambda+1})}}, \frac{2(1-\beta)}{\lambda+\mu+2\xi\delta}\right\} \tag{3.3}$$

and

$$|a_3| \leq \begin{cases} \min\left\{\dfrac{(1-\beta)\left(4+\frac{24\delta}{2\lambda+1}\right)}{(2\lambda+\mu+6\xi\delta)(1+\mu+\frac{12\delta}{2\lambda+1})}, \dfrac{4(1-\beta)^2}{(\lambda+\mu+2\xi\delta)^2} + \dfrac{2(1-\beta)}{2\lambda+\mu+6\xi\delta}\right\}; & 0 \leq \mu < 1 \\ \dfrac{2(1-\beta)}{2\lambda+\mu+6\xi\delta}; & \mu \geq 1 \end{cases} \tag{3.4}$$

Proof Let $f \in \mathscr{B}^\mu_\Sigma(\beta, \lambda, \delta)$. It follows from (3.1) and (3.2) that there exist $p, q \in \mathcal{P}$ such that

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^\mu + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} + \xi\delta z f''(z) = \beta + (1-\beta)p(z) \tag{3.5}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^\mu + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} + \xi\delta w g''(w) = \beta + (1-\beta)q(w), \tag{3.6}$$

where $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ and $q(w) = 1 + q_1 w + q_2 w^2 + \cdots$ in $\mathcal{P}$.

Now, equating the coefficients in (3.5) and (3.6), we get

$$(\lambda + \mu + 2\xi\delta) a_2 = (1-\beta)p_1, \tag{3.7}$$

$$(2\lambda + \mu)\left[\left(\frac{\mu-1}{2}\right)a_2^2 + \left(1 + \frac{6\delta}{2\lambda+1}\right)a_3\right] = (1-\beta)p_2, \tag{3.8}$$

and

$$-(\lambda + \mu + 2\xi\delta) a_2 = (1-\beta)q_1, \tag{3.9}$$

$$(2\lambda + \mu)\left[\left(\frac{\mu+3}{2} + \frac{12\delta}{2\lambda+1}\right)a_2^2 - \left(1 + \frac{6\delta}{2\lambda+1}\right)a_3\right] = (1-\beta)q_2. \tag{3.10}$$

From (3.7) and (3.9), we obtain

$$p_1 = -q_1, \tag{3.11}$$

and

$$2\left(\lambda + \mu + 2\xi\delta\right)^2 a_2^2 = (1 - \beta)^2 (p_1^2 + q_1^2). \tag{3.12}$$

By adding (3.8) to (3.10), we get

$$(2\lambda + \mu)\left(1 + \mu + \frac{12\delta}{2\lambda + 1}\right) a_2^2 = (1 - \beta)(p_2 + q_2). \tag{3.13}$$

From Eqs. (3.12) and (3.13), we get

$$|a_2|^2 \le \frac{(1 - \beta)^2}{2(\lambda + \mu + 2\xi\delta)^2}\left(|p_1|^2 + |q_1|^2\right) \tag{3.14}$$

and

$$|a_2|^2 \le \frac{(1 - \beta)}{(2\lambda + \mu)(1 + \mu + \frac{12\delta}{2\lambda+1})}\left(|p_2| + |q_2|\right), \tag{3.15}$$

respectively.

By considering Lemma 1.1 we get from (3.14) and (3.15) the desired inequality (3.3).

Next, by subtracting (3.10) from (3.8), we have

$$2(2\lambda + \mu)\left(1 + \frac{6\delta}{2\lambda + 1}\right) a_3 - 2(2\lambda + \mu)\left(1 + \frac{6\delta}{2\lambda + 1}\right) a_2^2 = (1 - \beta)(p_2 - q_2). \tag{3.16}$$

Further, in view of (3.12), it follows from (3.16) that

$$a_3 = \frac{(1 - \beta)^2}{2\left(\lambda + \mu + 2\xi\delta\right)^2}(p_1^2 + q_1^2) + \frac{(1 - \beta)}{2(2\lambda + \mu + 6\xi\delta)}\left(p_2 - q_2\right). \tag{3.17}$$

Applying Lemma 1.1 for (3.17), we get

$$|a_3| \le \frac{4(1 - \beta)^2}{(\lambda + \mu + 2\xi\delta)^2} + \frac{2(1 - \beta)}{2\lambda + \mu + 6\xi\delta}. \tag{3.18}$$

On other hand, by using the equation (3.13) in (3.16), we obtain

$$a_3 = \frac{1 - \beta}{2(2\lambda + \mu + 6\xi\delta)}\left[\left(\frac{3 + \mu + \frac{24\delta}{2\lambda+1}}{1 + \mu + \frac{12\delta}{2\lambda+1}}\right) p_2 + \left(\frac{1 - \mu}{1 + \mu + \frac{12\delta}{2\lambda+1}}\right) q_2\right]. \tag{3.19}$$

Applying Lemma 1.1 for (3.19), we get

$$|a_3| \leq \frac{1-\beta}{2\lambda + \mu + 6\xi\delta} \left(\frac{3 + \mu + \frac{24\delta}{2\lambda+1} + |1-\mu|}{1 + \mu + \frac{12\delta}{2\lambda+1}} \right). \tag{3.20}$$

Hence, by considering Eq. (3.18) we obtain from Eq. (3.20) for $0 \leq \mu < 1$ the first part of the desired inequality (3.4), and for $\mu \geq 1$ the skecond part of the desired inequality (3.4).

This completes the proof of Theorem 3.2.   □

If we take $\lambda = 1$, $\mu = 1$ and $\delta = 0$ in Theorem 3.2, we get the following consequence.

Corollary 3.3 [7] *Let the function $f(z)$ given by* (1.1) *be in the class $\mathscr{B}_\Sigma(\beta)$. Then*

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\beta)}{3}}; & 0 \leq \beta < \frac{1}{3} \\ 1 - \beta; & \frac{1}{3} \leq \beta < 1 \end{cases}$$

and

$$|a_3| \leq \frac{2(1-\beta)}{3}.$$

If we choose $\mu = 1$ and $\delta = 0$ in Theorem 3.2, we get the following consequence.

Corollary 3.4 [7] *Let the function $f(z)$ given by* (1.1) *be in the class $\mathscr{B}_\Sigma(\beta, \lambda)$. Then*

$$|a_2| \leq \min\left\{ \sqrt{\frac{2(1-\beta)}{2\lambda+1}}, \frac{2(1-\beta)}{\lambda+1} \right\}$$

and

$$|a_3| \leq \frac{2(1-\beta)}{2\lambda+1}.$$

If we choose $\delta = 0$ in Theorem 3.2, we get the following consequence.

Corollary 3.5 [7] *Let the function $f(z)$ given by* (1.1) *be in the class $\mathscr{B}_\Sigma^\mu(\beta, \lambda)$. Then*

$$|a_2| \leq \min\left\{ \sqrt{\frac{4(1-\beta)}{(2\lambda+\mu)(\mu+1)}}, \frac{2(1-\beta)}{(\lambda+\mu)} \right\}$$

and

$$|a_3| \leq \begin{cases} \min\left\{ \frac{4(1-\beta)}{(2\lambda+\mu)(\mu+1)}, \frac{4(1-\beta)^2}{(\lambda+\mu)^2} + \frac{2(1-\beta)}{2\lambda+\mu} \right\}; & 0 \leq \mu < 1 \\ \frac{2(1-\beta)}{2\lambda+\mu}; & \mu \geq 1 \end{cases}.$$

If we choose $\lambda = 1, \mu = 0$ and $\delta = 0$ in Theorem 3.2, we get the following consequence.

Corollary 3.6 [7] *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{S}_{\Sigma}^{*}(\beta)$. Then*

$$|a_2| \leq \sqrt{2(1-\beta)}$$

and

$$|a_3| \leq \begin{cases} 2(1-\beta); & 0 \leq \beta < \frac{3}{4} \\ (1-\beta)(5-4\beta); & \frac{3}{4} \leq \beta < 1 \end{cases}.$$

Availability of data and materials Not applicable.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interests.

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