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On the generalized nonlinear Camassa–Holm equation

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Abstract
In this paper, a generalized nonlinear Camassa–Holm equation with time- and space-dependent coefficients is considered. We show that the control of the higher order dispersive term is possible by using an adequate weight function to define the energy. The existence and uniqueness of solutions are obtained via a standard Picard iterative method, so that there is no loss of regularity of the solution with respect to the initial condition in some appropriate Sobolev space.

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1 Introduction

1.1 Presentation of the problem

In this paper, we study the Cauchy problem for the general nonlinear higher order Camassa–Holm-type equation:

$$\begin{cases} (1-m\partial_x^2)u_t + a_1(t,x,u)u_x + a_2(t,x,u,u_x)u_{xx} \\ \qquad +a_3(t,x,u)u_{xxx} + a_4(t,x)u_{xxxx} + a_5(t,x)u_{xxxxx} = f \\ \text{ for } \quad (t,x) \in (0,T] \times \mathbb{R} \\ u_{|_{t=0}} = u^0, \end{cases} \tag{1.1}$$

where $u = u(t,x)$, from $[0,T] \times \mathbb{R}$ into $\mathbb{R}$, is the unknown function of the problem, $m > 0$ and a_i, $1 \leq i \leq 5$, are real-valued smooth given functions where their exact regularities will be precised later. This equation covers several important unidirectional models for the water waves problems at different regimes which take into account the variations of the bottom. We have in view in particular the example of the Camassa–Holm equation (see [1,2]), which is more nonlinear then the KdV equation (see for

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instance [3–8]). However, the most prominent example that we have in mind is the Kawahara-type approximation (see [9,10]), in which case the coefficient a_5 does not vanish. The presence of the fifth order derivative term is very important, so that the equation describes both nonlinear and dispersive effects as does the Camassa–Holm equation in the case of special tension surface values for the development of models for water waves problem was initiated in order to gain insight into wave breaking (see [11,12]).

Looking for solutions of (1.1) plays an important and significant role in the study of unidirectional limits for water wave problems with variable depth and topographies. To our knowledge the problem (1.1) has not been analyzed previously. In the present paper, we prove the local well-posedness of the initial value problem (1.1) by a standard Picard iterative scheme and the use of adequate energy estimates under a condition of nondegeneracy of the higher dispersive coefficient a_5.

1.2 Notations and main result

In the following, C_0 denotes any nonnegative constant different than zero whose exact expression is of no importance. The notation $a \lesssim b$ means that $a \leq C_0\, b$.

We denote by $C(\lambda_1, \lambda_2, \dots)$ a nonnegative constant depending on the parameters $\lambda_1, \lambda_2, \dots$ and whose dependence on the λ_j is always assumed to be nondecreasing.

For any $s \in \mathbb{R}$, we denote $[s]$ the integer part of s.

Let p be any constant with $1 \leq p < \infty$ and denote $L^p = L^p(\mathbb{R})$ the space of all Lebesgue-measurable functions f with the standard norm

$$|f|_{L^p} = \left(\int_{\mathbb{R}} |f(x)|^p dx\right)^{1/p} < \infty.$$

The real inner product of any two functions f_1 and f_2 in the Hilbert space $L^2(\mathbb{R})$ is denoted by

$$(f_1, f_2) = \int_{\mathbb{R}} f_1(x) f_2(x) dx.$$

The space $L^\infty = L^\infty(\mathbb{R})$ consists of all essentially bounded and Lebesgue-measurable functions f with the norm

$$|f|_{L^\infty} = \sup |f(x)| < \infty.$$

We denote by $W^{1,\infty}(\mathbb{R}) = \{f, \text{ s.t. } f, \partial_x f \in L^\infty(\mathbb{R})\}$ endowed with its canonical norm.

For any real constant $s \geq 0$, $H^s = H^s(\mathbb{R})$ denotes the Sobolev space of all tempered distributions f with the norm $|f|_{H^s} = |\Lambda^s f|_{L^2} < \infty$, where Λ is the pseudo-differential operator $\Lambda = (1 - \partial_x^2)^{1/2}$. For any two functions $u = u(t, x)$ and $v(t, x)$ defined on $[0, T) \times \mathbb{R}$ with $T > 0$, we denote the inner product, the L^p-norm and especially the L^2-norm, as well as the Sobolev norm, with respect to the spatial variable x, by $(u, v) = (u(t,\cdot), v(t,\cdot))$, $|u|_{L^p} = |u(t,\cdot)|_{L^p}$, $|u|_{L^2} = |u(t,\cdot)|_{L^2}$ and $|u|_{H^s} = |u(t,\cdot)|_{H^s}$, respectively.

Let E be a given normed space we denote $L^\infty([0,T); E)$ the space of functions such that $u(t,\cdot)$ is controlled in E, uniformly for $t \in [0,T)$:

$$\|u\|_{L^\infty([0,T);E)} = ess \sup_{t\in[0,T)} |u(t,\cdot)|_E < \infty.$$

Let E be a given normed space we denote $C([0,T); E)$ the space of functions such that $u(t,\cdot)$ is controlled in E, uniformly for $t \in [0,T)$:

$$\|u\|_{L^\infty([0,T);E)} = \sup_{t\in[0,T)} |u(t,\cdot)|_E < \infty.$$

Let X be a given space, we denote $C([0,T); X)$ the space of functions such that $u(t,\cdot)$ is in X.

Finally, $C^k(\mathbb{R}^i)$, $i \geq 1$ denote the space of k-times continuously differentiable functions over $\mathbb{R}^i$.

For any closed operator T defined on a Banach space X of functions, the commutator $[T, f]$ is defined by $[T, f]g = T(fg) - fT(g)$ with f, g and fg belonging to the domain of T. The same notation is used for f as an operator mapping the domain of T into itself.

Actually, we admit without proof this lemma that presents some properties for the commutator operator.

If $f \in F$ and $g \in G$, F and G being two Banach spaces, the notation $|f|_F \lesssim |g|_G$ means that $|f|_F \leq C|g|_G$ for some constant C which does not depend on f nor g.

Here, $\mathcal{S}(\mathbb{R})$ denotes the Schwartz space of rapidly decaying functions, and for any distribution $f \in \mathcal{S}'(\mathbb{R})$, we write $\widehat{f}$ Fourier transform on $\mathcal{S}'(\mathbb{R})$.

We use the classical notation $f(D)$ to denote the Fourier multiplier, namely, $\widehat{f(D)u}(\cdot) = f(\cdot)\widehat{u}(\cdot)$.

We use the condensed notation

$$A_s = B_s + \langle C_s \rangle_{s>s_0} \tag{1.2}$$

to say that $A_s = B_s$ if $s \leq s_0$ and $A_s = B_s + C_s$ if $s > s_0$.

1.3 Product and commutator estimates in Sobolev spaces

Let us recall here some product as well as commutator estimates in Sobolev spaces, used throughout the present paper (see [11]).

Lemma 1.1 (product estimates) *Let $s \geq 0$, one has $\forall f, g \in H^s(\mathbb{R}) \bigcap L^\infty(\mathbb{R})$, one has*

$$|\, f \; g\, |_{H^s} \lesssim |\, f\, |_{L^\infty} |\, g\, |_{H^s} + |\, f\, |_{H^s} |\, g\, |_{L^\infty}.$$

If $s \geq s_0 > 1/2$, one deduces thanks to continuous embedding of Sobolev spaces,

$$\big| f\ g \big|_{H^s} \lesssim \big| f \big|_{H^s} \big| g \big|_{H^s}.$$

More generally, for $s \geq 0$ and $s_0 > 1/2$, one has $\forall f \in H^s(\mathbb{R}) \bigcap H^{s_0}(\mathbb{R})$, $g \in H^s(\mathbb{R})$,

$$\big| f\ g \big|_{H^s} \lesssim \big| f \big|_{H^{s_0}} \big| g \big|_{H^s} + \big\langle \big| f \big|_{H^s} \big| g \big|_{H^{s_0}} \big\rangle_{s>s_0}.$$

Let $F \in C^\infty(\mathbb{R})$ be a smooth function such that $F(0) = 0$. If $g \in H^s(\mathbb{R}) \bigcap L^\infty(\mathbb{R})$ with $s \geq 0$, one has $F(g) \in H^s(\mathbb{R})$ and

$$\big| F(g) \big|_{H^s} \leq C(\big|g\big|_{L^\infty}, F) \big| g \big|_{H^s}.$$

We know recall commutator estimate, mainly due to the Kato–Ponce [13], and recently improved by Lannes [11] (see Theorems 3 and 6):

Lemma 1.2 *(*commutator estimates*)*
For any $s \geq 0$, and $\partial_x f, g \in L^\infty(\mathbb{R}) \bigcap H^{s-1}(\mathbb{R})$, one has

$$\big| [\Lambda^s, f]g \big|_{L^2} \lesssim \big| \partial_x f \big|_{H^{s-1}} \big| g \big|_{L^\infty} + \big| \partial_x f \big|_{L^\infty} \big| g \big|_{H^{s-1}}.$$

Thanks to continuous embedding of Sobolev spaces, one has for $s \geq s_0 + 1$, $s_0 > \frac{1}{2}$,

$$\big| [\Lambda^s, f]g \big|_{L^2} \lesssim \big| \partial_x f \big|_{H^{s-1}} \big| g \big|_{H^{s-1}}.$$

More generally, for any $s \geq 0$ and $s_0 > 1/2$, $\partial_x f, g \in H^{s_0}(\mathbb{R}) \bigcap H^{s-1}(\mathbb{R})$, one has

$$\big| [\Lambda^s, f]g \big|_{L^2} \lesssim \big| \partial_x f \big|_{H^{s_0}} \big| g \big|_{H^{s-1}} + \big\langle \big| \partial_x f \big|_{H^{s-1}} \big| g \big|_{H^{s_0}} \big\rangle_{s>s_0+1}.$$

We conclude this section with the following remark Also, let us remark these continuous embedding.

Remark 1.1 Let $s > \frac{3}{2}$, then:

- $H^s(\mathbb{R}) \hookrightarrow W^{1,\infty}(\mathbb{R})$
- $H^{s-1}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$
- $H^s(\mathbb{R}) \hookrightarrow H^{s-1}(\mathbb{R})$.

Moreover, we define the following operators for $s > 0$: $\Lambda_m^s = (1 - m\partial_x^2)^{\frac{s}{2}}$ and its inverse Λ_m^{-s} such that the Fourier Transform is given as following:

$$\widehat{\Lambda_m^{-s}(u)} = (1 + m\xi^2)^{-\frac{s}{2}} \hat{u}.$$

Finally, we will study the local well-posedness of the initial value problem (1.1) in $H^s(\mathbb{R})$ endowed with canonical norm.

1.4 Main results

Let us now state our main result:

Theorem 1.1 *Let $s > \frac{5}{2}$ and $f \in C([0,T]; H^s(\mathbb{R}))$. We suppose that:*

- *a_1, a_2, a_3 are smooth mappings such that a_1, a_3 in $C([0,T], C^{[s]+1}(\mathbb{R}^2))$ and a_2 in $C([0,T], C^{[s]+1}(\mathbb{R}^3))$.*
- *$a_4 \in C([0,T]; H^{s+1}(\mathbb{R}))$, $\partial_t a_4 \in L^\infty(0,T, L^\infty(\mathbb{R}))$,*
- *$a_5 \in C([0,T], L^\infty(\mathbb{R}))$, $\partial_x a_5 \in C([0,T]; H^{s+2}(\mathbb{R}))$, with $\partial_t a_5 \in L^\infty(0,T; L^\infty(\mathbb{R}))$,*
- *$F(t,x) := \int_0^x \frac{a_4}{a_5} dy \in C([0,T]; L^\infty(\mathbb{R}))$ and $\partial_t F \in L^\infty(0,T; L^\infty(\mathbb{R}))$,*

Assume moreover that there is a positive constant $c_1 > 0$ such that $c_1 \le |a_5(t,x)|\ \forall (t,x) \in [0,T] \times \mathbb{R}$. Then for all $u^0 \in H^s(\mathbb{R})$, there exist a time $T^\star > 0$ and a unique solution u to (1.1)
in $C([0,T^\star]; H^s(\mathbb{R}))$.

Remark 1.2 There is no restriction on the signs of the coefficients a_2 and a_4; this means that our result handles also the case of the anti-diffusive terms, in which case these terms are controlled by dispersion.

2 Proof of the main results

Before we start the proof, we give the following useful lemma:

Lemma 2.1 *Let $m > 0$, $s > 0$ then the linear operator*
$\Lambda_m^2 \colon H^{s+2}(\mathbb{R}) \to H^s(\mathbb{R})$ is well defined, continuous, one-to-one and onto. If we suppose that $u = \Lambda_m^{-2} f$ for $f \in H^s(\mathbb{R})$ then:

$$|u|_{H^{s+2}} \le \frac{1}{m} |f|_{H^s} \quad \text{if} \quad 0 < m \le 1 \tag{2.1}$$

$$|u|_{H^{s+2}} \le |f|_{H^s} \quad \text{if} \quad m \ge 1. \tag{2.2}$$

Moreover,

$$\Lambda^s \Lambda_m^{-2} = \Lambda^{s-2} \Lambda_m^0 = \Lambda_m^0 \Lambda^{s-2},$$

where $\Lambda_m^0 \colon H^s(\mathbb{R}) \to H^s(\mathbb{R})$ is linear continuous one-to-one and onto operator defined by

$$\widehat{\Lambda_m^0 u(\xi)} = (1+\xi^2)(1+m\xi^2)^{-1} \hat{u}(\xi),$$

with

$$|\Lambda_m^0|_{H^s \to H^s} \le \max\left(\frac{1}{m}, 1\right), \tag{2.3}$$

$$|(\Lambda_m^0)^{-1}|_{H^s \to H^s} \le \max(m, 1). \tag{2.4}$$

Proof We have $\|\Lambda_m^{-2} f\|_{H^{s+2}} = \|(1+\xi^2)^{\frac{s}{2}+1}(1+m\xi^2)^{-1}\hat{f}\|_{L^2}$. If $m \geq 1$, then $1+m\xi^2 \geq 1+\xi^2$ and $\frac{1+\xi^2}{1+m\xi^2} \leq 1$, therefore

$$\begin{aligned}\|(1+\xi^2)^{\frac{s}{2}+1}(1+m\xi^2)^{-1}\hat{f}\|_{L^2} &= \|(1+\xi^2)^{\frac{s}{2}}(1+\xi^2)(1+m\xi^2)^{-1}\hat{f}\|_{L^2}\\ &\leq \|(1+\xi^2)^{\frac{s}{2}}\hat{f}\|_{L^2}.\end{aligned}$$

If $0 < m < 1$, we have $\frac{1+\xi^2}{1+m\xi^2} = 1 + (1-m)\frac{\xi^2}{1+m\xi^2} \leq 1 + \frac{(1-m)}{m} = \frac{1}{m}$, then

$$\|\Lambda_m^{-2} f\|_{H^{s+2}} \leq \frac{1}{m}\|f\|_{H^s}.$$

Now we have

$$\|\Lambda_m^0 f\|_{H^s} = \|\Lambda^2\Lambda_m^{-2} f\|_{H^s} = \|\Lambda_m^{-2} f\|_{H^{s+2}} \leq \max(1, \frac{1}{m})\|f\|_{H^s}.$$

and

$$\|(\Lambda_m^0)^{-1} f\|_{H^s} = \|(1+m\xi^2)(1+\xi^2)^{-1}(1+\xi^2)^{\frac{s}{2}}\hat{f}\|_{L^2}.$$

If $m \geq 1$, then $(1+m\xi^2)(1+\xi^2)^{-1} = 1 + (m-1)\frac{\xi^2}{1+\xi^2} \leq m$, therefore

$$\|(\Lambda_m^0)^{-1} f\|_{H^s} \leq m\|f\|_{H^s}.$$

If $0 < m < 1$, $(1+m\xi^2)(1+\xi^2)^{-1} \leq 1$, then

$$\|(\Lambda_m^0)^{-1} f\|_{H^s} \leq \|f\|_{H^s}.$$

Finally $\|(\Lambda_m^0)^{-1} f\|_{H^s} \leq \max(1, m)\|f\|_{H^s}$.

We will start the proof of Theorem 1.1 by studying a linearized problem associated to (1.1). □

2.1 Linear analysis

For any smooth enough v, we define the "linearized" operator:

$$\begin{aligned}\mathcal{L}(v, \partial) = {} & \Lambda_m^2\partial_t + a_1(t,x,v)\partial_x + a_2(t,x,v,v_x)\partial_x^2\\ & + a_3(t,x,v)\partial_x^3 + a_4(t,x)\partial_x^4 + a_5(t,x)\partial_x^5.\end{aligned}$$

and the following initial value problem:

$$\begin{cases}\mathcal{L}(v,\partial)u = f,\\ u_{|_{t=0}} = u^0.\end{cases} \tag{2.5}$$

Equation (2.5) is a linear equation which can be solved by a standard method (see [14]) in any time interval in which its coefficients are defined and regular enough. We first establish some precise energy-type estimates of the solution. We define the "energy" norm,

$$E^s(u)^2 = |w\Lambda^s u|_{L^2}^2,$$

where w is a weight function that will be chosen later. For the moment, we just require that there exists two positive numbers w_1, w_2 such that for all (t, x) in $(0, T] \times \mathbb{R}$,

$$w_1 \le w(t, x) \le w_2,$$

so that $E^s(u)$ is uniformly equivalent to the standard H^s-norm. Differentiating $\frac{1}{2}e^{-\lambda t}E^s(u)^2$ with respect to time, one gets using (2.5)

$$\begin{aligned}\frac{1}{2}e^{\lambda t}\partial_t(e^{-\lambda t}E^s(u)^2) = &-\frac{\lambda}{2}E^s(u)^2 - \left(\Lambda_m^0\Lambda^{s-2}(a_1u_x), w^2\Lambda^s u\right)\\ &-\left(\Lambda_m^0\Lambda^{s-2}(a_2u_{xx}), w^2\Lambda^s u\right) - \left(\Lambda_m^0\Lambda^{s-2}(a_3u_{xxx}), w^2\Lambda^s u\right)\\ &-\left(\Lambda_m^0\Lambda^{s-2}(a_4u_{xxxx}), w^2\Lambda^s u\right)\\ &-\left(\Lambda_m^0\Lambda^{s-2}(a_5u_{xxxxx}), w^2\Lambda^s u\right)\\ &+\left(\Lambda_m^0\Lambda^{s-2}f, w^2\Lambda^s u\right) + \left(ww_t\Lambda^s u, \Lambda^s u\right).\end{aligned}$$

We now turn to estimating the different terms of the r.h.s of the previous identity by using the needed estimates provided from Sect. 1.3

• Estimate of $\left(\Lambda^{s-2}(a_1u_x), \Lambda_m^0 w^2\Lambda^s u\right)$. By the Cauchy-Schwarz inequality and the Sect. 1.3 on the composite functions we have

$$\begin{aligned}|\left(\Lambda^{s-2}(a_1u_x), \Lambda_m^0 w^2\Lambda^s u\right)| \le &\frac{1}{m}|a_1(t, x, v) - a_1(t, x, 0)|_{H^{s-2}}\times\\ &|a_1(t, x, 0)u_x|_{H^{s-1}}|w^2\Lambda^s u|_{L^2}\\ \le &C(m^{-1}, a_1, \|v\|_{H^s}, |w|_{L^\infty})E^s(u)^2. \qquad (2.6)\end{aligned}$$

• Estimate of $\left(\Lambda^{s-2}(a_2u_{xx}), \Lambda_m^0 w^2\Lambda^s u\right)$. Similarly as the above estimation, we have

$$|\left(\Lambda^{s-2}(a_2u_{xx}), \Lambda_m^0 w^2\Lambda^s u\right)| \le C(m^{-1}, a_2, \|v\|_{H^s}, |w|_{L^\infty})E^s(u)^2.$$

•Estimate of $\left(\Lambda^{s-2}(a_3u_{xxx}), \Lambda_m^0 w^2\Lambda^s u\right)$. Since we have more than s derivative on u, we remark that one can write:

$$a_3u_{xxx} = \partial_x^2(a_3\partial_x u) - \partial_x^2 a_3\partial_x u - 2a_3\partial_x^2 u,$$

then

$$\Lambda^{s-2}(a_3u_{xxx}) = \Lambda^{s-2}(\partial_x^2(a_3\partial_x u)) - \Lambda^{s-2}(\partial_x^2 a_3\partial_x u) - 2\Lambda^{s-2}(\partial_x a_3\partial_x^2 u).$$

Now use the identity $\Lambda^2 = 1 - \partial_x^2$ to get that

$$\begin{aligned}\Lambda^{s-2}(\partial_x^2(a_3\partial_x u)) &= \Lambda^{s-2}\big((1-\Lambda^2)(a_3\partial_x u)\big)\\ &= \Lambda^{s-2}(a_3\partial_x u) - \Lambda^s(a_3\partial_x u)\\ &= \Lambda^{s-2}(a_3\partial_x u) - [\Lambda^s, a_3]\partial_x u - a_3\Lambda^s\partial_x u,\end{aligned}$$

then we obtain:

$$\begin{aligned}&\big(\Lambda^{s-2}(a_3u_{xxx}), \Lambda_m^0 w^2\Lambda^s u\big)\\ &\quad= \big(\Lambda^{s-2}(a_3\partial_x u), \Lambda_m^0 w^2\Lambda^s u\big) - \big([\Lambda^s, a_3]\partial_x u, \Lambda_m^0 w^2\Lambda^s u\big)\\ &\quad\quad - \big(a_3\Lambda^s\partial_x u, \Lambda_m^0 w^2\Lambda^s u\big) - \big(\Lambda^{s-2}(\partial_x^2 a_3\partial_x u), \Lambda_m^0 w^2\Lambda^s u\big)\\ &\quad\quad - 2\big(\Lambda^{s-2}(\partial_x a_3\partial_x^2 u), \Lambda_m^0 w^2\Lambda^s u\big).\end{aligned}$$

By integration by parts, the third term of the last equality becomes:

$$\big(a_3\Lambda^s\partial_x u, \Lambda_m^0 w^2\Lambda^s u\big) = -\frac{1}{2}\big(\partial_x(\Lambda_m^0 w^2 a_3), (\Lambda^s u)^2\big),$$

Now by Cauchy Schwarz we have:

$$\begin{aligned}|\big(\Lambda^{s-2}(a_3u_{xxx}), \Lambda_m^0 w^2\Lambda^s u\big)| \le \frac{1}{m}\big(&\|a_3\partial_x u\|_{H^{s-2}}E^s(u)\\ &+\|\partial_x a_3\|_{H^{s-1}}\|\partial_x u\|_{H^{s-1}}E^s(u)\\ &+\|w^2 a_3\|_{W^{1,\infty}}E^s(u)^2\\ &+\|\partial_x^2 a_3\partial_x u\|_{H^{s-2}}E^s(u) + \|a_3\partial_x^2 u\|_{H^{s-2}}E^s(u)\big)\\ \le C(m^{-1}, a_3,& \|v\|_{H^s}, \|w\|_{W^{1,\infty}})E^s(u)^2.\end{aligned}$$

• Estimate of $\big([\Lambda^{s-2}, a_4]\partial_x^4 u, \Lambda_m^0 w^2\Lambda^s u\big) + \big(a_4\Lambda^{s-2}\partial_x^4 u, \Lambda_m^0 w^2\Lambda^s u\big)$:

$$\begin{aligned}a_4\Lambda^{s-2}\partial_x^4 u &= a_4\Lambda^{s-2}(1-\Lambda^2)\partial_x^2 u = a_4(\Lambda^{s-2} - \Lambda^s)\partial_x^2 u\\ &= a_4\Lambda^{s-2}\partial_x^2 u - a_4\Lambda^s\partial_x^2 u,\end{aligned}$$

then:

$$\big(a_4\Lambda^{s-2}\partial_x^4 u, \Lambda_m^0 w^2\Lambda^s u\big) = \big(a_4\Lambda^{s-2}\partial_x^2 u, \Lambda_m^0 w^2\Lambda^s u\big) - \big(a_4\Lambda^s\partial_x^2 u, \Lambda_m^0 w^2\Lambda^s u\big)$$

By Cauchy Schwarz, the first term of the last equality is controlled by:

$$|\big(a_4\Lambda^{s-2}\partial_x^2 u, \Lambda_m^0 w^2\Lambda^s u\big)| \le \frac{1}{m}|a_4\Lambda^{s-2}\partial_x^2 u|_{L^2}E^s(u) \le C(m^{-1}, |a_4|_{L^\infty})E^s(u)^2.$$

$\big(a_4\Lambda^s\partial_x^2u,\, \Lambda_m^0 w^2\Lambda^s u\big) = -\big(a_4\Lambda_m^0 w^2,\, (\partial_x\Lambda^s u)^2\big) + Q_1$, where

$$|Q_1| \leq C(m, s, |w|_{W^{1,\infty}}, |\partial_x a_4|_{L^\infty}) E^s(u)^2.$$

Now, using the first order Poisson brackets : (see [15] for more details)

$$\{\Lambda^{s-2}, a_4\}_1 = -(s-2)\partial_x(a_4)\Lambda^{s-2}\partial_x,$$

we get:

$$([\Lambda^{s-2}, a_4]\partial_x^4 u,\, \Lambda_m^0 w^2\Lambda^s u) = (s-2)(\partial_x(a_4)\Lambda^s\partial_x u,\, \Lambda_m^0 w^2\Lambda^s u) + Q_2,$$

Where

$$|Q_2| \leq C(m, s, |w|_{W^{2,\infty}}, |a_4|_{H^{s+1}}) E^s(u)^2.$$

Now, by integration by parts we have:

$$(s-2)(\partial_x(a_4)\Lambda^s\partial_x u,\, \Lambda_m^0 w^2\Lambda^s u) = -\frac{(s-2)}{2}(\partial_x(\partial_x(a_4)\Lambda_m^0 w^2)\Lambda^s u,\, \Lambda^s u),$$

then

$$|([\Lambda^{s-2}, a_4]\partial_x^4 u,\, \Lambda_m^0 w^2\Lambda^s u)| \leq C(m, s, |w|_{W^{2,\infty}}, |a_4|_{H^{s+1}}) E^s(u)^2.$$

• Estimate of $\big([\Lambda^{s-2}, a_5]\partial_x^5 u,\, \Lambda_m^0 w^2\Lambda^s u\big) + \big(a_5\Lambda^{s-2}\partial_x^5 u,\, \Lambda_m^0 w^2\Lambda^s u\big)$:

$$a_5\Lambda^{s-2}\partial_x^5 u = a_5\Lambda^{s-2}(1-\Lambda^2)\partial_x^3 u = a_5\Lambda^{s-2}\partial_x^3 u - a_5\Lambda^s\partial_x^3 u = \\ a_5\Lambda^{s-2}\partial_x u - a_5\Lambda^s\partial_x u - a_5\Lambda^s\partial_x^3 u.$$

Therefore,

$$\big(a_5\Lambda^{s-2}\partial_x^5 u,\, \Lambda_m^0 w^2\Lambda^s u\big) = \big(a_5\Lambda^{s-2}\partial_x u,\, \Lambda_m^0 w^2\Lambda^s u\big) - \big(a_5\Lambda^s\partial_x u,\, \Lambda_m^0 w^2\Lambda^s u\big) - \\ \big(a_5\Lambda^s\partial_x^3 u,\, \Lambda_m^0 w^2\Lambda^s u\big).$$

The first two terms can be easily controlled by $E^s(u)^2$ as above. Now,

$$\begin{aligned}\big(a_5\partial_x^3\Lambda^s u,\, \Lambda_m^0 w^2\Lambda^s u\big) = &-\frac{1}{2}\big(\partial_x^3(a_5\Lambda_m^0 w^2)\Lambda^s u,\, \Lambda^s u\big)\\ &-\frac{3}{2}\big(\partial_x^2(w^2\Lambda_m^0 a_5)\Lambda^s\partial_x u,\, \Lambda^s u\big)\\ &-\frac{3}{2}\big(\partial_x(\Lambda_m^0 w^2 a_5)\Lambda^s u,\, \Lambda^s\partial_x^2 u\big).\end{aligned}$$

By integration by parts, we obtain

$$-\frac{3}{2}\big(\partial_x(\Lambda_m^0 w^2 a_5)\Lambda^s u, \Lambda^s \partial_x^2 u\big) = \frac{3}{2}\big(\partial_x^2(\Lambda_m^0 w^2 a_5)\Lambda^s u, \Lambda^s \partial_x u\big) + \frac{3}{2}\big(\partial_x(a_5 \Lambda_m^0 w^2), (\Lambda^s \partial_x u)^2\big).$$

Now:

$$[\Lambda^{s-2}, a_5]\partial_x^5 u = \{\Lambda^{s-2}, a_5\}_2 \partial_x^5 u + Q_3 \partial_x^5 u,$$

where $\{\cdot,\cdot\}_2$ stands for the second order Poisson brackets,

$$\{\Lambda^{s-2}, a_5\}_2 = -(s-2)\partial_x(a_5)\Lambda^{s-4}\partial_x + \frac{1}{2}[(s-2)\partial_x^2(a_5)\Lambda^{s-4} - (s-4)(s-2)\partial_x^2(a_5)\Lambda^{s-6}\partial_x^2]$$

and Q_3 is an operator of order $s-5$ that can be controlled by the general commutator estimates (see [15]). We thus get

$$|\big(Q_3\partial_x^5 u, \Lambda_m^0 w^2 \Lambda^s u\big)| \leq C(m, |\partial_x a_5|_{H^{s+1}}) E^s(u)^2.$$

We now use the fact that $H^1(\mathbb{R})$ is continuously embedded in $L^\infty(\mathbb{R})$ to get

$$|\big([s\partial_x^2(a_5)\Lambda^{s-4} - (s-4)(s-2)\partial_x^2(a_5)\Lambda^{s-6}\partial_x^2]\partial_x^5 u, \Lambda_m^0 w^2 \Lambda^s u\big)| \leq C(m, s, |\partial_x a_5|_{H^{s+1}}, |w|_{W^{1,\infty}}) E^s(u)^2.$$

This leads to the expression

$$\big([\Lambda^{s-2}, a_5]\partial_x^5 u, \Lambda_m^0 w^2 \Lambda^s u\big) = -(s-2)\big(\partial_x(a_5)\Lambda^s \partial_x^2 u, \Lambda_m^0 w^2 \Lambda^s u\big) + Q_4,$$

where $|Q_4| \leq C(m, s, |w|_{W^{1,\infty}}, |a_5|_{H^{s+1}}) E^s(u)^2$. Remarking now, by integration by parts

$$-(s-2)\big(\partial_x(a_5)\Lambda^s \partial_x^2 u, \Lambda_m^0 w^2 \Lambda^s u\big) = (s-2)\big(\partial_x(\partial_x(a_5)\Lambda_m^0 w^2)\Lambda^s \partial_x u, \Lambda^s u\big) + (s-2)\big(\partial_x(a_3)\Lambda_m^0 w^2, (\Lambda^s \partial_x u)^2\big). \tag{2.7}$$

We now choose w such that

$$-(s-2)\big(\partial_x(a_5)\Lambda_m^0 w^2, (\Lambda^s \partial_x u)^2\big) + \frac{3}{2}\big(\partial_x(a_5 \Lambda_m^0 w^2), (\Lambda^s \partial_x u)^2\big) + \big(a_4 \Lambda_m^0 w^2, (\partial_x \Lambda^s u)^2\big) = 0; \tag{2.8}$$

therefore, if we take $w = (\Lambda_m^0)^{-1}\Big(|a_5|^{\left(\frac{2s-7}{6}\right)}\exp(-\frac{1}{3}\int_0^x \frac{a_4}{a_5}dy)\Big)$ we easily obtain (2.8). Finally, one has

$$\begin{aligned}&\big([\Lambda^{s-2}, a_5]\partial_x^5 u, \Lambda_m^0 w^2\Lambda^s u\big) + \big(a_5\partial_x^5\Lambda^{s-2}u, \Lambda_m^0 w^2\Lambda^s u\big)\\ &\quad = Q_4 + (s-2)\big(\partial_x(\partial_x(a_5)\Lambda_m^0 w^2)\Lambda^s\partial_x u, \Lambda^s u\big) - \frac{1}{2}\big(\partial_x^3(a_5\Lambda_m^0 w^2)\Lambda^s u, \Lambda^s u\big)\\ &\quad -\frac{3}{2}\big(\partial_x^2(a_5\Lambda_m^0 w^2)\Lambda^s\partial_x u, \Lambda^s u\big) + \frac{3}{2}\big(\partial_x^2(a_5\Lambda_m^0 w^2)\Lambda^s\partial_x u, \Lambda^s u\big);\end{aligned}$$

therefore,

$$\begin{aligned}&|\big([\Lambda^{s-2}, a_5]\partial_x^5 u, \Lambda_m^0 w^2\Lambda^s u\big) + \big(a_5\partial_x^5\Lambda^{s-2}u, \Lambda_m^0 w^2\Lambda^s u\big)|\\ &\leq C(s, m, |\partial_x a_5|_{H^{s+1}})E^s(u)^2.\end{aligned}$$

• Estimate of $\big(w_t\Lambda^{s-2}u, \Lambda_m^0 w\Lambda^s u\big)$: Using the Cauchy-Schwarz inequality we obtain

$$|\big(w_t\Lambda^s u, w\Lambda^s u\big)| \leq C(m, |w_t|_{L^\infty}, |w|_{L^\infty})E^s(u)^2.$$

Gathering the information provided by the above estimates, since one has

$$|\big(\Lambda^{s-2}f, \Lambda_m^0 w^2\Lambda^s u\big)| \leq \frac{1}{m}E^s(f)E^s(u).$$

If we assemble the previous estimates and using Gronwall's lemma we obtain the following estimate:

$$e^{\lambda t}\partial_t(e^{-\lambda t}E^s(u)^2) \leq \big(C(E^s(v)) - \lambda\big)E^s(u)^2 + 2E^s(f)E^s(u).$$

Taking $\lambda = \lambda_T$ large enough (how large depends on $\sup_{t\in[0,T]} C(E^s(v(t))$ for the first term of the right hand side of the above inequality to be negative for all $t \in [0, T]$, we deduce that

$$E^s(u(t)) \leq e^{\lambda_T t}E^s(u^0) + 2\int_0^t e^{\lambda_T(t-t')}E^s(f(t'))dt'.$$

2.2 Proof of the theorem

Thanks to this energy estimate, we classically conclude (see e.g. [16]) the existence of a time

$$T^* = T^*(E^s(u^0)) > 0,$$

and a unique solution $u \in C([0, T^*]; H^s(\mathbb{R})) \cap C^1([0, T^*]; H^{s-3}(\mathbb{R}))$ to (1.1) as the limit of the iterative scheme

$$u_0 = u^0, \quad \text{and} \quad \forall n \in \mathbb{N}, \quad \begin{cases} \mathcal{L}(u^n, \partial)u^{n+1} = f, \\ u^{n+1}_{|t=0} = u^0. \end{cases}$$

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interests.

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