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New blow-up criterion for the Degasperis–Procesi equation with weak dissipation

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Abstract
In this paper, we investigate the Cauchy problem of the Degasperis–Procesi equation with weak dissipation. We establish a new local-in-space blow-up criterion of the dissipative Degasperis–Procesi equation on line $\mathbb{R}$ and on circle S, respectively.

Keywords Degasperis–Procesi equation · Blow-up · Local-in-space · Weak dissipation

Mathematics Subject Classification 35B44 · 35G55 · 37K10

1 Introduction

In this paper, we are concerned with the following initial value problem of the Degasperis–Procesi (DP) equation with weak dissipation [26–28]

$$\begin{cases} u_t - u_{xxt} + 4uu_x + \lambda(u - u_{xx}) = 3u_x u_{xx} + uu_{xxx}, \\ u_0(x) = u_0(x), \end{cases} \tag{1}$$

where $\lambda(1 - \partial_x^2)u$ is the dissipative term with a positive constant $\lambda > 0$.

If $\lambda = 0$, then Eq. (1) becomes the following well-known DP equation [8–10,19]

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \tag{2}$$

which can be regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as the Camassa-Holm equation [4]. There is a rather large literature on the research of this equation. For instance, the global existence of strong

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solutions and global weak solutions to the DP Eq. (2) were shown in [22,29,30]. The local well-posedness for the Cauchy problem of DP Eq. (2) was elaborated in [31] for non periodic case, and in [32] for the periodic case. With respect to blow-up criteria on the line and the unit cicle, please refer to [16,33]. As for some other issue of integrability, traveling wave solutions, solitons and peakon as well as its stability, please see [5–7,13–15,18,21,23–25] and references therein for more literature about DP equation. It should be noted that the DP equation has its own peculiarities, although it shares some common properties with the Camassa-Holm equation. A specific feature is that it has not only peakon solutions of the form $u(t,x) = ce^{-|x-ct|}$, $c > 0$, but also shock peakon solitons of the form $u(t,x) = \frac{1}{t+k} sign(x) e^{-|x-ct|}$, $k > 0$. For details, please see [11,12,17].

Recently, Wu and Yin [26–28] studied the blow-up and the decay of the solution to the weakly dissipative DP Eq. (1) on the line and on the circle. They found that Eq. (1) has the same blow-up rate as the DP Eq. (2), which shows that the blow-up rate of the DP equation is not affected by the additional weakly dissipative term. However, they also pointed out that the occurrence of blow-up of (1) is affected by the dissipative parameter λ.

In this paper, we would like to further investigate the Cauchy problem of the weakly dissipative DP Eq. (1). More specially, we rather focus on blow-up criteria as well as the estimates about the lifespan of the solutions. It should be noted here that in the references [26–28], the blow-up condition on the initial datum u_0 typically involves the computation of the norms $||u_0||_{L^2}$ and $||u_0||_{L^\infty}$. The aim of this paper is to present a new blow-up result for the weakly dissipative DP Eq. (1). Motivated by the works of [1–3], we will establish a new local-in-space blowup criterion for Eq. (1) on the line and on the circle, i.e., a blowup condition involving only the properties of u_0 in a neighborhood of a single point $x_0 \in \mathbb{R}$ or $x_0 \in \mathbb{S}$. We shall see that such criterion is more general than earlier blowup results.

This paper is organized as follows. In the next section we recall the local well-posedness of the Cauchy problem to Eq. (1) on the line $\mathbb{R}$ or on the circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$, and several useful results from [26–28] which are needed for our purpose. Sects. 3 and 4 is devoted to establishing a new local-in-space blow-up result for Eq. (1) on the line $\mathbb{R}$ and on the circle $\mathbb{S}$, respectively.

Notation. Throughout this paper, we denote the norm of the Lebesgue space L^p by $||\cdot||_{L^p}$, $1 \le p \le \infty$. We denote by $*$ the spatial convolution.

2 Preliminaries

In this section, we recall the local well-posedness result of the Cauchy problem to Eq. (1) on the line $\mathbb{R}$ or on the circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$, and some useful properties of strong solutions to Eq. (1) from [26–28].

Let $m = u - u_{xx}$ be the momentum variable, then Eq. (1) can be reformulated as the form:

$$\begin{cases} m_t + um_x + 3u_x m + \lambda m = 0, \\ u(0,x) = u_0(x). \end{cases} \tag{3}$$

Note that if $p(x) := \frac{1}{2}e^{-|x|}$ for $x \in \mathbb{R}$ (or if $p(x) := \frac{\cosh(x-[x]-\frac{1}{2})}{2\sinh\frac{1}{2}}$ for $x \in \mathbb{S}$, where $[x]$ stands for the largest integer part of $x \in \mathbb{R}$), then we have $(1-\partial_x^2)^{-1}f = p * f$ for all $f \in L^2(\mathbb{R})$(or $f \in L^2(\mathbb{S})$) and $p * m = u$. Thus, Eq. (3) can be rewritten as follows:

$$\begin{cases} u_t + uu_x + \partial_x p * (\frac{3}{2}u^2) + \lambda u = 0, \\ u_0(x) = u_0(x). \end{cases} \tag{4}$$

By applying the Kato's theorem [20], one can obtain the following local well-posedness result.

Theorem 1 [26–28] *Given $u_0 \in H^s, s > \frac{3}{2}$, there exists a maximal $T^* > 0$ and a unique solution u to* (3)(or (4))*, such that*

$$u = u(\cdot, u_0) \in C\left([0, T^*); H^s\right) \cap C^1([0, T^*); H^{s-1}).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping $u_0 \mapsto u(\cdot, u_0) : H^s \to C([0, T^*); H^s) \cap C^1([0, T^*); H^{s-1})$ is continuous and the maximal time of existence T^* is independent of s.

By the above local well-posedness result and energy estimates, one can readily obtain the following precise blow-up scenario.

Theorem 2 [26–28] *Given $u_0 \in H^s, s > \frac{3}{2}$, the solution u of* (3)(or (4)) *blows up in a finite time $T > 0$ if and only if*

$$\liminf_{t\to T}\{\inf_{x\in\mathbb{R}(x\in\mathbb{S})}[u_x(t,x)]\} = 0.$$

Next, we introduce the particle trajectory $q(t,x) \in C^1([0,T)\times\mathbb{R},\mathbb{R})$, defined by

$$\begin{cases} q_t = u(t,q), \quad t > 0, \\ q(0,x) = x. \end{cases} \tag{5}$$

By simple analysis, we can obtain the following result on q which is crucial in the proof of blow-up solutions.

Lemma 1 [26–28] *Let $u_0 \in H^s, s \geq 3$, and let $T > 0$ be the maximal existence time of the corresponding solution u to* Eq. (4)*. Then* Eq. (5) *has a unique solution $q \in C^1([0,T)\times\mathbb{R},\mathbb{R})$. Moreover, the map $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with*

$$q_x(t,x) = \exp\left(\int_0^t u_x(s,q(s,x))ds\right) > 0, \forall (t,x) \in [0,T)\times\mathbb{R}.$$

Furthermore, we have

$$m(t,q(t,x))q_x^3(t,x) = m_0(x)e^{-\lambda t}, \forall (t,x) \in [0,T)\times\mathbb{R}.$$

Lemma 2 [26–28] *If $u_0 \in H^s$, $s \geq \frac{3}{2}$, then as long as the solution $u(t,x)$ given by Theorem 1 exists, we have*

$$\int_{\mathbb{R}(\mathbb{S})} m(t,x)v(t,x)dx = e^{-2\lambda t}\int_{\mathbb{R}(\mathbb{S})} m_0(x)v_0(x)dx,$$

where $m(t,x) = u(t,x) - u_{xx}(t,x)$ and $v(t,x) = (4-\partial_x^2)^{-1}u(t,x)$. Moreover, we have the following norm estimate

$$\frac{1}{4}e^{-2\lambda t}||u_0||_{L^2}^2 \leq ||u(t)||_{L^2}^2 \leq 4e^{-2\lambda t}||u_0||_{L^2}^2. \tag{6}$$

3 Blow-up result of Eq. (1) on the line $\mathbb{R}$

In this section, we will establish a new blow-up result for the solutions to Eq. (1) on the line $\mathbb{R}$. Our main result can be formulated as follows.

Theorem 3 *Let $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. Assume that there is $x_0 \in \mathbb{R}$ such that*

$$u_0'(x_0) < -\sqrt{\frac{3}{2}}|u_0(x_0)| - \frac{\lambda}{2} - C, \tag{7}$$

where

$$C := \sqrt{\frac{\lambda^2}{4} + 3(\sqrt{\frac{3}{2}} - 1)||u_0||_{L^2}^2}. \tag{8}$$

Then the solution u of (4) blows up in finite time. Moreover, the lifespan T^* is estimated above by

$$T^* \leq \frac{1}{2C}\ln\frac{\sqrt{u_0'(x_0)^2 - \frac{3}{2}u_0^2(x_0)} - \frac{\lambda}{2} + C}{\sqrt{u_0'(x_0)^2 - \frac{3}{2}u_0^2(x_0)} - \frac{\lambda}{2} - C},$$

Proof Using the identity $\partial_x^2(p * f) = p * f - f$, we take the derivative with respect to x in (4) which yields

$$\begin{cases} u_{tx} + uu_{xx} = -u_x^2 - \lambda u_x + \frac{3}{2}u^2 - p * (\frac{3}{2}u^2), \\ u(0,x) = u_0(x). \end{cases} \tag{9}$$

According to Lemma 1, we can know that the flow map $q(t,x)$ introduced in Eq. (5) is indeed well defined in the interval $[0, T^*)$ with $q \in C^1([0,T^*) \times \mathbb{R}, \mathbb{R})$.

Then we have

$$\frac{d}{dt}[u_x(t,q(t,x))] = [u_{tx} + uu_{xx}](t,q(t,x))$$

$$= -u_x^2 - \lambda u_x + \frac{3}{2}u^2 - p * \left(\frac{3}{2}u^2\right).$$

Inspired by [1–3], we now introduce

$$A(t,x) = \left[-u_x + \sqrt{\frac{3}{2}}u\right](t, q(t,x)),$$

and

$$B(t,x) = \left[-u_x - \sqrt{\frac{3}{2}}u\right](t, q(t,x)).$$

Recalling that the kernel p satisfies the identity

$$p = p_+ + p_-, \quad p_x = p_- - p_+,$$

where

$$p_+ * f(x) = \frac{e^{-x}}{2}\int_{-\infty}^{x} e^y f(y)dy,$$
$$p_- * f(x) = \frac{e^{x}}{2}\int_{x}^{+\infty} e^{-y} f(y)dy.$$

Then we have

$$\begin{aligned}
\frac{d}{dt}A(t,x) &= \sqrt{\frac{3}{2}}(u_t + uu_x) - (u_{tx} + uu_{xx}) \\
&= u_x^2 + \lambda u_x - \sqrt{\frac{3}{2}}\lambda u - \frac{3}{2}u^2 + \left(p - \sqrt{\frac{3}{2}}p_x\right) * \left(\frac{3}{2}u^2\right) \\
&= AB - \lambda A + \left(1 + \sqrt{\frac{3}{2}}\right) p_+ * \left(\frac{3}{2}u^2\right) - \left(\sqrt{\frac{3}{2}} - 1\right) p_- * \left(\frac{3}{2}u^2\right) \\
&\geq AB - \lambda A - \left(\sqrt{\frac{3}{2}} - 1\right) p_- * \left(\frac{3}{2}u^2\right).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\frac{d}{dt}B(t,x) &= -\sqrt{\frac{3}{2}}(u_t + uu_x) - (u_{tx} + uu_{xx}) \\
&= u_x^2 + \lambda u_x + \sqrt{\frac{3}{2}}\lambda u - \frac{3}{2}u^2 + \left(p + \sqrt{\frac{3}{2}}p_x\right) * \left(\frac{3}{2}u^2\right)
\end{aligned}$$

$$= AB - \lambda B + \left(1 + \sqrt{\frac{3}{2}}\right) p_- * \left(\frac{3}{2}u^2\right) - \left(\sqrt{\frac{3}{2}} - 1\right) p_+ * \left(\frac{3}{2}u^2\right)$$
$$\geq AB - \lambda B - \left(\sqrt{\frac{3}{2}} - 1\right) p_+ * \left(\frac{3}{2}u^2\right).$$

By using the Young inequality and the norm estimate (6) presented in Lemma 2, we can derive that

$$p_\pm * (\frac{3}{2}u^2) \leq ||p_\pm||_{L^\infty} \|\frac{3}{2}u^2\|_{L^1} = \frac{3}{4}\|u\|_{L^2}^2 \leq 3e^{-2\lambda t}\|u_0\|_{L^2}^2 \leq 3\|u_0\|_{L^2}^2.$$

Thus, we have

$$\frac{dA}{dt} \geq AB - \lambda A - 3\left(\sqrt{\frac{3}{2}} - 1\right)\|u_0\|_{L^2}^2, \tag{10}$$

and

$$\frac{dB}{dt} \geq AB - \lambda B - 3\left(\sqrt{\frac{3}{2}} - 1\right)\|u_0\|_{L^2}^2. \tag{11}$$

The initial condition (7):

$$u_0'(x_0) < -\sqrt{\frac{3}{2}}|u_0(x_0)| - \frac{\lambda}{2} - C,$$

is equivalent to

$$A(0) > \frac{\lambda}{2} + C > 0, \quad B(0) > \frac{\lambda}{2} + C > 0, \tag{12}$$

where we denote $A(t) = A(t, x_0)$, $B(t) = B(t, x_0)$ and C is defined in (8). Hence we can know that

$$A(0)B(0) > C^2,$$
$$A(0)[B(0) - \lambda] - 3\left(\sqrt{\frac{3}{2}} - 1\right)\|u_0\|_{L^2}^2 > \left(\frac{\lambda}{2} + C\right)\left(-\frac{\lambda}{2} + C\right) - 3\left(\sqrt{\frac{3}{2}} - 1\right)\|u_0\|_{L^2}^2 = 0,$$
$$B(0)[A(0) - \lambda] - 3\left(\sqrt{\frac{3}{2}} - 1\right)\|u_0\|_{L^2}^2 > \left(\frac{\lambda}{2} + C\right)\left(-\frac{\lambda}{2} + C\right) - 3\left(\sqrt{\frac{3}{2}} - 1\right)\|u_0\|_{L^2}^2 = 0.$$

This implies that

$$
\begin{aligned}
A'(0) &\geq A(0)B(0) - \lambda A(0) - 3\left(\sqrt{\frac{3}{2}} - 1\right)\|u_0\|_{L^2}^2 > 0, \\
B'(0) &\geq A(0)B(0) - \lambda B(0) - 3\left(\sqrt{\frac{3}{2}} - 1\right)\|u_0\|_{L^2}^2 > 0.
\end{aligned} \tag{13}
$$

We now claim that over the time of existence it always holds that

$$
A'(t) > 0, \quad B'(t) > 0. \tag{14}
$$

If this claim is not true, then there exists $t_0 \in [0, T^*)$ such that

$$
t_0 = \min\{t \in [0, T^*) | A'(t) = 0 \quad or \quad B'(t) = 0\}. \tag{15}
$$

It is easy to see from (13) that $t_0 > 0$. In view of (10)–(11) and the definition of t_0 presented in (15), we have

$$
\begin{aligned}
& 0 = A'(t_0) \geq A(t_0)B(t_0) - \lambda A(t_0) - 3\left(\sqrt{\frac{3}{2}} - 1\right)\|u_0\|_{L^2}^2, \\
or \quad & 0 = B'(t_0) \geq A(t_0)B(t_0) - \lambda B(t_0) - 3\left(\sqrt{\frac{3}{2}} - 1\right)\|u_0\|_{L^2}^2.
\end{aligned} \tag{16}
$$

However, we can derive that

$$
A(t_0) \geq A(0) > \frac{\lambda}{2} + C > 0, \quad B(t_0) \geq B(0) > \frac{\lambda}{2} + C > 0,
$$

since $A'(t) \geq 0$ and $B'(t) \geq 0$ for $t \in [0, t_0]$. Thus, we have

$$
\begin{aligned}
A(t_0)B(t_0) - \lambda A(t_0) - 3\left(\sqrt{\frac{3}{2}} - 1\right)\|u_0\|_{L^2}^2 &> \left(\frac{\lambda}{2} + C\right)\left(-\frac{\lambda}{2} + C\right) - 3\left(\sqrt{\frac{3}{2}} - 1\right)\|u_0\|_{L^2}^2 = 0, \\
A(t_0)B(t_0) - \lambda B(t_0) - 3\left(\sqrt{\frac{3}{2}} - 1\right)\|u_0\|_{L^2}^2 &> \left(\frac{\lambda}{2} + C\right)\left(-\frac{\lambda}{2} + C\right) - 3\left(\sqrt{\frac{3}{2}} - 1\right)\|u_0\|_{L^2}^2 = 0.
\end{aligned}
$$

which is a contradiction to (16). Therefore (14) is true for all $t \in [0, T^*)$. In other words, it means that $A(\cdot, x_0)$, $B(\cdot, x_0)$ and $AB(\cdot, x_0)$ are all positive and increasing during the whole existence time $[0, T^*)$.

To conclude the proof, we consider $h(t) = \sqrt{AB(t, x_0)}$. By computing the time derivative of h, we get

$$\begin{aligned}\frac{d}{dt}h(t) &= \frac{A_tB + AB_t}{2\sqrt{AB}}(t, x_0)\\ &\geq \frac{\left(AB - 3\left(\sqrt{\frac{3}{2}} - 1\right)\|u_0\|_{L^2}^2\right)(A+B) - 2\lambda AB}{2\sqrt{AB}}(t, x_0)\\ &\geq AB - 3\left(\sqrt{\frac{3}{2}} - 1\right)\|u_0\|_{L^2}^2 - \lambda\sqrt{AB}\\ &= h^2(t) - \lambda h(t) - 3\left(\sqrt{\frac{3}{2}} - 1\right)\|u_0\|_{L^2}^2,\\ &= \left[h(t) - \frac{\lambda}{2}\right]^2 - C^2,\end{aligned}$$

where we have used the geometric-arithmetic mean inequality $A + B \geq 2\sqrt{AB} = 2h(t)$. Solving the above differential inequality, we get

$$h(t) \geq \frac{\lambda}{2} + \frac{C[h(0) - \frac{\lambda}{2} + C + (h(0) - \frac{\lambda}{2} - C)e^{2Ct}]}{h(0) - \frac{\lambda}{2} + C - (h(0) - \frac{\lambda}{2} - C)e^{2Ct}}.$$

It is thereby inferred that

$$-u_x(t, q(t, x_0)) = \frac{A+B}{2} \geq h(t) \to +\infty, \quad as \quad t \to \frac{1}{2C}\ln\frac{h(0) - \frac{\lambda}{2} + C}{h(0) - \frac{\lambda}{2} - C},$$

which implies that the solution u blows up at a finite time and the lifespan T^* is estimated above by

$$T^* \leq \frac{1}{2C}\ln\frac{\sqrt{u_0'(x_0)^2 - \frac{3}{2}u_0^2(x_0)} - \frac{\lambda}{2} + C}{\sqrt{u_0'(x_0)^2 - \frac{3}{2}u_0^2(x_0)} - \frac{\lambda}{2} - C},$$

The proof of Theorem 3 is completed.

□

4 Blow-up result of Eq. (1) on the circle $\mathbb{S}$

In this section, we shall present a new blow-up result for the solutions to Eq. (1) on the circle $\mathbb{S}$. Our main result can be formulated as follows.

Theorem 4 *Let $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$. Assume that there is $x_0 \in \mathbb{S}$ such that*

$$u_0'(x_0) < -\sqrt{\frac{3}{2}}|u_0(x_0)| - \lambda, \tag{17}$$

then the solution u of (4) *blows up in finite time. Moreover, the lifespan* T^* *is estimated above by*

$$T^* \leq \frac{1}{\lambda} \ln \frac{\sqrt{u_0'(x_0)^2 - \frac{3}{2}u_0^2(x_0)}}{\sqrt{u_0'(x_0)^2 - \frac{3}{2}u_0^2(x_0)} - \lambda},$$

Proof We employ the same notation as in the preceding proof, but now the Green function $p(x)$ is given by

$$p(x) := \frac{\cosh(x - [x] - \frac{1}{2})}{2\sinh\frac{1}{2}}, \tag{18}$$

where $[x]$ stands for the largest integer part of $x \in \mathbb{R}$.

As before, we again introduce

$$A(t,x) = \left[-u_x + \sqrt{\frac{3}{2}}u\right](t, q(t,x)),$$

and

$$B(t,x) = \left[-u_x - \sqrt{\frac{3}{2}}u\right](t, q(t,x)).$$

Then we have

$$\begin{aligned} \frac{d}{dt}A(t,x) &= \sqrt{\frac{3}{2}}(u_t + uu_x) - (u_{tx} + uu_{xx}) \\ &= u_x^2 + \lambda u_x - \sqrt{\frac{3}{2}}\lambda u - \frac{3}{2}u^2 + \left(p - \sqrt{\frac{3}{2}}p_x\right) * \left(\frac{3}{2}u^2\right) \\ &= AB - \lambda A + \left(p - \sqrt{\frac{3}{2}}p_x\right) * \left(\frac{3}{2}u^2\right). \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt}B(t,x) &= -\sqrt{\frac{3}{2}}(u_t + uu_x) - (u_{tx} + uu_{xx}) \\ &= u_x^2 + \lambda u_x + \sqrt{\frac{3}{2}}\lambda u - \frac{3}{2}u^2 + \left(p + \sqrt{\frac{3}{2}}p_x\right) * \left(\frac{3}{2}u^2\right) \\ &= AB - \lambda B + \left(p + \sqrt{\frac{3}{2}}p_x\right) * \left(\frac{3}{2}u^2\right). \end{aligned}$$

For the kernel function $p(x)$ given by (18), we know that

$$p(x) \pm \alpha p_x(x) \geq 0 \Leftrightarrow |\alpha| \leq \coth \frac{1}{2}.$$

Notice that $\sqrt{\frac{3}{2}} < \coth \frac{1}{2}$, so we have

$$\begin{aligned} \frac{d}{dt} A(t,x) &\geq AB - \lambda A\,, \\ \frac{d}{dt} B(t,x) &\geq AB - \lambda B\,. \end{aligned}$$

The initial condition

$$u_0'(x_0) < -\sqrt{\frac{3}{2}}|u_0(x_0)| - \lambda$$

is equivalent to

$$A(0, x_0) > \lambda > 0, \quad B(0, x_0) > \lambda > 0.$$

By using similar proof method as before, we can infer that

$$A'(t, x_0) > 0, \ \ B'(t, x_0) > 0\,,$$

that is, both $A(\cdot, x_0)$ and $B(\cdot, x_0)$ are increasing during the whole existence time $t \in [0, T^*)$.

Next, we prove that $T^* < \infty$. Consider $h(t) = \sqrt{AB(t, x_0)}$. Computing the time derivative of h yields

$$\begin{aligned} \frac{d}{dt} h(t) &= \frac{A_t B + AB_t}{2\sqrt{AB}}(t, x_0) \\ &\geq \frac{AB(A+B) - 2\lambda AB}{2\sqrt{AB}}(t, x_0) \\ &\geq h^2(t) - \lambda h(t), \end{aligned}$$

Note that $h(0) = \sqrt{A(0, x_0)B(0, x_0)} > \lambda > 0$, hence the solution blows up in finite time. By solving the above differential inequality, we find that

$$h(t) \to +\infty, \quad as \quad t \to T^* \leq \frac{1}{\lambda} \ln \frac{h(0)}{h(0) - \lambda}.$$

We notice the fact that $-u_x(t, q(t, x_0)) \geq h(t)$, which implies that $u_x \to -\infty$ as $t \to T^*$. Thus, the proof of Theorem 4 is completed. □

Remark Compared with Theorem 2.3 presented in [27], Theorem 3.1 in [28] and Theorem 3.2 in [26], we found that checking the blow-up conditions involves the computation of two norm quantities $||u_0||_{L^2}$ and $||u_0||_{L^\infty}$, while our blow-up condition in Theorem 3 only involves the computation of $||u_0||_{L^2}$. Particularly interesting is that our blow-up condition in Theorem 4 does not involve any norms of u_0 at all. This shows that our blow-up results considerably extend the previous results established in [26–28].

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