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Conservation laws and nonlocally related systems of the Hunter–Saxton equation for liquid crystal

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Abstract
Conservation laws of the Hunter–Saxton equation for liquid crystal are constructed by using multipliers. Based on the obtained conservation laws, we construct a tree of partial differential equations systems nonlocally related to the Hunter–Saxton equation. Many new local and nonlocal symmetries for these systems are found. The equivalence transformations of two potential systems are obtained. A symmetry-based method is employed to construct nonlocally related inverse potential systems. The symmetry-based method does not rely on the existence of conservation laws for the original equation.

Keywords Hunter–Saxton equation · Nonlocally related systems · Inverse potential systems · Conservation laws

Mathematics Subject Classification 76M60 · 70S10 · 54H15

1 Introduction

The nonlinear partial differential equations (PDEs) are useful in analyzing nonlinear phenomena in engineering and scientific problems. In the past decades, many effective methods for investigating properties of PDEs have been developed, such as the bilinear method [1–4], Riemann–Bäcklund method [5–7], inverse scattering method [8,9] algebraic geometry method [10–12] and Fokas method [13,14]. Symmetry analysis method is one of the most effective method for analyzing PDEs [15–20]. Any symmetry transforms the solutions of a PDE to the solutions of the same equation. On the basis of the symmetry theory, one can construct conservation laws of PDEs. Many method for deriving conservation laws of PDEs have been developed, such as

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Noether's approach [21–23] direct method [24–27], Ibragimovs method [28,29] and the mixed method [30]. The direct method is able to find all conservation laws for any given system of PDEs. In contrast, Noether's method is limited to variational systems, while Ibragimov's method and the mixed method are merely special cases of the multiplier method [31–33]. The problem of finding all conservation laws for a given PDEs is equivalent to the problem of finding all infinitesimal symmetries. Therefore, there is no need to derive conservation laws with the aid of special methods [33]. Once a PDE's conservation laws are constructed, the nonlocally related systems of this PDE can be established [25]. The nonlocally related systems are equivalent to the given PDE system [34]. Nonlocally related systems play an important role in finding the nonlocal symmetries and nonlocal conservation laws [35–38]. However, the conservation law-based method for constructing nonlocally related systems is not valid to the equation that has no nontrivial local conservation laws. It is notable that Bluman et al. proposed a symmetry-based method to find nonlocally related PDE systems [39]. Each point symmetry can yield a nonlocally related PDE system (inverse potential system). The symmetry-based method can also be used to construct trees of nonlocally related PDE system.

In the paper [40], based on the polynomial recursion formalism, Hou et al. derive the HS hierarchy. The first equation of this hierarchy is written as

$$U(x,t,u)=0: \quad u_{xxt}+4u_xu_{xx}+2uu_{xxx}=0. \tag{1}$$

This equation is an important physical model which can be used to describe the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal director field. The liquid crystal state is a distinct phase of matter observed between the solid and liquid states. The director field of the liquid crystal is usually floating [41]. Equation (1) is useful in studying the dynamics of director field since it can be used to model crucial point for nematic liquid crystals. Eq. (1) is Hunter–Saxton (HS) equation. HS equation is a short-wave limit of the Camassa–Holm equation [42]. This paper aims at constructing conservation laws and nonlocally related PDE systems of this equation.

This paper is organized as follows. In Sect. 2, the conservation laws of HS equation are constructed by using direct method. The conservation law-based method is employed to find the nonlocally related PDE systems of Eq. (1). Many new local and nonlocal symmetries for these systems are found. In Sect. 3, the equivalence transformations of two potential systems are investigated. In Sect. 4, the inverse potential systems arising from each Lie point symmetries are presented. A tree of inverse potential systems of Eq. (1) is also constructed. Finally, some conclusions are given in the last section.

2 Nonlocally related systems

Consider a k-order system of PDEs $\mathcal{R}_\alpha[u]$ with n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$

$$\mathcal{R}_\alpha [u] = \mathcal{R}_\alpha \left(x, u, u_{(1)}, \ldots, u_{(k)}\right) = 0, \quad \alpha = 1, 2, \ldots, m, \tag{2}$$

where $u_{(k)}$ is kth-order derivative. A local divergence-type conservation law of the PDE system (2) is a divergence expression of the form

$$D_i \Phi^i [u] = D_1 \Phi^1 [u] + \cdots + D_n \Phi^n [u] \tag{3}$$

in terms of total derivative operators holding on solutions of (2). There exists a set of conservation law multipliers

$$\Lambda_\alpha [u] = \Lambda_\alpha \left(x, u, \partial u, \ldots, \partial^l u\right), \quad \alpha = 1, 2, \ldots m, \tag{4}$$

such that

$$D_i \Phi^i [u] \equiv \Lambda_\alpha [u] \mathcal{R}_\alpha [u] \tag{5}$$

holds for arbitrary u.

For any divergence expression $D_i \Phi^i [u]$, one has

$$E_{u^j} \left(D_i \Phi^i [u]\right) \equiv 0, \quad j = 1, 2, \ldots m, \tag{6}$$

where $E_{u^j} = \frac{\partial}{\partial u^j} - D_i \frac{\partial}{\partial u^j_i} + \cdots + (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^j_{i_1 \cdots i_s}} + \cdots$ is Euler operator with respect to u^j.

A set of local multipliers $\Lambda_\alpha \left(x, u, \partial u, \ldots, \partial^l u\right)$ yields a divergence expression for PDE system (2) if and only if

$$E_{u^j} \left(\Lambda_\alpha \left(x, u, \partial u, \ldots, \partial^l u\right) R_\alpha \left(x, u, \partial u, \ldots, \partial^k u\right)\right) \equiv 0, \quad j = 1, 2, \ldots m \tag{7}$$

holds for arbitrary u [16].

Consider the conservation law multipliers $\Lambda[u] = \Lambda (t, x, u)$ to HS equation. Then

$$E_u (\Lambda (x, t, u) (u_{xxt} + 4u_x u_{xx} + 2u u_{xxx})) \equiv 0. \tag{8}$$

Splitting with Eq. (8) respect to third derivatives of u yields the following determining system

$$\begin{array}{l} -\Lambda_{uu} = 0, \ -2\Lambda_{xu} = 0, \\ -\Lambda_{txx} - 2u\Lambda_{xxx} = 0, \\ -6u\Lambda_{xu} - \Lambda_{tu} - 2\Lambda_x = 0, \\ -6u\Lambda_{xuu} - \Lambda_{tuu} - 4\Lambda_{xu} = 0, \\ -6u\Lambda_{xxu} - 2\Lambda_{txu} - 2\Lambda_{xx} = 0. \end{array} \tag{9}$$

The solution of the determining system (9) is given by

$$\Lambda (x, t, u, u_x, u_t) = x\mathcal{F}'(t) + u (-2\mathcal{F} (t) + c_1) + \mathcal{G} (t), \tag{10}$$

Table 1 Conservation laws of HS equation

CL	Multipliers	Conservation laws
V_1	$\Lambda = u$	$\Phi^t[u] = uu_{xx} + \frac{1}{2}u_x^2$, $\Phi^x[u] = 2u^2u_{xx} - u_t u_x$
V_2	$\Lambda = \mathcal{G}(t)$	$\Phi^t[u] = \mathcal{G}(t)u_{xx}$, $\Phi^x[u] = 2\mathcal{G}(t)uu_{xx} + \mathcal{G}(t)u_x^2 - \mathcal{G}'(t)u_x$
V_3	$\Lambda = x\mathcal{F}'(t) - 2u\mathcal{F}(t)$	$\Phi^t[u] = -\mathcal{F}(t)u_x^2 + \mathcal{F}'(t)u_x - 2\mathcal{F}(t)uu_{xx} + x\mathcal{F}'(t)u_{xx}$, $\Phi^x[u] = -4\mathcal{F}(t)u^2u_{xx} + x\mathcal{F}'(t)u_x^2 + 2\mathcal{F}(t)u_tu_x - x\mathcal{F}''(t)u_x - \mathcal{F}'(t)u_t + 2\mathcal{F}'(t)xuu_{xx}$

where c_1 is an arbitrary constant and $\mathcal{G}(t)$ and $\mathcal{F}(t)$ are arbitrary differential functions about t. The solution yields three local conservation laws multipliers

$$(1)\ \Lambda = u, \quad (2)\ \Lambda = \mathcal{G}(t), \quad (3)\ \Lambda = x\mathcal{F}'(t) - 2u\mathcal{F}(t). \tag{11}$$

Each multiplier determines a corresponding flux as Table 1 by using direct method with the aid of GeM [43,44].

The three conservation laws in Table 1 result in the following potential systems

$$UV_1\{x,t,u,v_1\} = 0: \begin{cases} v_{1x} = uu_{xx} + \frac{1}{2}u_x^2, \\ v_{1t} = u_tu_x - 2u^2u_{xx}, \end{cases} \tag{12}$$

$$UV_2\{x,t,u,v_2\} = 0: \begin{cases} v_{2x} = \mathcal{G}(t)u_{xx}, \\ v_{2t} = -\left(2\mathcal{G}(t)uu_{xx} + \mathcal{G}(t)u_x^2 - \mathcal{G}'(t)u_x\right), \end{cases} \tag{13}$$

$$UV_3\{x,t,u,v_3\} = \\ 0: \begin{cases} v_{3x} = -\mathcal{F}(t)u_x^2 + \mathcal{F}'(t)u_x - 2\mathcal{F}(t)uu_{xx} + x\mathcal{F}'(t)u_{xx}, \\ v_{3t} = -\left(-4\mathcal{F}(t)u^2u_{xx} + x\mathcal{F}'(t)u_x^2 + 2\mathcal{F}(t)u_tu_x \right. \\ \qquad \left. -x\mathcal{F}''(t)u_x - \mathcal{F}'(t)u_t + 2\mathcal{F}'(t)xuu_{xx}\right). \end{cases} \tag{14}$$

The three conservation laws in Table 1 yield up to $2^3 - 1 = 7$ nonlocally related PDE systems. Therefore, the following theorem can be established.

Theorem 1 *For the Hunter–Saxton equation, the set of locally inequivalent potential systems arising from multipliers depending on x, t and u is established by the following systems:*

- *Three potential systems* (12), (13) *and* (14) *involving single potentials.*
- *Three couplets* $UV_1V_2\{x,t,u,v_1,v_2\}$ [(12), (13)], $UV_1V_3\{x,t,u,v_1,v_3\}$ [(12), (14)] *and* $UV_2V_3\{x,t,u,v_2,v_3\}$ [(13), (14)].
- *One triplet* $UV_1V_2V_3\{x,t,u,v_1,v_2,v_3\}$ [(12), (13), (14)].

A tree of nonlocally related PDE system for the Hunter–Saxton equation is presented in Fig. 1. In what follows, we shall investigate the Lie point and nonlocal symmetries of the nonlocally related PDE systems. On the basis of the Lie symmetry analysis, the Lie point symmetries of UV_1 *are given*

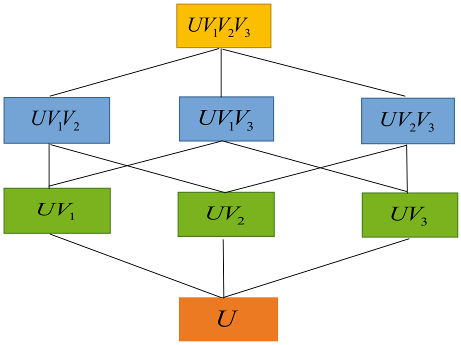

Fig. 1 A tree of nonlocally related PDE system for the Hunter–Saxton equation

Table 2 Point symmetry classification of the potential system UV_2

$\mathcal{G}(t)$	No.	Point symmetries
Arbitrary	2	$Y_1 = \frac{\partial}{\partial x},\ Y_2 = \frac{\partial}{\partial v_2}$
t	4	$Y_1 = \frac{\partial}{\partial x},\ Y_2 = \frac{\partial}{\partial v_2},\ Y_3 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}$ $Y_4 = t\ln(t)\frac{\partial}{\partial t} + \frac{-tu\ln(t) - tu + \frac{1}{2}}{t}\frac{\partial}{\partial u} - v_2\frac{\partial}{\partial v_2}$
$t^\alpha,\ \alpha \neq 1$	4	$Y_1 = \frac{\partial}{\partial x},\ Y_2 = \frac{\partial}{\partial v_2},\ Y_5 = t^\alpha\frac{\partial}{\partial t} + \frac{1}{2}\frac{\alpha\left(-2ut^{\alpha+1} + (\alpha-1)t^\alpha\right)}{t^2}\frac{\partial}{\partial u}$ $Y_6 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} + (\alpha-1)v_2\frac{\partial}{\partial v_2}$
e^t	4	$Y_1 = \frac{\partial}{\partial x},\ Y_2 = \frac{\partial}{\partial v_2},\ Y_7 = \frac{\partial}{\partial t} + v_2\frac{\partial}{\partial v_2}$ $Y_8 = -2e^t\frac{\partial}{\partial t} + (2u-1)e^t\frac{\partial}{\partial u}$

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial v_1},\\
X_4 &= -t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} + v_1\frac{\partial}{\partial v_1},\\
X_5 &= t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}.
\end{aligned} \tag{15}
$$

The symmetry classification of system UV_2 and UV_3 are given by Tables 2 and 3 respectively. Table 4 presents the symmetry classification of other potential systems of the tree of nonlocally related system (Fig. 1).

Theorem 2 *If $\mathcal{F} = e^t$ the symmetry Z_{10} of the system UV_1V_3 is the nonlocal symmetry of the system UV_1.*

Proof For the Lie point symmetry Z_{10},

$$\begin{aligned}&\xi^1(x,t,u,v_1,v_3)=2e^t,\ \ \xi^2(x,t,u,v_1,v_3)=2xe^t,\\&\eta^1(x,t,u,v_1,v_3)=xe^t,\ \ \eta^2(x,t,u,v_1,v_3)=v_3,\\&\eta^3(x,t,u,v_1,v_3)=xe^{2t}.\end{aligned} \tag{16}$$

Then

$$\left(\frac{\partial\xi^1}{\partial v_3}\right)^2+\left(\frac{\partial\xi^2}{\partial v_3}\right)^2+\left(\frac{\partial\eta^1}{\partial v_3}\right)^2+\left(\frac{\partial\eta^2}{\partial v_3}\right)^2=1>0.$$

So the symmetry Z_{10} is the nonlocal symmetry of the system UV_1. □

Remark 1 We can conclude that Z_{11} and Z_{12} are the nonlocal symmetries of system UV_3 when $\mathcal{F}=\ln(t)$ as the same analysis as Theorem 2. In addition, Z_{15} is the nonlocal symmetry of UV_1 and Z_{17} is the nonlocal symmetry of UV_3. Z_{21} is the nonlocal symmetry of the system UV_1 and UV_1V_2. Z_{22} is the nonlocal symmetry of the system UV_2, UV_1V_2 and UV_2V_3. Finally, Z_{23} is the nonlocal symmetry of the system UV_3 and UV_2V_3 when $\mathcal{G}=e^t$ and $\mathcal{F}=c$.

Remark 2 In this section, three local conservation laws of the Hunter–Saxton equation are constructed by limiting the multipliers to lowest-order. This class will miss some conservation laws. For the HS equation, the three-order multiplier $\Lambda(x,t,u,u_x,u_{xx},u_{xxx})$ is $x\mathcal{F}'(t)-2u\mathcal{F}(t)+c_1u+\mathcal{G}(t)+c_2\sqrt{u_{xx}}$. The term $\sqrt{u_{xx}}$ will yield new conservation laws by using the direct method. However, the $\sqrt{u_{xx}}$ is not a continuous function. It cannot split the flux continuously. For the multiplier $\Lambda(x,t,u,u_x,u_{xx},u_{xxx},u_{xxxx},u_{xxxxx})$, it will appear new term $\frac{u_{xxxx}}{(u_{xx})^{\frac{5}{2}}}-\frac{5}{4}\frac{u_{xxx}^2}{(u_{xx})^{\frac{7}{2}}}$. It is also hard to determine the conserved densities. Thus we don't consider the high-order multiplier in this paper.

Remark 3 The obtained nonlocally related systems of theorem 1 are not exhaustive. It is a fact that linear combinations of the starting conservation laws may yield additional systems [45]. The most general form of potential system can be written as

$$UV\{x,t,u,v\}=0:$$
$$\begin{cases}v_x=c_1\left(uu_{xx}+\frac{1}{2}u_x^2\right)+c_2\left(\mathcal{G}(t)u_{xx}\right)\\\qquad+c_3\left(-\mathcal{F}(t)u_x^2+\mathcal{F}'(t)u_x-2\mathcal{F}(t)uu_{xx}+x\mathcal{F}'(t)u_{xx}\right)\\v_t=c_1\left(u_tu_x-2u^2u_{xx}\right)+c_2\left(-2\mathcal{G}(t)uu_{xx}-\mathcal{G}(t)u_x^2+\mathcal{G}'(t)u_x\right)\\\qquad-c_3\left(-4\mathcal{F}(t)u^2u_{xx}+x\mathcal{F}'(t)u_x^2+2\mathcal{F}(t)u_tu_x\right.\\\qquad\left.-x\mathcal{F}''(t)u_x-\mathcal{F}'(t)u_t+2\mathcal{F}'(t)xuu_{xx}\right).\end{cases}$$

Together with the potential systems in Theorem 1, they exhaust all possible inequivalent potential systems.

Table 3 Point symmetry classification of the potential system UV_3

$\mathcal{F}(t)$	No.	Point symmetries
Arbitrary	2	$W_1=\frac{\partial}{\partial v_3},\ W_2=x\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}+v_3\frac{\partial}{\partial v_3}$
t	5	$W_1=\frac{\partial}{\partial v_3},\ W_2=x\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}+v_3\frac{\partial}{\partial v_3}$ $W_3=t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x},\ W_4=2t\frac{\partial}{\partial x}+\frac{\partial}{\partial u}$ $W_5=t\ln(t)\frac{\partial}{\partial t}+x(\ln(t)+2)\frac{\partial}{\partial x}+\left(u+\frac{x}{2t}\right)\frac{\partial}{\partial u}+\frac{x}{2t}\frac{\partial}{\partial v_3}$
$t^{\beta},\beta\neq 1$	5	$W_1=\frac{\partial}{\partial v_3},\ W_2=x\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}+v_3\frac{\partial}{\partial v_3},\ W_6=t\frac{\partial}{\partial t}+(\beta-2)x\frac{\partial}{\partial x}-(\beta-1)u\frac{\partial}{\partial u}$ $W_7=\frac{4t^{\beta}}{\beta^2}\frac{\partial}{\partial x}+\frac{2t^{\beta-1}}{\beta}\frac{\partial}{\partial u}+t^{2\beta-2}\frac{\partial}{\partial v_3}$ $W_8=2t^{\beta}\frac{\partial}{\partial t}+2\beta x t^{\beta-1}\frac{\partial}{\partial x}+\beta(\beta-1)xt^{\beta-2}\frac{\partial}{\partial u}+\beta^2(\beta-1)xt^{2\beta-3}\frac{\partial}{\partial v_3}$
e^t	5	$W_1=\frac{\partial}{\partial v_3},\ W_2=x\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}+v_3\frac{\partial}{\partial v_3}$ $W_9=\frac{\partial}{\partial t}-x\frac{\partial}{\partial x}-u\frac{\partial}{\partial u},\ W_{10}=4e^t\frac{\partial}{\partial x}+2e^t\frac{\partial}{\partial u}+e^{2t}\frac{\partial}{\partial v_3}$ $W_{11}=e^t\frac{\partial}{\partial t}-xe^t\frac{\partial}{\partial x}+\frac{1}{2}xe^t\frac{\partial}{\partial u}+\frac{1}{2}xe^{2t}\frac{\partial}{\partial v_3}$

3 Equivalence transformations of potential systems UV_2 and UV_3

An equivalence transformation transforms an equation that has arbitrary functions to an equation preserving the same differential structure but with different arbitrary functions [46,47]. We shall use Lie's infinitesimal criterion to derive the equivalence transformations of potential systems UV_2 and UV_3. For the system (13), the equivalence transformation is obtained by seeking an infinitesimal operator of the Lie algebra

$$E=\tau\frac{\partial}{\partial t}+\xi\frac{\partial}{\partial x}+\zeta^1\frac{\partial}{\partial u}+\zeta^2\frac{\partial}{\partial v_2}+\zeta^3\frac{\partial}{\partial \mathcal{G}}. \tag{17}$$

The one-parameter group of equivalence transformation is given by

$$\begin{aligned}\tilde{t}&=t+\varepsilon\tau(t,x,u,v_2)+O\left(\varepsilon^2\right),\\ \tilde{x}&=x+\varepsilon\xi(t,x,u,v_2)+O\left(\varepsilon^2\right),\\ \tilde{u}&=u+\varepsilon\zeta^1(t,x,u,v_2)+O\left(\varepsilon^2\right),\\ \tilde{v}_2&=v_2+\varepsilon\zeta^2(t,x,u,v_2)+O\left(\varepsilon^2\right),\\ \tilde{\mathcal{G}}&=\mathcal{G}+\varepsilon\zeta^3(t,x,u,v_2,\mathcal{G})+O\left(\varepsilon^2\right),\end{aligned} \tag{18}$$

where ε is the group parameter. The equivalence transformation operator (17) leaves not only the invariance of (13) but also the invariance of $\mathcal{G}_x=\mathcal{G}_u=\mathcal{G}_{v_2}=0$. Then the invariance criterion yields an overdetermined system for $\tau,\ \xi,\ \zeta^1,\ \zeta^2$ and ζ^3. Solving this system one has following operators

Table 4 Point symmetry classification of the potential system UV_1V_2, UV_1V_3, UV_2V_3 and $UV_1V_2V_3$

Potential systems	$\mathcal{G}, \mathcal{F}$	No.	Point symmetries
UV_1V_2	Arbitrary	3	$Z_1 = \frac{\partial}{\partial x},\ Z_2 = \frac{\partial}{\partial v_1},\ \ Z_3 = \frac{\partial}{\partial v_2}$
	$\mathcal{G} = t^\alpha$	4	$Z_1,\ Z_2,\ Z_3,\ Z_4 = -t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + 2v_1\frac{\partial}{\partial v_1} + (1-\alpha)\, v_2\frac{\partial}{\partial v_2}$
	$\mathcal{G} = e^t$	4	$Z_1,\ Z_2,\ Z_3,\ Z_5 = \frac{\partial}{\partial t} + v_2\frac{\partial}{\partial v_2}$
UV_1V_3	Arbitrary	2	$Z_2,\ Z_6 = \frac{\partial}{\partial v_3}$
	$\mathcal{F} = t^\beta$	3	$Z_2,\ Z_6,\ Z_7 = -t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + 2v_1\frac{\partial}{\partial v_1} + (2-\beta)\, v_3\frac{\partial}{\partial v_3}$
	$\mathcal{F} = e^t$	5	$Z_2,\ Z_6,\ Z_8 = \frac{\partial}{\partial t} + v_3\frac{\partial}{\partial v_3}, Z_9 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + v_1\frac{\partial}{\partial v_1} + v_3\frac{\partial}{\partial v_3}$ $Z_{10} = 2e^t\frac{\partial}{\partial t} + 2xe^t\frac{\partial}{\partial x} + xe^t\frac{\partial}{\partial u} + v_3\frac{\partial}{\partial v_1} + xe^{2t}\frac{\partial}{\partial v_3}$
	$\mathcal{F} = \ln(t)$	4	$Z_2,\ Z_6, Z_{11} = -t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + 2v_1\frac{\partial}{\partial v_1} + (2v_1 + 2v_3)\frac{\partial}{\partial v_3}$ $Z_{12} = t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} - 2v_1\frac{\partial}{\partial v_3}$
	$\mathcal{F} = c$	7	$Z_1,\ Z_6, Z_{13} = \frac{\partial}{\partial t}, Z_{14} = f_1(-2cv_1 - v_3)\frac{\partial}{\partial v_3}$ $Z_{15} = f_2(2cv_1 + v_3)\frac{\partial}{\partial v_1}, Z_{16} = t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}$ $Z_{17} = -2t\frac{\partial}{\partial t} - 2x\frac{\partial}{\partial x} + 2v_1\frac{\partial}{\partial v_1} + (-2cv_1 + v_3)\frac{\partial}{\partial v_3}$
UV_2V_3	Arbitrary	2	$Z_3,\ Z_6$
	$\mathcal{G}=t^\alpha, \mathcal{F}=t^\beta$	3	$Z_3,\ Z_6,\ Z_{18} = -t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + (1-\alpha)\, v_2\frac{\partial}{\partial v_2} + (2-\beta)\, v_3\frac{\partial}{\partial v_3}$
	$\mathcal{G} = \mathcal{F} = e^t$	3	$Z_3,\ Z_6,\ Z_{19} = \frac{\partial}{\partial t} + v_2\frac{\partial}{\partial v_2} + v_3\frac{\partial}{\partial v_3}$
$UV_1V_2V_3$	Arbitrary	3	$Z_2,\ Z_3,\ Z_6$
	$\mathcal{G}=t^\alpha, \mathcal{F}=t^\beta$	4	$Z_2,\ Z_3,\ Z_6$ $Z_{20} = -t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + 2v_1\frac{\partial}{\partial v_1} + (1-\alpha)\, v_2\frac{\partial}{\partial v_2} + (2-\beta)\, v_3\frac{\partial}{\partial v_3}$
	$\mathcal{G}=e^t, \mathcal{F}=c$	5	$Z_1,\ Z_5,\ Z_{21} = f_3(2cv_1 + v_3)\frac{\partial}{\partial v_1}$ $Z_{22} = f_4(2cv_1 + v_3)\frac{\partial}{\partial v_2}, Z_{23} = f_5(2cv_1 + v_3)\frac{\partial}{\partial v_3}$

$$
\begin{aligned}
&E_1 = \frac{\partial}{\partial t}, \quad E_2 = \frac{\partial}{\partial x}, \quad E_3 = \frac{\partial}{\partial v_2}, \\
&E_4 = v_2\frac{\partial}{\partial v_2} + \mathcal{G}\frac{\partial}{\partial \mathcal{G}}, \\
&E_5 = -t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} - \mathcal{G}\frac{\partial}{\partial \mathcal{G}},
\end{aligned} \tag{19}
$$

where $\mathcal{G} = \mathcal{G}(t)$ is arbitrary function. Thus the five-parameter equivalence group associated with above five generators is given by

$$\begin{aligned} &E_1: \ \tilde{t}=t+a_1,\ \tilde{x}=x,\ \tilde{u}=u,\ \tilde{v}_2=v_2,\ \tilde{\mathcal{G}}=\mathcal{G},\\ &E_2: \ \tilde{t}=t,\ \tilde{x}=x+a_2,\ \tilde{u}=u,\ \tilde{v}_2=v_2,\ \tilde{\mathcal{G}}=\mathcal{G},\\ &E_3: \ \tilde{t}=t,\ \tilde{x}=x,\ \tilde{u}=u,\ \tilde{v}_2=v_2+a_3,\ \tilde{\mathcal{G}}=\mathcal{G},\\ &E_4: \ \tilde{t}=t,\ \tilde{x}=x,\ \tilde{u}=u,\ \tilde{v}_2=e^{a_4}v_2,\ \tilde{\mathcal{G}}=e^{a_4}\mathcal{G},\\ &E_5: \ \tilde{t}=e^{-a_5}t,\ \tilde{x}=x,\ \tilde{u}=e^{a_5}u,\ \tilde{v}_2=v_2,\ \tilde{\mathcal{G}}=e^{-a_5}\mathcal{G}. \end{aligned} \tag{20}$$

Therefore, the following theorem is established.

Theorem 3 *Any transformation of the form*

$$\begin{aligned} &\tilde{t}=a_1+e^{-a_5}t, \quad \tilde{x}=a_2+x, \quad \tilde{u}=e^{a_5}u,\\ &\tilde{v}_2=a_3+e^{a_4}v_2, \quad \tilde{\mathcal{G}}=e^{a_4-a_5}\mathcal{G}, \end{aligned}$$

where $a_1,\ldots,a_5$ are arbitrary constants, maps the potential systems UV_2 (13) to the PDE system with same form

$$\begin{cases} \tilde{v}_{2\tilde{x}}=\tilde{\mathcal{G}}\left(\tilde{t}\right)\tilde{u}_{\tilde{x}\tilde{x}},\\ \tilde{v}_{2\tilde{t}}=-\left(2\tilde{\mathcal{G}}\left(\tilde{t}\right)\tilde{u}\tilde{u}_{\tilde{x}\tilde{x}}+\tilde{\mathcal{G}}\left(\tilde{t}\right)\tilde{u}_{\tilde{x}}^2-\tilde{\mathcal{G}}'\left(\tilde{t}\right)\tilde{u}_{\tilde{x}}\right). \end{cases}$$

Then we can obtain the equivalence transformation theorem for the potential system UV_3 (14) similar to the process of the derivation of Theorem 3.

Theorem 4 *Any transformation of the form*

$$\begin{aligned} &\tilde{t}=a_1+e^{-a_3+a_4+a_5}t,\ \tilde{x}=e^{-a_3+2a_4+a_5}x,\ \tilde{u}=e^{a_4}u,\\ &\tilde{v}_1=a_2+e^{a_3}v_1,\ \tilde{\mathcal{F}}=e^{a_5}\mathcal{F}, \end{aligned}$$

where $a_1,\ldots,a_5$ are arbitrary constants, maps the potential systems UV_3 (14) to the PDE system with same form

$$\begin{cases} \tilde{v}_{3\tilde{x}}=-\tilde{\mathcal{F}}\left(\tilde{t}\right)\tilde{u}_{\tilde{x}}^2+\tilde{\mathcal{F}}'\left(\tilde{t}\right)\tilde{u}_{\tilde{x}}-2\tilde{\mathcal{F}}\left(\tilde{t}\right)\tilde{u}\tilde{u}_{\tilde{x}\tilde{x}}+\tilde{x}\tilde{\mathcal{F}}'\left(\tilde{t}\right)\tilde{u}_{\tilde{x}\tilde{x}},\\ \tilde{v}_{3\tilde{t}}=-\left(-4\tilde{\mathcal{F}}\left(\tilde{t}\right)\tilde{u}^2\tilde{u}_{\tilde{x}\tilde{x}}+\tilde{x}\tilde{\mathcal{F}}'\left(\tilde{t}\right)\tilde{u}_{\tilde{x}}^2+2\tilde{\mathcal{F}}\left(\tilde{t}\right)\tilde{u}_{\tilde{t}}\tilde{u}_{\tilde{x}}\right.\\ \quad\left.-\tilde{x}\tilde{\mathcal{F}}''\left(\tilde{t}\right)\tilde{u}_{\tilde{x}}-\tilde{\mathcal{F}}'\left(\tilde{t}\right)\tilde{u}_{\tilde{t}}+2\tilde{\mathcal{F}}'\left(\tilde{t}\right)\tilde{x}\tilde{u}\tilde{u}_{\tilde{x}\tilde{x}}\right). \end{cases}$$

4 Inverse potential systems arising from Lie point symmetries

In this section, a symmetry-based method is employed to construct inverse potential systems of HS equation. The symmetry group will be generated by the vector field of the form

$$X=\xi^1(t,x,u)\frac{\partial}{\partial t}+\xi^2(t,x,u)\frac{\partial}{\partial x}+\eta(t,x,u)\frac{\partial}{\partial u}, \tag{21}$$

then X (21) must satisfy Lie's symmetry condition

$$pr^{(3)}X(\Delta)|_{\Delta=0}=0, \tag{22}$$

where $\Delta=u_{xxt}+4u_xu_{xx}+2uu_{xxx}=0$. The Lie symmetry condition yields an overdetermined system of partial differential equations about $\xi^1,\ \xi^2$ and η

$$\begin{array}{l}\eta_{xx}=0,\quad \eta_{xu}=0,\quad \xi^1_{tt}=2\eta_x,\\ \eta_{tu}=0,\ \eta_{uu}=0,\quad \xi^1_x=0,\\ \xi^2_x=\xi^1_t+\eta_u,\quad \xi^2_t=-2u\eta_u+2\eta,\\ \xi^1_u=0,\quad \xi^2_u=0.\end{array}$$

Solving this system, one can get

$$\begin{array}{l}\xi^1=c_2t+c_3+\int 2g'(t)dt,\\ \xi^2=(c_2+c_1)x+c_4+\int\left(2g'(t)x+2f'(t)\right)dt,\\ \eta=g'(t)x+c_1u+f'(t).\end{array} \tag{23}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants and $f(t)$ and $g(t)$ are arbitrary differential functions. Hence the infinitesimal symmetries of (1) form the infinite dimensional Lie algebra L spanned by the following vector fields

$$\begin{array}{l}X_1=\dfrac{\partial}{\partial t},\quad X_2=\dfrac{\partial}{\partial x},\quad X_3=x\dfrac{\partial}{\partial x}+u\dfrac{\partial}{\partial u},\\ X_4=t\dfrac{\partial}{\partial t}+x\dfrac{\partial}{\partial x},\quad X_5=2f(t)\dfrac{\partial}{\partial x}+f'(t)\dfrac{\partial}{\partial u},\\ X_6=2g(t)\dfrac{\partial}{\partial t}+2g(t)x\dfrac{\partial}{\partial x}+g'(t)x\dfrac{\partial}{\partial u}.\end{array} \tag{24}$$

4.1 Inverse potential system from X_1

For the symmetry X_1, it maps into the canonical form $P=\frac{\partial}{\partial v}$ by introducing canonical coordinates

$$\begin{array}{l}r=x,\\ s=u,\\ v(r,s)=t.\end{array} \tag{25}$$

At the same time, the Eq. (1) is mapped to an invertibly equivalent equation

$$\begin{array}{l}12sv_rv_s^2v_{sr}^2-18sv_sv_r^2v_{sr}v_{ss}-6sv_s^3v_{sr}v_{rr}\\ \quad -6sv_rv_s^3v_{srr}+6sv_r^3v_{ss}^2+6sv_s^2v_rv_{ss}v_{rr}+6sv_s^2v_r^2v_{ssr}\\ \quad -2sv_sv_r^3v_{sss}+2sv_s^4v_{rrr}+8v_s^2v_r^2v_{sr}\\ \quad -4v_sv_r^3v_{ss}-4v_rv_s^3v_{rr}-2v_s^2v_{sr}^2+6v_rv_sv_{sr}v_{ss}+v_s^3v_{srr}\\ \quad -3v_r^2v_{ss}^2-v_s^2v_{ss}v_{rr}-2v_rv_s^2v_{ssr}+v_sv_r^2v_{sss}=0.\end{array} \tag{26}$$

Introducing the new variable $\phi=v_r$ and $\psi=v_s$, one can obtain the locally related intermediate system

$$
\begin{aligned}
&\phi = v_r,\\
&\psi = v_s,\\
&12s\phi\psi^2\psi_r^2 - 18s\psi\phi^2\psi_r\psi_s - 6s\psi^3\psi_r\phi_r - 6s\phi\psi^3\psi_{rr} + 6s\phi^3\psi_s^2\\
&\quad +6s\psi^2\phi\psi_s\phi_r + 6s\psi^2\phi^2\psi_{sr} - 2s\psi\phi^3\psi_{ss}\\
&\quad +2s\psi^4\phi_{rr} + 8\psi^2\phi^2\psi_r - 4\psi\phi^3\psi_s - 4\phi\psi^3\phi_r - 2\psi^2\psi_r^2\\
&\quad +6\phi\psi\psi_r\psi_s + \psi^3\psi_{rr} - 3\phi^2\psi_s^2 - \psi^2\psi_s\phi_r\\
&\quad -2\phi\psi^2\psi_{sr} + \psi\phi^2\psi_{ss} = 0.
\end{aligned}
\tag{27}
$$

Eliminating v from the system (27), one obtains an inverse potential system (IP_1) of Eq. (1)

$$
\begin{aligned}
&\phi_s = \psi_r,\\
&12s\phi\psi^2\psi_r^2 - 18s\psi\phi^2\psi_r\psi_s - 6s\psi^3\psi_r\phi_r - 6s\phi\psi^3\psi_{rr}\\
&\quad +6s\phi^3\psi_s^2 + 6s\psi^2\phi\psi_s\phi_r + 6s\psi^2\phi^2\psi_{sr} - 2s\psi\phi^3\psi_{ss}\\
&\quad +2s\psi^4\phi_{rr} + 8\psi^2\phi^2\psi_r - 4\psi\phi^3\psi_s - 4\phi\psi^3\phi_r - 2\psi^2\psi_r^2\\
&\quad +6\phi\psi\psi_r\psi_s + \psi^3\psi_{rr} - 3\phi^2\psi_s^2 - \psi^2\psi_s\phi_r\\
&\quad -2\phi\psi^2\psi_{sr} + \psi\phi^2\psi_{ss} = 0.
\end{aligned}
\tag{28}
$$

Due to the inverse potential system (28) is nonlocally related to the intermediate system (27), the inverse potential system (28) is nonlocally related to Eq. (1). The transformation (25) establishes a one-to-one mapping between the solutions of (28) and (1). As the process of construction of the nonlocally related system by using symmetry X_1, one can construct the nonlocally related systems, which are based on X_2 to X_6.

4.2 Inverse potential system from X_2

For the symmetry X_2, it maps into the canonical form $P = \frac{\partial}{\partial v}$ by introducing canonical coordinates

$$
\begin{aligned}
r &= t,\\
s &= u,\\
v(r,s) &= x.
\end{aligned}
\tag{29}
$$

At the same time, the Eq. (1) is mapped to an invertibly equivalent equation

$$
v_s^2 v_{ssr} - 3v_s v_{sr} v_{ss} + 3v_r v_{ss}^2 - 6s v_{ss}^2 - v_r v_s v_{sss} + 2s v_s v_{sss} + 4v_s v_{ss} = 0 \tag{30}
$$

Introducing the new variable $\phi = v_r$ and $\psi = v_s$, one can obtain the locally related intermediate system

$$
\begin{aligned}
&\phi = v_r,\\
&\psi = v_s,\\
&v_s^2 v_{ssr} - 3v_s v_{sr} v_{ss} + 3v_r v_{ss}^2 - 6s v_{ss}^2 - v_r v_s v_{sss} + 2s v_s v_{sss} + 4v_s v_{ss} = 0.
\end{aligned}
\tag{31}
$$

Eliminating v from the system (31), one obtains an inverse potential system (IP_2) of Eq. (1)

$$\begin{aligned} &\phi_s = \psi_r, \\ &\psi^2\psi_{sr} - 3\psi\psi_r\psi_s + 3\phi\psi_s^2 - 6s\psi_s^2 - \phi\psi\psi_{ss} + 2s\psi\psi_{ss} + 4\psi\psi_s = 0. \end{aligned} \tag{32}$$

4.3 Inverse potential system from X_3

For the symmetry X_3, it maps into the canonical form $P = \frac{\partial}{\partial v}$ by introducing canonical coordinates

$$\begin{aligned} r &= t, \\ s &= \frac{u}{x}, \\ v(r,s) &= x. \end{aligned} \tag{33}$$

At the same time, the Eq. (1) is mapped to an invertibly equivalent equation

$$\begin{aligned} &2sv_s^4 - v_{sr}v_s^3 + v_{ss}v_rv_s^2 - 4sv_{ss}v_s^2 + 3v_sv_{sr}v_{ss} - v_s^2v_{ssr} - 3v_rv_{ss}^2 + 6sv_{ss}^2 \\ &\quad + v_rv_sv_{sss} - 2sv_sv_{sss} + 4v_s^3 - 4v_sv_{ss} = 0. \end{aligned} \tag{34}$$

Introducing the new variable $\phi = v_r$ and $\psi = v_s$, one can obtain the locally related intermediate system

$$\begin{aligned} &\phi = v_r, \\ &\psi = v_s, \\ &2s\psi^4 - \psi_r\psi^3 + \phi\psi^2\psi_s - 4s\psi^2\psi_s + 3\psi\psi_r\psi_s - \psi^2\psi_{sr} - 3\phi\psi_s^2 + 6s\psi_s^2 \\ &\quad + \phi\psi\psi_{ss}^2 - 2s\psi\psi_{ss} + 4\psi^3 - 4\psi\psi_s = 0. \end{aligned} \tag{35}$$

Eliminating v from the system (35), one obtains an inverse potential system (IP_3) of Eq. (1)

$$\begin{aligned} &\phi_s = \psi_r, \\ &2s\psi^4 - \psi_r\psi^3 + \phi\psi^2\psi_s - 4s\psi^2\psi_s + 3\psi\psi_r\psi_s - \psi^2\psi_{sr} - 3\phi\psi_s^2 + 6s\psi_s^2 \\ &\quad + \phi\psi\psi_{ss}^2 - 2s\psi\psi_{ss} + 4\psi^3 - 4\psi\psi_s = 0. \end{aligned} \tag{36}$$

4.4 Inverse potential system from X_4

For the symmetry X_4, it maps into the canonical form $P = \frac{\partial}{\partial v}$ by introducing canonical coordinates

$$\begin{aligned} r &= \frac{t}{x}, \\ s &= u, \\ v(r,s) &= \ln(x). \end{aligned} \tag{37}$$

Equation (1) is mapped to an invertibly equivalent equation

$$
\begin{aligned}
&-6sr^3v_rv_s^3v_{srr}+6sr^3v_s^2v_r^2v_{ssr}-2sr^3v_sv_r^3v_{sss}-24sr^2v_rv_s^3v_{sr}\\
&\quad+12sr^2v_s^2v_r^2v_{ss}-3r^2v_rv_s^2v_{rr}v_{ss}+6sr^2v_s^2v_{rr}v_{ss}\\
&\quad+9r^2v_sv_r^2v_{ss}v_{sr}+12sr^2v_rv_s^2v_{ssr}-6sr^2v_sv_r^2v_{sss}\\
&\quad+18rsv_rv_s^2v_{ss}+12rv_sv_rv_{ss}v_{sr}\\
&\quad-18rsv_sv_{ss}v_{sr}-6rsv_sv_rv_{sss}-6sr^3v_s^3v_{sr}v_{rr}+12sr^3v_rv_s^2v_{sr}^2\\
&\quad-2rv_s^2v_{ss}v_{rr}+18rsv_rv_{ss}^2-4rv_rv_s^2v_{ssr}\\
&\quad+6srv_s^2v_{ssr}+2rv_sv_r^2v_{sss}-12rv_sv_rv_{ss}+2sr^3v_s^4v_{rrr}+6sr^3v_r^3v_{ss}^2\\
&\quad-4r^3v_rv_s^3v_{rr}+12sr^2v_s^4v_{rr}+8r^3v_s^2v_r^2v_{sr}\\
&\quad-4r^3v_sv_r^3v_{ss}+3r^2v_s^3v_{sr}v_{rr}-6r^2v_rv_s^2v_{sr}^2\\
&\quad+12sr^2v_s^2v_{sr}^2+3r^2v_rv_s^3v_{srr}-6sr^2v_s^3v_{srr}\\
&\quad+18sr^2v_r^2v_{ss}^2-3r^2v_s^2v_r^2v_{ssr}+r^2v_sv_r^3v_{sss}+12rsv_rv_s^4+8rv_rv_s^3v_{sr}\\
&\quad+16r^2v_rv_s^2v_{sr}-18srv_s^3v_{sr}-4rv_s^2v_r^2v_{ss}\\
&\quad-12r^2v_sv_r^2v_{ss}+6sr^3v_rv_s^2v_{ss}v_{rr}-18sr^3v_sv_r^2v_{ss}v_{sr}\\
&\quad-36sr^2v_sv_rv_{ss}v_{sr}-4v_s^3-v_s^2v_{ssr}\\
&\quad-4v_sv_{ss}-3v_rv_{ss}^2-2v_rv_s^4+4sv_s^4+3v_s^3v_{sr}\\
&\quad+6sv_{ss}^2+v_sv_rv_{sss}-2sv_sv_{sss}+2rv_s^3v_{srr}-6rv_r^2v_{ss}^2\\
&\quad-12rv_rv_s^3+8rv_s^2v_{sr}-3v_rv_s^2v_{ss}+6sv_s^2v_{ss}+3v_sv_{sr}v_{ss}\\
&\quad-r^2v_s^4v_{rrr}-3r^2v_r^3v_{ss}^2-8r^2v_r^2v_s^3-4rv_s^4v_{rr}\\
&\quad-4r^2v_s^3v_{rr}-4rv_s^2v_{sr}^2=0.
\end{aligned}
\tag{38}
$$

Introducing the new variable $\phi=v_r$ and $\psi=v_s$ and eliminating v from the locally related intermediate system, one obtains the inverse potential system (IP_4)

$$
\begin{aligned}
&\phi_s=\psi_r,\\
&-6sr^3\phi\psi^3\psi_{rr}+6sr^3\psi^2\phi^2\psi_{sr}-2sr^3\psi\phi^3\psi_{ss}\\
&\quad-24sr^2\phi\psi^3\psi_r+12sr^2\psi^2\phi^2\psi_s-3r^2\phi\psi^2\phi_r\psi_s\\
&\quad+6sr^2\psi^2\phi_r\psi_s+9r^2\psi\phi^2\psi_s\psi_r+12sr^2\phi\psi^2\psi_{sr}\\
&\quad-6sr^2\psi\phi^2\psi_{ss}+18rs\phi\psi^2\psi_s+12r\psi\phi\psi_s\psi_r\\
&\quad-18rs\psi\psi_s\psi_r-6rs\psi\phi\psi_{ss}-6sr^3\psi^3\psi_r\phi_r\\
&\quad+12sr^3\phi\psi^2\psi_r^2-2r\psi^2\psi_s\phi_r+18rs\phi\psi_s^2-4r\phi\psi^2\psi_{sr}\\
&\quad+6sr\psi^2\psi_{sr}+2r\psi\phi^2\psi_{ss}-12r\psi\phi\psi_s+2sr^3\psi^4\phi_{rr}\\
&\quad+6sr^3\phi^3\psi_s^2-4r^3\phi\psi^3\phi_r+12sr^2\psi^4\phi_r\\
&\quad+8r^3\psi^2\phi^2\psi_r-4r^3\psi\phi^3\psi_s+3r^2\psi^3\psi_r\phi_r-6r^2\phi\psi^2\psi_r^2\\
&\quad+12sr^2\psi^2\psi_r^2+3r^2\phi\psi^3\psi_{rr}-6sr^2\psi^3\psi_{rr}\\
&\quad+18sr^2\phi^2\psi_s^2-3r^2\psi^2\phi^2\psi_r+r^2\psi\phi^3\psi_{ss}+12rs\phi\psi^4\\
&\quad+8r\phi\psi^3\psi_r+16r^2\phi\psi^2\psi_r-18sr\psi^3\psi_r\\
&\quad-4r\psi^2\phi^2\psi_s-12r^2\psi\phi^2\psi_s+6sr^3\phi\psi^2\psi_s\phi_r\\
&\quad-18sr^3\psi\phi^2\psi_s\psi_r-36sr^2\psi\phi\psi_s\psi_r-4\psi^3-\psi^2\psi_{sr}\\
&\quad-4\psi\psi_s-3\phi\psi_s^2-2\phi\psi^4+4s\psi^4+3\psi^3\psi_r+6s\psi_s^2\\
&\quad+\psi\phi\psi_{ss}-2s\psi\psi_{ss}+2r\psi^3\psi_{rr}-6r\phi^2\psi_s^2\\
&\quad-12r\phi\psi^3+8r\psi^2\psi_r-3\phi\psi^2\psi_s+6s\psi^2\psi_s\\
&\quad+3\psi\psi_r\psi_s-r^2\psi^4\phi_{rr}-3r^2\phi^3\psi_s^2-8r^2\phi^2\psi^3-4r\psi^4\phi_r\\
&\quad-4r^2\psi^3\phi_r-4r\psi^2\psi_r^2=0.
\end{aligned}
\tag{39}
$$

4.5 Inverse potential system from X_5

When $f(t) = t$ for X_5, canonical coordinates induced by X_5 are given by

$$\begin{aligned} r &= t, \\ s &= u - \frac{1}{2}\frac{x}{t}, \\ v(r, s) &= \frac{1}{2}\frac{x}{t}. \end{aligned} \tag{40}$$

Transformation (40) maps Eq. (1) to the equation

$$r v_s^2 v_{ssr} - r v_r v_s v_{sss} + 3r v_r v_{ss}^2 - 3r v_s v_{sr} v_{ss} + s v_s v_{sss} - 3s v_{ss}^2 + 2 v_s v_{ss} = 0. \tag{41}$$

which is invertibly related to Eq. (1). Introducing the new variable $\phi = v_r$ and $\psi = v_s$ and eliminating v from the locally related intermediate system, one obtains the inverse potential system (IP_5)

$$\begin{aligned} &\phi_s = \psi_r, \\ &r\psi^2\psi_{sr} - r\phi\psi\psi_{ss} + 3r\phi\psi_s^2 - 3r\psi\psi_r\psi_s + s\psi\psi_{ss} - 3s\psi_s^2 + 2\psi\psi_s = 0. \end{aligned} \tag{42}$$

4.6 Inverse potential system from X_6

When $g(t) = e^t$ for X_6, canonical coordinates induced by X_6 are given by

$$\begin{aligned} r &= -\ln(x) + t, \\ s &= -\frac{1}{2}x + u, \\ v(r, s) &= -\frac{1}{2}e^{-t}. \end{aligned} \tag{43}$$

Transformation (43) maps Eq. (1) to the equation

$$\begin{aligned} &\big(12s v_r^3 v_{ss}^2 - 36s v_s v_r^2 v_{sr} v_{ss} + 12s v_s^2 v_r^2 v_{ss} + 12s v_r v_s^2 v_{ss} v_{rr} \\ &\quad + 24s v_r v_s^2 v_{sr}^2 - 24s v_s^3 v_r v_{sr} \\ &\quad - 12s v_s^3 v_{sr} v_{rr} - 4s v_s v_r^3 v_{sss} + 12s v_s^2 v_r^2 v_{ssr} + 8s v_s^4 v_r \\ &\quad - 12s v_r v_s^3 v_{srr} + 12s v_{rr} v_s^4 + 4s v_s^4 v_{rrr} s \\ &\quad - 8v_s v_r^3 v_{ss} + 16 v_s^2 v_r^2 v_{sr} - 8v_s^3 v_r^2 - 8v_r v_s^3 v_{rr}\big) e^r \\ &\quad + 3v_r^2 v_{ss}^2 - 6v_s v_r v_{ss} v_{sr} + v_r v_s^2 v_{ss} + v_s^2 v_{ss} v_{rr} \\ &\quad + 2v_s^2 v_{sr}^2 - v_s^3 v_{sr} - v_s v_r^2 v_{sss} + 2v_r v_s^2 v_{ssr} - v_s^3 v_{srr} = 0. \end{aligned} \tag{44}$$

Introducing the new variable $\phi = v_r$ and $\psi = v_s$ and eliminating v from the locally related intermediate system, one obtains the inverse potential system (IP_6)

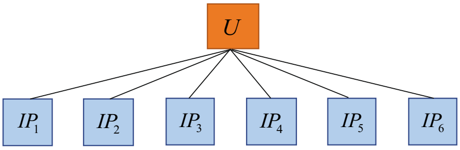

Fig. 2 A tree of inverse potential systems for the Hunter–Saxton equation

$$
\begin{aligned}
&\phi_s = \psi_r,\\
&\big(12s\phi^3\psi_s^2-36s\psi\phi^2\psi_r\psi_s+12s\psi^2\phi^2\psi_s+12s\phi\psi^2\psi_s\phi_r+24s\phi\psi^2\psi_r^2-24s\psi^3\phi\psi_r\\
&\quad -12s\psi^3\psi_r\phi_r-4s\psi\phi^3\psi_{ss}+12s\psi^2\phi^2\psi_{sr}\\
&\quad +8s\psi^4\phi-12s\phi\psi^3\psi_{rr}+12s\phi_r\psi^4+4s\psi^4\phi_{rr}\\
&\quad -8\psi\phi^3\psi_s+16\psi^2\phi^2\psi_r-8\psi^3\phi^2-8\phi\psi^3\phi_{rr}\big)\,e^r+3\phi^2\psi_s^2-6\psi\phi\psi_s\psi_r+\phi\psi^2\psi_s\\
&\quad +\psi^2\psi_s\phi_r+2\psi^2\psi_r^2-\psi^3\psi_r-\psi\phi^2\psi_{ss}+2\phi\psi^2\psi_{sr}-\psi^3\psi_{rr}=0.
\end{aligned}
\tag{45}
$$

Remark 4 The inverse potential system $IP_1, IP_2, IP_3, IP_4, IP_5$ and IP_6 play an important role in analyzing HS equation, which are all equivalent to the HS equation. The relationship between the solutions of inverse potential system and HS equation is one-to-one. All the inverse potential systems are nonlocally related to HS equation. Figure 2 presents a tree of inverse potential systems arising from Lie point symmetries, which further extend the tree of nonlocally related systems (see Fig. 1) form conservation law-based method.

5 Conclusions

In this paper Lie symmetry analysis method is performed on the HS equation. The direct method is used to derive local conservation laws of the HS equation. The nonlocally related PDE systems of HS equation are constructed with the aid of conservation law-based method. Based on the symmetry classification of the potential systems, we obtain many new local and nonlocal symmetries. A tree of nonlocally related PDE system for HS equation is presented in Fig. 1. Two equivalence transformation theorems of the potential systems are established. In order to extend the tree we established a tree of the inverse potential systems (Fig. 2) by using a symmetry-based method. The results of this paper are helpful for further analysis of the properties of the HS equation.

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Compliance with ethical standards

Conflict of interest The author declares that he has no conflict of interest.

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