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Two simple projection-type methods for solving variational inequalities

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Abstract

In this paper we study a classical monotone and Lipschitz continuous variational inequality in real Hilbert spaces. Two projection type methods, Mann and its viscosity generalization are introduced with their strong convergence theorems. Our methods generalize and extend some related results in the literature and their main advantages are: the strong convergence and the adaptive step-size usage which avoids the need to know apriori the Lipschitz constant of variational inequality associated operator. Primary numerical experiments in finite and infinite dimensional spaces compare and illustrate the behaviors of the proposed schemes.

Keywords Projection-type method · Variational inequality · Mann-type method · Viscosity method · Projection and contraction method

Mathematics Subject Classification 47H09 · 47J20 · 65K15 · 90C25

Dedicated to Professor Le Dung Muu on the Occasion of his 70th Birthday.

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1 Introduction

In this paper, we study the classical Variational Inequality (VI) of Fichera [14,15] in real Hilbert spaces. The VI is formulated as follows: Find a point $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0 \ \ \forall x \in C, \tag{1}$$

where $C \subseteq H$ is a nonempty, closed and convex set of a real Hilbert space H and $A : H \rightarrow H$ is a given mapping. We denote by $VI(C, A)$ the solution set of the VI (1).

Variational inequalities are fundamental problems which stand the core of diverse applied fields such as in economics, engineering mechanics, transportation, and many more, see for example, [2,3,20], just to name a few. In the last decades, many iterative methods have been constructed for solving variational inequalities and their related optimization problems, see for example the excellent book of Facchinei and Pang [13], Konnov [20] and the many references therein.

The first simplest method for solving VIs, derived from optimization theory, is known as the *gradient method* (GM). The iterative step of this method requires the calculation of the orthogonal projection onto the feasible set of the VI, that is C, per each iteration. Given the current iterate x_n, the algorithm's iterative step has the following form.

$$x_{n+1} = P_C(x_n - \tau Ax_n), \tag{2}$$

where $\tau \in (0, \frac{1}{L})$, L is the Lipschitz constant of A and P_C denotes the metric projection onto C. It is shown that gradient method (2) convergence under Lipschitz continuity and some restrictive monotonicity assumption, such as strong monotonicity or inverse strongly monotone, see for example [18]. Korpelevich [21] (also Antipin [1] independently) proposed a double-projection method, known as the *extragradient method* (EM) which enable to obtain convergence in Euclidean spaces under Lipschitz continuity and just monotonicity. Given the current iterate x_n, the algorithm's iterative step has the following form.

$$\begin{cases} y_n = P_C(x_n - \tau Ax_n), \\ x_{n+1} = P_C(x_n - \tau Ay_n), \end{cases} \tag{3}$$

where τ and P_C are as above. This method is studied intensively and extended and improved in various ways, for example see, e.g. [6–10,25,26,31,34,35] and the references therein.

Although the extragradient method converges under weaker monotonicity assumption than the gradient method, it requires to calculate two projections onto C per each iteration. So, in case that the set C is not "easy" to project onto it, a minimum distance subproblem has to be solved twice per each iteration in order to evaluate P_C, a fact which might affect the applicability and computational complexity of the method.

In a direction to overcome this obstacle, Censor et al. [7–9] introduced the so-called *subgradient extragradient method* (SEM). In this algorithm, the second projection onto

C is replaced by an easy and constructible projection onto some super set which contains C. Given the current iterate x_n, the algorithm's iterative step has the following form.

$$\begin{cases} y_n = P_C(x_n - \tau A x_n), \\ T_n = \{x \in H \mid \langle x_n - \tau A x_n - y_n, x - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \tau A y_n), \end{cases} \tag{4}$$

where $\tau \in (0, 1/L)$.

Another method which uses only one projection onto C is *projection and contraction method* (PC) of He [17] (see also Sun [32]). In this method, the point y_n in calculated in the same spirit of (3), but the next iterate x_{n+1} is calculated via some adaptive step size rules. Given the current iterate x_n, the algorithm's iterative step has the following form.

$$y_n = P_C(x_n - \tau_n A x_n),$$

and then the next iterate x_{n+1} is generated via the following PC-algorithms:

$$x_{n+1} = x_n - \gamma \eta_n d(x_n, y_n), \tag{5}$$

where $\gamma \in (0, 2)$, $\tau_n \in (0, 1/L)$ (or τ_n is updated by some self-adaptive rule),

$$d(x_n, y_n) := x_n - y_n - \tau_n(A x_n - A y_n),$$

and

$$\eta_n := \frac{\langle x_n - y_n, d(x_n, y_n) \rangle}{\|d(x_n, y_n)\|^2}.$$

Recently, projection and contraction type methods for solving VIs have received great attention by many authors, see, e.g., [4,11,12], just to name a few.

Since the SEM and PC algorithms originally introduced in Euclidean spaces, a natural question which was studied is how to extend the method to infinite dimensional spaces and obtain strong convergence. In 2012, Censor et al. [8] proposed two subgradient extragradient variants, which converge strongly in real Hilbert spaces. One of the SEM variant has the following form. Given the current iterate x_n, the next iterate x_{n+1} is calculated via the following.

$$\begin{cases} y_n = P_C(x_n - \tau A x_n), \\ T_n = \{x \in H \mid \langle x_n - \tau A x_n - y_n, x - y_n \rangle \leq 0\}, \\ z_n = \alpha_n x_n + (1 - \alpha_n) P_{T_n}(x_n - \tau A y_n), \\ C_n = \{w \in H \mid \|z_n - w\| \leq \|x_n - w\|\}, \\ Q_n = \{w \in H \mid \langle x_n - w, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0. \end{cases} \tag{6}$$

Inspired by the results in [8], Kraikaew and Saejung [22] combined the subgradient extragradient method and the Halpern-type method and propose the so-called *Halpern subgradient extragradient method*.Given the current iterate x_n, the next iterate x_{n+1} is calculated via the following.

$$\begin{cases} y_n = P_C(x_n - \tau A x_n), \\ T_n = \{x \in H \mid \langle x_n - \tau A x_n - y_n, x - y_n \rangle \leq 0\}, \\ z_n = P_{T_n}(x_n - \tau A y_n), \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) z_n, \quad \forall n \geq 0, \end{cases} \tag{7}$$

where $\tau \in (0, \frac{1}{L})$, $\{\alpha_n\} \subset (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = +\infty$ and $x_0 \in H$. Similar to (6) of Censor et al. [8], (7) converges strongly to a specific point $p = P_{VIP(C,A)} x_0$.

Another two very recent and related (viscosity type methods) which are also used as comparison with our methods in Sect. 4 are Shehu and Iyiola [30, Algorithm 3.1] and Thong and Hieu [33, Algorithm 3].

The setting of Shehu and Iyiola [30, Algorithm 3.1] is as follows. Given $\rho, \mu \in (0, 1)$ and let $\{\alpha_n\}_{n=0}^{\infty} \subset (0, 1)$, f a contraction and choose an arbitrary starting point $x_1 \in H$. Given the current iterate x_n, calculate.

$$y_n = P_C(x_n - \lambda_n A x_n),$$

where $\lambda_n = \rho l_n$ and l_n is the smallest nonnegative integer l such that

$$\lambda_n \|x_n - y_n\| \leq \mu \|r_{\rho l_n}(x_n)\|$$

where $r_{\rho l_n}(x_n) := x_n - P_C(x_n - \rho l_n A x_n)$. Construct the set T_n as in (4) and compute

$$z_n = P_{T_n}(x_n - \lambda_n A y_n),$$

and calculate the next iterate as follows.

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) z_n. \tag{8}$$

The setting of Thong and Hieu [33, Algorithm 3] is as follows. Given $\rho \in [0, 1)$, $\mu, l \in (0, 1)$ and $\gamma > 0$. Let $\{\alpha_n\}_{n=0}^{\infty} \subset (0, 1)$, f a contraction and choose an arbitrary starting point $x_1 \in H$. Given the current iterate x_n, calculate.

$$y_n = P_C(x_n - \lambda_n A x_n),$$

where λ_n is chosen to be the largest $\lambda \in \{\gamma, \gamma l, \gamma l^2, \cdots\}$ satisfying

$$\lambda \|A x_n - A y_n\| \leq \mu \|x_n - y_n\|.$$

Calculate the next iterate as follows.

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) z_n \tag{9}$$

where $z_n = y_n - \lambda_n (A y_n - A x_n)$.

Motivated and inspired by the above results and the ongoing research in these directions, we suggest two modified projection-type methods, Man-type [27] and viscosity type [28], for solving monotone and Lipschitz continuous variational inequalities which converge strongly in real Hilbert spaces and does not require the knowledge of the Lipschitz constant of A a-priori.

The paper is organized as follows. We first recall some basic definitions and results in Sect. 2. Our algorithms are presented and analysed in Sect. 3. In Sect. 4 we present some numerical experiments which demonstrate the algorithms performances as well as provide a preliminary computational overview by comparing it with some related algorithms. Final remarks and conclusions are given in Sect. 5.

2 Preliminaries

Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. The weak convergence of $\{x_n\}_{n=1}^{\infty}$ to x is denoted by $x_n \rightharpoonup x$ as $n \to \infty$, while the strong convergence of $\{x_n\}_{n=1}^{\infty}$ to x is written as $x_n \to x$ as $n \to \infty$. For each $x, y \in H$ and $\alpha \in \mathbb{R}$, we have

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle. \tag{10}$$

$$\|\alpha x + (1 - \alpha) y\|^2 = \alpha \|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2. \tag{11}$$

$$\begin{aligned} \|\alpha x + \beta y + \gamma z\|^2 = \alpha \|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 \\ -\alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2 \end{aligned} \tag{12}$$

for all $x, y, z \in H$ and for all $\alpha, \beta, \gamma \in [0; 1]$ with $\alpha + \beta + \gamma = 1$.

Definition 2.1 Let $T : H \to H$ be an operator. Then

1. the operator T is called L-`Lipschitz continuous` with $L > 0$ if

$$\|Tx - Ty\| \leq L\|x - y\| \quad \forall x, y \in H. \tag{13}$$

if $L = 1$ then the operator T is called `nonexpansive` and if $L \in (0, 1)$, T is called `contraction`.

2. T is called `monotone` if

$$\langle Tx - Ty, x - y\rangle \geq 0 \quad \forall x, y \in H. \tag{14}$$

3. the `fixed point set` of T, denoted by $Fix(T)$ is defined as follows.

$$Fix(T) := \{x \in H \mid Tx = x\}. \tag{15}$$

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that $\|x - P_C x\| \leq \|x - y\| \; \forall y \in C$. P_C is called the *metric projection* of H onto C. It is known that P_C is nonexpansive.

Lemma 2.1 [16] *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$. Then $z = P_C x \iff \langle x - z, z - y \rangle \geq 0 \;\; \forall y \in C$.*

Lemma 2.2 [16] *Let C be a closed and convex subset in a real Hilbert space H, $x \in H$. Then*

i) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle \; \forall y \in C$;
ii) $\|P_C x - y\|^2 \leq \|x - y\|^2 - \|x - P_C x\|^2 \; \forall y \in C$;
iii) $\langle (I - P_C)x - (I - P_C)y, x - y \rangle \geq \|(I - P_C)x - (I - P_C)y\|^2 \; \forall y \in C$.

For properties of the metric projection, the interested reader could be referred to Section 3 in [16].

The following Lemmas are useful for the convergence of our proposed methods.

Lemma 2.3 [22] *Let $A : H \to H$ be a monotone and L-Lipschitz continuous mapping on C. Let $S = P_C(I - \tau A)$, where $\tau > 0$. If $\{x_n\}$ is a sequence in H satisfying $x_n \rightharpoonup q$ and $x_n - Sx_n \to 0$ then $q \in VI(C, A) = Fix(S)$.*

Lemma 2.4 [24] *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that $a_{n_j} < a_{n_j+1}$ for all $j \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

In fact, m_k is the largest number n in the set $\{1, 2, \cdots, k\}$ such that $a_n < a_{n+1}$.

The next technical lemma is very useful and used by many authors, for example Liu [23] and Xu [36]. Furthermore, a variant of Lemma 2.5 has already been used by Reich in [29].

Lemma 2.5 *Let $\{a_n\}$ be sequence of nonnegative real numbers such that:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{b_n\}$ is a sequence such that
a) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
b) $\limsup_{n\to\infty} b_n \leq 0$.
Then $\lim_{n\to\infty} a_n = 0$.

3 Main results

In this section we introduce our two modified projection-type methods for solving VIs. For the convergence analysis of the methods, we assume the following conditions.

Condition 3.1 The VI (1) associated operator $A : H \to H$ is monotone and L-Lipschitz continuous on H.

Condition 3.2 The solution set of the VI (1) is nonempty, that is $VI(C, A) \neq \emptyset$.

Condition 3.3 Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $(0, 1)$ such that $\{\beta_n\} \subset (a, b) \subset (0, 1 - \alpha_n)$ for some $a > 0, b > 0$ and

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty.$$

3.1 Mann-type projection algorithm

Algorithm 3.1

Initialization: *Given* $\lambda > 0, l \in (0, 1), \mu \in (0, 1), \gamma \in (0, 2)$. *Let* $x_0 \in H$ *be arbitrary*

Iterative Steps: *Given the current iterate* x_n, *calculate* x_{n+1} *as follows:*

Step 1. *Compute*

$$y_n = P_C(x_n - \tau_n Ax_n),$$

where τ_n *is chosen to be the largest* $\tau \in \{\lambda, \lambda l, \lambda l^2, ...\}$ *satisfying*

$$\tau \|Ax_n - Ay_n\| \le \mu\|x_n - y_n\|. \tag{16}$$

If $x_n = y_n$ *then stop and* y_n *is a solution of* $VI(C, A)$. *Otherwise*
Step 2. *Compute*

$$z_n = x_n - \gamma\eta_n d_n,$$

where

$$d_n := x_n - y_n - \tau_n(Ax_n - Ay_n),$$

and

$$\eta_n := (1 - \mu)\frac{\|x_n - y_n\|^2}{\|d_n\|^2}.$$

Step 3. *Compute*

$$x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \beta_n z_n.$$

Set $n := n + 1$ *and go to* ***Step 1***.

We start the analysis of the algorithm's convergence by proving the validity of the stopping criterion.

Lemma 3.1 *Assume that Conditions 3.1–3.2 hold. The Armijo-like search rule (16) is well defined and*

$$\min\left\{\lambda, \frac{\mu l}{L}\right\} \leq \tau_n \leq \lambda.$$

Proof See e.g., Lemma 3.1 in [33]. □

Lemma 3.2 *Let $\{d_n\}$ be a sequence generated by Algorithm 3.1. Then $d_n = 0$ if and only if $x_n = y_n$.*

Proof Indeed, we will prove that

$$(1-\mu)\|x_n - y_n\| \leq \|d_n\| \leq (1+\mu)\|x_n - y_n\|. \tag{17}$$

We have

$$\begin{aligned}\|d_n\| &= \|x_n - y_n - \tau_n(Ax_n - Ay_n)\| \\ &\geq \|x_n - y_n\| - \tau_n\|Ax_n - Ay_n\| \\ &\geq \|x_n - y_n\| - \mu\|x_n - y_n\| \\ &= (1-\mu)\|x_n - y_n\|.\end{aligned} \tag{18}$$

and it is also easy to see that

$$\|d_n\| \leq (1+\mu)\|x_n - y_n\|. \tag{19}$$

Combining (18) and (19) we obtain

$$(1-\mu)\|x_n - y_n\| \leq \|d_n\| \leq (1+\mu)\|x_n - y_n\|.$$

It implies from (17) that $d_n = 0$ if and only if $x_n = y_n$. □

Remark 3.1 From Lemma 3.2 we show that if $d_n = 0$ then stop and y_n is a solution of $VI(C, A)$.

Lemma 3.3 *Assume that Conditions 3.1 and 3.2 hold. Let $\{z_n\}$ be a sequence generated by Algorithm 3.1. Then*

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 - \frac{2-\gamma}{\gamma}\|x_n - z_n\|^2 \quad \forall p \in VI(C, A). \tag{20}$$

Proof Using (16) we have

$$\begin{aligned}\langle x_n - p, d_n\rangle &= \langle x_n - y_n, d_n\rangle + \langle y_n - p, d_n\rangle \\ &= \langle x_n - y_n, x_n - y_n - \tau_n(Ax_n - Ay_n)\rangle + \langle y_n - p, x_n - y_n \\ &\quad - \tau_n(Ax_n - Ay_n)\rangle\end{aligned}$$

$$
\begin{aligned}
&= \|x_n - y_n\|^2 - \tau_n \langle x_n - y_n, Ax_n - Ay_n \rangle \\
&\quad + \langle y_n - p, x_n - y_n - \tau_n (Ax_n - Ay_n) \rangle \\
&\geq \|x_n - y_n\|^2 - \tau_n \|x_n - y_n\| \|Ax_n - Ay_n\| \\
&\quad + \langle y_n - p, x_n - y_n - \tau_n (Ax_n - Ay_n) \rangle \\
&\geq \|x_n - y_n\|^2 - \mu \|x_n - y_n\|^2 \\
&\quad + \langle y_n - p, x_n - y_n - \tau_n (Ax_n - Ay_n) \rangle. \qquad (21)
\end{aligned}
$$

On the other hand, since $y_n = P_C(x_n - \tau_n Ax_n)$ we get

$$
\langle x_n - y_n - \tau_n Ax_n, y_n - p \rangle \geq 0, \qquad (22)
$$

By the monotonicity of A and $p \in VI(C, A)$ we have

$$
\langle Ay_n, y_n - p \rangle \geq \langle Ap, y_n - p \rangle \geq 0. \qquad (23)
$$

Adding (22) and (23) we get

$$
\langle y_n - p, x_n - y_n - \tau_n (Ax_n - Ay_n) \rangle \geq 0 \qquad (24)
$$

Combining (21) and (24) we get

$$
\langle x_n - p, d_n \rangle \geq (1 - \mu) \|x_n - y_n\|^2. \qquad (25)
$$

On the other hand, we have

$$
\begin{aligned}
\|z_n - p\|^2 &= \|x_n - \gamma \eta_n d_n - p\|^2 \\
&= \|x_n - p\|^2 - 2\gamma \eta_n \langle x_n - p, d_n \rangle + \gamma^2 \eta_n^2 \|d_n\|^2. \qquad (26)
\end{aligned}
$$

It implies from (25) and (26) that

$$
\|z_n - p\|^2 \leq \|x_n - p\|^2 - 2\gamma \eta_n (1 - \mu) \|x_n - y_n\|^2 + \gamma^2 \eta_n^2 \|d_n\|^2.
$$

Since $\eta_n = (1 - \mu) \dfrac{\|x_n - y_n\|^2}{\|d_n\|^2}$, it implies that $\|x_n - y_n\|^2 = \dfrac{\eta_n \|d_n\|^2}{1 - \mu}$. Thus,

$$
\begin{aligned}
\|z_n - p\|^2 &\leq \|x_n - p\|^2 - 2\gamma \eta_n^2 \|d_n\|^2 + \gamma^2 \eta_n^2 \|d_n\|^2 \\
&= \|x_n - p\|^2 - \gamma (2 - \gamma) \|\eta_n d_n\|^2 \\
&= \|x_n - p\|^2 - \frac{2 - \gamma}{\gamma} \|\gamma \eta_n d_n\|^2 \\
&= \|x_n - p\|^2 - \frac{2 - \gamma}{\gamma} \|x_n - z_n\|^2.
\end{aligned}
$$

□

Lemma 3.4 *Assume that Conditions 3.1–3.2 hold and let the sequence* $\{x_n\}$ *be generated by Algorithm 3.1. Then*

$$\|x_n - y_n\|^2 \leq \frac{(1+\mu)^2}{[(1-\mu)\gamma]^2}\|x_n - z_n\|^2. \tag{27}$$

Proof We have

$$\begin{aligned}\|x_n - y_n\|^2 &= \frac{1}{1-\mu}.\eta_n\|d_n\|^2 = \frac{1}{\eta_n(1-\mu)}\|\eta_n d_n\|^2 \\ &= \frac{1}{\eta_n(1-\mu)\gamma^2}\|x_n - z_n\|^2.\end{aligned} \tag{28}$$

On the other hand, from (17) we get

$$\eta_n = (1-\mu)\frac{\|x_n - y_n\|^2}{\|d_n\|^2} \geq \frac{1-\mu}{(1+\mu)^2},$$

thus,

$$\frac{1}{\eta_n} \leq \frac{(1+\mu)^2}{1-\mu} \tag{29}$$

It implies from (28) and (29) that

$$\|x_n - y_n\|^2 \leq \frac{(1+\mu)^2}{[(1-\mu)\gamma]^2}\|x_n - z_n\|^2.$$

□

Theorem 3.1 *Assume that Conditions 3.1–3.3 hold. Then any sequence* $\{x_n\}$ *generated by Algorithm 3.1 converges strongly to* $p \in VI(C, A)$*, where* $\|p\| = \min\{\|z\| : z \in VI(C, A)\}$.

Proof Thanks to Lemma 3.3 we get

$$\|z_n - p\| \leq \|x_n - p\| \;\; \forall n. \tag{30}$$

Claim 1. We prove that the sequence $\{x_n\}$ is bounded. We have

$$\begin{aligned}\|x_{n+1} - p\| &= \|(1-\alpha_n-\beta_n)x_n + \beta_n z_n - p\| \\ &= \|(1-\alpha_n-\beta_n)(x_n - p) + \beta_n(z_n - p) - \alpha_n p\| \\ &\leq \|(1-\alpha_n-\beta_n)(x_n - p) + \beta_n(z_n - p)\| + \alpha_n\|p\|.\end{aligned} \tag{31}$$

On the other hand, using (30) we get

$\|(1-\alpha_n - \beta_n)(x_n - p) + \beta_n(z_n - p)\|^2$

$$
\begin{aligned}
&= (1-\alpha_n-\beta_n)^2\|x_n-p\|^2 + 2(1-\alpha_n-\beta_n)\beta_n\langle x_n-p, z_n-p\rangle \\
&\quad + \beta_n^2\|z_n-p\|^2 \\
&\le (1-\alpha_n-\beta_n)^2\|x_n-p\|^2 + 2(1-\alpha_n-\beta_n)\beta_n\|z_n-p\|\|x_n-p\| \\
&\quad + \beta_n^2\|z_n-p\|^2 \\
&\le (1-\alpha_n-\beta_n)^2\|x_n-p\|^2 + 2(1-\alpha_n-\beta_n)\beta_n\|x_n-p\|^2 + \beta_n^2\|x_n-p\|^2 \\
&= (1-\alpha_n)^2\|x_n-p\|^2.
\end{aligned}
$$

This implies that

$$
\|(1-\alpha_n-\beta_n)(x_n-p) + \beta_n(z_n-p)\| \le (1-\alpha_n)\|x_n-p\| \;\; \forall n. \tag{32}
$$

From (31) and (32) we get

$$
\begin{aligned}
\|x_{n+1}-p\| &\le (1-\alpha_n)\|x_n-p\| + \alpha_n\|p\| \\
&\le \max\{\|x_n-p\|, \|p\|\} \\
&\le \cdots \le \max\{\|x_0-p\|, \|p\|\}.
\end{aligned}
$$

That is, the sequence $\{x_n\}$ is bounded and $\{z_n\}$ is also.

Claim 2. We show that

$$
\beta_n\frac{2-\gamma}{\gamma}\|x_n-z_n\|^2 \le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + \alpha_n\|p\|^2. \tag{33}
$$

Indeed, using (12) we have

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \|(1-\alpha_n-\beta_n)x_n + \beta_n z_n - p\|^2 \\
&= \|(1-\alpha_n-\beta_n)(x_n-p) + \beta_n(z_n-p) + \alpha_n(-p)\|^2 \\
&= (1-\alpha_n-\beta_n)\|x_n-p\|^2 + \beta_n\|z_n-p\|^2 + \alpha_n\|p\|^2 - \beta_n(1 \\
&\quad - \alpha_n-\beta_n)\|x_n-z_n\|^2 \\
&\quad - \alpha_n(1-\alpha_n-\beta_n)\|x_n\|^2 - \alpha_n\beta_n\|z_n\|^2 \\
&\le (1-\alpha_n-\beta_n)\|x_n-p\|^2 + \beta_n\|z_n-p\|^2 + \alpha_n\|p\|^2,
\end{aligned} \tag{34}
$$

which, together Lemma 3.3 we obtain

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\le (1-\alpha_n-\beta_n)\|x_n-p\|^2 + \beta_n\|x_n-p\|^2 \\
&\quad - \beta_n\frac{2-\gamma}{\gamma}\|x_n-z_n\|^2 + \alpha_n\|p\|^2 \\
&= (1-\alpha_n)\|x_n-p\|^2 - \beta_n\frac{2-\gamma}{\gamma}\|x_n-z_n\|^2 + \alpha_n\|p\|^2 \\
&\le \|x_n-p\|^2 - \beta_n\frac{2-\gamma}{\gamma}\|x_n-z_n\|^2 + \alpha_n\|p\|^2.
\end{aligned} \tag{35}
$$

Therefore, we get

$$\beta_n \frac{2-\gamma}{\gamma} \|x_n - z_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n \|p\|^2.$$

Claim 3. We show that

$$\|x_{n+1} - p\|^2 \leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n[2\beta_n \|x_n - z_n\| \|x_{n+1} - p\| \\ + 2\langle p, p - x_{n+1}\rangle]. \tag{36}$$

Indeed, setting $t_n = (1-\beta_n)x_n + \beta_n z_n$. We have

$$\begin{aligned} \|t_n - p\| &= \|(1-\beta_n)(x_n - p) + \beta_n(z_n - p)\| \\ &= (1-\beta_n)\|x_n - p\| + \beta_n \|z_n - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n \|x_n - p\| \\ &= \|x_n - p\|, \end{aligned} \tag{37}$$

and

$$\|t_n - x_n\| = \beta_n \|x_n - z_n\|. \tag{38}$$

Using (37) and (38) we get

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(1-\alpha_n-\beta_n)x_n + \beta_n z_n - p\|^2 \\ &= \|(1-\beta_n)x_n + \beta_n z_n - \alpha_n x_n - p\|^2 \\ &= \|(1-\alpha_n)(t_n - p) - \alpha_n(x_n - t_n) - \alpha_n p\|^2 \\ &\leq (1-\alpha_n)^2 \|t_n - p\|^2 - 2\langle \alpha_n(x_n - t_n) + \alpha_n p, x_{n+1} - p\rangle \\ &= (1-\alpha_n)^2 \|t_n - p\|^2 + 2\alpha_n \langle x_n - t_n, p - x_{n+1}\rangle + 2\alpha_n \langle p, p - x_{n+1}\rangle \\ &\leq (1-\alpha_n)\|t_n - p\|^2 + 2\alpha_n \|x_n - t_n\| \|x_{n+1} - p\| + 2\alpha_n \langle p, p - x_{n+1}\rangle \\ &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n[2\beta_n \|x_n - z_n\| \|x_{n+1} - p\| \\ &\quad + 2\langle p, p - x_{n+1}\rangle]. \end{aligned}$$

Claim 4. Now, we will show that the sequence $\{\|x_n - p\|^2\}$ converges to zero by considering two possible cases on the sequence $\{\|x_n - p\|^2\}$.

Case 1: There exists an $N \in \mathbb{N}$ such that $\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2$ for all $n \geq N$. This implies that $\lim_{n\to\infty} \|x_n - p\|^2$ exists. It implies from Claim 2 that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0,$$

which, together with Lemma 3.4, we get

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

We also have

$$\|x_{n+1} - x_n\| \le \alpha_n \|x_n\| + \beta_n \|x_n - z_n\| \to 0 \text{ as } n \to \infty.$$

Since $\{x_n\}$ is bounded we assume that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup q$ and

$$\limsup_{n\to\infty} \langle p, p - x_n\rangle = \lim_{j\to\infty} \langle p, p - x_{n_j}\rangle = \langle p, p - q\rangle.$$

We have $x_{n_j} \rightharpoonup q$, $\min\{\lambda, \frac{\mu l}{L}\} \le \tau_n \le \lambda$ and $\|x_n - y_n\| = \|x_n - P_C(x_n - \tau_n A x_n)\| \to 0$, by Lemma 2.3 we get $q \in VI(C, A)$.

Since $q \in VI(C, A)$ and $\|p\| = \min\{\|z\| : z \in VI(C, A)\}$, that is $p = P_{VI(C,A)}0$ we obtain

$$\limsup_{n\to\infty} \langle p, p - x_n\rangle = \langle p, p - q\rangle \le 0.$$

By $\|x_{n+1} - x_n\| \to 0$ we get

$$\limsup_{n\to\infty} \langle p, p - x_{n+1}\rangle \le 0.$$

Therefore by Claim 3 and Lemma 2.5 we get $\lim_{n\to\infty} \|x_n - p\|^2 = 0$, that is $x_n \to p$.

Case 2: There exists a subsequence $\{\|x_{n_j} - p\|^2\}$ of $\{\|x_n - p\|^2\}$ such that $\|x_{n_j} - p\|^2 < \|x_{n_j+1} - p\|^2$ for all $j \in \mathbb{N}$. In this case, it follows from Lemma 2.4 that there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following inequalities hold for all $k \in \mathbb{N}$:

$$\|x_{m_k} - p\|^2 \le \|x_{m_k+1} - p\|^2 \text{ and } \|x_k - p\|^2 \le \|x_{m_k+1} - p\|^2. \tag{39}$$

Since $\{\beta_n\} \subset (a, b)$ and Claim 2, we have

$$\begin{aligned} a\frac{2-\gamma}{\gamma}\|x_{m_k} - z_{m_k}\|^2 &\le \beta_{m_k}\frac{2-\gamma}{\gamma}\|x_{m_k} - z_{m_k}\|^2 \\ &\le \|x_{m_k} - p\|^2 - \|x_{m_k+1} - p\|^2 + \alpha_{m_k}\|p\|^2 \\ &\le \alpha_{m_k}\|p\|^2. \end{aligned}$$

Therefore, we get

$$\lim_{k\to\infty} \|x_{m_k} - z_{m_k}\| = 0. \tag{40}$$

As proved in the first case, we obtain

$$\|x_{m_k+1} - x_{m_k}\| \to 0$$

and

$$\limsup_{k\to\infty}\langle p, p - x_{m_k+1}\rangle \le 0.$$

Since Claim 3 we have

$$\begin{aligned}\|x_{m_k+1} - p\|^2 \le\ & (1-\alpha_{m_k})\|x_{m_k} - p\|^2 \\ & + \alpha_{m_k}[2\beta_{m_k}\|x_{m_k} - z_{m_k}\|\|x_{m_k+1} - p\| + 2\langle p, p - x_{m_k+1}\rangle] \\ \le\ & (1-\alpha_{m_k})\|x_{m_k+1} - p\|^2 \\ & + \alpha_{m_k}[2\beta_{m_k}\|x_{m_k} - z_{m_k}\|\|x_{m_k+1} - p\| + 2\langle p, p - x_{m_k+1}\rangle].\end{aligned}$$

This implies that

$$\|x_k - p\|^2 \le \|x_{m_k+1} - p\|^2 \le 2\beta_{m_k}\|x_{m_k} - z_{m_k}\|\|x_{m_k+1} - p\| + 2\langle p, p - x_{m_k+1}\rangle.$$

Therefore, we obtain $\limsup_{k\to\infty}\|x_k - p\| \le 0$, that is $x_k \to p$. The proof is completed. □

3.2 Viscosity projection type algorithm

In this section, we propose our viscosity projection type algorithm for solving variational inequalities, with the usage of a ρ-contraction $f : H \to H$.

Algorithm 3.2

Initialization: *Given* $\lambda > 0, l \in (0,1), \mu \in (0,1), \gamma \in (0,2)$. *Let* $x_0 \in H$ *be arbitrary*

Iterative Steps: *Given the current iterate* x_n, *calculate the next iterate* x_{n+1} *as follows:*

Step 1. *Compute*

$$y_n = P_C(x_n - \tau_n A x_n),$$

where τ_n *is chosen to be the largest* $\tau \in \{\lambda, \lambda l, \lambda l^2, ...\}$ *satisfying*

$$\tau\|Ax_n - Ay_n\| \le \mu\|x_n - y_n\|. \tag{41}$$

If $x_n = y_n$ *then stop and* y_n *is a solution of* $VI(C, A)$. *Otherwise*
Step 2. *Compute*

$$z_n = x_n - \gamma\eta_n d_n,$$

where

$$\eta_n := (1-\mu)\frac{\|x_n - y_n\|^2}{\|d_n\|^2},$$

and

$$d_n := x_n - y_n - \tau_n(Ax_n - Ay_n).$$

Step 3. *Compute*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)z_n.$$

Set $n := n + 1$ *and go to* ***Step 1***.

Theorem 3.2 *Assume that Conditions 3.1–3.2 hold and given a* ρ*-contraction* $f : H \to H$. *Assume that* $\{\alpha_n\}$ *is a real sequence in* $(0, 1)$ *such that*

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then any sequence $\{x_n\}$ *generated by Algorithm 3.2 converges strongly to an element* $p \in VI(C, A)$, *where* $p = P_{VI(C,A)} \circ f(p)$.

Proof **Claim 1.** We prove that the $\{x_n\}$ is bounded. Indeed, According to Lemma 3.3 we have

$$\|z_n - p\| \le \|x_n - p\|. \tag{42}$$

Using (42) we obtain

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n f(x_n) + (1 - \alpha_n)z_n - p\| \\
&= \|\alpha_n(f(x_n) - p) + (1 - \alpha_n)(z_n - p)\| \\
&\le \alpha_n\|f(x_n) - p\| + (1 - \alpha_n)\|z_n - p\| \\
&\le \alpha_n\|f(x_n) - f(p)\| + \alpha_n\|f(p) - p\| + (1 - \alpha_n)\|z_n - p\| \\
&\le \alpha_n\rho\|x_n - p\| + \alpha_n\|f(p) - p\| + (1 - \alpha_n)\|x_n - p\| \\
&\le [1 - \alpha_n(1 - \rho)]\|x_n - p\| + \alpha_n(1 - \rho)\frac{\|f(p) - p\|}{1 - \rho} \\
&\le \max\{\|x_n - q\|, \frac{\|f(p) - p\|}{1 - \rho}\} \\
&\le \cdots \le \max\left\{\|x_0 - p\|, \frac{\|f(p) - p\|}{1 - \rho}\right\}.
\end{aligned}$$

This implies that the sequence $\{x_n\}$ is bounded. Consequently, $\{f(x_n)\}$, $\{y_n\}$ and $\{z_n\}$ are bounded.

Claim 2. We show that

$$(1 - \alpha_n)\frac{2 - \gamma}{\gamma}\|x_n - z_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n\|f(x_n) - p\|^2.$$

Indeed, using (11) and Lemma 3.3 we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n(f(x_n)-p)+(1-\alpha_n)(z_n-p)\|^2\\
&= \alpha_n\|f(x_n)-p\|^2+(1-\alpha_n)\|z_n-p\|^2-\alpha_n(1-\alpha_n)\|f(x_n)-z_n\|^2\\
&\le \alpha_n\|f(x_n)-p\|^2+(1-\alpha_n)\|z_n-p\|^2\\
&\le \alpha_n\|f(x_n)-p\|^2+(1-\alpha_n)\|x_n-p\|^2\\
&\quad -(1-\alpha_n)\beta_n\frac{2-\gamma}{\gamma}\|x_n-z_n\|^2\\
&\le \alpha_n\|f(x_n)-p\|^2+\|x_n-p\|^2-(1-\alpha_n)\frac{2-\gamma}{\gamma}\|x_n-z_n\|^2.
\end{aligned}$$

This implies that

$$(1-\alpha_n)\frac{2-\gamma}{\gamma}\|x_n-z_n\|^2 \le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\|f(x_n)-p\|^2.$$

Claim 3. We show that

$$\|x_{n+1}-p\|^2 \le (1-(1-\rho)\alpha_n)\|x_n-p\|^2+(1-\rho)\alpha_n.\frac{2}{1-\rho}\langle f(p)-p, x_{n+1}-p\rangle.$$

Indeed, using (10) and (42) we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n f(x_n)+(1-\alpha_n)z_n-p\|^2\\
&= \|\alpha_n(f(x_n)-f(p))+(1-\alpha_n)(z_n-p)+\alpha_n(f(p)-p)\|^2\\
&\le \|\alpha_n(f(x_n)-f(p))+(1-\alpha_n)(z_n-p)\|^2\\
&\quad +2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle\\
&\le \alpha_n\|f(x_n)-f(p)\|^2+(1-\alpha_n)\|z_n-p\|^2\\
&\quad +2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle\\
&\le \alpha_n\rho\|x_n-p\|^2+(1-\alpha_n)\|x_n-p\|^2+2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle\\
&= (1-(1-\rho)\alpha_n)\|x_n-p\|^2+(1-\rho)\alpha_n\\
&\quad .\frac{2}{1-\rho}\langle f(p)-p, x_{n+1}-p\rangle.
\end{aligned} \tag{43}$$

Claim 4. Now, we will show that the sequence $\{\|x_n-p\|^2\}$ converges to zero by considering two possible cases on the sequence $\{\|x_n-p\|^2\}$.

Case 1: There exists an $N\in\mathbb{N}$ such that $\|x_{n+1}-p\|^2 \le \|x_n-p\|^2$ for all $n\ge N$. This implies that $\lim_{n\to\infty}\|x_n-p\|^2$ exists.

Since the Claim 2 and $\lim_{n\to\infty}\alpha_n=0$ we get

$$\lim_{n\to\infty}\|x_n-z_n\|=0, \tag{44}$$

and by Lemma 3.4

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{45}$$

We also have

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|\alpha_n f(x_n) + (1-\alpha_n)z_n - x_n\| \\ &\le \alpha_n \|f(x_n) - x_n\| + (1-\alpha_n)\|z_n - x_n\| \to 0. \end{aligned}\tag{46}$$

Since the sequence $\{x_n\}$ is bounded, it implies that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that weak convergence to some $z \in H$ such that

$$\begin{aligned}\limsup_{n\to\infty}\langle f(p) - p, x_n - p\rangle &= \lim_{k\to\infty}\langle f(p) - p, x_{n_k} - p\rangle \\ &= \langle f(p) - p, z - p\rangle. \end{aligned}\tag{47}$$

From (45) and Lemma 2.3 we have $z \in VI(C, A)$.

By the definition of p and $z \in VI(C, A)$ we have

$$\limsup_{n\to\infty}\langle f(p) - p, x_n - p\rangle = \langle f(p) - p, z - p\rangle \le 0. \tag{48}$$

which, together with (46) and (47) we get

$$\begin{aligned}\limsup_{n\to\infty}\langle f(p) - p, x_{n+1} - p\rangle &\le \limsup_{n\to\infty}\langle f(p) - p, x_{n+1} - x_n\rangle \\ &\quad + \limsup_{n\to\infty}\langle f(p) - p, x_n - p\rangle \\ &= \langle f(p) - p, z - p\rangle \le 0. \end{aligned}\tag{49}$$

Using Lemma 2.5, (49) and Claim 3 we obtain $x_n \to p$.

Case 2. There exists a subsequence $\{\|x_{n_j} - p\|^2\}$ of $\{\|x_n - p\|^2\}$ such that $\|x_{n_j} - p\|^2 < \|x_{n_j+1} - p\|^2$ for all $j \in \mathbb{N}$. In this case, it follows from Lemma 2.4 that there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following inequalities hold for all $k \in \mathbb{N}$:

$$\|x_{m_k} - p\|^2 \le \|x_{m_k+1} - p\|^2, \tag{50}$$

and

$$\|x_k - p\|^2 \le \|x_{m_k} - p\|^2. \tag{51}$$

According to Claim 2 we get

$$(1-\alpha_{m_k})\frac{2-\gamma}{\gamma}\|x_{m_k} - z_{m_k}\|^2 \le \|x_{m_k} - p\|^2 - \|x_{m_k+1} - p\|^2$$

$$+ \alpha_{m_k} \| f(x_{m_k}) - p\|^2$$
$$\leq \alpha_{m_k} \| f(x_{m_k}) - p\|^2.$$

We obtain

$$\lim_{k\to\infty} \|x_{m_k} - z_{m_k}\| = 0, \tag{52}$$

and by Lemma 3.4 we get

$$\lim_{k\to\infty} \|x_{m_k} - y_{m_k}\| = 0. \tag{53}$$

Using the same arguments as in the proof of Case 1, we obtain

$$\limsup_{k\to\infty} \langle f(p) - p, x_{m_k+1} - p\rangle \leq 0. \tag{54}$$

Thanks to Claim 3, we have

$$\begin{aligned}\|x_{m_k+1} - p\|^2 &\leq (1-(1-\rho)\alpha_{m_k})\|x_{m_k} - p\|^2 \\ &\quad + (1-\rho)\alpha_{m_k} \cdot \frac{2}{1-\rho} \langle f(p) - p, x_{m_k+1} - p\rangle,\end{aligned} \tag{55}$$

together with (50), we deduce that

$$\begin{aligned}\|x_{m_k+1} - p\|^2 &\leq (1-(1-\rho)\alpha_{m_k})\|x_{m_k+1} - p\|^2 \\ &\quad +(1-\rho)\alpha_{m_k} \cdot \frac{2}{1-\rho} \langle f(p) -, x_{m_k+1} - p\rangle.\end{aligned}$$

This follows that

$$\|x_{m_k+1} - p\|^2 \leq \frac{2}{1-\rho} \langle f(p) - p, x_{m_k+1} - p\rangle. \tag{56}$$

Combining (51), (54) and (56) we get

$$\limsup_{k\to\infty} \|x_k - p\| \leq 0, \tag{57}$$

that is $x_k \to p$. The proof is completed. □

4 Numerical illustrations

In this section we present two numerical experiments which demonstrate the performances of our Mann-type and viscosity-type projection algorithm (Algorithms 3.1 and 3.2) in finite and infinite dimensional spaces. In both experiments the parameters are chosen as $\lambda = 7.55,\ l = 0.5,\ \mu = 0.85$ and $\gamma = 1.99$, $\alpha_k = 1/k$, $\beta_k = (k-1)/2k$.

Table 1 Algorithm 3.1 with different Cases

$x_1(t)$	No. of Iterations	CPU time
$\frac{1}{600}\left[sin(-3t)+cos(-10t)\right]$	13	0.0625
$\frac{1}{525}\left[t^2-e^{-t}\right]$	13	0.078125

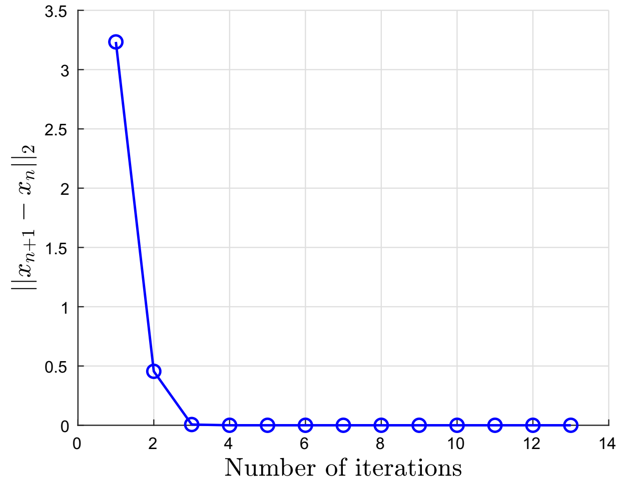

Fig. 1 $x_1(t) = \frac{1}{600}\left[sin(-3t)+cos(-10t)\right]$

Example 1 Suppose that $H = L^2([0, 1])$ with norm $\|x\| := \left(\int_0^1 |x(t)|^2 dt\right)^{\frac{1}{2}}$ and inner product $\langle x, y\rangle := \int_0^1 x(t)y(t)dt, \quad \forall x, y \in H$. Let $C := \{x \in H \mid \|x\| \leq 1\}$ be the unit ball. Define operator $A : C \to H$ by $(Ax)(t) = \max(0, x(t))$. Then it can be easily verified that A is 2-Lipschitz continuous and monotone on C (see [19]). With these given C and A, the set of solution to the variational inequality is $\{0\} \neq \emptyset$. It is known that, see for example [5]

$$P_C(x) = \begin{cases} \frac{x}{\|x\|_{L^2}}, & \text{if } \|x\|_{L^2} > 1, \\ x, & \text{if } \|x\|_{L^2} \leq 1, \end{cases}$$

We implement our algorithm with different starting point $x_1(t)$. We choose the stopping criterion $||x_{n+1} - x_n|| < \varepsilon$ with $\varepsilon = 10^{-30}$. The results are presented in Table 1 and Figs. 1 and 2.

Example 2 In this example we consider a nonlinear variational inequality with $A : \mathbb{R}^m \to \mathbb{R}^m$ which is defined as $Ax = Mx + Fx + q$, with M an $m \times m$ symmetric semi-definite matrix, q is a vector in $\mathbb{R}^m$ and Fx is the proximal mapping of the function $g(x) = \frac{1}{4}||x||^4$, i.e.,

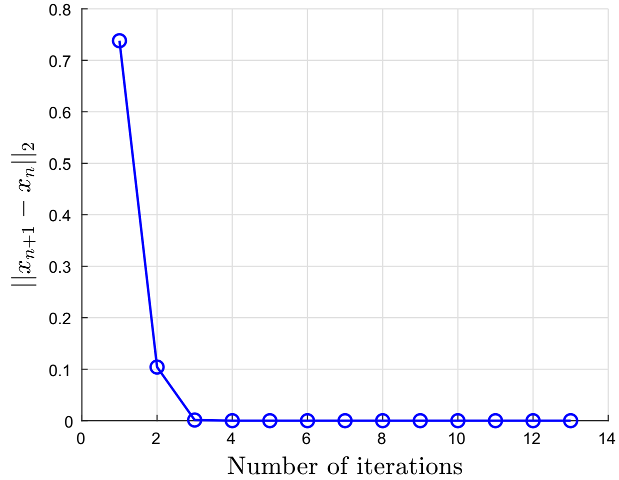

Fig. 2 $x_1(t) = \frac{1}{525}\left[t^2 - e^{-t}\right]$

$$Fx = \arg\min\left\{\frac{||y||^4}{4} + \frac{1}{2}||y - x||^2 \mid y \in \mathbb{R}^m\right\}.$$

The feasible set is a polyhedral convex set, given by $C = \{x \in \mathbb{R}^m \mid Qx \leq b\}$, where $Q \in \mathbb{R}^{r\times m}$ and $b \in \mathbb{R}^l$. In this case, A is monotone and Lipschitz continuous with $L = ||M|| + 1$. All the entries of Q, M, q are generated randomly in $(-2, 2)$ and b in $(0, 1)$, $m = 100$, $r = 10$ and we choose the stopping criterion $||x_n - y_n|| < \varepsilon$ with $\varepsilon = 10^{-5}$. The starting point is $x_0 = (1, 1, \ldots, 1) \in \mathbb{R}^m$. The projections onto C and the evaluation of F are computed by using the MATLAB solvers `fmincon`. For comparison we choose two very recent viscosity type methods, Shehu and Iyiola [30, Algorithm 3.1] and Thong and Hieu [33, Algorithm 3]. In all algorithms we take the contractions $f(x) = x/2$. The numerical results are showed in Fig. 3 with respect to the logarithmic scale. In Fig. 4 we illustrate the performances of Algorithm 3.2 for different choices of the contraction $f(x) = 0.9x, 0.75x, 0.5x, 0.25x$.

5 Conclusions

In this paper we proposed two projection-type methods, Mann and viscosity schemes methods [27,28] for solving variational inequalities in real Hilbert spaces. Both algorithms converge strongly under monotonicity and Lipschitz continuity of the VI associated mapping A. The algorithms require the calculation of only one projection onto the VI's feasible set C per each iteration and by using the projection and contraction technique there is no need to know the Lipschitz constant of A in advance. These two properties emphasize the applicability and advantages over several existing results in the literature. Numerical experiments in finite and infinite dimensional spaces compare and illustrate the performance of the our new schemes.

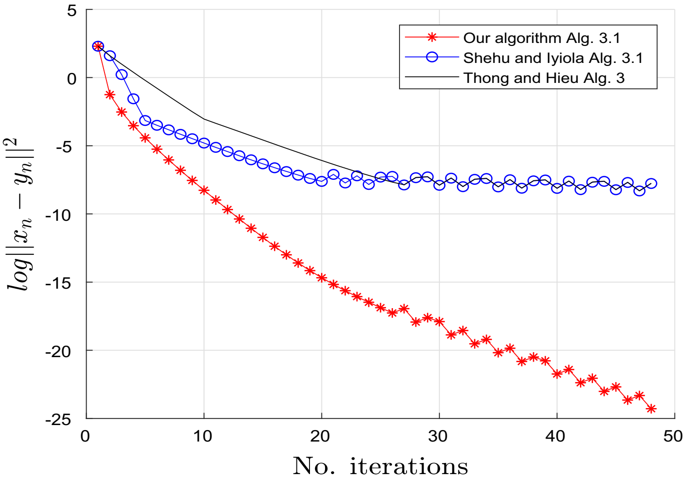

Fig. 3 Comparison between Algorithm 3.2 and [30, Algorithm 3.1] and [33, Algorithm 3]

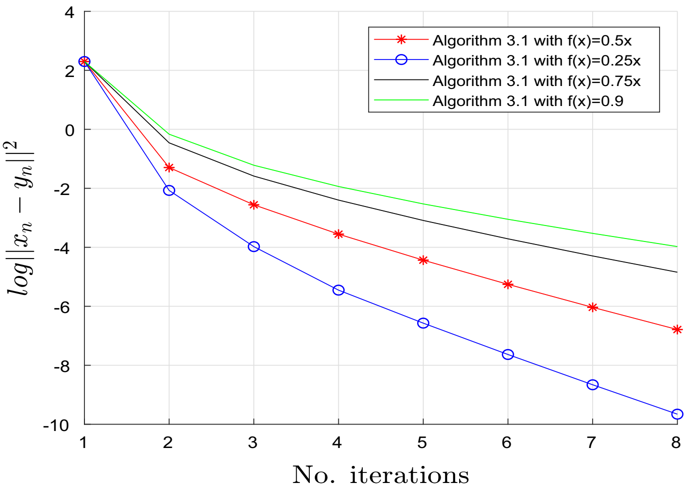

Fig. 4 The performances of Algorithm 3.2 for different choices of the contraction $f(x) = 0.9x, 0.75x, 0.5x, 0.25x$

Compliance with ethical standards

Conflict of interest The authors declare no conflict of interest.

References

1. Antipin, A.S.: On a method for convex programs using a symmetrical modification of the Lagrange function. Ekonomika i Mat. Metody. **12**, 1164–1173 (1976)
2. Aubin, J.P., Ekeland, I.: Applied Nonlinear Analysis. Wiley, New York (1984)

3. Baiocchi, C., Capelo, A.: Variational and Quasivariational Inequalities, Applications to Free Boundary Problems. Wiley, New York (1984)
4. Cai, X., Gu, G., He, B.: On the $O(1/t)$ convergence rate of the projection and contraction methods for variational inequalities with Lipschitz continuous monotone operators. Comput. Optim. Appl. **57**, 339–363 (2014)
5. Cegielski, A.: Iterative Methods for Fixed Point Problems in Hilbert Spaces. Lecture Notes in Mathematics, vol. 2057. Springer, Berlin (2012)
6. Ceng, L.C., Hadjisavvas, N., Wong, N.C.: Strong convergence theorem by a hybrid extragradient-like approximation method for variational inequalities and fixed point problems. J. Glob. Optim. **46**, 635–646 (2010)
7. Censor, Y., Gibali, A., Reich, S.: The subgradient extragradientmethod for solving variational inequalities in Hilbert space. J. Optim. Theory Appl. **148**, 318–335 (2011)
8. Censor, Y., Gibali, A., Reich, S.: Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space. Optim. Meth. Softw. **26**, 827–845 (2011)
9. Censor, Y., Gibali, A., Reich, S.: Extensions of Korpelevich's extragradient method for the variational inequality problem in Euclidean space. Optimization **61**, 1119–1132 (2011)
10. Censor, Y., Gibali, A., Reich, S.: Algorithms for the split variational inequality problem. Numer. Algorithms **56**, 301–323 (2012)
11. Dong, Q.L., Gibali, A., Jiang, D., Ke, S.H.: Convergence of projection and contraction algorithms with outer perturbations and their applications to sparse signals recovery. J. Fixed Point Theory Appl. **20**, 16 (2018). https://doi.org/10.1007/s11784-018-0501-1
12. Dong, L.Q., Cho, J.Y., Zhong, L.L., Rassias, MTh: Inertial projection and contraction algorithms for variational inequalities. J. Glob. Optim. **70**, 687–704 (2018)
13. Facchinei, F., Pang, J.S.: Finite-Dimensional Variational Inequalities and Complementarity Problems. Springer Series in Operations Research, vols. I and II. Springer, New York (2003)
14. Fichera, G.: Sul problema elastostatico di Signorini con ambigue condizioni al contorno. Atti Accad. Naz. Lincei, VIII Ser. Rend. Cl. Sci. Fis. Mat. Nat. **34**, 138–142 (1963)
15. Fichera, G.: Problemi elastostatici con vincoli unilaterali: il problema di Signorini con ambigue condizioni al contorno. Atti Accad. Naz. Lincei, Mem. Cl. Sci. Fis. Mat. Nat. Sez. I, VIII. Ser. **7**, 91–140 (1964)
16. Goebel, K., Reich, S.: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings. Marcel Dekker, New York (1984)
17. He, B.S.: A class of projection and contraction methods for monotone variational inequalities. Appl. Math. Optim. **35**, 69–76 (1997)
18. He, B.S., Liao, L.Z.: Improvements of some projection methods for monotone nonlinear variational inequalities. J. Optim. Theory Appl. **112**, 111–128 (2002)
19. Hieu, D.V., Anh, P.K., Muu, L.D.: Modified hybrid projection methods for finding common solutions to variational inequality problems. Comput. Optim. Appl. **66**, 75–96 (2017)
20. Konnov, I.V.: Combined Relaxation Methods for Variational Inequalities. Springer, Berlin (2001)
21. Korpelevich, G.M.: The extragradient method for finding saddle points and other problems. Ekonomika i Mat. Metody. **12**, 747–756 (1976)
22. Kraikaew, R., Saejung, S.: Strong convergence of the Halpern subgradient extragradient method for solving variational inequalities in Hilbert spaces. J. Optim. Theory Appl. **163**, 399–412 (2014)
23. Liu, L.S.: Ishikawa and Mann iteration process with errors for nonlinear strongly accretive mappings in Banach space. J. Math. Anal. Appl. **194**, 114–125 (1995)
24. Maingé, P.E.: A hybrid extragradient-viscosity method for monotone operators and fixed point problems. SIAM J. Control Optim. **47**, 1499–1515 (2008)
25. Malitsky, Y.V.: Projected reflected gradient methods for monotone variational inequalities. SIAM J. Optim. **25**, 502–520 (2015)
26. Malitsky, Y.V., Semenov, V.V.: A hybrid method without extrapolation step for solving variational inequality problems. J. Glob. Optim. **61**, 193–202 (2015)
27. Mann, W.R.: Mean value methods in iteration. Proc. Am. Math. Soc. **4**, 506–510 (1953)
28. Moudafi, A.: Viscosity approximating methods for fixed point problems. J. Math. Anal. Appl. **241**, 46–55 (2000)
29. Reich, S.: Constructive Techniques for Accretive and Monotone Operators. Applied Nonlinear Analysis, pp. 335–345. Academic Press, New York (1979)

30. Shehu, Y., Iyiola, O.S.: Strong convergence result for monotone variational inequalities. Numer. Algorithms **76**, 259–282 (2017)
31. Solodov, M.V., Svaiter, B.F.: A new projection method for variational inequality problems. SIAM J. Control Optim. **37**, 765–776 (1999)
32. Sun, D.F.: A class of iterative methods for solving nonlinear projection equations. J. Optim. Theory Appl. **91**, 123–140 (1996)
33. Thong, D.V., Hieu, D.V.: Weak and strong convergence theorems for variational inequality problems. Numer. Algorithms **78**, 1045–1060 (2018)
34. Thong, D.V., Hieu, D.V.: Modified subgradient extragradient method for variational inequality problems. Numer. Algorithms **79**, 597–610 (2018)
35. Thong, D.V., Hieu, D.V.: Inertial extragradient algorithms for strongly pseudomonotone variational inequalities. J. Comput. Appl. Math. **341**, 80–98 (2018)
36. Xu, H.K.: Iterative algorithms for nonlinear operators. J. Lond. Math. Soc. **66**, 240–256 (2002)