Analysis and Mathematical Physics (2019) 9:821–827
https://doi.org/10.1007/s13324-019-00320-y

Non-tangential limits for analytic Lipschitz functions

Stephen Deterding[1,2]

Received: 1 October 2018 / Accepted: 23 November 2018 / Published online: 23 May 2019

Abstract
Let U be a bounded open subset of the complex plane. Let $0 < \alpha < 1$ and let $A_\alpha(U)$ denote the space of functions that satisfy a Lipschitz condition with exponent α on the complex plane, are analytic on U and are such that for each $\epsilon > 0$, there exists $\delta > 0$ such that for all $z, w \in U$, $|f(z) - f(w)| \leq \epsilon|z - w|^\alpha$ whenever $|z - w| < \delta$. We show that if a boundary point x_0 for U admits a bounded point derivation for $A_\alpha(U)$ and U has an interior cone at x_0 then one can evaluate the bounded point derivation by taking a limit of a difference quotient over a non-tangential ray to x_0. Notably our proofs are constructive in the sense that they make explicit use of the Cauchy integral formula.

Keywords Non-tangential limits · Analytic · Lipschitz condition · Point derivation

Mathematics Subject Classification 30E25 · 30H99 · 46J10

1 Background and statement of results

In this paper, we consider the behavior of Lipschitz functions which are analytic on a bounded open subset of the complex plane and how much analyticity extends to the boundary of the domain. Let U be an open subset in the complex plane and let $0 < \alpha < 1$. A function $f : U \to \mathbb{C}$ satisfies a Lipschitz condition with exponent α on U if there exists $k > 0$ such that for all $z, w \in U$

$$|f(z) - f(w)| \leq k|z - w|^\alpha \tag{1}$$

Let $\text{Lip}\alpha(U)$ denote the space of functions that satisfy a Lipschitz condition with exponent α on U. $\text{Lip}\alpha(U)$ is a Banach space with norm given by $||f||_{Lip\alpha(U)} =$

✉ Stephen Deterding
stephen.deterding@westliberty.edu

1 University of Kentucky, Lexington, KY 40506, USA

2 Present Address: West Liberty University, West Liberty, WV 26074, USA

$\sup_U |f| + k(f)$, where $k(f)$ is the smallest constant that satisfies (1). If we let $||f||'_{Lip\alpha(U)} = k(f)$ then $||f||'_{Lip\alpha(U)}$ is a seminorm on Lip$\alpha(U)$.

An important subspace of Lip$\alpha(U)$ is the little Lipschitz class, lip$\alpha(U)$, which consists of those functions in Lip$\alpha(U)$ that also satisfy the additional property that for each $\epsilon > 0$, there exists $\delta > 0$ such that for all z, w in U, $|f(z) - f(w)| \leq \epsilon|z - w|^\alpha$ whenever $|z - w| < \delta$.

The importance of lip$\alpha(U)$ is illustrated by the following result of De Leeuw [1]. Let Δ be a closed disk. Then the restriction spaces Lip$\alpha(\Delta) = \{f|\Delta : f \in$ Lip$\alpha(\mathbb{C})\}$ and lip$\alpha(\Delta) = \{f|\Delta : f \in$ lip$\alpha(\mathbb{C})\}$ are Banach spaces and lip$\alpha^{**}(\Delta)$ is isometrically isomorphic to Lip$\alpha(\Delta)$. Thus the weak-star topology can be applied to Lip$\alpha(\Delta)$ as the dual of lip$\alpha^*(\Delta)$.

Let U be a bounded open subset of the complex plane. We will restrict our study to those functions in lip$\alpha(\mathbb{C})$ which are analytic on U. Let $A_\alpha(U) = \{f \in$ lipα : $\overline{\partial} f = 0$ on $U\}$, where $\overline{\partial} f = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$. For an arbitrary set $E \subset \mathbb{C}$, let $A_\alpha(E) = \bigcup\{A_\alpha(U) : U$ open , $E \subset U\}$.

While the functions in $A_\alpha(U)$ are differentiable on the interior of U, they need not be differentiable on the boundary of U. In this paper, we consider the question of how close the functions in $A_\alpha(U)$ come to being differentiable at boundary points of U. To answer this question we will make use of the concept of a bounded point derivation. For $x_0 \in \mathbb{C}$, it is known that $A_\alpha(U \cup \{x_0\})$ is dense in $A_\alpha(U)$. [4, Lemma 1.1] Thus we say that $A_\alpha(U)$ admits a bounded point derivation at x_0 if the map $f \to f'(x_0)$ extends from $A_\alpha(U \cup \{x_0\})$ to a bounded linear functional on $A_\alpha(U)$. Equivalently, $A_\alpha(U)$ admits a bounded point derivation at x_0 if and only if there exists a constant $C > 0$ such that

$$|f'(x_0)| \leq C||f||_{Lip\alpha(\mathbb{C})}, \tag{2}$$

for all f in $A_\alpha(U \cup \{x_0\})$.

The existence of a bounded point derivation at x_0 shows that the functions in $A_\alpha(U)$ possess some semblance of analytic structure at x_0. If, in addition, U has an interior cone at x_0, a more explicit description of this analytic structure can be obtained. We say that U has an interior cone at x_0 if there is a segment J ending at x_0 and a constant $k > 0$ such that dist$(x, \partial U) \geq k|x - x_0|$ for all x in J. The segment J is called a non-tangential ray to x_0. It is a result of O'Farrell [5] that if U has an interior cone at a boundary point x_0, then a bounded point derivation on $A_\alpha(U)$ at x_0 can be evaluated by taking the limit of the difference quotient over a non-tangential ray to x_0. To be precise, O'Farrell has proven the following theorem.

Theorem 1 *Let $0 < \alpha < 1$, and let U be an open set with x_0 in ∂U. Suppose that U has an interior cone at x_0 and that J is a non-tangential ray to x_0. If $A_\alpha(U)$ admits a bounded point derivation D at x_0, then for every f in $A_\alpha(U)$,*

$$Df = \lim_{x \to x_0, x \in J} \frac{f(x) - f(x_0)}{x - x_0}.$$

Thus the difference quotient for boundary points that admit bounded point derivations for $A_\alpha(U)$ converges when taken over a non-tangential ray to the point. This illustrates the additional analytic structure of functions in $A_\alpha(U)$ at these points.

O'Farrell comments that the methods used in his proof of Theorem 1 are nonconstructive, involving abstract measures and duality arguments from functional analysis as opposed to using the Cauchy integral formula directly, and suggests that it should be possible to give a proof using constructive techniques. In this paper we present a constructive proof of Theorem 1, which confirms O'Farrell's conjecture. In Sect. 2 we review some key properties of $A_\alpha(U)$ and in Sect. 3 we prove Theorem 1 using constructive techniques.

2 Preliminary results

We begin by reviewing the Hausdorff content of a set, which is defined using measure functions. A measure function is a monotone nondecreasing function $h : [0, \infty) \to [0, \infty)$. For example, r^β is a measure function for $0 \leq \beta < \infty$. If h is a measure function then the Hausdorff content M_h associated to h is defined by

$$M_h(E) = \inf \sum h(\text{diam } B),$$

where the infimum is taken over all countable coverings of E by balls and the sum is taken over all the balls in the covering. If $h(r) = r^\beta$ then we denote M_h by M^β. The lower $1 + \alpha$ dimensional Hausdorff content $M_*^{1+\alpha}(E)$ is defined by

$$M_*^{1+\alpha}(E) = \sup M_h(E),$$

where the supremum is taken over all measurable functions h such that $h(r) \leq r^{1+\alpha}$ and $r^{-1-\alpha}h(r)$ converges to 0 as r tends to 0. The lower $1+\alpha$ dimensional Hausdorff content is a monotone set function; i.e. if $E \subseteq F$ then $M_*^{1+\alpha}(E) \leq M_*^{1+\alpha}(F)$.

In [4], Lord and O'Farrell gave necessary and sufficient conditions for the existence of bounded point derivations on $A_\alpha(U)$ in terms of Hausdorff contents. There are similar conditions for bounded point derivations defined on other function spaces. ([2,3]).

Theorem 2 *Let U be an open subset of the complex plane with x_0 on the boundary of U. Let $0 < \alpha < 1$. Then $A_\alpha(U)$ has a bounded point derivation at x_0 if and only if*

$$\sum_{n=1}^{\infty} 4^n M_*^{1+\alpha}(A_n(x_0)\backslash U) < \infty.$$

Another key lemma is the following Cauchy theorem for Lipschitz functions which also appears in the paper of Lord and O'Farrell [4, pg.110].

Lemma 1 *Let Γ be a piecewise analytic curve bounding a region $\Omega \in \mathbb{C}$, and suppose that Γ is free of outward pointing cusps. Let $0 < \alpha < 1$ and suppose that $f \in lip\alpha(\mathbb{C})$. Then there exists a constant $\kappa > 0$ such that*

$$\left|\int f(z)dz\right| \leq \kappa \cdot M_*^{1+\alpha}(\Omega \cap S) \cdot ||f||'_{Lip\alpha(\Omega)}.$$

The constant κ only depends on α and the equivalence class of Γ under the action of the conformal group of $\mathbb{C}$. In particular this means that κ is the same for any curve obtained from Γ by rotation or scaling.

3 The proof of the main theorem

To prove Theorem 1, we first note that by translation invariance we may suppose that $x_0 = 0$. Moreover by replacing f by $f - f(0)$ if needed, we may suppose that $f(0) = 0$. In addition, we may suppose that U is contained in the unit disk. Let J be a non-tangential ray to x_0 and for each x in J, define a linear functional L_x by $L_x(f) = \dfrac{f(x)}{x} - Df$. Then to prove Theorem 1 it suffices to show that L_x tends to the 0 functional as $x \to 0$ through J. We make the following claim.

Lemma 2 *The collection $\{L_x : x \in J\}$ is a family of bounded linear functionals on $A_\alpha(U)$; that is there exists a constant $C > 0$ that does not depend on x or f such that $|L_x(f)| \leq C||f||_{Lip\alpha(\mathbb{C})}$ for all f in $A_\alpha(U)$ and all $x \in J$.*

Proof We will first prove Lemma 2 for the case when f belongs to $A_\alpha(U \cup \{0\})$ and then extend to the general case. It follows from (2) that it is enough to show that $\left|\dfrac{f(x)}{x}\right| \leq C||f||_{Lip\alpha(\mathbb{C})}$ where the constant C does not depend on f or x. If f belongs to $A(U \cup \{0\})$, then there is a neighborhood Ω of 0 such that f is analytic on Ω. We can further suppose that $U \subseteq \Omega$. Let B_n denote the ball centered at 0 with radius 2^{-n}. Then there exists an integer $N > 0$ such that Ω contains B_N and hence f is analytic inside the ball B_N. In addition, there exists an integer M such that $\Omega \subseteq B_M$. Since J is a non-tangential ray to x_0, it follows that there is a sector in $\mathring{U}$ with vertex at x_0 that contains J. Let C denote this sector. It follows from the Cauchy integral formula that

$$\frac{f(x)}{x} = \frac{1}{2\pi i}\int_{\partial(C \bigcup B_N)} \frac{f(z)}{z(z-x)}dz$$

where the boundary is oriented so that the interior of $C \bigcup B_N$ lies always to the left of the path of integration. (See Fig. 1.) Let $D_n = A_n \backslash C$. Then

$$\frac{f(x)}{x} = \frac{1}{2\pi i}\sum_{n=M}^{N}\int_{\partial D_n} \frac{f(z)}{z(z-x)}dz + \frac{1}{2\pi i}\int_{|z|=2^{-M}} \frac{f(z)}{z(z-x)}dz.$$

Since x lies on J, which is a non-tangential ray to x_0, there exists a constant $k > 0$ such that for $z \notin U$, $\dfrac{|x|}{|z-x|} \leq k^{-1}$. Thus for $z \notin U$, $\dfrac{|z|}{|z-x|} \leq 1 + \dfrac{|x|}{|z-x|} \leq 1 + k^{-1}$.

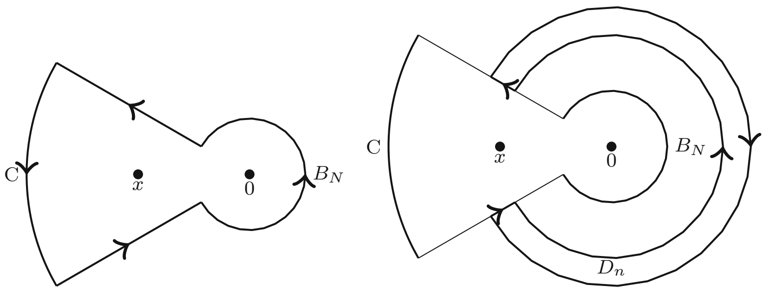

Fig. 1 The contour of integration

Hence $\dfrac{1}{|z|\cdot|z-x|} \le \dfrac{1+k^{-1}}{|z|^2}$ and therefore

$$\frac{|f(x)|}{|x|} \le \frac{1}{2\pi}\sum_{n=M}^{N}\left|\int_{\partial D_n}\frac{f(z)}{z(z-x)}dz\right| + \frac{4^M(1+k^{-1})}{2\pi}||f||_\infty. \tag{3}$$

Since $\dfrac{f(z)}{z(z-x)}$ is analytic on $D_n\backslash U$ for $M \le n \le N$, an application of Lemma 1 shows that

$$\left|\int_{\partial D_n}\frac{f(z)}{z(z-x)}dz\right| \le \kappa M_*^{1+\alpha}(D_n\backslash U)\cdot\left\|\frac{f(z)}{z(z-x)}\right\|'_{Lip\alpha(D_n)}. \tag{4}$$

Recall that the constant κ is the same for curves in the same equivalence class. Since the regions D_n differ from each other by a scaling it follows that κ doesn't depend on n in (4).

We now show that $\left\|\dfrac{f(z)}{z(z-x)}\right\|'_{Lip\alpha(D_n)}$ can be bounded by a constant independent of f and x. It follows from the definition of the Lipschitz seminorm that

$$\begin{aligned}\left\|\frac{f(z)}{z(z-x)}\right\|'_{Lip\alpha(D_n)} &= \sup_{z\neq w;z,w\in D_n}\frac{\left|\dfrac{f(z)}{z(z-x)}-\dfrac{f(w)}{w(w-x)}\right|}{|z-w|^\alpha}\\ &= \sup_{z\neq w;z,w\in D_n}\frac{|w(w-x)f(z)-z(z-x)f(w)|}{|z|\cdot|z-x|\cdot|w|\cdot|w-x|\cdot|z-w|^\alpha}.\end{aligned}$$

Thus it follows from the triangle inequality that

$$\begin{aligned}\left\|\frac{f(z)}{z(z-x)}\right\|'_{Lip\alpha(D_n)} \le& \sup_{z\neq w;z,w\in D_n}\frac{|w(w-x)f(z)-w(w-x)f(w)|}{|z|\cdot|z-x|\cdot|w|\cdot|w-x|\cdot|z-w|^\alpha}\\ &+\sup_{z\neq w;z,w\in D_n}\frac{|w(w-x)f(w)-z(z-x)f(w)|}{|z|\cdot|z-x|\cdot|w|\cdot|w-x|\cdot|z-w|^\alpha}.\end{aligned} \tag{5}$$

We first bound the first term on the right of (5)

$$\sup_{z\neq w; z,w\in D_n} \frac{|w(w-x)f(z)-w(w-x)f(w)|}{|z|\cdot|z-x|\cdot|w|\cdot|w-x|\cdot|z-w|^\alpha} \leq \sup_{z\in D_n} \frac{1}{|z|\cdot|z-x|}\cdot||f||'_{Lip\alpha(D_n)}.$$

Since $z\notin U$, $\dfrac{1}{|z|\cdot|z-x|} < \dfrac{1+k^{-1}}{|z|^2}$, and therefore,

$$\sup_{z\neq w; z,w\in D_n} \frac{|w(w-x)f(z)-w(w-x)f(w)|}{|z|\cdot|z-x|\cdot|w|\cdot|w-x|\cdot|z-w|^\alpha} \leq C4^n||f||'_{Lip\alpha(D_n)}. \tag{6}$$

We now bound the second term on the right side of (5). Since $f(0)=0$ it follows that for $w\in\mathbb{C}$, $\dfrac{|f(w)|}{|w|^\alpha} \leq ||f||'_{Lip\alpha(\mathbb{C})}$. Moreover, a computation shows that $w(w-x)-z(z-x)=(w-z)(z+w-x)$. Hence

$$\sup_{z\neq w; z,w\in D_n} \frac{|w(w-x)f(w)-z(z-x)f(w)|}{|z|\cdot|z-x|\cdot|w|\cdot|w-x|\cdot|z-w|^\alpha} \leq \left(\sup_{z\neq w; z,w\in D_n} \frac{|w-z|^{1-\alpha}}{|z-x|\cdot|w|^{1-\alpha}\cdot|w-x|} + \frac{|w-z|^{1-\alpha}}{|z|\cdot|z-x|\cdot|w|^{1-\alpha}}\right)\cdot||f||'_{Lip\alpha(\mathbb{C})}. \tag{7}$$

Since x lies on J, there exists a constant $k>0$ such that $\dfrac{1}{|z-x|} < \dfrac{1+k^{-1}}{|z|}$ and $\dfrac{1}{|w-x|} < \dfrac{1+k^{-1}}{|w|}$. Hence

$$\sup_{z\neq w; z,w\in D_n} \frac{|w-z|^{1-\alpha}}{|z-x|\cdot|w|^{1-\alpha}\cdot|w-x|} \leq C\frac{2^n\cdot(2^n)^{2-\alpha}}{(2^n)^{1-\alpha}} = C4^n, \tag{8}$$

and

$$\sup_{z\neq w; z,w\in D_n} \frac{|w-z|^{1-\alpha}}{|z|\cdot|z-x|\cdot|w|^{1-\alpha}} \leq C\frac{4^n\cdot(2^n)^{1-\alpha}}{(2^n)^{1-\alpha}} = C4^n. \tag{9}$$

Then (7), (8), and (9) yield

$$\sup_{z\neq w; z,w\in D_n} \frac{|w(w-x)f(w)-z(z-x)f(w)|}{|z|\cdot|z-x|\cdot|w|\cdot|w-x|\cdot|z-w|^\alpha} \leq C4^n||f||'_{Lip\alpha(\mathbb{C})}, \tag{10}$$

and it follows from (5), (6), and (10) that

$$\left\| \frac{f(z)}{z(z-x)} \right\|'_{Lip\alpha(D_n)} \leq C4^n ||f||'_{Lip\alpha(\mathbb{C})}. \tag{11}$$

Thus (3), (4), and (11) together yield

$$\frac{|f(x)|}{|x|} \leq C \sum_{n=1}^{\infty} 4^n M_*^{1+\alpha}(D_n \backslash U) \cdot ||f||'_{Lip\alpha(\mathbb{C})}.$$

Since Hausdorff content is monotone, $M_*^{1+\alpha}(D_n \backslash U) \leq M_*^{1+\alpha}(A_n \backslash U)$ and hence

$$\frac{|f(x)|}{|x|} \leq C \sum_{n=1}^{\infty} 4^n M_*^{1+\alpha}(A_n \backslash U) \cdot ||f||_{Lip\alpha(\mathbb{C})},$$

and it follows from Theorem 2 that

$$\frac{|f(x)|}{|x|} \leq C||f||_{Lip\alpha(U)},$$

where C does not depend on x or f. Thus $L_x(f) \leq C||f||_{Lip\alpha(\mathbb{C})}$ for $f \in A_\alpha(U \cup \{0\})$ and since $A_\alpha(U \cup \{0\})$ is dense in $A_\alpha(U)$, it follows that L_x is a family of uniformly bounded linear functionals on $A_\alpha(U)$. □

To complete the proof of Theorem 1, since $A_\alpha(U \cup 0)$ is dense in $A_\alpha(U)$, there exists a sequence $\{f_j\}$ in $A_\alpha(U \cup 0)$ such that $f_j \to f$ in the Lipschitz norm. Since each f_j is analytic in a neighborhood of 0 and since $Df_j = f_j'(0)$, it follows that for each j, $L_x(f_j) \to 0$ as $x \to 0$. It follows from the claim that $|L_x(f) - L_x(f_j)| \leq C||f - f_j||_{Lip\alpha(U)}$. By first choosing j sufficiently large, the right hand side can be made arbitrarily small. Then by choosing x sufficiently close to 0, $L_x(f_j)$ can be made arbitrarily close to 0. Thus $L_x(f) \to 0$ as $x \to 0$ through J, which proves Theorem 1.

Compliance with ethical standards

Conflict of interest The author declares that he has no conflict of interest.

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