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On the complete integrability of the geodesic flow of pseudo-H-type Lie groups

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Abstract
Pseudo-H-type groups $G_{r,s}$ form a class of step-two nilpotent Lie groups with a natural pseudo-Riemannian metric. In this paper the question of complete integrability in the sense of Liouville is studied for the corresponding (pseudo-)Riemannian geodesic flow. Via the isometry group of $G_{r,s}$ families of first integrals are constructed. A modification of these functions gives a set of dim $G_{r,s}$ functionally independent smooth first integrals in involution. The existence of a lattice L in $G_{r,s}$ is guaranteed by recent work of K. Furutani and I. Markina. The complete integrability of the pseudo-Riemannian geodesic flow of the compact nilmanifold $L\backslash G_{r,s}$ is proved under additional assumptions on the group $G_{r,s}$.

Keywords Pseudo-Riemannian metric · Hamilton's equation · Killing vector fields · Pseudo-H-type nilmanifolds

Mathematics Subject Classification 53C30 · 22E25 · 37K10

Dedicated to the memory of Alexander Vasil'ev.

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1 Introduction

A classical aspect in the study of the geodesic flow of a complete smooth Riemannian manifold (M, g) of dimension n is the question of the complete integrability in the sense of Liouville. If one can find n smooth functions $f_1, \ldots, f_n$ (first integrals) on the cotangent bundle T^*M which are functionally independent and in involution with respect to the natural Poisson structure, i.e. $\{f_i, f_j\} = 0$, then the level set $\Phi^{-1}(c) = \mathcal{M}_c$ with a regular value $c \in \mathbb{R}^n$ of the map

$$\Phi = (f_1 \ldots, f_n)^T : T^*M \to \mathbb{R}^n$$

forms a Lagrangian submanifold of T^*M. The geodesic flow preserves $\mathcal{M}_c$ and, from a physical point of view, Φ represents a set of n conservation laws including the Hamiltonian, or the kinetic energy. If $\mathcal{M}_c$ is compact and connected then by the famous *Liouville-Arnol'd Theorem* it is known to be diffeomorphic to a torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

In particular, the integrability of the geodesic flow of semi-simple Lie groups $M = G$ with a left-invariant metric has been studied intensively during the last decades (see e.g. [8]). The corresponding problem in the case of nilpotent Lie groups G and nilmanifolds (quotient of G by a lattice) seems to be less well understood, even if the group is assumed to be of step-two. Besides the abstract proof [20] of the existence of a maximal Poisson commuting ring of functions on general nilpotent Lie algebras, an important source of information is [3] where the complete integrability for compact nilmanifolds over step-two nilpotent Lie groups G with Lie algebra $\mathfrak{g}$ of Heisenberg–Reiter type (see Definition 5.1) is proved. On the other hand, the paper [4] gives a negative result. Assume that $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$ is decomposed into the center $\mathfrak{z}$ and its orthogonal complement $\mathfrak{v}$ with respect to a non-degenerate scalar product $\langle \cdot, \cdot \rangle$. Let $j(Z) : \mathfrak{v} \to \mathfrak{v}$ for all $Z \in \mathfrak{z}$ be defined through the relation:

$$\big\langle [U, V], Z \big\rangle = \big\langle j(Z)U, V \big\rangle, \qquad \textit{for all} \;\; U, V \in \mathfrak{v}.$$

If $j(Z)$ is invertible for all $0 \neq Z \in \mathfrak{z}$, then $\mathfrak{g}$ is called *non-singular*, cf [7, Lem. 1.8]. In [4] the notion of a step-two *non-integrable Lie algebra* $\mathfrak{g}$ is defined. It is shown that for any co-compact subgroup L of G and any left-invariant metric g on G the geodesic flow of $(L\backslash G, g)$ is not completely integrable in the sense of Liouville. As is known, a non-singular step-two nilpotent Lie algebra cannot be non-integrable. In the present paper singular and non-singular Lie algebras are considered. In the former cases complete integrability cannot be excluded by Butler's result in [4].

More recently and in the case of the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}_{2n+1}$ a set of $(2n+1)$ Poisson commuting first integrals induced by the isometry group of $\mathbb{H}_{2n+1}$ has been constructed explicitly in [14]. In the present paper the analysis in [14] is further generalized and some of the results in [3] are extended. Instead of the Heisenberg group one considers the wider class of pseudo-H-type Lie groups $G_{r,s}$ which have been introduced and intensively studied in [5,9,10]. These groups form a subclass of all step-two nilpotent Lie groups and are non-singular in the case $s = 0$. The bracket relations of the corresponding Lie algebra are linked to a Clifford module action of the Clifford algebra $C\ell_{r,s}$ with non-negative integers (signature) r, s, cf [5,16]. In the case $s > 0$ these groups are naturally equipped with a pseudo-Riemannian metric which induces a pseudo-Riemannian geodesic flow on the cotangent bundle. In this framework a set of first integrals is explicitly constructed and the complete integrability is proved.

Examples of step-two nilpotent Lie groups without a lattice (co-compact discrete subgroup) have been constructed by P. Eberlein, cf [6]. However, due to the results in [9], the existence of a lattice L in any pseudo-H-type group is guaranteed. Under additional assumptions on the Lie algebra $\mathcal{N}_{r,s}$ of $G_{r,s}$ the complete integrability is proved for the pseudo-Riemannian geodesic flow of the compact pseudo-H-type nilmanifold $L\backslash G_{r,s}$ (Theorem 5.6). There is presented an example of a Lie algebra $\mathcal{N}_{r,s}$ which is not of Heisenberg-Reiter type (which is assumed in [3]) and for which the pseudo-Riemannian geodesic flow of the quotient $L\backslash G_{r,s}$ remains to be completely Liouville integrable.

Different aspects of the complete integrability for compact nilmanifolds have been studied by various authors. As examples, one can point out the question whether the complete integrability of the geodesic flow is determined by the Laplace spectrum (a negative answer is given in [19]) or the examples in [17] on the non-integrable sub-Riemannian geodesic flow on Carnot groups of step larger than two.

The paper is organized as follows. In Sect. 2 the notations are fixed and the geometric setting is explained. In particular, the notion of a pseudo-H-type group is recalled. Via the isometry group of a step-two nilpotent Lie group G a family of (in general not Poisson commuting) first integrals is derived for the pseudo-Riemannian geodesic flow in Sect. 3. In general, one cannot select dim G first integrals in involution from this family of functions. Section 4 contains the proof of the main result on the complete integrability of the pseudo-Riemannian geodesic flow. In Sect. 5, under additional assumptions on $G_{r,s}$, sufficiently many first integrals are shown to descend from $G_{r,s}$ to the compact nilmanifold $L\backslash G_{r,s}$ and the complete integrability is proved. In particular, these assumptions imply that the Lie algebra $\mathcal{N}_{r,s}$ of $G_{r,s}$ is of Heisenberg-Reiter type. In case of the Riemannian geodesic flow this problem has been solved in a more general framework in the work by Butler, cf [3]. However, there is also given a nilmanifold $L\backslash G_{r,s}$ for which $\mathcal{N}_{r,s}$ is not of Heisenberg-Reiter type but the complete integrability of the geodesic flow can be verified directly.

2 Notation and definitions

In this section, we fix the notation and explain some basic definitions of the Hamiltonian formalism of the geodesic flow of Lie groups equipped with a left-invariant metric. The problem of complete integrability in the sense of Liouville is explained for Hamiltonian systems on the tangent bundle of a Lie group. Then, we recall the basic definition of the (pseudo-)H-type groups introduced by Kaplan [13] (and by Ciatti [5]) as a generalization of Heisenberg groups. An explicit description of Hamilton's equations for the geodesic flow is given for the (pseudo-)H-type groups.

Let $G = (G, *)$ be a Lie group with identity element e_G and Lie algebra $\mathfrak{g}$. The tangent and the cotangent bundle to G admit the left-actions by G through

$$\mathrm{d}L_g : TG \supset T_pG \ni X \mapsto \left(\mathrm{d}L_g\right)_p X \in T_{g*p}G \subset TG,$$
$$\left(L_{g^{-1}}\right)^* : T^*G \supset T_p^*G \ni \xi \mapsto \left(L_{g^{-1}}\right)_p^* \xi \in T_{g*p}^*G \subset T^*G,$$

where $g \in G$ and $L_g : G \ni p \mapsto g * p \in G$ stands for the left-multiplication. The tangent and the cotangent bundle of G are trivialized as $TG \cong G \times \mathfrak{g}$ and $T^*G \cong G \times \mathfrak{g}^*$ through the left-trivializations

$$\lambda\colon TG \supset T_pG \ni X \mapsto \left(p, \left(\mathrm{d}L_{p^{-1}}\right)_p X\right) \in G \times \mathfrak{g},$$
$$\lambda^*\colon T^*G \supset T_p^*G \ni \xi \mapsto \left(p, \left(L_p\right)_p^* \xi\right) \in G \times \mathfrak{g}^*.$$

Through these left-trivializations, left-invariant functions on TG and on T^*G are identified with the functions defined on $\mathfrak{g}$ and on $\mathfrak{g}^*$, respectively. Similarly, we identify the elements $Y \in \mathfrak{g} \equiv T_{e_G}G$ and $\eta \in \mathfrak{g}^* \equiv T_{e_G}^*G$ with the left-invariant vector field $G \ni p \mapsto \left(\mathrm{d}L_p\right)_{e_G} Y \in T_pG \subset TG$ and with the left-invariant differential one-form $G \ni p \mapsto \left(L_{p^{-1}}\right)_{e_G}^* \eta \in T_p^*G \subset T^*G$, respectively.

We now take a non-degenerate scalar product[1] $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$. Through the left-trivialization $T^*G \otimes T^*G \cong G \times (\mathfrak{g}^* \otimes \mathfrak{g}^*)$, the scalar product $\langle\cdot,\cdot\rangle$ induces a left-invariant (pseudo-)Riemannian metric on G which we denote by the same symbol. We write the induced scalar product on T_pG as $\langle\cdot,\cdot\rangle_p$ and set $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{e_G}$ for simplicity. By means of the left-invariant (pseudo-)Riemannian metric $\langle\cdot,\cdot\rangle$ on G, the cotangent bundle $T^*G \cong G \times \mathfrak{g}^*$ can be identified with the tangent bundle $TG \cong G\times\mathfrak{g}$. This identification is compatible to the one $\mathfrak{g} \ni Y \mapsto \langle Y, \cdot\rangle \in \mathfrak{g}^*$ between the Lie algebra and its dual. We further have the induced (pseudo-)Riemannian metric on $TG \cong G \times \mathfrak{g}$ described as

$$\left\langle (U, V), (U', V')\right\rangle = \left\langle U, U'\right\rangle + \left\langle V, V'\right\rangle,$$

where $(U, V), (U', V') \in \mathfrak{g}\times\mathfrak{g} \cong T_pG \times \mathfrak{g} \cong T_{(p,Y)}(TG)$. We will keep using these identifications in the notation below. The gradient vector field $\mathrm{grad} F$ for $F \in \mathcal{C}^\infty(TG)$ is defined through

$$\left\langle \mathrm{grad}_{(p,Y)} F, (U', V')\right\rangle = (\mathrm{d}F)_{(p,Y)}\left(U', V'\right), \tag{1}$$

where $(U', V') \in \mathfrak{g} \times \mathfrak{g}$.

Through the identification of $TG \cong G \times \mathfrak{g}$ and $T^*G \cong G \times \mathfrak{g}^*$, we have the canonical one-form Θ and the canonical symplectic form Ω on $TG \cong G \times \mathfrak{g}$, induced by those on T^*G. More precisely, we have at $(p, Y) \in G \times \mathfrak{g} \cong TG$

$$\Theta_{(p,Y)}\left((U, V)\right) = \langle Y, U\rangle,$$
$$\Omega_{(p,Y)}\left((U, V), (U', V')\right) = -\left\langle V, U'\right\rangle + \left\langle V', U\right\rangle + \left\langle Y, \left[U, U'\right]\right\rangle,$$

[1] By a scalar product we mean a symmetric bilinear form on $\mathfrak{g}$ which not necessarily needs to be positive-definite.

for $(U, V), (U', V') \in \mathfrak{g} \times \mathfrak{g}$. (These formulae can be deduced from [1, Prop. 4.4.1].) The Hamiltonian vector field Ξ_F for the *Hamiltonian* $F \in \mathcal{C}^\infty(TG)$ is written as

$$(\Xi_F)_{(p,Y)} = \left(V, (\mathrm{ad}_V)^{\mathrm{T}} Y - U\right) \in \mathfrak{g} \times \mathfrak{g},$$

where $\mathrm{grad}_{(p,Y)} F = (U, V) \in \mathfrak{g} \times \mathfrak{g}$ and $(\mathrm{ad}_V)^{\mathrm{T}} : \mathfrak{g} \to \mathfrak{g}$ stands for the adjoint operator of ad_V with respect to $\langle \cdot, \cdot \rangle$:

$$\left\langle (\mathrm{ad}_V)^{\mathrm{T}} U', U'' \right\rangle = \left\langle U', \mathrm{ad}_V \left(U''\right) \right\rangle, \quad U', U'' \in \mathfrak{g}.$$

The Poisson bracket for $F, F' \in \mathcal{C}^\infty(TG)$ is expressed as

$$\left\{F, F'\right\}(p, Y) = -\left\langle V, U' \right\rangle + \left\langle V', U \right\rangle - \left\langle Y, \left[V, V'\right] \right\rangle,$$

where $\mathrm{grad}_{(p,Y)} F = (U, V)$, $\mathrm{grad}_{(p,Y)} F' = \left(U', V'\right) \in \mathfrak{g} \times \mathfrak{g}$. If F is left-invariant, we have $(\Xi_F)_{(p,Y)} = \left(V, (\mathrm{ad}_V)^{\mathrm{T}} Y\right) \in \mathfrak{g} \times \mathfrak{g}$. Hamilton's equations are written as

$$\begin{cases} \frac{\mathrm{d}p}{\mathrm{d}t} = \left(\mathrm{d}L_p\right)_{e_G} U, \\ \frac{\mathrm{d}Y}{\mathrm{d}t} = (\mathrm{ad}_U)^{\mathrm{T}} Y. \end{cases}$$

Recall that the second equation is usually called *Euler-Poincaré equation* (cf [18, Thm 6.6]). A smooth function f on the cotangent bundle $T^*G \cong G \times \mathfrak{g}^*$ is called a *first integral of the geodesic flow* if it is constant along the integral curves of the geodesic flow or, equivalently, if f Poisson commutes with the Hamiltonian. More generally, in this paper we consider pseudo-H-type Lie groups G (cf Definition 2.2 below) which are naturally equipped with a pseudo-Riemannian metric. In this setting we may form the Hamiltonian and the induced (pseudo-Riemannian) geodesic flow with respect to the pseudo-Riemannian metric, which is non-degenerate but not necessarily positive-definite. In some of our results the notion of first integrals and complete integrability is used in this more general framework. For completeness we mention the definition of complete integrability in the sense of Liouville (cf [1, Def. 5.2.20]).

Definition 2.1 The Hamiltonian system $(T^*G \cong TG \cong G \times \mathfrak{g}^*, \Omega, F)$ is called *completely integrable in the sense of Liouville* if there exist n functions $F_1 (= F)$, $F_2, \ldots, F_n$ in $\mathcal{C}^\infty(TG)$ which are functionally independent, i.e.

$$(\mathrm{d}F_1)_{(p,Y)}, \ldots, (\mathrm{d}F_n)_{(p,Y)} \in T_{(p,Y)}(TG)$$

are linearly independent for (p, Y) in an open dense subset of TG, and Poisson commute: $\left\{F_i, F_j\right\} = 0$ for all $i, j = 1, \ldots, n$, where $n = \dim G$. □

From now on, we assume that G is a connected, simply connected step-two nilpotent Lie group. In this case, the corresponding Lie algebra $\mathfrak{g}$ satisfies $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{z}$, where $\mathfrak{z} \subset \mathfrak{g}$ is the center of the Lie algebra. Recall that, under such assumptions, the exponential mapping $\exp : \mathfrak{g} \to G$ is a diffeomorphism. We denote by $\mathfrak{v}$ the orthogonal complement $\mathfrak{v} = \mathfrak{z}^{\perp} \subset \mathfrak{g}$ to the center $\mathfrak{z}$ with respect to the scalar product $\langle \cdot, \cdot \rangle$. Assume that the restriction $\langle \cdot, \cdot \rangle_{\mathfrak{z}}$ of $\langle \cdot, \cdot \rangle$ to $\mathfrak{z}$ is non-degenerate such that one has the orthogonal direct sum decomposition $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$. In the following the restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{v}$ is denoted by $\langle \cdot, \cdot \rangle_{\mathfrak{v}}$. Given an element $Y \in \mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$, its two components are denoted by $Y_{\mathfrak{v}} \in \mathfrak{v}$ and $Y_{\mathfrak{z}} \in \mathfrak{z}$: $Y = Y_{\mathfrak{v}} + Y_{\mathfrak{z}}$.

In this case, one can associate a skew-symmetric linear operator $j(Z) : \mathfrak{v} \to \mathfrak{v}$ to any element $Z \in \mathfrak{z}$ through the formula

$$\big\langle [U, V], Z \big\rangle = \big\langle j(Z) U, V \big\rangle, \tag{2}$$

where $U, V \in \mathfrak{v}$ are arbitrary. The skew-symmetry of the Lie bracket implies the skew-symmetry of the linear operator $j(Z)$ with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{v}}$.

Using the operator j, the canonical symplectic form on $G \times \mathfrak{g} \cong TG$ for a step-two nilpotent Lie group G is written as

$$\Omega_{(p,Y)}\big((U, V), (U', V') \big) = \langle U, V' \rangle - \langle V, U' \rangle + \big\langle j(Y_{\mathfrak{z}}) U_{\mathfrak{v}}, U'_{\mathfrak{v}} \big\rangle_{\mathfrak{v}},$$

for $(U, V), (U', V') \in \mathfrak{g} \times \mathfrak{g} \cong T_pG \times \mathfrak{g} \cong T_{(p,Y)}(TG)$, while the Hamiltonian vector field and Poisson bracket are given as

$$\begin{aligned} (\Xi_F)_{(p,Y)} &= \big(V, j(Y_{\mathfrak{z}}) V_{\mathfrak{v}} - U\big) \in \mathfrak{g} \times \mathfrak{g}, \\ \{F, F'\}(p, Y) &= \langle V', U \rangle - \langle V, U' \rangle - \big\langle j(Y_{\mathfrak{z}}) V_{\mathfrak{v}}, V'_{\mathfrak{v}} \big\rangle_{\mathfrak{v}} \\ &= \langle V', U \rangle - \langle V, U' \rangle - \big\langle Y_{\mathfrak{z}}, [V_{\mathfrak{v}}, V'_{\mathfrak{v}}] \big\rangle_{\mathfrak{z}}, \end{aligned} \tag{3}$$

where $\mathrm{grad}_{(p,Y)} F = (U, V)$, $\mathrm{grad}_{(p,Y)} F' = (U', V') \in \mathfrak{g} \times \mathfrak{g}$. In particular, for left-invariant functions $g, g' \in \mathcal{C}^{\infty}(TG)$, we have

$$\{g, g'\}(p, Y) = -\big\langle j(Y_{\mathfrak{z}}) V_{\mathfrak{v}}, V'_{\mathfrak{v}} \big\rangle_{\mathfrak{v}} = -\big\langle Y_{\mathfrak{z}}, [V_{\mathfrak{v}}, V'_{\mathfrak{v}}] \big\rangle_{\mathfrak{z}}, \tag{4}$$

where $\mathrm{grad}_{(p,Y)} g = (0, V)$ and $\mathrm{grad}_{(p,Y)} g' = (0, V')$. As an immediate result, if $g = g(Y_{\mathfrak{z}})$ depends only on the component $Y_{\mathfrak{z}} \in \mathfrak{z}$ in the center $\mathfrak{z}$, then it Poisson commutes with all left-invariant differentiable functions $g' = g'(Y)$:

$$\{g, g'\} = -\big\langle Y_{\mathfrak{z}}, [0, V'_{\mathfrak{v}}] \big\rangle = 0. \tag{5}$$

The Hamiltonian for the geodesic flow with respect to the left-invariant pseudo-Riemannian metric $\langle \cdot, \cdot \rangle$ is given as

$$H(p, Y) = \frac{1}{2} \langle Y, Y \rangle \quad \textit{where } (p, Y) \in G \times \mathfrak{g} \cong TG.$$

The associated Hamiltonian vector field Ξ_H is calculated at $(p, Y) \in G \times \mathfrak{g} \cong TG$ as

$$(\Xi_H)_{(p,Y)} = \big(Y, j(Y_{\mathfrak{z}}) Y_{\mathfrak{v}}\big).$$

We recall the notion of a pseudo-H-type nilpotent Lie algebra (group), cf [2,5,9].

Definition 2.2 The step-two nilpotent Lie algebra $\mathfrak{g}$ equipped with the non-degenerate scalar product $\langle\cdot,\cdot\rangle$ is called *pseudo-H-type (nilpotent) Lie algebra*, if the operator $j:\mathfrak{z}\to\mathfrak{so}\,(\mathfrak{v},\langle\cdot,\cdot\rangle_{\mathfrak{v}})$ satisfies the orthogonality condition

$$\left\langle j(Z)V, j(Z)V'\right\rangle_{\mathfrak{v}} = \langle Z,Z\rangle_{\mathfrak{z}}\cdot\langle V,V'\rangle_{\mathfrak{v}}, \tag{6}$$

for all $V, V'\in\mathfrak{v}$, $Z\in\mathfrak{z}$. The corresponding connected, simply connected step-two nilpotent Lie group G is called *pseudo-H-type (nilpotent) Lie group*. □

Here $\mathfrak{g}=\mathfrak{v}\oplus\mathfrak{z}$ may be singular or non-singular and the Clifford relations

$$j(Z)j(Z')+j(Z')j(Z) = -2\langle Z,Z'\rangle_{\mathfrak{z}}\mathrm{id}_{\mathfrak{v}} \tag{7}$$

hold, where $Z, Z'\in\mathfrak{z}$. Hence the map j extends to the Clifford algebra $C\ell(\mathfrak{z},\langle\cdot,\cdot\rangle_{\mathfrak{z}})$ of $\mathfrak{z}$ with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathfrak{z}}$. It defines a Clifford representation

$$j : C\ell(\mathfrak{z},\langle\cdot,\cdot\rangle_{\mathfrak{z}})\to\mathrm{End}(\mathfrak{v}),$$

which, for simplicity, we assume to be *minimal admissible*, i.e. a module, which attains the minimal dimension, among the $C\ell(\mathfrak{z},\langle\cdot,\cdot\rangle_{\mathfrak{z}})$-modules satisfying the skew-symmetry condition $\left\langle j\,(Z)\,V,V'\right\rangle_{\mathfrak{v}} = -\left\langle V, j\,(Z)\,V'\right\rangle_{\mathfrak{v}}$ (cf [9, p.980]). It is known that a minimal admissible $C\ell(\mathfrak{z},\langle\cdot,\cdot\rangle_{\mathfrak{z}})$-module is either irreducible or double of an irreducible module (cf [5]). The dimension of $\mathfrak{v}$ is even and we denote it by $\dim\mathfrak{v}=2m$. With (r,s) being the signature of $\langle\cdot,\cdot\rangle_{\mathfrak{z}}$, the Clifford algebra $C\ell\big(\mathfrak{z},\langle\cdot,\cdot\rangle_{\mathfrak{z}}\big)$ is isomorphic to $C\ell_{r,s}$ generated by $\mathbb{R}^{r,s}=\big(\mathbb{R}^{r+s},\langle\cdot,\cdot\rangle_{r,s}\big)$, where

$$\langle z,z\rangle_{r,s}=\sum_{i=1}^{r}z_i^2-\sum_{j=1}^{s}z_{j+r}^2$$

for $z=(z_1,\dots,z_{r+s})\in\mathbb{R}^{r+s}$, as a consequence of the universality of Clifford algebras (cf [16, Prop. 1.1]). We use the following notation (cf [10, §§2.3, Def. 1]).

Definition 2.3 For an admissible $C\ell_{r,s}$-module $\mathfrak{v}$ with the representation

$$j : C\ell_{r,s}\to\mathrm{End}\,(\mathfrak{v}),$$

the pseudo-H-type Lie algebra $\mathfrak{v}\oplus\mathbb{R}^{r,s}$ whose Lie bracket $[\cdot,\cdot]:\mathfrak{v}\times\mathfrak{v}\to\mathbb{R}^{r,s}$ is defined through (2) where $Z\in\mathbb{R}^{r,s}$, $U,V\in\mathfrak{v}$, is denoted by $\mathcal{N}_{r,s}\,(\mathfrak{v})$. If $\mathfrak{v}$ is minimal admissible, we write $\mathcal{N}_{r,s}$. We denote the connected, simply connected pseudo-H-type Lie group corresponding to $\mathcal{N}_{r,s}$ by $G_{r,s}$.

Note that $\mathcal{N}_{r,s}$ and $G_{r,s}$ are unique up to isomorphisms (see [10, §6] for the details). If $\langle\cdot,\cdot\rangle_{\mathfrak{z}}$ is positive-definite, i.e. if $s=0$, then the Lie algebra $\mathfrak{g}=\mathfrak{v}\oplus\mathfrak{z}$ is called *of H-type*. Such Lie algebras were first considered in [11,12].

3 Construction of first integrals

In the present section we construct a family $\mathcal{F}$ of first integrals on step-two nilpotent Lie groups G with a left-invariant Riemannian metric and induced by *Killing vector fields*. More precisely, $\mathcal{F}$ is obtained as the range of an injective Lie algebra homomorphism on the semi-direct product $\mathfrak{k} \ltimes_\tau \mathfrak{g}$ where $\mathfrak{k}$ is the Lie algebra of the isotropy group of G. A priori, the elements in $\mathcal{F}$ do not Poisson commute and it may not even be possible to choose a subset $\mathcal{S} \subset \mathcal{F}$ of $|\mathcal{S}| = \dim G$ Poisson commuting first integrals in $\mathcal{F}$. However, under further assumptions on G and by modifying the construction below we prove the complete integrability of the geodesic flow in Sect. 4. We remark that the functions constructed in this section define first integrals even if the scalar product on G is not positive-definite. In particular, we obtain first integrals of the pseudo-Riemannian geodesic flow for pseudo-H-type groups $G_{r,s}$.

Recall that a vector field X^* on G is called *Killing vector field* if it induces a flow of continuous isometries on G. Shortly, $\mathcal{L}_{X^*} g = 0$ where g is the (pseudo-Riemannian) metric on G and $\mathcal{L}_{X^*}$ the Lie derivative of X^*. Lemma 3.1 serves as a source of first integrals:

Lemma 3.1 *Let X^* be a Killing vector field on G. Then the function*

$$F_{X^*}(p, Y) := \big\langle X^*, Y \big\rangle_p, \qquad \textit{where } \ Y \in T_p G$$

is a first integral of the geodesic flow. Here we write $\langle \cdot, \cdot \rangle_g := g_p(\cdot, \cdot)$.

Proof We fix a geodesic $\gamma(t)$ on G and by ∇ we denote the Levi-Civita connection. Then

$$\begin{aligned} \frac{d}{dt}\big\langle X^* \circ \gamma(t), \gamma'(t) \big\rangle_{\gamma(t)} &= \big\langle \nabla_{\gamma'(t)} X^* \circ \gamma(t), \gamma'(t) \big\rangle_{\gamma(t)} + \big\langle X^* \circ \gamma(t), \underbrace{\nabla_{\gamma'(t)} \gamma'(t)}_{=0} \big\rangle_{\gamma(t)} \\ &= \big\langle \nabla_{\gamma'(t)} X^* \circ \gamma(t), \gamma'(t) \big\rangle_{\gamma(t)}. \end{aligned}$$

From $\mathcal{L}_{X^*} g = 0$ we have $\langle \nabla_Y X^* \circ \gamma(t), Z \rangle_{\gamma(t)} = -\langle Y, \nabla_Z X^* \circ \gamma(t) \rangle_{\gamma(t)}$ for all $Y, Z \in T_{\gamma(t)} G$. In particular, choosing $Y = Z = \gamma'(t)$ in the above equation gives:

$$\frac{d}{dt}\big\langle X^* \circ \gamma(t), \gamma'(t) \big\rangle_{\gamma(t)} = 0.$$

Therefore F_{X^*} is constant along the integral curves of the geodesic flow. □

Lemma 3.1 serves a motivation for the derivation of the explicit first integrals below. We will not explicitly make use of it since all Poisson brackets between the functions constructed below via Killing vector fields are collected in Proposition 3.8 and follow by a direct calculation. Let $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$ be the decomposition of $\mathfrak{g}$ from Sect. 2. For the moment we do not assume that the non-degenerate scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ is positive-definite. Choose bases $[X_1, \ldots, X_{2m}]$ of $\mathfrak{v}$ and $[Z_1, \ldots, Z_d]$ of $\mathfrak{z}$ with

$$\langle X_i, X_{i'} \rangle_{\mathfrak{v}} = \pm\delta_{i,i'} \qquad \textit{and} \qquad \langle Z_\ell, Z_{\ell'} \rangle_{\mathfrak{z}} = \pm\delta_{\ell,\ell'}. \tag{8}$$

If these conditions are satisfied, we call $[X_1, \dots, X_{2m}]$ and $[Z_1, \dots, Z_d]$ *orthonormal* for brevity. Expanding elements $Y \in \mathfrak{g}$ with respect to the above basis

$$Y = \sum_{i=1}^{2m} x_i X_i + \sum_{\ell=1}^{d} z_\ell Z_\ell, \qquad (x_i, z_\ell \in \mathbb{R}),$$

defines coordinates $(x_1, \dots, x_{2m}) \in \mathbb{R}^{2m} \cong \mathfrak{v}$ and $(z_1, \dots, z_d) \in \mathbb{R}^d \cong \mathfrak{z}$ and gives an identification $\mathfrak{g} \cong \mathbb{R}^{2m+d}$.

Throughout this section we assume that $j(Z_\ell)$ is invertible on $\mathfrak{v}$ for $\ell = 1, \dots, d$. Recall that the latter condition is fulfilled for a *non-singular nilpotent Lie algebra* in the sense of [7, Def. 1.4]. Even in the case of a pseudo-H-type Lie algebra $\mathcal{N}_{r,s}$ where $s > 0$ (and therefore $\mathcal{N}_{r,s}$ is not non-singular) the invertibility of $j(Z_\ell)$ follows from the relations (8) and (7).

The left-multiplication on G by $\exp(-tX_i)$ and $\exp(-tZ_\ell)$ induces flows on G. Hence we obtain vector fields $X_i^{(r)}$ on G which—being defined by a left-multiplication—are right-invariant. In the case where $\langle \cdot, \cdot \rangle$ is positive-definite we can interpret $X_i^{(r)}$ as Killing vector fields. Since G is of step-two, the *Baker–Campbell–Hausdorff formula* implies for given $p = \exp(W) \in G$:

$$\exp\big(-tX_i\big) * p = \exp\left(-tX_i + W - \frac{t}{2}\big[X_i, W\big]\right).$$

Let $f \in \mathcal{C}^\infty(G)$, then $X_i^{(r)}$ acts as:

$$\begin{aligned}\Big[X_i^{(r)} f\Big](p) &:= \frac{\mathsf{d}}{\mathsf{d}t}\bigg|_{t=0} f\Big(\exp\big(-tX_i\big) * p\Big)\\ &= \frac{\mathsf{d}}{\mathsf{d}t}\bigg|_{t=0} f\left(\exp\big(-tX_i + W - \frac{t}{2}[X_i, W]\big)\right).\end{aligned} \tag{9}$$

Expanding the Lie bracket $[X_i, W] \in \mathfrak{z}$ with respect to the basis $[Z_1, \dots, Z_d]$ gives:

$$[X_i, W] = \sum_{\ell=1}^{d} \frac{\big\langle [X_i, W], Z_\ell \big\rangle}{\langle Z_\ell, Z_\ell \rangle} Z_\ell = \sum_{\ell=1}^{d} \frac{\big\langle j(Z_\ell) X_i, W_{\mathfrak{v}} \big\rangle}{\langle Z_\ell, Z_\ell \rangle} Z_\ell.$$

Moreover, $Z_\ell^{(r)} = \frac{\partial}{\partial z_\ell}$ and (9) leads to the following differential expressions of $X_i^{(r)}$:

$$X_i^{(r)} = -\frac{\partial}{\partial x_i} - \frac{1}{2} \sum_{\ell=1}^{d} \frac{\big\langle j(Z_\ell) X_i, W_{\mathfrak{v}} \big\rangle}{\langle Z_\ell, Z_\ell \rangle} \frac{\partial}{\partial z_\ell}, \qquad (i = 1, \dots, 2m). \tag{10}$$

We replace the left-multiplication by $\exp(-tX_i)$ in (9) by a right-multiplication with the element $\exp(tX_i)$. Similarly one obtains left-invariant vector fields X_i and Z_ℓ

which in the standard way are identified with the basis elements of $\mathfrak{v}$ and $\mathfrak{z}$, respectively. We do not change the notation and simply write:

$$X_i = \frac{\partial}{\partial x_i} - \frac{1}{2}\sum_{\ell=1}^{d} \frac{\langle j(Z_\ell)X_i, W_\mathfrak{v}\rangle}{\langle Z_\ell, Z_\ell\rangle}\frac{\partial}{\partial z_\ell}. \tag{11}$$

Comparing (10) and (11) and using Lemma 3.1 implies:

Proposition 3.2 *In terms of the left-invariant vector fields X_i and Z_ℓ the Killing vector fields $X_i^{(r)}$ at $p = \exp(W) \in G$ can be expressed as:*

$$X_i^{(r)}(p) = -\big(dL_p\big)_{e_G}\left(X_i + \sum_{\ell=1}^{d}\frac{\langle j(Z_\ell)X_i, W_\mathfrak{v}\rangle}{\langle Z_\ell, Z_\ell\rangle}Z_\ell\right). \tag{12}$$

Let $(p, Y) \in TG \cong G \times \mathfrak{g}$. A set of $\dim G = 2m + d$ *first integrals is obtained by*

$$F_{X_i^{(r)}}(p, Y) = \langle X_i^{(r)}(p), \big(dL_p\big)_{e_G} Y\rangle_p = \langle X_i, j(Y_\mathfrak{z})W_\mathfrak{v} - Y\rangle, \qquad i = 1, \dots, 2m,$$
$$F_{Z_\ell}(p, Y) = \langle Z_\ell, Y\rangle, \qquad \ell = 1, \dots, d.$$

Proof $F_{X_i^{(r)}}$ is obtained by inserting (12) into $\langle X_i^{(r)}(p), \big(\mathrm{d}L_p\big)_{e_G} Y\rangle_p$. □

For the moment let us assume that G carries a Riemannian metric, i.e. the scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ is positive-definite. We can extend the construction in Proposition 3.2 by replacing the left-translation on G by the full isometry group $I(G)$ of G. Then Lemma 3.1 can be applied and induces an enlarged class of first integrals. As is well-known $I(G)$ is obtained as a semi-direct product of G (acting by left-multiplication) with the *isotropy subgroup* K of $I(G)$ which is identified with

$$K = \Big\{(\Phi, T) \in O(\mathfrak{z}, \langle\cdot,\cdot\rangle_\mathfrak{z}) \times O(\mathfrak{v}, \langle\cdot,\cdot\rangle_\mathfrak{v}) \;:\; Tj(Z)T^{-1} = j\left(\Phi(Z)\right),\; Z \in \mathfrak{z}\Big\},$$

(see [15,21] for details). Here $O(\mathfrak{z}, \langle\cdot,\cdot\rangle_\mathfrak{z})$ and $O(\mathfrak{v}, \langle\cdot,\cdot\rangle_\mathfrak{v})$ denote the isometries of $\mathfrak{z}$ and $\mathfrak{v}$, respectively.

Lemma 3.3 *Let $U, V \in \mathfrak{g}$ and $(\Phi, T) \in K$. Then, we have the following relations:*

(a) $T^{-1}j(Z)T = j\big(\Phi^{-1}(Z)\big)$ *for all $Z \in \mathfrak{z}$,*
(b) $[TU_\mathfrak{v}, TV_\mathfrak{v}] = \Phi\big[U, V\big]$.

Proof We only show (b). For any $Z \in \mathfrak{z}$ we have

$$\langle Z, \Phi[U, V]\rangle = \langle\Phi^{-1}Z, [U, V]\rangle = \langle\Phi^{-1}Z, [U_\mathfrak{v}, V_\mathfrak{v}]\rangle = \langle j(\Phi^{-1}Z)U_\mathfrak{v}, V_\mathfrak{v}\rangle,$$
$$\langle Z, [TU_\mathfrak{v}, TV_\mathfrak{v}]\rangle = \langle j(Z)TU_\mathfrak{v}, TV_\mathfrak{v}\rangle = \langle T^{-1}j(Z)TU_\mathfrak{v}, V_\mathfrak{v}\rangle.$$

Since Z was chosen arbitrarily and $\langle\cdot,\cdot\rangle_\mathfrak{z}$ is non-degenerate on $\mathfrak{z}$, (a) implies (b). □

We summarize the above statements in the next proposition.

Proposition 3.4 (see [15,21]) *The isometry group $I(G)$ of G is given by the semi-direct product $I(G) = K \ltimes G$. The Lie algebra $\mathfrak{k}$ of K is identified with*

$$\mathfrak{k} = \Big\{(A, B) \in \mathfrak{so}(\mathfrak{z}, \langle\cdot,\cdot\rangle_{\mathfrak{z}}) \times \mathfrak{so}(\mathfrak{v}, \langle\cdot,\cdot\rangle_{\mathfrak{v}}) \ : \ Bj(Z) - j(Z)B = j(AZ), \ Z \in \mathfrak{z}\Big\}. \tag{13}$$

Here $\mathfrak{so}(E)$ denotes the skew-symmetric operators on a scalar product space E.

In the following we write $\exp_K : \mathfrak{k} \to K$ and $\exp_G : \mathfrak{g} \to G$ for the exponential maps of $\mathfrak{k}$ and $\mathfrak{g}$, respectively. Let $\pi_{\mathfrak{z}} : K \to O(\mathfrak{z}, \langle\cdot,\cdot\rangle_{\mathfrak{z}})$ and $\pi_{\mathfrak{v}} : K \to O(\mathfrak{v}, \langle\cdot,\cdot\rangle_{\mathfrak{v}})$ denote the projections onto the first and second component of K. Fix $k = (A, B) \in \mathfrak{k}$ and define a family of maps $\rho_{k,s} : G \to G$ depending on a real parameter $s \in \mathbb{R}$ by

$$\rho_{k,s}\big(\exp_G(U)\big) := \exp_G\Big(\big(\pi_{\mathfrak{z}} \circ \exp_K(sk)\big)U_{\mathfrak{z}} + \big(\pi_{\mathfrak{v}} \circ \exp_K(sk)\big)U_{\mathfrak{v}}\Big), \tag{14}$$

where $U \in \mathfrak{g}$ (recall that $\exp_G$ is a diffeomorphism). Then:

Proposition 3.5 *For each $k \in \mathfrak{k}$ the map $s \to \rho_{k,s}$ is a one-parameter group of isometries of G. In other words: it is the flow of a Killing vector field on G.*

Proof We show that $\rho_{k,s} : G \to G$ defines a homomorphism. Let $U, V \in \mathfrak{g}$ and recall that $\exp_G(U) * \exp_G(V) = \exp_G(U + V + [U, V]/2)$. Note that:

$$\begin{aligned}\rho_{k,s}\big(\exp_G(U)\big) * \rho_{k,s}\big(\exp_G(V)\big) &= \exp_G\Big(\big(\pi_{\mathfrak{z}} \circ \exp_K(sk)\big)(U_{\mathfrak{z}} + V_{\mathfrak{z}}) \\ &\quad + \big(\pi_{\mathfrak{v}} \circ \exp_K(sk)\big)(U_{\mathfrak{v}} + V_{\mathfrak{v}}) + \frac{1}{2}\Big[\big(\pi_{\mathfrak{v}} \circ \exp_K(sk)\big)U_{\mathfrak{v}}, \big(\pi_{\mathfrak{v}} \circ \exp_K(sk)\big)V_{\mathfrak{v}}\Big]\Big) \\ &= I.\end{aligned}$$

We compare this expression with

$$\begin{aligned}\rho_{k,s}\big(\exp_G(U) * \exp_G(V)\big) &= \exp_G\Big(\big(\pi_{\mathfrak{z}} \circ \exp_K(sk)\big)(U_{\mathfrak{z}} + V_{\mathfrak{z}}) \\ &\quad + \big(\pi_{\mathfrak{v}} \circ \exp_K(sk)\big)(U_{\mathfrak{v}} + V_{\mathfrak{v}}) + \frac{1}{2}\big(\pi_{\mathfrak{z}} \circ \exp_K(sk)\big)[U, V]\Big) = II.\end{aligned}$$

Now Lemma 3.3, (b) shows that $I = II$.

We show that $\rho := \rho_{k,s}$ is an isometry of G, which means that for each $p \in G$ the map

$$\mathrm{d}\Big(L_{\rho(p)^{-1}} \circ \rho \circ L_p\Big)_{e_G} : \mathfrak{g} \longrightarrow \mathfrak{g}$$

is isometric on $\mathfrak{g}$. Since ρ is a homomorphism, we have with $p, p' \in G$:

$$L_{\rho(p)^{-1}} \circ \rho \circ L_p(p') = L_{\rho(p)^{-1}} \circ \rho(p * p') = L_{\rho(p)^{-1}}\Big(\rho(p) * \rho(p')\Big) = \rho(p').$$

From the definition of ρ, note that

$$(\mathrm{d}\rho)_{e_G} = \begin{pmatrix} \pi_{\mathfrak{z}} \circ \exp_K(sk) & 0 \\ 0 & \pi_{\mathfrak{v}} \circ \exp_K(sk) \end{pmatrix}.$$

Since the matrices on the diagonal are isometries of $\mathfrak{z}$ and $\mathfrak{v}$, respectively, it follows that $(\mathrm{d}\rho)_{e_G}$ is an isometry of $\mathfrak{g}$. A direct calculation show that $s \mapsto \rho_{k,s}$ defines a homomorphism on $(\mathbb{R}, +)$. In particular, $\rho_{k,s}$ is bijective since $\rho_{k,0} = \mathrm{Id}$. □

Let $k = (A, B) \in \mathfrak{k}$. Proposition 3.5 allows to calculate a corresponding Killing vector field X_k^* which induces the flow $(\rho_{k,s})_s$ on G. From the Baker–Campbell–Hausdorff formula we have:

$$\exp_G(U + V) = \exp_G(U) \exp_G\left(V - \frac{1}{2}[U, V]\right), \quad \textit{for all } U, V \in \mathfrak{g}$$

and therefore

$$\big(\mathrm{d}\exp_G\big)_{U_{\mathfrak{v}}}(V) = \mathrm{d}L_{\exp_G(U_{\mathfrak{v}})}\Big(V - \frac{1}{2}\big[U_{\mathfrak{v}}, V\big]\Big).$$

Let $p = \exp_G(W) \in G$, then:

$$\begin{aligned}
X_k^*(p) &:= \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \rho_{k,s}(p) \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \exp_G\Big(\big(\pi_{\mathfrak{z}} \circ \exp_K(sk)\big)W_{\mathfrak{z}}\Big) \exp_G\Big(\big(\pi_{\mathfrak{v}} \circ \exp_K(sk)\big)W_{\mathfrak{v}}\Big) \\
&= \mathrm{d}L_{\exp_G(W_{\mathfrak{z}})}\mathrm{d}L_{\exp_G(W_{\mathfrak{v}})}\Big(BW_{\mathfrak{v}} - \frac{1}{2}\big[W_{\mathfrak{v}}, BW_{\mathfrak{v}}\big]\Big) \\
&\quad + \mathrm{d}L_{\exp_G(W_{\mathfrak{v}})}\mathrm{d}L_{\exp_G(W_{\mathfrak{z}})}\big(AW_{\mathfrak{z}}\big) \\
&= \mathrm{d}L_p\Big(BW_{\mathfrak{v}} - \frac{1}{2}\big[W_{\mathfrak{v}}, BW_{\mathfrak{v}}\big] + AW_{\mathfrak{z}}\Big).
\end{aligned}$$

This calculation leads to another family of first integral of the geodesic flow.

Proposition 3.6 *The Killing vector field on G corresponding to $k = (A, B) \in \mathfrak{k}$ evaluated at the point $p = \exp_G(W) \in G$ is given by:*

$$X_k^*(p) = dL_p\left(BW_{\mathfrak{v}} - \frac{1}{2}\big[W_{\mathfrak{v}}, BW_{\mathfrak{v}}\big] + AW_{\mathfrak{z}}\right). \tag{15}$$

According to Lemma 3.1, X_k^* induces a first integral $F_{X_k^*} : TG \cong G \times \mathfrak{g} \to \mathbb{R}$:

$$F_{X_k^*}(p, Y) = \big\langle X_k^*(p), \big(dL_p\big)_{e_G} Y\big\rangle_p = \Big\langle BW - \frac{1}{2}[W, BW] + AW, Y\Big\rangle, \tag{16}$$

where we have extended A and B from $\mathfrak{z}$ and $\mathfrak{v}$ to $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$ by zero, respectively.

Next we present the Poisson brackets between two first integrals $F_{X_k^*}$ and $F_{X_{k'}^*}$ in Proposition 3.6. According to the identifications in Sect. 2 we need to determine the gradient of $F_{X_k^*}$, which is defined through (1). With $p = \exp_G(W) \in G$ the differential of $F_{X_k^*}$ applied to $(U, V) \in \mathfrak{g} \times \mathfrak{g} \cong T_{(p,Y)}(TG)$ has the form:

$$\begin{aligned}
&\mathrm{d}F_{X_k^*}(p, Y) \cdot (U, V)\\
&\quad = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} F_{X_k^*}\Big(p * \exp_G(\epsilon U), Y + \epsilon V\Big)\\
&\quad = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} F_{X_k^*}\Big(\exp_G\big(W + \epsilon U + \frac{\epsilon}{2}[W, U]\big), Y + \epsilon V\Big)\\
&\quad = \Big\langle BW - \frac{1}{2}\big[W, BW\big] + AW, V\Big\rangle\\
&\qquad + \Big\langle BU + AU + \frac{1}{2}A[W, U] - \frac{1}{2}\big[W_{\mathfrak{v}}, BU_{\mathfrak{v}}\big] - \frac{1}{2}\big[U_{\mathfrak{v}}, BW_{\mathfrak{v}}\big], Y\Big\rangle.
\end{aligned}$$

Using $\langle [W, U], Z\rangle = \langle j(Z_{\mathfrak{z}})W_{\mathfrak{v}}, U\rangle$ for all $W, U, Z \in \mathfrak{g}$ and comparing with (1) shows:

$$\begin{aligned}
\mathrm{grad}_{(p,W)} F_{X_k^*} = \Big(&- BY - AY - \frac{1}{2}j(AY_{\mathfrak{z}})W + \frac{1}{2}\Big(Bj(Y_{\mathfrak{z}})W + j(Y_{\mathfrak{z}})BW\Big),\\
&BW - \frac{1}{2}\big[W, BW\big] + AW\Big).
\end{aligned}$$

Since $(A, B) \in \mathfrak{k}$ we can use $Bj\big(Y_{\mathfrak{z}}\big) = j(Y_{\mathfrak{z}})B + j\big(AY_{\mathfrak{z}}\big)$ to simplify the expression.

Lemma 3.7 *Let $k = (A, B) \in \mathfrak{k}$, $(p, Y) \in TG \cong G \times \mathfrak{g}$, and $\exp_G W = p$. Then:*

$$\mathit{grad}_{(p,Y)} F_{X_k^*} = \Big(-BY_{\mathfrak{v}} + j(Y_{\mathfrak{z}})BW_{\mathfrak{v}} - AY_{\mathfrak{z}}, BW_{\mathfrak{v}} - \frac{1}{2}\big[W_{\mathfrak{v}}, BW_{\mathfrak{v}}\big] + AW_{\mathfrak{z}}\Big). \tag{17}$$

For $i = 1, \ldots, 2m$, we have

$$\mathit{grad}_{(p,Y)} F_{X_i^{(r)}} = \Big(-j\big(Y_{\mathfrak{z}}\big)X_i, -X_i + [W, X_i]\Big). \tag{18}$$

Formula (18) can be proved by similar calculations applying (1) and Proposition 3.2. Based on Lemma 3.7 and (3) we obtain the Poisson brackets between the above first integrals. Let $k = (A, B)$, $k' = \big(A', B'\big) \in \mathfrak{k}$ and by $\big[k, k'\big] = \big(\big[A, A'\big], \big[B, B'\big]\big)$ denote the Lie bracket in $\mathfrak{k}$.

Proposition 3.8 *By $F_{X_i^{(r)}}$ and F_{Z_ℓ} for $i = 1, \ldots, 2m$, $\ell = 1, \ldots, d$, we denote the first integrals in Proposition 3.2. Let*

$$g : TG \cong G \times \mathfrak{g} \to \mathbb{R}$$

be a left-invariant differentiable function with grad $g(p, Y) = (0, V')$. With $p = \exp_G(W)$ we have:

(a) $\{F_{X_k^*}, F_{X_{k'}^*}\} = F_{X^*_{[k,k']}}$,
(b) $\big\{F_{X_k^*}, F_{X_i^{(r)}}\big\}(p, Y) = \big\langle BX_i, j(Y_{\mathfrak{z}})W_{\mathfrak{v}} - Y_{\mathfrak{v}}\big\rangle = F_{(BX_i)^{(r)}}(p, Y)$,
(c) $\big\{F_{X_i^{(r)}}, F_{X_{i'}^{(r)}}\big\}(p, Y) = \big\langle [X_i, X_{i'}], Y\big\rangle = F_{[X_i, X_{i'}]}(p, Y)$,
(d) $\big\{F_{X_k^*}, g\big\}(p, Y) = \big\langle Y_{\mathfrak{v}}, BV'_{\mathfrak{v}}\big\rangle + \big\langle Y_{\mathfrak{z}}, AV'_{\mathfrak{z}}\big\rangle$,
(e) $\big\{F_{X_k^*}, F_{Z_\ell}\big\}(p, Y) = \big\langle Y_{\mathfrak{z}}, AZ_\ell\big\rangle = F_{(AZ_\ell)}(p, Y)$,
(f) $\big\{F_{X_i^{(r)}}, g\big\}(p, Y) = 0$.

In particular, the Hamiltonian H of the geodesic flow Poisson commutes with $F_{X_k^*}$ as well as with $F_{X_i^{(r)}}$ where $i = 1, \dots, 2m$ and F_{Z_ℓ}, $\ell = 1, \dots, d$.

Proof The formulae in (a)–(f) follow by a direct calculation. We only show the short proof of the last statement which directly follows from Lemma 3.1. Recall that grad $H(p, Y) = (0, Y)$ and therefore (d) implies:

$$\{F_{X_k^*}, H\} = \langle Y_{\mathfrak{v}}, BY_{\mathfrak{v}}\rangle + \langle Y_{\mathfrak{z}}, AY_{\mathfrak{z}}\rangle.$$

Since A and B are skew-symmetric the Poisson bracket vanishes. □

Combining the statements in Propositions 3.2 and 3.6 we assign first integrals of the geodesic flow to arbitrary elements of the semi-direct product $\mathfrak{k} \ltimes_\tau \mathfrak{g}$.

Denote by $\mathrm{Der}(\mathfrak{g})$ the Lie algebra of derivations on $\mathfrak{g}$. A Lie algebra homomorphism is obtained by:

$$\tau : \mathfrak{k} \to \mathrm{Der}(\mathfrak{g}) : (A, B) \mapsto \Big[\mathfrak{g} \ni U = U_{\mathfrak{z}} + U_{\mathfrak{v}} \mapsto AU_{\mathfrak{z}} + BU_{\mathfrak{v}} \in \mathfrak{g}\Big], \tag{19}$$

i.e. by a direct calculation using $j(AZ) = Bj(Z) - j(Z)B$ for all $Z \in \mathfrak{z}$ one finds:

$$\tau(A, B)\big[U, W\big] = \big[U, \tau(A, B)W\big] + \big[\tau(A, B)U, W\big], \qquad U, W \in \mathfrak{g}.$$

Recall that via the map τ we can form the semi-direct product $\mathfrak{k} \ltimes_\tau \mathfrak{g}$ retaining the brackets in $\mathfrak{k}$ and $\mathfrak{g}$ and satisfying:

$$\big[(A, B), U\big] = \tau(A, B)(U) \quad \textit{where } (A, B) \in \mathfrak{k}, \ U \in \mathfrak{g}. \tag{20}$$

We use the notation in Proposition 3.2. Consider $\Psi : \mathfrak{k} \oplus \mathfrak{g} \to \mathcal{C}^\infty(TG)$ defined by

$$\Psi\big(k, U\big) := F_{X_k^*} + \sum_{i=1}^{2m} a_i F_{X_i^{(r)}} + \sum_{\ell=1}^{d} b_\ell F_{Z_\ell}, \ \ k = (A, B) \in \mathfrak{k}, \tag{21}$$

with $U = \sum_{i=1}^{2m} a_i X_i + \sum_{\ell=1}^{d} b_\ell Z_\ell$. Theorem 3.9 below extends Theorem 3.6. in [14].

Theorem 3.9 *The map* Ψ *in* (21) *defines an injective Lie algebra homomorphism*

$$\Psi : \mathfrak{k} \ltimes_\tau \mathfrak{g} \to \Big(C^\infty(TG), \{\cdot, \cdot\}\Big). \tag{22}$$

Moreover, functions in the range of Ψ *are first integrals of the geodesic flow.*

Proof To prove the first statement we apply Proposition 3.8. For $i, i' = 1, \ldots, 2m$ and $\ell = 1, \ldots, d$ and $k, k' \in \mathfrak{k}$ we have:

$$\begin{aligned}
\Psi\Big(\big[(k,0),(0,X_i)\big]\Big) &= \Psi\big(\tau(k)(X_i)\big) = \Psi(BX_i) = F_{(BX_i)^{(r)}} = \big\{F_{X_k^*}, F_{X_i^{(r)}}\big\},\\
\Psi\Big(\big[(k,0),(0,Z_\ell)\big]\Big) &= \Psi\big(\tau(k)(Z_\ell)\big) = \Psi(AZ_\ell) = F_{AZ_\ell} = \big\{F_{X_k^*}, F_{Z_\ell}\big\},\\
\Psi\big([k,k']\big) &= F_{X^*_{[k,k']}} = \big\{F_{X_k^*}, F_{X_{k'}^*}\big\},\\
\Psi\Big([X_i, X_{i'}]\Big) &= F_{[X_i, X_{i'}]} = \big\{F_{X_i^{(r)}}, F_{X_{i'}^{(r)}}\big\}.
\end{aligned}$$

Hence (22) defines a Lie algebra homomorphism. It remains to prove the injectivity of Ψ. Assume that for all $(p, Y) \in G \times \mathfrak{g}$:

$$\Psi(k,U)(p,Y) = F_{X_k^*}(p,Y) + \sum_{i=1}^{2m} a_i F_{X_i^{(r)}}(p,Y) + \sum_{\ell=1}^{d} b_\ell F_{Z_\ell}(p,Y) = 0. \tag{23}$$

Choose $p = \exp(Z_\ell)$ and $Y = Z_\ell$ with $\ell \in \{1, \ldots, d\}$. Proposition 3.2 shows:

$$F_{Z_{\ell'}}(p, Z_\ell) = \langle Z_{\ell'}, Z_\ell\rangle = \delta_{\ell',\ell} \quad \textit{and} \quad F_{X_i^{(r)}}(p, Z_\ell) = -\big\langle X_i, Z_\ell\big\rangle = 0.$$

Proposition 3.6 implies that $F_{X_k^*}(p, Z_\ell) = \langle AZ_\ell, Z_\ell\rangle_{\mathfrak{z}} = 0$, since A is skew-symmetric. Therefore all the coefficients b_ℓ in (23) must vanish.

Now we consider (23) at points (p, Z_ℓ) where $\ell \in \{1, \ldots, d\}$ and $p = \exp(W) \in G$ is arbitrary. Using again Propositions 3.2 and 3.6 shows:

$$0 = \Big\langle -\frac{1}{2}[W_{\mathfrak{v}}, BW_{\mathfrak{v}}] + AW_{\mathfrak{z}}, Z_\ell\Big\rangle_{\mathfrak{z}} + \sum_{i=1}^{2m} a_i \Big\langle X_i, j(Z_\ell)W_{\mathfrak{v}}\Big\rangle. \tag{24}$$

Choosing $W = Z_r$ for $r = 1, \ldots, d$ gives $\langle AZ_r, Z_\ell\rangle_{\mathfrak{z}} = 0$ and therefore $A = 0$. In particular, it follows for all $Z \in \mathfrak{z}$:

$$0 = j(AZ) = Bj(Z) - j(Z)B. \tag{25}$$

If we replace $W_{\mathfrak{v}}$ by $tW_{\mathfrak{v}}$ with $t \in \mathbb{R}$, then the first term on the right of (24) is of quadratic order in t whereas the second summand is of linear order. Therefore:

$$0 = \Big\langle \sum_{i=1}^{2m} a_i X_i, j(Z_\ell)W_{\mathfrak{v}}\Big\rangle \qquad \textit{for all} \;\; W_{\mathfrak{v}} \in \mathfrak{v}, \tag{26}$$

$$0 = \Big\langle \big[W_{\mathfrak{v}}, BW_{\mathfrak{v}} \big], Z_\ell \Big\rangle = \big\langle j(Z_\ell) W_{\mathfrak{v}}, BW_{\mathfrak{v}} \big\rangle = -\big\langle Bj(Z_\ell) W_{\mathfrak{v}}, W_{\mathfrak{v}} \big\rangle. \tag{27}$$

First we use (26). Since $j(Z_\ell)$ is assumed to be invertible on $\mathfrak{v}$ we have $\sum_{i=1}^{2m} a_i X_i = 0$ showing that $a_i = 0$ for $i = 1, \ldots, 2m$ and therefore $U = 0$.

From (25) together with the skew-symmetry of B and $j(Z)$ we see that $Bj(Z)$ is symmetric with respect to $\langle \cdot, \cdot \rangle$. Now (27), the polarization identity, and the non-degeneracy of $\langle \cdot, \cdot \rangle$ show that $Bj(Z_\ell) = 0$ which (by invertibility of $j(Z_\ell)$) implies that $B = 0$. We conclude that $k = (A, B) = 0$ showing the injectivity of Ψ. □

Remark 3.10 In calculating the functions $F_{X_k^*}$ in Proposition 3.6 we have assumed that the metric $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ is positive-definite. In this case $\mathfrak{k}$ is defined as the Lie algebra of the isotropy group K of $I(G)$. However, in the more general setting where $\langle \cdot, \cdot \rangle$ is only assumed to be a non-degenerate bilinear form (e.g. $\mathfrak{g} = \mathcal{N}_{r,s}$ the Lie algebra of a pseudo-H-type group $G = G_{r,s}$ with $s > 0$) we can use the right hand side of (13) as the definition of $\mathfrak{k}$. The functions $F_{X_k^*}$ are then defined by the expression (16). A direct calculation shows that the Poisson bracket relations in Proposition 3.8 remain true and also Theorem 3.9 holds in this more general setting.

4 Statement and proof of the main result

In this section, we prove the complete integrability of the (pseudo-)Riemannian geodesic flow of G for the left-invariant (pseudo-)Riemannian metric $\langle \cdot, \cdot \rangle$. Motivated by the first integrals $F_{X_i^{(r)}}$, $i = 1, \ldots, 2m$, constructed in Sect. 3, we introduce the function $F_\alpha \in \mathcal{C}^\infty(TG)$ defined through

$$F_\alpha(p, Y) := \sum_{i=1}^{2m} \alpha_i \left(Y_{\mathfrak{z}} \right) F_{X_i^{(r)}}(p, Y), \qquad (p, Y) \in G \times \mathfrak{g} \cong TG,$$

associated with an arbitrary differentiable mapping $\alpha : \mathfrak{z} \to \mathfrak{v}$. Here, $\alpha_i \in \mathcal{C}^\infty(\mathfrak{z})$ is the coefficients of the linear combination $\alpha\left(Y_{\mathfrak{z}}\right) = \sum_{i=1}^{2m} \alpha_i \left(Y_{\mathfrak{z}}\right) X_i$ in the basis $[X_1, \ldots, X_{2m}]$ of $\mathfrak{v}$. Then, F_α Poisson commutes with any left-invariant function $g \in \mathcal{C}^\infty(TG)$, i. e. $\{F_\alpha, g\} = 0$, and hence with the Hamiltonian H. This can be proved easily by Leibniz rule of Poisson bracket, (5), and Proposition 3.8.

Proposition 4.1 *If $\alpha, \beta : \mathfrak{z} \to \mathfrak{v}$ are differentiable mappings, we have*

$$\left\{ F_\alpha, F_\beta \right\}(p, Y) = \left\langle j\left(Y_{\mathfrak{z}}\right) \alpha\left(Y_{\mathfrak{z}}\right), \beta\left(Y_{\mathfrak{z}}\right) \right\rangle = \left\langle Y_{\mathfrak{z}}, \left[\alpha\left(Y_{\mathfrak{z}}\right), \beta\left(Y_{\mathfrak{z}}\right) \right] \right\rangle. \tag{28}$$

Proof In view of the linear combinations $\alpha = \sum_{i=1}^{2m} \alpha_i X_i$, $\beta = \sum_{i'=1}^{2m} \beta_{i'} X_{i'}$, we have

$$\begin{aligned}\left\{F_\alpha, F_\beta\right\}(p, Y) &= \sum_{i,i'=1}^{2m} \alpha_i\left(Y_{\mathfrak{z}}\right)\beta_{i'}\left(Y_{\mathfrak{z}}\right)\left\{F_{X_i^{(r)}}, F_{X_{i'}^{(r)}}\right\}(p, Y) \\ &= \sum_{i,i'=1}^{2m} \alpha_i\left(Y_{\mathfrak{z}}\right)\beta_{i'}\left(Y_{\mathfrak{z}}\right)\left\langle j\left(Y_{\mathfrak{z}}\right)X_i, X_{i'}\right\rangle = \left\langle j\left(Y_{\mathfrak{z}}\right)\alpha\left(Y_{\mathfrak{z}}\right), \beta\left(Y_{\mathfrak{z}}\right)\right\rangle,\end{aligned}$$

by Proposition 3.8. □

Now, we construct the first integrals for the geodesic flow concretely, choosing an appropriate set of first integrals F_α. The key idea in the construction of a sufficient number of first integrals is based on the normalization of the operator $j(Z)$, $Z \in \mathfrak{z}$. We consider the case of pseudo-H-type Lie groups, where we assume that the scalar product $\langle \cdot, \cdot \rangle$ is not positive-definite. Then, we address the case of H-type Lie groups.

Complete integrability of pseudo-H-type Lie groups. We consider the complete integrability of the geodesic flow of pseudo-H-type nilpotent Lie groups. Assuming that the scalar product $\langle \cdot, \cdot \rangle_{\mathfrak{z}}$ is not positive-definite, we see that $\langle \cdot, \cdot \rangle_{\mathfrak{v}}$ is neutral in the sense that the signature of $\langle \cdot, \cdot \rangle_{\mathfrak{v}}$ is (m, m) (cf [9, Prop. 2.1, p. 984]). Thus, we can take an orthonormal basis (according to the notation in (8)) $[X_1, \ldots, X_{2m}]$ of $\mathfrak{v}$ such that

$$\langle X_i, X_{i'} \rangle_{\mathfrak{v}} = \begin{cases} \delta_{i,i'}, & \text{if } i, i' = 1, \ldots, m, \\ -\delta_{i,i'}, & \text{if } i, i' = m+1, \ldots, 2m, \\ 0, & \text{otherwise.} \end{cases} \tag{29}$$

Now, we fix $Z \in \mathfrak{z}$ and suppose that $\langle Z, Z \rangle_{\mathfrak{z}} \neq 0$. We show that m is even and that, for Z in an open dense subset of $\mathfrak{z}$, we can construct a suitable basis $[w_1, \ldots, w_{2m}]$ of $\mathfrak{v}$, such that $\left\langle w_q, w_{q'} \right\rangle_{\mathfrak{v}} = 0$ if $q \neq q'$,

$$\begin{aligned} &\begin{cases} j(Z)\, w_{2i-1} = \sqrt{\left|\langle Z, Z \rangle_{\mathfrak{z}}\right|}\, w_{2i}, \\ j(Z)\, w_{2i} = -\frac{\langle Z, Z \rangle_{\mathfrak{z}}}{\sqrt{\left|\langle Z, Z \rangle_{\mathfrak{z}}\right|}} w_{2i-1}, \end{cases} \\ \langle w_{2i-1}, w_{2i-1} \rangle_{\mathfrak{v}} &= \begin{cases} 1 & \text{if } i = 1, \ldots, m/2, \\ -1 & \text{if } i = 1 + m/2, \ldots, m, \end{cases} \\ \langle w_{2i}, w_{2i} \rangle_{\mathfrak{v}} &= \frac{\langle Z, Z \rangle_{\mathfrak{z}}}{\left|\langle Z, Z \rangle_{\mathfrak{z}}\right|} \langle w_{2i-1}, w_{2i-1} \rangle_{\mathfrak{v}}, \end{aligned} \tag{30}$$

for $i = 1, \ldots, m$.

If $\langle Z, Z \rangle_{\mathfrak{z}} > 0$, the matrix representation of $j(Z)$ with respect to the basis $[w_1, \ldots, w_{2m}]$, which satisfies (30), is

$$\sqrt{\langle Z, Z \rangle_{\mathfrak{z}}}\, \mathrm{diag}\,(S_2, \ldots, S_2), \quad where \quad S_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and if $\langle Z, Z\rangle_{\mathfrak{z}} < 0$, the one with respect to the basis

$$\big[w_1, w_3, \ldots, w_{m-1}, w_{m+2}, w_{m+4}, \ldots, w_{2m}, w_2, w_4, \\ \ldots, w_m, w_{m+1}, w_{m+3}, \ldots, w_{2m-1}\big],$$

which satisfies (30), is

$$\sqrt{-\langle Z, Z\rangle_{\mathfrak{z}}}\begin{pmatrix} 0 & E_m \\ E_m & 0 \end{pmatrix},$$

where E_m is the $m \times m$ unit matrix.

The existence of the basis $[w_1, \ldots, w_{2m}]$ of $\mathfrak{v}$ satisfying (30) can be verified by direct calculations as follows:

We fix $Z \in \mathfrak{z}$ such that $\langle Z, Z\rangle_{\mathfrak{z}} \neq 0$. We set $w_1 := X_1$ and $w_2 := \dfrac{j(Z)X_1}{\sqrt{|\langle Z, Z\rangle_{\mathfrak{z}}|}}$. Then, (30) is satisfied for $i = 1$. Reordering $X_2, \ldots, X_{2m}$ if necessary, we can assume that $[w_1, w_2, X_3, \ldots, X_{2m}]$ is a basis of $\mathfrak{v}$.

Next, we assume that we have constructed orthogonal vectors $w_1, \ldots, w_{2i'} \in \mathfrak{v}$ such that (30) is satisfied for $i = 1, \ldots, i'$ and that $\big[w_1, \ldots, w_{2i'}, X_{2i'+1}, \ldots, X_{2m}\big]$ is a basis of $\mathfrak{v}$. If we set

$$X'_{2i'+1} := X_{2i'+1} - \sum_{q=1}^{2i'} \frac{\langle X_{2i'+1}, w_q\rangle_{\mathfrak{v}}}{\langle w_q, w_q\rangle_{\mathfrak{v}}} w_q,$$

then $\Big\langle X'_{2i'+1}, w_q\Big\rangle_{\mathfrak{v}} = 0$, $q = 1, \ldots, 2i'$. Note that

$$\langle X'_{2i'+1}, X'_{2i'+1}\rangle_{\mathfrak{v}} = \langle X_{2i'+1}, X_{2i'+1}\rangle_{\mathfrak{v}} - \sum_{q=1}^{2i'} \frac{\langle X_{2i'+1}, w_q\rangle_{\mathfrak{v}}^2}{\langle w_q, w_q\rangle_{\mathfrak{v}}}$$

is a rational polynomial in the components of Z and we denote it by $P_{i'}(Z)$. We assume that $P_{i'}(Z) \neq 0$ and set

$$w_{2i'+1} := \frac{X'_{2i'+1}}{\sqrt{\Big|\Big\langle X'_{2i'+1}, X'_{2i'+1}\Big\rangle_{\mathfrak{v}}\Big|}} \qquad \textit{and} \qquad w_{2i'+2} := \frac{j(Z)\,w_{2i'+1}}{\sqrt{|\langle Z, Z\rangle_{\mathfrak{z}}|}}.$$

It is easy to check that (30) is satisfied for $i = 1, \ldots, i' + 1$. Reordering $X_{2i'+2}, \ldots, X_{2m}$ if necessary, we see that $\big[w_1, \ldots, w_{2i'+2}, X_{2i'+3}, \ldots, X_{2m}\big]$ is a basis of $\mathfrak{v}$. Inductively, we obtain a basis $[w_1, \ldots, w_{2m}]$ of $\mathfrak{v}$ with the desired property (30). Note that, by (6), we see that m must be even.

We write $Y = Y_{\mathfrak{z}} + Y_{\mathfrak{v}}$, $Y_{\mathfrak{v}} = \sum_{q=1}^{2m} y_q w_q \left(Y_{\mathfrak{z}}\right)$, $Y_{\mathfrak{z}} = \sum_{\ell=1}^{d} y_\ell^{\mathfrak{z}} Z_\ell$, where $w_q = w_q\left(Y_{\mathfrak{z}}\right)$ is regarded as a $\mathfrak{v}$-valued function of $Y_{\mathfrak{z}}$. Note that each vector w_q, $q = 1, \dots, 2m$, is a rational polynomial in $y_1^{\mathfrak{z}}, \dots, y_d^{\mathfrak{z}}$ and $\left\langle Y_{\mathfrak{z}}, Y_{\mathfrak{z}}\right\rangle_{\mathfrak{z}}$ which have no pole unless

$$\left\langle Y_{\mathfrak{z}}, Y_{\mathfrak{z}}\right\rangle_{\mathfrak{z}} \cdot P\left(Y_{\mathfrak{z}}\right) = 0, \qquad \textit{where}\ \ P\left(Y_{\mathfrak{z}}\right) = \prod_{i=1}^{m-1} P_i\left(Y_{\mathfrak{z}}\right),$$

and hence $F_{w_{2i-1}}$, $i = 1, \dots, m$, is differentiable if $\left\langle Y_{\mathfrak{z}}, Y_{\mathfrak{z}}\right\rangle \cdot P\left(Y_{\mathfrak{z}}\right) \neq 0$, while it may be singular if $\left\langle Y_{\mathfrak{z}}, Y_{\mathfrak{z}}\right\rangle \cdot P\left(Y_{\mathfrak{z}}\right) = 0$. To construct globally defined differentiable functions from $F_{u_{2i-1}}$, $i = 1, \dots, m$, we define $\psi \in \mathcal{C}^\infty(\mathbb{R})$ by

$$\psi(x) = \begin{cases} \exp\left(-\frac{1}{x}\right) & \text{if } x > 0, \\ 0 & \text{if } x \leq 0 \end{cases}$$

and consider the differentiable functions

$$\tilde{F}_{w_{2i-1}}(p, Y) := \psi\left(\left\langle Y_{\mathfrak{z}}, Y_{\mathfrak{z}}\right\rangle_{\mathfrak{z}}^2 \cdot P\left(Y_{\mathfrak{z}}\right)^2\right) F_{w_{2i-1}}(p, Y)$$

on $G \times \mathfrak{g} \cong TG$. We easily see that $\tilde{F}_{w_{2i-1}}$, $i = 1, \dots, m$, mutually Poisson commute by (28), (30), and Proposition 3.8.

We suppose that $\left\langle Y_{\mathfrak{z}}, Y_{\mathfrak{z}}\right\rangle_{\mathfrak{z}} > 0$ and consider the left-invariant functions $g_i^+(p, Y) := y_{2i-1}^2 + y_{2i}^2$, $i = 1, \dots, m$. Then, we have $\mathrm{grad}_{(p,Y)} g_i^+ = (0, 2\left(y_{2i-1} w_{2i-1} + y_{2i} w_{2i}\right))$. Multiplication by ψ gives m differentiable functions

$$\tilde{g}_i^+(p, Y) := \psi\left(\left\langle Y_{\mathfrak{z}}, Y_{\mathfrak{z}}\right\rangle_{\mathfrak{z}} \cdot P\left(Y_{\mathfrak{z}}\right)^2\right) g_i^+(p, Y),$$

globally defined on $G \times \mathfrak{g} \cong TG$. Note that $\tilde{g}_i^+(p, Y) = 0$ if $\left\langle Y_{\mathfrak{z}}, Y_{\mathfrak{z}}\right\rangle_{\mathfrak{z}} \leq 0$. We then have $\left\{\tilde{g}_i^+, \tilde{g}_{i'}^+\right\} = 0$ for all $i, i' = 1, \dots, m$ by (4), (5), and (30). Note that the Hamiltonian for the geodesic flow can be written as

$$H(p, Y) = \frac{1}{2}\langle Y, Y\rangle = \frac{1}{2}\left(\sum_{i=1}^{m/2}\left(g_i^+(p, Y) - g_{i+m/2}^+(p, Y)\right) + \left\langle Y_{\mathfrak{z}}, Y_{\mathfrak{z}}\right\rangle_{\mathfrak{z}}\right),$$

from which we see that it Poisson commutes with $\tilde{g}_i^+$, $i = 1, \dots, m$.

If $\left\langle Y_{\mathfrak{z}}, Y_{\mathfrak{z}}\right\rangle_{\mathfrak{z}} < 0$, we think of the left-invariant functions $g_i^-(p, Y) := y_{2i-1}^2 - y_{2i}^2$, $i = 1, \dots, m$. We have $\mathrm{grad}_{(p,Y)} g_i^- = (0, 2\left(y_{2i-1} w_{2i-1} - y_{2i} w_{2i}\right))$. Using the function ψ, we have the differentiable function

$$\tilde{g}_i^-(p, Y) := \psi\left(-\left\langle Y_{\mathfrak{z}}, Y_{\mathfrak{z}}\right\rangle_{\mathfrak{z}} \cdot P\left(Y_{\mathfrak{z}}\right)^2\right) g_i^-(p, Y),$$

globally defined on $G \times \mathfrak{g} \cong TG$. Note that $\tilde{g}_i^-(p, Y) = 0$ if $\langle Y_\mathfrak{z}, Y_\mathfrak{z} \rangle_\mathfrak{z} \geq 0$. We have $\{\tilde{g}_i^-, \tilde{g}_{i'}^-\} = 0$ for all $i, i' = 1, \ldots, m$ by (4), (5), and (30). The Hamiltonian is written in this case as

$$H(p, Y) = \frac{1}{2} \langle Y, Y \rangle = \frac{1}{2} \left(\sum_{i=1}^{m/2} \left(g_i^-(p, Y) - g_{i+m/2}^-(p, Y) \right) + \langle Y_\mathfrak{z}, Y_\mathfrak{z} \rangle_\mathfrak{z} \right).$$

We can see that H Poisson commutes with $\tilde{g}_i^-$, $i =, 1 \ldots, m$.

To construct functionally independent first integrals on TG, we further consider the function

$$\tilde{g}_i(p, Y) := \tilde{g}_i^+(p, Y) + \tilde{g}_i^-(p, Y),$$

where $i = 1, \ldots, m$. In addition, we take the first integrals $h_\ell(Y_\mathfrak{z}) = y_\ell^\mathfrak{z}$, $\ell = 1, \ldots, d$, where $Y_\mathfrak{z} = \sum_{\ell=1}^{d} y_\ell^\mathfrak{z} Z_\ell \in \mathfrak{z}$.

Theorem 4.2 *The functions $\tilde{F}_{w_{2i-1}}$, $\tilde{g}_i$, $i = 1, \ldots, m$, h_ℓ, $\ell = 1, \ldots, d$, in $\mathcal{C}^\infty(TG)$ are functionally independent and Poisson commuting first integrals for the pseudo-Riemannian geodesic flow of the pseudo-H-type Lie group $G = G_{r,s}$ where $s > 0$.*

Complete integrability on H-type Lie groups. In the case of H-type groups, the scalar product $\langle \cdot, \cdot \rangle$, as well as $\langle \cdot, \cdot \rangle_\mathfrak{z}$ and $\langle \cdot, \cdot \rangle_\mathfrak{v}$, is positive-definite. Starting with an orthonormal basis $[X_1, \ldots, X_{2m}]$ of $\mathfrak{v}$, we can normalize the operator $j(Z)$, $Z \in \mathfrak{z}$. with respect to a suitable orthonormal basis $w_1, \ldots, w_{2m}$ exactly as in the case of pseudo-H-type groups under the condition $\langle Z, Z \rangle_\mathfrak{z} > 0$. The existence of such an orthonormal basis of $\mathfrak{v}$ is guaranteed by the normal form of skew-symmetric matrices, for which an equivalent but sophisticated description is given in [3, Lem. 2.3, p.777]. Note that m not necessarily is even in the case of H-type groups.

The first integrals $\tilde{F}_{w_{2i-1}}$, $\tilde{g}_i^+$, $i = 1, \ldots, m$, and $h_\ell^\mathfrak{z}$, $\ell = 1, \ldots, d$, are globally defined differentiable functions on TG and, in particular, they are functionally independent on TG. The Hamiltonian of the geodesic flow for the H-type groups is given as

$$H(p, Y) = \frac{1}{2} \langle Y, Y \rangle = \frac{1}{2} \left(\sum_{i=1}^{m} g_i^+(p, Y) + \langle Y_\mathfrak{z}, Y_\mathfrak{z} \rangle_\mathfrak{z} \right),$$

from which we can conclude the following theorem.

Theorem 4.3 *The functions $\tilde{F}_{w_{2i-1}}$, $\tilde{g}_i^+$, $i = 1, \ldots, m$, and h_ℓ, $\ell = 1, \ldots, d$ on TG, are functionally independent and Poisson commuting first integrals for the geodesic flow of the H-type nilpotent Lie group $G = G_{r,0}$.*

5 Complete integrability on nilmanifolds

It has been shown in [9] that each pseudo-H-type Lie group $G_{r,s}$ contains a lattice L (co-compact discrete subgroup). In the present section we explicitly construct commuting first integrals on the compact pseudo-H-type nilmanifold $M = L \setminus G_{r,s}$ with respect to the pseudo-Riemannian metric descended from $G_{r,s}$. In cases where the Lie algebra $\mathcal{N}_{r,s}$ of $G_{r,s}$ is of Heisenberg-Reiter type (HR-type) we descend sufficiently many first integrals to M to prove the complete integrability of the pseudo-Riemannian geodesic flow of M. We may as well assume that $G_{r,s}$ is equipped with a left-invariant Riemannian metric g and $\mathcal{N}_{r,s}$ is of HR-type. In [3] Butler has shown that under these assumptions the geodesic flow of g is smoothly Liouville integrable on $T^*(L \setminus G_{r,s})$ (see also [14] for the case of the Heisenberg group). However, it seems that a complete classification of pseudo-H-type Lie algebras of HR-type is not known and we leave this problem for a future investigation.

Definition 5.1 A step-two nilpotent Lie algebra $\mathfrak{g}$ is called a *Heisenberg-Reiter Lie algebra*, if $\mathfrak{g}$ admits a decomposition

$$\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{n} \oplus \mathfrak{z} \tag{31}$$

such that $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{z}$, $[\mathfrak{z}, \mathfrak{g}] = 0$, $[\mathfrak{r}, \mathfrak{r}] = 0$, and $[\mathfrak{n}, \mathfrak{n}] = 0$. The decomposition (31) is called a *presentation* of $\mathfrak{g}$.

Example 5.2 Let $r = 0$ and consider $G_{0,s}$ with Lie algebra $\mathcal{N}_{0,s} = \mathfrak{v} \oplus \mathbb{R}^{0,s}$ and center $\mathfrak{z} = \mathbb{R}^{0,s}$. As is shown in [5], $\mathfrak{v}$ has a positive-definite subspace $\mathfrak{v}_+$ and negative-definite subspace $\mathfrak{v}_-$ of the same dimension, i.e. $\langle \cdot, \cdot \rangle_{\mathfrak{v}}$ is positive-definite on $\mathfrak{v}_+$ and negative-definite on $\mathfrak{v}_-$. We can choose $\mathfrak{v}_+$ and $\mathfrak{v}_-$ orthogonal to each other. Hence we have the decomposition:

$$\mathcal{N}_{0,s} = \mathfrak{v}_+ \oplus_\perp \mathfrak{v}_- \oplus_\perp \mathbb{R}^{0,s}.$$

Since each $0 \neq Z \in \mathbb{R}^{0,s} = \mathfrak{z}$ is negative it can be shown that $j(Z) : \mathfrak{v} \to \mathfrak{v}$ maps $\mathfrak{v}_+$ to $\mathfrak{v}_-$ and vice versa. In fact, let $X \in \mathfrak{v}_+$, then (6) implies:

$$\big\langle j(Z)X, j(Z)X \big\rangle_{\mathfrak{v}} = \big\langle Z, Z \big\rangle_{\mathfrak{z}} \cdot \big\langle X, X \big\rangle_{\mathfrak{v}} < 0.$$

If we put $\mathfrak{r} = \mathfrak{v}_+$ and $\mathfrak{n} = \mathfrak{v}_-$ in (31), then $[\mathcal{N}_{0,s}, \mathcal{N}_{0,s}] \subset \mathbb{R}^{0,s} = \mathfrak{z}$ and $[\mathfrak{z}, \mathcal{N}_{0,s}] = 0$. Moreover, with $X_1, X_2 \in \mathfrak{v}_+$ and $Y_1, Y_2 \in \mathfrak{v}_-$ we have for all $Z \in \mathbb{R}^{0,s}$:

$$\big\langle [X_1, X_2], Z \big\rangle = \big\langle j(Z)X_1, X_2 \big\rangle = 0,$$
$$\big\langle [Y_1, Y_2], Z \big\rangle = \big\langle j(Z)Y_1, Y_2 \big\rangle = 0,$$

showing that $[\mathfrak{v}_+, \mathfrak{v}_+] = 0 = [\mathfrak{v}_-, \mathfrak{v}_-]$. Hence $\mathcal{N}_{0,s}$ is of Heisenberg-Reiter type.

The next example shows that the condition $r = 0$ is sufficient but not necessary for $\mathcal{N}_{r,s,}$ being of HR-type.

Example 5.3 Consider the pseudo-H-type Lie algebra $\mathcal{N}_{1,1} = \mathfrak{v} \oplus \mathbb{R}^{1,1}$ and choose a basis $\{Z_1, Z_2\}$ of $\mathbb{R}^{1,1} = \mathfrak{z}$ such that

$$\langle Z_1, Z_1 \rangle_{\mathfrak{z}} = 1, \quad \langle Z_2, Z_2 \rangle_{\mathfrak{z}} = -1, \quad \textit{and} \quad \langle Z_1, Z_2 \rangle_{\mathfrak{z}} = 0.$$

An admissible $C\ell_{1,1}$-module $\mathfrak{v}$ has dimension 4. Let $v \in \mathfrak{v}$ with $\langle v, v \rangle_{\mathfrak{v}} = 1$ and choose an integral basis (cf [2,9,10]) $[X_1, \ldots, X_4]$ of $\mathfrak{v}$ as follows:

$$X_1 = v, \quad X_2 = j(Z_1)j(Z_2)v, \quad X_3 = j(Z_1)v \quad \textit{and} \quad X_4 = j(Z_2)v.$$

Note that X_1, X_3 are positive and X_2, X_4 are negative. Moreover, from (2) we obtain the following table of commutation relations:

	X_1	X_2	X_3	X_4
X_1	0	0	Z_1	Z_2
X_2	0	0	Z_2	Z_1
X_3	$-Z_1$	$-Z_2$	0	0
X_4	$-Z_2$	$-Z_1$	0	0

If we define $\mathfrak{r} = \text{span}\{X_1, X_2\}$ and $\mathfrak{n} = \text{span}\{X_3, X_4\}$, then we obtain a decomposition of $\mathfrak{g}$ in the form (31) which shows that $\mathcal{N}_{1,1}$ is of HR-type.

We pass from the pseudo-H-type Lie group $G_{r,s}$ to the compact quotient $L \backslash G_{r,s}$ by a standard lattice (co-compact discrete subgroup) L in $G_{r,s}$. Recall that by *Malćev's Theorem* the existence of L is guaranteed if the pseudo-H-type Lie algebra $\mathcal{N}_{r,s}$ admits a basis with rational structure coefficients. This fact is proved in [9]:

Theorem 5.4 ([9]) *Let $\mathcal{N}_{r,s} = \mathfrak{v} \oplus_{\perp} \mathbb{R}^{r,s}$ be a pseudo-H-type Lie algebra. For each orthonormal basis $\{Z_\ell\}$ in the center $\mathfrak{z} \cong \mathbb{R}^{r,s}$, there exists an orthonormal basis $\{X_i\}$ in $\mathfrak{v}$ with respect to which the structure constants $c_{ii'}^{\ell}$ in*

$$[X_i, X_{i'}] = \sum_{\ell} c_{ii'}^{\ell} Z_\ell$$

only take the values $\{0, \pm 1\}$. Moreover, if $[X_i, X_{i'}]$ does not vanish, then there is a unique element $Z_{k(i,i')}$ such that

$$[X_i, X_{i'}] = \epsilon_{k(i,i')} Z_{k(i,i')}, \quad \textit{where} \ \ \epsilon_{k(i,i')} \in \{-1, 1\}. \tag{32}$$

Let $[X_1, \ldots, X_{2m}, Z_1, \ldots, Z_{r+s}]$ denote an (integral) orthonormal basis of $\mathcal{N}_{r,s}$ as described in Theorem 5.4 above. Then the lattice in $G_{r,s}$ generated by the group elements $\{\exp(X_i), \exp(Z_\ell) \ : \ i = 1, \ldots, 2m, \ell = 1, \ldots, r+s\}$ is given by:

$$L = \exp\Big\{ \sum_{i=1}^{2m} \gamma_i X_i + \frac{1}{2} \sum_{\ell=1}^{r+s} \beta_\ell Z_\ell \ : \ \gamma_i, \beta_\ell \in \mathbb{Z} \Big\}. \tag{33}$$

With $i = 1, \dots, 2m$ we consider the set of first integrals $F_{X_i^{(r)}}$ in Proposition 3.2 induced from Killing vector fields of right-invariant vector fields

$$F_{X_i^{(r)}}(p, Y) = \big\langle X_i^{(r)}(p), \big(\mathrm{d}L_p\big)_{e_G} Y\big\rangle_p = \big\langle X_i, j(Y_\mathfrak{z})W_\mathfrak{v} - Y\big\rangle \ \textit{with} \ p = \exp(W) \in G.$$

Fix an element $g = \exp(V) \in L$ in the lattice with $V \in \mathcal{N}_{r,s}$ and $V_\mathfrak{v} = \sum_{i=1}^{2m} \gamma_i X_i$ where $\gamma_i \in \mathbb{Z}$. A direct calculation using $(\log(g * p))_\mathfrak{v} = V_\mathfrak{v} + W_\mathfrak{v}$ implies

$$F_{X_i^{(r)}}(g * p, Y) = F_{X_i^{(r)}}(p, Y) + \big\langle X_i, j(Y_\mathfrak{z})V_\mathfrak{v}\big\rangle = F_{X_i^{(r)}}(p, Y) + G_{i,g}(Y_\mathfrak{z}), \tag{34}$$

where we use the notation

$$G_{i,g}(Y_\mathfrak{z}) := \big\langle X_i, j(Y_\mathfrak{z})V_\mathfrak{v}\big\rangle = \sum_{i'=1}^{2m} \gamma_{i'}\big\langle X_i, j(Y_\mathfrak{z})X_{i'}\big\rangle \quad \textit{with} \quad i = 1, \dots, 2m.$$

Consider the matrix-valued function $M(Y_\mathfrak{z}) := (\langle X_i, j(Y_\mathfrak{z})X_{i'}\rangle)_{i,i'=1}^{2m}$. We may write (34) in the more compact form:

$$\begin{pmatrix} F_{X_1^{(r)}} \\ \vdots \\ F_{X_{2m}^{(r)}} \end{pmatrix} (g * p, Y) = \begin{pmatrix} F_{X_1^{(r)}} \\ \vdots \\ F_{X_{2m}^{(r)}} \end{pmatrix} (p, Y) + M(Y_\mathfrak{z}) \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_{2m} \end{pmatrix}. \tag{35}$$

In the following let

$$\mathcal{N}_\mathfrak{z} := \big\{z \in \mathfrak{z} \ : \ \langle z, z\rangle_\mathfrak{z} = 0\big\} \subset \mathfrak{z}$$

denote the set of all null-vectors in the center $\mathfrak{z}$. If $Y_\mathfrak{z} \notin \mathcal{N}_\mathfrak{z}$, then $M(Y_\mathfrak{z})$ is invertible since $j(Y_\mathfrak{z})^2 = -\langle Y_\mathfrak{z}, Y_\mathfrak{z}\rangle_\mathfrak{z} \mathrm{id}_\mathfrak{v}$. Multiplying both sides of (35) with $M(Y_\mathfrak{z})^{-1}$ gives

$$M(Y_\mathfrak{z})^{-1} \begin{pmatrix} F_{X_1^{(r)}} \\ \vdots \\ F_{X_{2m}^{(r)}} \end{pmatrix} (g * p, Y) = M(Y_\mathfrak{z})^{-1} \begin{pmatrix} F_{X_1^{(r)}} \\ \vdots \\ F_{X_{2m}^{(r)}} \end{pmatrix} (p, Y) + \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_{2m} \end{pmatrix}.$$

From the latter relation we construct $2m$ functions on $G_{r,s} \times (\mathcal{N}_{r,s} \setminus \mathcal{N}_\mathfrak{z})$ which are invariant under multiplication from the left by elements $g \in L$ in the $G_{r,s}$-component.

Proposition 5.5 *Define an* $\mathbb{R}^{2m}$*-valued function* $\widetilde{F}(p, Y)$ *on* $G_{r,s} \times (\mathcal{N}_{r,s} \setminus \mathcal{N}_\mathfrak{z})$ *by:*

$$\widetilde{F}(p, Y) := M(Y_\mathfrak{z})^{-1} \begin{pmatrix} F_{X_1^{(r)}} \\ \vdots \\ F_{X_{2m}^{(r)}} \end{pmatrix} (p, Y) = \begin{pmatrix} \widetilde{F}_1 \\ \vdots \\ \widetilde{F}_{2m} \end{pmatrix} (p, Y).$$

Then for $i = 1, \ldots, 2m$ the functions f_i on $G_{r,s} \times \mathcal{N}_{r,s}$ defined as

$$f_i(p, Y) := \begin{cases} e^{-\frac{1}{\langle Y_3, Y_3\rangle_3^2}} \cdot \sin\Big(2\pi \widetilde{F}_i(p, Y)\Big), & \textit{if} \quad Y_3 \notin \mathcal{N}_3, \\ 0, & \textit{if} \quad Y_3 \in \mathcal{N}_3 \end{cases}$$

are invariant under the left-action of L, i.e. $f_i(g * p, Y) = f_i(p, Y)$ for all $g \in L$. Moreover, if in the above construction the functions

$$\Big\{F_{X_{i_1}^{(r)}}, \ldots, F_{X_{i_j}^{(r)}}\Big\} \subset \mathcal{C}^\infty(TG_{r,s}), \qquad i_\ell \in \{1, \ldots, 2m\},$$

pairwise Poisson commute, then $\{f_{i_1}, \ldots, f_{i_j}\}$ pairwise Poisson commute, as well.

Proof We only prove the last statement. For each $\varepsilon > 0$ consider a cut-off function $\chi_\varepsilon \in \mathcal{C}^\infty(\mathbb{R})$ with $0 \leq \chi_\varepsilon \leq 1$ and

$$\chi_\varepsilon(x) = \begin{cases} 0, & \text{if } |x| \leq \frac{\varepsilon}{2}, \\ 1, & \text{if } |x| \geq \varepsilon. \end{cases}$$

Assume that $\{F_{X_i^{(r)}}, F_{X_{i'}^{(r)}}\} = 0$ where $i, i' \in \{1, \ldots, 2m\}$. Since the Poisson bracket is a local expression, it is sufficient to prove that for all $\varepsilon > 0$:

$$\Big\{e^{\frac{-1}{\langle Y_3, Y_3\rangle^2}} \sin\Big(2\pi \widetilde{F}_i(p, Y)\chi_\varepsilon(\langle Y_3, Y_3\rangle)\Big), e^{\frac{-1}{\langle Y_3, Y_3\rangle^2}} \sin\Big(2\pi \widetilde{F}_{i'}(p, Y)\chi_\varepsilon(\langle Y_3, Y_3\rangle)\Big)\Big\}$$
$$= 0.$$

According to the Leibniz rule this equality follows if for each $\varepsilon > 0$ we can show that

$$0 = \Big\{e^{\frac{-1}{\langle Y_3, Y_3\rangle^2}}, \sin\Big(2\pi \widetilde{F}_{i'}(p, Y)\chi_\varepsilon(\langle Y_3, Y_3\rangle)\Big)\Big\} =: K_1,$$
$$0 = \Big\{\sin\Big(2\pi \widetilde{F}_i(p, Y)\chi_\varepsilon(\langle Y_3, Y_3\rangle)\Big), \sin\Big(2\pi \widetilde{F}_{i'}(p, Y)\chi_\varepsilon(\langle Y_3, Y_3\rangle)\Big)\Big\} =: K_2.$$

We now use the standard relation

$$\big\{\Phi(F_1), \Psi(F_2)\big\} = \Phi'(F_1)\Psi'(F_2)\big\{F_1, F_2\big\}, \tag{36}$$

where $\Psi, \Phi : \mathbb{R} \to \mathbb{R}$ are smooth functions and $F_1, F_2 \in \mathcal{C}^\infty(T^*G_{r,s})$. Below we choose $\Psi(t) = t$ and $\Phi(t) = \sin(2\pi t)$. By definition we have in the case of $Y_3 \notin \mathcal{N}_3$:

$$\widetilde{F}_{i'}(p,Y) = e_{i'}^T \cdot M(Y_\mathfrak{z})^{-1} \begin{pmatrix} F_{X_1^{(r)}} \\ \vdots \\ F_{X_{2m}^{(r)}} \end{pmatrix}$$
$$= -\frac{1}{\langle Y_\mathfrak{z}, Y_\mathfrak{z}\rangle_\mathfrak{z}} e_{i'}^T \cdot M(Y_\mathfrak{z}) \begin{pmatrix} F_{X_1^{(r)}} \\ \vdots \\ F_{X_{2m}^{(r)}} \end{pmatrix} = -\frac{1}{\langle Y_\mathfrak{z}, Y_\mathfrak{z}\rangle_\mathfrak{z}} \sum_{q=1}^{2m} M_{i',q}(Y_\mathfrak{z}) F_{X_q^{(r)}},$$

where we write $e_{i'} = (0, \dots, 0, 1, 0, \dots, 0)^T \in \mathbb{R}^{2m}$ and the non-zero entry appears at position i'. Hence

$$K_1 = -\cos\Big(2\pi \widetilde{F}_{i'}(p,Y)\chi_\varepsilon(\langle Y_\mathfrak{z}, Y_\mathfrak{z}\rangle)\Big) \frac{2\pi\,\chi_\varepsilon(\langle Y_\mathfrak{z}, Y_\mathfrak{z}\rangle)}{\langle Y_\mathfrak{z}, Y_\mathfrak{z}\rangle} \sum_{q=1}^{2m} M(Y_\mathfrak{z})_{i',q}$$
$$\times \Big\{ e^{\frac{-1}{\langle Y_\mathfrak{z}, Y_\mathfrak{z}\rangle^2}}, F_{X_q^{(r)}} \Big\} = 0.$$

In the last equation we have used Proposition 3.8, which implies that for $q = 1, \dots, 2m$ the Poisson brackets $\{e^{\frac{-1}{\langle Y_\mathfrak{z}, Y_\mathfrak{z}\rangle^2}}, F_{X_q^{(r)}}\}$ vanish. In order to prove that $K_2 = 0$ it suffices [because of (36)] to show that

$$\widetilde{K}_2 := \Big\{ \widetilde{F}_i(p,Y)\chi_\varepsilon(\langle Y_\mathfrak{z}, Y_\mathfrak{z}\rangle), \widetilde{F}_{i'}(p,Y)\chi_\varepsilon(\langle Y_\mathfrak{z}, Y_\mathfrak{z}\rangle) \Big\} = 0.$$

We insert the definition of $\widetilde{F}_i, \widetilde{F}_\ell$ and again use $M(Y_\mathfrak{z})^{-1} = -\langle Y_\mathfrak{z}, Y_\mathfrak{z}\rangle_\mathfrak{z}^{-1} M(Y_\mathfrak{z})$ for $Y_\mathfrak{z} \notin \mathcal{N}_\mathfrak{z}$. Applying Proposition 3.8, (c) again gives:

$$\widetilde{K}_2 = \frac{\chi_\varepsilon\big(\langle Y_\mathfrak{z}, Y_\mathfrak{z}\rangle_\mathfrak{z}\big)^2}{\langle Y_\mathfrak{z}, Y_\mathfrak{z}\rangle_\mathfrak{z}^2} \sum_{q_1,q_2=1}^{2m} M(Y_\mathfrak{z})_{i,q_1} M(Y_\mathfrak{z})_{i',q_2} \Big\{ F_{X_{q_1}^{(r)}}, F_{X_{q_2}^{(r)}} \Big\}.$$

Recall that

$$\Big\{ F_{X_{q_1}^{(r)}}, F_{X_{q_2}^{(r)}} \Big\} = \big\langle [X_{q_1}, X_{q_2}], Y \big\rangle = \big\langle j(Y_\mathfrak{z}) X_{q_1}, X_{q_2} \big\rangle = -M(Y_\mathfrak{z})_{q_1,q_2}. \tag{37}$$

Applying the relation $M(Y_\mathfrak{z})^2 = -\langle Y_\mathfrak{z}, Y_\mathfrak{z}\rangle_\mathfrak{z}$ we have:

$$\widetilde{K}_2 = \frac{\chi_\varepsilon\big(\langle Y_\mathfrak{z}, Y_\mathfrak{z}\rangle_\mathfrak{z}\big)^2}{\langle Y_\mathfrak{z}, Y_\mathfrak{z}\rangle_\mathfrak{z}^2} \Big(M(Y_\mathfrak{z})^3 \Big)_{i,i'}$$
$$= -\frac{\chi_\varepsilon\big(\langle Y_\mathfrak{z}, Y_\mathfrak{z}\rangle_\mathfrak{z}\big)^2}{\langle Y_\mathfrak{z}, Y_\mathfrak{z}\rangle_\mathfrak{z}} M(Y_\mathfrak{z})_{i,i'} = \frac{\chi_\varepsilon\big(\langle Y_\mathfrak{z}, Y_\mathfrak{z}\rangle_\mathfrak{z}\big)^2}{\langle Y_\mathfrak{z}, Y_\mathfrak{z}\rangle_\mathfrak{z}} \underbrace{\Big\{ F_{X_i^{(r)}}, F_{X_{i'}^{(r)}} \Big\}}_{=0} = 0.$$

This proves the assertion. $\square$

Since the functions $(f_i)_{i=1}^{2m}$ in Proposition 5.5 are invariant under left-multiplication by elements in the lattice L we can descend them to functions $(\overline{F}_i)_{i=1}^{2m}$ on $T(L\backslash G_{r,s})$. This tangent bundle is identified with $(L\backslash G_{r,s}) \times \mathcal{N}_{r,s}$. More precisely, put:

$$\overline{F}_i(Lp, Y) := f_i(p, Y) \quad \textit{where} \quad (Lp, Y) \in (L\backslash G_{r,s}) \times \mathcal{N}_{r,s}.$$

In addition, all left-invariant functions g on $T(G_{r,s}) \cong G_{r,s} \times \mathcal{N}_{r,s}$ do not depend on the $G_{r,s}$-coordinate and therefore they descend to functions on $T(L\backslash G_{r,s})$, as well, which we denote by $\overline{g}$. Let

$$\pi : T(G_{r,s}) \cong G_{r,s} \times \mathcal{N}_{r,s} \to T(L\backslash G_{r,s}) \cong (L\backslash G_{r,s}) \times \mathcal{N}_{r,s}$$

be the projection which is a Poisson map, i.e. for all $f, g \in \mathcal{C}^\infty(T(L\backslash G_{r,s}))$ we have:

$$\{f \circ \pi, g \circ \pi\} = \{f, g\} \circ \pi.$$

Let $[X_1, \dots, X_{2m}, Z_1, \dots, Z_{r+s}]$ denote an (integral) orthonormal basis of $\mathcal{N}_{r,s}$ as described in Theorem 5.4 and let $L \subset G_{r,s}$ be the corresponding lattice in (33). Now we can state and prove the main result of the present section.

Theorem 5.6 *Assume that the matrix of commutation relations has the form*

$$C := \left([X_i, X_{i'}]\right)_{i,i'=1}^{2m} = \begin{pmatrix} 0_m & A_m \\ -A_m & 0_m \end{pmatrix} \in \mathbb{R}^{2m\times 2m}, \quad \textit{where} \quad 0_m, A_m \in \mathbb{R}^{m\times m} \tag{38}$$

and 0_m *denotes the matrix with zero-entries. Then the geodesic flow on* $T^*(L\backslash G_{r,s})$ *is completely integrable in the sense of Liouville with smooth first integrals.*

Proof From the form of the matrix C in (38) and Proposition 3.8, (c) it is clear that the functions $[F_{X_1^{(r)}}, \dots, F_{X_m^{(r)}}]$ pairwise Poisson commute. According to Proposition 5.5 they descend to Poisson commuting function $[\overline{F}_1, \dots, \overline{F}_m]$ on $T(L\backslash G_{r,s}) \cong T^*(L\backslash G_{r,s})$. In addition, the mutually Poisson commuting first integrals $\tilde{g}_1, \dots, \tilde{g}_m$, $y_1^3, \dots, y_{r+s}^3$ constructed in Theorem 4.2 descend to $T(L\backslash G_{r,s})$ as well and Poisson commute with the former ones. In total we have found $2m + (r+s) = \dim G_{r,s}$ Poisson commuting smooth first integrals of $T^*(L\backslash G_{r,s})$. □

The result in Theorem 5.6 is not sharp. We present an example of an integral basis in $\mathcal{N}_{1,2}$ that does not induce commutation relations as in (38) However, we can still prove complete integrability of the geodesic flow on $T(L\backslash G_{1,2})$:

Example 5.7 Consider the pseudo-H-type algebra $\mathcal{N}_{1,2} = \mathbb{R}^{2,2} \oplus_\perp \mathbb{R}^{1,2}$ which can be shown to be not of HR-type. We can take basis elements $X_1, \dots, X_4 \in \mathbb{R}^{2,2}$ and $Z_1, Z_2, Z_3 \in \mathbb{R}^{1,2}$ such that $\langle X_1, X_1\rangle = \langle X_2, X_2\rangle = 1$, $\langle X_3, X_3\rangle = \langle X_4, X_4\rangle = -1$, $\langle Z_1, Z_1\rangle = 1$, $\langle Z_2, Z_2\rangle = \langle Z_3, Z_3\rangle = -1$. The matrix of commutation relations can be computed from [2, Table 6, p.570]:

$$\big([X_i, X_{i'}]\big)_{i,i'=1}^4 = \begin{pmatrix} 0 & Z_1 & Z_2 & Z_3 \\ -Z_1 & 0 & Z_3 & -Z_2 \\ -Z_2 & -Z_3 & 0 & -Z_1 \\ -Z_3 & Z_2 & Z_1 & 0 \end{pmatrix}, \tag{39}$$

showing that it is not of the form (38). From (37) and (39) we observe that:

$$\Big\{F_{X_1^{(r)}}, F_{X_2^{(r)}}\Big\} + \Big\{F_{X_3^{(r)}}, F_{X_4^{(r)}}\Big\} = 0 = \Big\{F_{X_1^{(r)}}, F_{X_4^{(r)}}\Big\} - \Big\{F_{X_2^{(r)}}, F_{X_3^{(r)}}\Big\}. \tag{40}$$

Consider the new functions $S_1 := F_{X_1^{(r)}} + F_{X_3^{(r)}}$ and $S_2 := F_{X_2^{(r)}} + F_{X_4^{(r)}}$. By using (40) one finds that $\{S_1, S_2\} = 0$. As in Proposition 5.5 (and with the notation there) we construct Poisson commuting smooth functions $f_j \in \mathcal{C}^\infty(TG_{1,2})$ for $j = 1, \ldots, 4$ which are invariant under the left-multiplication by L in the $G_{1,2}$-component. Put

$$s_1 := f_1 + f_3 \quad \textit{and} \quad s_2 := f_2 + f_4.$$

As in the proof of Proposition 5.5 one finds that s_1 and s_2 Poisson commute. Moreover, s_1 and s_2 descend to $T(L\backslash G_{1,2})$ and as in Theorem 5.6 we can complement them by five first integrals descended from left-invariant functions on $T(G_{1,2}) \cong T^*(G_{1,2})$ to prove the complete integrability of the pseudo-Riemannian geodesic flow. on $T^*(L\backslash G_{1,2})$.

Problem: Finally, we would like to mention two open problems related to the analysis in this paper:

(1) Give a complete classification of pseudo-H-type Lie groups $\mathcal{N}_{r,s}$ of Heisenberg-Reiter type.
(2) Completely characterize the class of pseudo-H-type nilmanifolds $L\backslash G_{r,s}$ with completely Liouville integrable pseudo-Riemannian geodesic flow.

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