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Approximation by trigonometric polynomials in variable exponent Morrey spaces

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Abstract We investigate the direct and inverse theorems for trigonometric polynomials in the Morrey space $\mathcal{M}^{p(\cdot),\lambda(\cdot)}$ with variable exponents. For this space, we obtain estimates of the K-functional in terms of the modulus of smoothness and the Bernstein type inequality for trigonometric polynomials.

Keywords Variable exponent Morrey spaces · Bernstein inequality · Steklov operator · Trigonometric polynomial

Mathematics Subject Classification 41A10 · 41A25 · 42A10 · 46E30 · 46E35

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1 Introduction

Many approximation problems including the convergence of the Fourier series hinge upon the local property of the functions as it is seen from the Dini criterion. We aim to show that the function spaces with variable exponents are useful in this direction of research. Studying function spaces with variable exponent is now an extensively developed field after the advent of two books [6,8] on variable exponent Lebesgue and Sobolev spaces. Nowadays many mathematicians solved many problems about the boundedness of various operators of harmonic analysis in these spaces including a number of weighted counterparts. Among others there are also various advances in Morrey spaces with variable exponent, but to a less extent than in Lebesgue spaces with variable exponent.

Morrey spaces emerged in close connection with the local behavior of the solutions of elliptic differential equations and they describe local regularity more precisely than Lebesgue spaces; see, for example [11–13,31]. Morrey spaces, introduced by C. Morrey in 1938, have been studied intensively by various authors. For classical Morrey spaces we refer to the books [12,18,27] and the recent survey papers [15,21]; in the last reference we can find information on various versions of variable exponent Morrey spaces. Let X be a metric measure space. The Morrey space $\mathcal{M}^{p(\cdot),\lambda(\cdot)}(X)$ with variable exponents $p(\cdot)$ and $\lambda(\cdot)$ on the Euclidean spaces or on metric measure spaces was introduced and studied in [4,10,17,20].

Meanwhile, a considerable number of mathematicians has studied variable exponent Lebesgue spaces during last three decades. In this direction, the authors [4] introduced the Morrey space $\mathcal{M}^{p(\cdot),\lambda(\cdot)}(\Omega)$ with variable exponents over an open set $\Omega \subset \mathbb{R}^n$. In these spaces, the boundedness of the maximal, potential and singular integral operators are obtained; see [14]. We aim to study approximation properties of trigonometric polynomials in the Morrey space $\mathcal{M}^{p(\cdot),\lambda(\cdot)}[0, 2\pi]$ with variable exponents. In the theory of approximation, variable exponent spaces are useful to show that the approximation is essentially local. For example, the Fourier series of $f \in L^1[0, 2\pi]$ converges back to $f(x)$ when $x \in (0, 2\pi)$ satisfies the Dini condition

$$\int_{[x-1,x+1]\cap[0,2\pi]} \frac{|f(t)|}{|x-t|}\,dt < \infty.$$

We show that this idea is applicable to many practical approximations using variable exponent spaces.

One of the main results of the paper is the boundedness of the Steklov operator s_h with $0 < h \le 2\pi$ given by:

$$s_h(f)(x) = \frac{1}{h}\int_0^h f(x+t)dt \quad (x \in [0, 2\pi]) \tag{1.1}$$

within the framework of the Morrey space $\mathcal{M}^{p(\cdot),\lambda(\cdot)}[0, 2\pi]$ with variable exponents; see Theorem 1.2. Here and below for any $f \in L^1[0, 2\pi]$, we define $f(x) = f(x-2\pi)$ for $x \in (2\pi, 4\pi]$ and $f(x) = f(x+2\pi)$ for $x \in [-2\pi, 0)$, so that (1.1) makes sense.

Before we recall the definition of $\mathcal{M}^{p(\cdot),\lambda(\cdot)}[0, 2\pi]$, we recall the definition of the classical Morrey space $\mathcal{M}^{p,\lambda}[0, 2\pi]$. Let $0 \le \lambda \le 1$ and $1 \le p < \infty$. We consistently write

$$I_0 = [0, 2\pi], \quad I(x, r) = (x - r, x + r) \subset \mathbb{R}, \quad \tilde{I}(x, r) = I(x, r) \cap I_0$$

for intervals in this paper. The classical Morrey space $\mathcal{M}^{p,\lambda}(I_0)$ is defined as the set of all functions $f \in L^p(I_0)$ such that

$$\|f\|_{\mathcal{M}^{p,\lambda}(I_0)} = \sup\{r^{-\frac{\lambda}{p}} \|f\|_{L^p(\tilde{I}(x,r))} : x \in I_0, 0 < r < 2\pi\} < \infty.$$

Under this definition we learn $\mathcal{M}^{p,\lambda}(I_0)$ is a Banach space; moreover, for $\lambda = 0$ it coincides with $L^p(I_0)$ and for $\lambda = 1$ with $L^\infty(I_0)$. If $\lambda < 0$ or $\lambda > 1$, then it is easy to see that $\mathcal{M}^{p,\lambda}(I_0) = \Theta(I_0)$, where $\Theta(I_0)$ denotes the set of all functions equivalent to 0 on I_0.

Moreover, $\mathcal{M}^{p,\lambda_2}(I_0) \subset \mathcal{M}^{p,\lambda_1}(I_0)$ for $0 \le \lambda_1 \le \lambda_2 \le 1$. If $f \in \mathcal{M}^{p,\lambda}(I_0)$, then $f \in L^p(I_0)$ and hence $f \in L^1(I_0)$.

Compared to Lebesgue spaces, Morrey spaces have the following remarkable features: Let $1 < p < \infty$ and $0 < \lambda \le 1$.

(1) The function $f(x) = x^{-(1-\lambda)/p}$ is in $\mathcal{M}^{p,\lambda}(I_0)$.
(2) The Morrey space $\mathcal{M}^{p,\lambda}(I_0)$ is not reflexive; see [22, Example 5.2] and [30, Theorem 1.3].
(3) Denote by $C^\infty(I_0)$ the set of all functions that are realized as the restriction to I_0 of elements in $C^\infty(\mathbb{R})$. The Morrey space $\mathcal{M}^{p,\lambda}(I_0)$ does not have $C^\infty(I_0)$ as a dense closed subspace; see [29, Proposition 2.16].
(4) The Morrey space $\mathcal{M}^{p,\lambda}(I_0)$ is not separable; see [29, Proposition 2.16].

If $\lambda = 0$, all of these properties above fail to hold, since $\mathcal{M}^{p,0}(I_0) = L^p(I_0)$ with norm coincidence. Based on these properties, we define $\widetilde{\mathcal{M}}^{p,\lambda}(I_0)$ to be the closure of $C^\infty(I_0)$ in $\mathcal{M}^{p,\lambda}(I_0)$. Equipped with two parameters, Morrey spaces can describe the local regularity and the global regularity more precisely than the Lebesgue spaces. Our experience show that p describes the local regularity, while λ describes the global regularity. As it is hinted by the example of the Fourier series, we feel that p plays an essential role. This fact is verified in this paper.

To express our idea clearly, we now define the Morrey space $\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)$ with variable exponents. Let $p(\cdot) : I_0 \to [1, \infty)$ be a continuous function such that

$$1 \le p_- = \|p(\cdot)^{-1}\|_{L^\infty(I_0)}{}^{-1} \le p_+ = \|p(\cdot)\|_{L^\infty(I_0)} < \infty \tag{1.2}$$

and $\lambda(\cdot) : I_0 \to [0, 1]$ be a measurable function. Following the convention, we add $(\cdot)$ to indicate that the parameters are actually dependent on the position. Note that $p(\cdot)$ is required to be continuous while $\lambda(\cdot)$ is allowed to be merely measurable and bounded. This implies that the local regularity is essential in the theory of approximation. We remark that this fact is observed in [19, Theorem 4.4], [23, Theorem 4.1] and [24, Theorem 3.3].

We define the Morrey space $\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)$ with variable exponents is the space of measurable functions such that the modular

$$I_{p(\cdot),\lambda(\cdot)}(f) = \sup_{\substack{x\in I_0\\ 0<r<2\pi}} r^{-\lambda(x)} \int_{\tilde{I}(x,r)} |f(y)|^{p(y)}dy \tag{1.3}$$

is finite. The norm is defined by

$$\|f\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} = \|f\|_1 = \inf\left\{\eta > 0 : I_{p(\cdot),\lambda(\cdot)}\left(\frac{f}{\eta}\right) < 1\right\}.$$

In the setting of variable exponents, we adopt the following definition.

Definition 1.1 Let $p(\cdot) : I_0 \to [1,\infty)$ be a continuous function satisfying (1.2), and let $\lambda(\cdot) : I_0 \to [0, 1]$ be a measurable function. Denote by $\widetilde{\mathcal{M}}^{p(\cdot),\lambda(\cdot)}(I_0)$ the closure of the set of all trigonometric polynomials in $\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)$.

In addition of the above condition on $p(\cdot)$, we postulate

$$|p(x) - p(y)| \le \frac{A}{-\log|x-y|}, \quad 0 < |x-y| \le \frac{1}{2}, \quad x, y \in I_0. \tag{1.4}$$

Based on the definition above, we prove the uniform boundedness of the Steklov operators in Morrey spaces with variable exponents under the log-condition on $p(\cdot)$. In case of $\lambda(\cdot) = 0$, this result reduces to boundedness of the Steklov operators which proved by I.I. Sharapudinov [25, Lemma 3.1].

Theorem 1.2 *Let $p(\cdot)$ and $\lambda(\cdot)$ be measurable functions such that $0 \le \lambda_- \le \lambda_+ < 1$. Assume that $p(\cdot)$ satisfies conditions (1.2) and (1.4). Then the family of operators $\{s_h\}_{h\in(0,2\pi]}$, defined by (1.1) is uniformly bounded in $\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)$.*

We organize this paper as follows: We shall recall necessary definitions and auxiliary results on the boundedness of the Steklov operator in Sect. 2. We plan to compare the boundedness of the Hardy-Littlewood maximal operator with the one of the Steklov operator. Recall that the Hardy-Littlewood maximal function $Mf(x)$ on I_0 is defined as follows:

$$M(f)(x) = \sup_{r>0} \frac{1}{|I(x,r)|} \int_{\tilde{I}(x,r)} |f(y)|dy \quad (x \in I_0). \tag{1.5}$$

In Sect. 3, we prove the boundedness of the Steklov operator given by (1.1) in Morrey spaces with variable exponents under the log-continuity on $p(\cdot)$. Sections 4 and 5 contain the properties of the Jackson operators and the Bernstein type inequality in Morrey spaces with variable exponents, respectively. Finally, the last section, Sect. 6, is devoted to the direct and inverse approximation theorems in Morrey spaces with variable exponents.

2 Preliminaries

2.1 Morrey norms with variable exponents

There is another plausible definition of the norm: We may define the Morrey norm by:

$$\|f\|_2 = \sup_{\substack{x\in I_0 \\ 0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}} \|f\chi_{\tilde{I}(x,r)}\|_{L^{p(\cdot)}(I_0)},$$

However, Lemma 2.2 below shows that these norms are equivalent. The following lemmas were proved in [4]:

Lemma 2.1 [4, Lemma 2] *If $p(\cdot)$ be a measurable function on I_0 with values in $[1,\infty)$, and let $\lambda(\cdot)$ be a measurable function on I_0 with values in $[0,1)$, then for every $f \in \mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)$, the inequalities*

$$\begin{aligned} \|f\|_i^{p+} \le I_{p(\cdot),\lambda(\cdot)}(f) \le \|f\|_i^{p-} \quad \textit{if} \quad \|f\|_i \le 1 \\ \|f\|_i^{p-} \le I_{p(\cdot),\lambda(\cdot)}(f) \le \|f\|_i^{p+} \quad \textit{if} \quad \|f\|_i \ge 1 \end{aligned}$$

are valid for $i = 1, 2$.

We also have

Lemma 2.2 [4, Lemma 3] *For every $f \in \mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)$, $\|f\|_1 = \|f\|_2$.*

By the coincidence of the norms we can put

$$\|f\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} = \|f\|_1 = \|f\|_2.$$

2.2 Steklov function

Now, we introduce the Steklov function for a function $f \in L^1[0,2\pi]$. One defines $f(x) = f(x-2\pi)$ for $x \in (2\pi, 4\pi]$ and $f(x) = f(x+2\pi)$ for $x \in [-2\pi, 0)$. For $h > 0$ and $f \in L^1[0,2\pi]$, we define the Steklov operator by

$$f_h(x) = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} f(x+t)dt, \quad s_h(f)(x) = f_h\left(x+\frac{h}{2}\right) = \frac{1}{h}\int_0^h f(x+t)dt \tag{2.1}$$

for $x \in [0,2\pi]$. For $f \in \mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)$ and $\delta < 2\pi$, we define

$$\sigma_\delta(f) = \frac{1}{\delta}\int_0^\delta |s_t(f)(x) - f(x)|dt = \frac{1}{\delta}\int_0^\delta \left| f_t\left(x+\frac{t}{2}\right) - f(x)\right| dt. \tag{2.2}$$

We refer to the textbook [28] for this direction of research.

Let $W^{p(\cdot),\lambda(\cdot)}(I_0)$ be the linear space of all $f \in \mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)$ such that $f' \in \mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)$. Here the derivative is understood in the weak sense. If $f \in W^{p(\cdot),\lambda(\cdot)}(I_0)$, then we claim

$$\|\sigma_\delta(f)\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \le C\|f'\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}$$

with constant $C > 0$ independent of f, in other words $\sigma_\delta(f) \in \mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)$ provided $f \in W^{p(\cdot),\lambda(\cdot)}(I_0)$. The proof of this assertion will be given in Lemma 3.4.

Let us compare the properties of the Steklov function and the Hardy-Littlewood maximal function. So, we recall the corresponding boundedness of the Hardy-Littlewood maximal operator in $L^{p(\cdot)}(I_0)$ that was proved by Diening in 2002 in [7]:

Theorem A [7, Lemma 2.9] *Let $p(\cdot)$ be a measurable function on I_0 assuming its values in $[1,\infty)$, and suppose that $p(\cdot)$ satisfies conditions:*

$$1 < p_- \le p_+ < \infty, \tag{2.3}$$

and (1.4). Then the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(I_0)$.

A. Almeida, J. Hasanov and S. Samko proved the counterpart to the above theorem for Morrey spaces variable exponents proved [4]. Note that $\lambda(\cdot)$ need not be continuous.

Theorem B [4, Theorem 2] *Let $\lambda(\cdot)$ and $p(\cdot)$ satisfy $0 \le \lambda_- \le \lambda_+ < 1$, (1.4) and (2.3). Then the Hardy-Littlewood maximal operator M is bounded on the Morrey space $\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)$ with variable exponents.*

However for the problem of approximation of functions by the trigonometric polynomials Sharapudinov in [26] proved that the Steklov operator in $L^{p(\cdot)}(I_0)$ is bounded and he used it to define modulus of continuity. We remark that we assume $p(x) \ge 1$ as in [26] instead of assumption $1 < p_-$; see [1–3,16] for comparison.

Theorem C [26, Lemma 1] *Let $p(\cdot)$ be a measurable function on I_0 with $1 \le p_- \le p_+ < \infty$. Assume that $p(\cdot)$ satisfies conditions (1.2) and (1.4). Then the family of the Steklov operators $\{s_h\}_{0<h\le 1}$, defined by (1.1), is uniformly bounded on $L^{p(\cdot)}([0,1])$.*

3 Uniform boundness of the Steklov operator

In this section, we prove Theorem 1.2.

Proof Let $f \in \mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)$. We need to show that

$$r^{-\frac{\lambda(x)}{p(x)}}\|s_h(f)\|_{L^{p(\cdot)}(\tilde{I}(x,r))} \le C\|f\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \tag{3.1}$$

for all $r > 0$, $h > 0$ and $x \in I_0$, where C is independent of f and x.

If $0 < r \le h$, then for a nonnegative function 2π-periodic f on $\mathbb{R}$, the following trivial estimate holds:

$$|s_h(f)(z)| \le \frac{3}{h}\int_{\max\{x-r-h,0\}}^{\min\{x-r+2h,2\pi\}} |f(y)|\,dy$$

holds for all $z \in \tilde{I}(x,r)$. Thus by the fact that

$$\frac{1-\lambda(x)}{p(x)} \ge 0 \quad (x \in I_0)$$

as well as the Hölder inequality for variable Lebesgue spaces,

$$\begin{aligned} r^{-\frac{\lambda(x)}{p(x)}}\|s_h(f)\|_{L^{p(\cdot)}(\tilde{I}(x,r))} &\le Cr^{\frac{1-\lambda(x)}{p(x)}}\frac{1}{h}\int_{\max\{x-r-h,0\}}^{\min\{x-r+2h,2\pi\}} |f(y)|\,dy \\ &\le Ch^{\frac{1-\lambda(x)}{p(x)}-1}\int_{\max\{x-r-h,0\}}^{\min\{x-r+2h,2\pi\}} |f(y)|\,dy \\ &\le C\|f\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \end{aligned}$$

and (3.1) is obtained.

If $0 < h \le r$, then we use Theorem C to obtain

$$\begin{aligned} r^{-\frac{\lambda(x)}{p(x)}}\|s_h(f)\|_{L^{p(\cdot)}(\tilde{I}(x,r))} &= r^{-\frac{\lambda(x)}{p(x)}}\|s_h(\chi_{[\max\{x-r,0\},\min\{x+2r,2\pi\}]}f)\|_{L^{p(\cdot)}(\tilde{I}(x,r))} \\ &\le Cr^{-\frac{\lambda(x)}{p(x)}}\|f\|_{L^{p(\cdot)}([\max\{x-r,0\},\min\{x+2r,2\pi\}])} \\ &\le C\|f\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}. \end{aligned}$$

Thus, we obtain the desired result. □

We give the following definition:

Definition 3.1 Maintain the same conditions as Theorem C on $\lambda(\cdot)$ and $p(\cdot)$. Define by σ_δ by (2.2). For any $f \in \mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)$, the function $\Omega_{p(\cdot),\lambda(\cdot)}(f,\cdot) : (0,\infty] \to [0,\infty)$, defined by

$$\Omega_{p(\cdot),\lambda(\cdot)}(f,h) = \sup_{0<\delta\le\min(2\pi,h)} \|\sigma_\delta(f)\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}, \tag{3.2}$$

is called the modulus of smoothness of f in $\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)$.

Lemma 3.2 *Let $p(\cdot)$ and $\lambda(\cdot)$ satisfy the same conditions as Theorem C.*

(1) For $f_1, f_2 \in \mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)$ and $h > 0$,

$$\Omega_{p(\cdot),\lambda(\cdot)}(f_1+f_2,h) \le \Omega_{p(\cdot),\lambda(\cdot)}(f_1,h) + \Omega_{p(\cdot),\lambda(\cdot)}(f_2,h). \tag{3.3}$$

(2) For $f \in \widetilde{\mathcal{M}}^{p(\cdot),\lambda(\cdot)}(I_0)$,

$$\lim_{\delta\downarrow 0} \Omega_{p(\cdot),\lambda(\cdot)}(f,\delta) = 0. \tag{3.4}$$

Proof Inequality (3.3) is clear from the triangle inequality of $\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)$. For (3.4), first we prove in the case of trigonometric polynomials, i.e. we assume that g is a trigonometric polynomial and hence g is uniformly continuous. Let $\varepsilon > 0$ be fixed. Let $C_0 > 0$, whose precise value will be made clear shortly. Writing out

$$\mathrm{I} = I_{p(\cdot),\lambda(\cdot)}\left(\frac{\sigma_\delta(g)}{C_0 2^{\frac{1}{p_-}}\varepsilon}\right)$$

fully, we have

$$\begin{aligned}\mathrm{I} &= \sup_{\substack{x\in I_0\\0<r<2\pi}} r^{-\lambda(x)} \int_{\widetilde{I}(x,r)} \left(\frac{1}{2^{\frac{1}{p_-}}\varepsilon\delta}\int_0^\delta |s_\tau(g)(y)-g(y)|d\tau\right)^{p(y)} dy\\ &= \sup_{\substack{x\in I_0\\0<r<2\pi}} r^{-\lambda(x)} \int_{\widetilde{I}(x,r)} \left(\frac{1}{2^{\frac{1}{p_-}}\varepsilon\delta}\int_0^\delta \left|\frac{1}{\tau}\int_0^\tau (g(y+h)-g(y))dh\right| d\tau\right)^{p(y)} dy.\end{aligned}$$

For any $\varepsilon > 0$, there exists $\delta_0 = \delta_0(\varepsilon) > 0$ such that

$$|g(x) - g(x+t)| < \varepsilon \tag{3.5}$$

for $0 \le t \le \delta_0$ and $x \in I_0$. Hence for $0 \le h \le \delta_0$, using (3.5), we have

$$\mathrm{I} \le \frac{1}{2} \sup_{\substack{x\in I_0\\0<r<2\pi}} r^{-\lambda(x)} \left|\widetilde{I}(x,r)\right| \le \sup_{\substack{x\in I_0\\0<r<2\pi}} r^{1-\lambda(x)}.$$

If we let

$$C_0 = \sup_{\substack{x\in I_0\\0<r<2\pi}} r^{1-\lambda(x)},$$

then we obtain

$$\|\sigma_\delta(g)\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} < C_0 2^{\frac{1}{p_-}}\varepsilon \tag{3.6}$$

for $0 \le h < \delta_0$.

By the triangle inequality, we have

$$\begin{aligned}&\|f - s_t(f)\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}\\ &\quad\le \|f-g\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} + \|g - s_t(g)\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}\\ &\qquad+ \|s_t(g) - s_t(f)\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}\end{aligned} \tag{3.7}$$

for any $f \in \widetilde{\mathcal{M}}^{p(\cdot),\lambda(\cdot)}(I_0)$. Now for given $\varepsilon > 0$, we choose a trigonometric function g such that

$$\|f - g\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \leq \varepsilon. \tag{3.8}$$

By (3.6), for any trigonometric function g, we can find $\delta_0 = \delta_0(\varepsilon)$ such that

$$\|g - s_t(g)\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \leq \varepsilon,$$

for $0 \leq t \leq \delta_0$. Finally, by Theorem 2.1 and (3.8) we have

$$\|s_t(g) - s_t(f)\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \leq c(p)\varepsilon \tag{3.9}$$

and by (3.7) and (3.9) we have

$$\|f - s_t(f)\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \leq c(p)\varepsilon, \qquad 0 \leq t \leq \delta_0(\varepsilon). \tag{3.10}$$

Combining (3.1) and (3.2), we have (3.4) for any $f \in \widetilde{\mathcal{M}}^{p(\cdot),\lambda(\cdot)}(I_0)$. □

The following lemma is a generalization of Minkowski's inequality:

Lemma 3.3 *Let $p(\cdot)$ and $\lambda(\cdot)$ be measurable functions on I_0 with $1 \leq p_- \leq p_+ < \infty$ and $0 \leq \lambda_- \leq \lambda_+ < 1$, and f be a measurable function defined on $I_0 \times I_0$. Then the following inequality is valid:*

$$\left\| \int_{I_0} f(\cdot, \tau) d\tau \right\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \leq \int_{I_0} \|f(\cdot, \tau)\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} d\tau.$$

Proof We have

$$\left\| \int_{I_0} f(\cdot, \tau) d\tau \right\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} = \sup_{\substack{x \in I_0 \\ 0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}} \left\| \int_{I_0} f(\cdot, \tau) d\tau \right\|_{L^{p(\cdot)}(\tilde{I}(x,r))} \tag{3.11}$$

by the definition of the Morrey norm $\|\cdot\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}$. Now by [15], we have

$$\sup_{\substack{x \in I_0 \\ 0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}} \left\| \int_{I_0} f(\cdot, \tau) d\tau \right\|_{L^{p(\cdot)}(\tilde{I}(x,r))} \leq \sup_{\substack{x \in I_0 \\ 0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}} \int_{I_0} \left\| f(\cdot, \tau) \right\|_{L^{p(\cdot)}(\tilde{I}(x,r))} d\tau. \tag{3.12}$$

By combining (3.11) and (3.12), we have

$$\begin{aligned}\Big\| \int_{I_0} f(\cdot,\tau)d\tau \Big\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} &\le \sup_{\substack{x\in I_0\\0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}} \int_{I_0} \big\| f(\cdot,\tau)\big\|_{L^{p(\cdot)}(\tilde{I}(x,r))} d\tau \\ &= \int_{I_0} \sup_{\substack{x\in I_0\\0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}} \big\| f(\cdot,\tau)\big\|_{L^{p(\cdot)}(\tilde{I}(x,r))} d\tau \\ &= \int_{I_0} \big\| f(\cdot,\tau)\big\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} d\tau.\end{aligned}$$

□

Lemma 3.4 *Let $p(\cdot)$ and $\lambda(\cdot)$ be measurable functions on I_0 satisfying $1 \le p_- \le p_+ < \infty$ and $0 \le \lambda_- \le \lambda_+ < 1$, Let also $f \in W^{p(\cdot),\lambda(\cdot)}(I_0)$. Assume in addition that $p(\cdot)$ satisfies conditions (1.2) and (1.4). Then*

$$\|\sigma_\delta(f)\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \le C(p(\cdot))\delta\, \|f^{'}\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}.$$

with the constant $C(p(\cdot)) > 0$ independent of f.

Proof Let $f \in W^{p(\cdot),\lambda(\cdot)}(I_0)$. Then $f^{'} \in \mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)$. Inserting the related definitions used to define $\sigma_\delta(f)$, we obtain

$$\begin{aligned}\Big\| \frac{\sigma_\delta(f)}{\delta}\Big\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} &= \sup_{\substack{x\in I_0\\0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}} \Big\| \frac{\sigma_\delta(f)}{\delta}\chi_{\widetilde{I}(x,r)}(\cdot)\Big\|_{L^{p(\cdot)}(I_0)} \\ &= \sup_{\substack{x\in I_0\\0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}} \Big\| \frac{1}{\delta^2}\Big(\int_0^\delta \Big|f_t\Big(\cdot+\frac{t}{2}\Big)-f\Big|dt\Big)\chi_{\widetilde{I}(x,r)}(\cdot)\Big\|_{L^{p(\cdot)}(I_0)} \\ &= \sup_{\substack{x\in I_0\\0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}} \Big\| \frac{1}{\delta^2}\Big(\int_0^\delta \Big|\frac{1}{t}\int_0^t (f(\cdot+h)-f)dh\Big|dt\Big)\chi_{\widetilde{I}(x,r)}(\cdot)\Big\|_{L^{p(\cdot)}(I_0)} \\ &= \sup_{\substack{x\in I_0\\0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}} \Big\| \frac{1}{\delta^2}\Big(\int_0^\delta \Big|\frac{1}{t}\int_0^t \int_{\cdot}^{\cdot+h} f'(\tau)d\tau dh\Big|dt\Big)\chi_{\widetilde{I}(x,r)}(\cdot)\Big\|_{L^{p(\cdot)}(I_0)}.\end{aligned} \tag{3.13}$$

Let $y \in I_0$. By the Fubini theorem, we have

$$\begin{aligned}\int_0^t \Big(\int_y^{y+h} f'(\tau)d\tau\Big)dh &= \int_0^t \Big(\int_0^t f'(y+\tau)\chi_{(0,h)}(\tau)d\tau\Big)dh \\ &= \int_0^t \Big(\int_0^t \chi_{(0,h)}(\tau)dh\Big) f'(y+\tau)d\tau \\ &= \int_0^t f'(y+\tau)(t-\tau)d\tau.\end{aligned} \tag{3.14}$$

By substituting (3.14) into (3.13) we obtain

$$\begin{aligned}
&\left\|\frac{\sigma_\delta(f)}{\delta}\right\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \\
&\leq \sup_{\substack{x\in I_0\\0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}}\left\|\left(\frac{1}{\delta^2}\int_0^\delta\left|\int_0^t\frac{t-\tau}{t}f'(\cdot+\tau)d\tau\right|dt\right)\ \chi_{\widetilde{I}(x,r)}(\cdot)\right\|_{L^{p(\cdot)}(I_0)} \\
&\leq \sup_{\substack{x\in I_0\\0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}}\left\|\left(\frac{1}{\delta^2}\int_0^\delta\int_0^t\left|f'(\cdot+\tau)\right|d\tau dt\right)\chi_{\widetilde{I}(x,r)}(\cdot)\right\|_{L^{p(\cdot)}(I_0)}.
\end{aligned}$$

By the triangle inequality for integrals we have

$$\begin{aligned}
\left\|\frac{\sigma_\delta(f)}{\delta}\right\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} &\leq \sup_{\substack{x\in I_0\\0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}}\left\|\frac{1}{\delta^2}\left(\int_0^\delta t\,s_t(|f'|)\,dt\right)\chi_{\widetilde{I}(x,r)}\right\|_{L^{p(\cdot)}(I_0)} \\
&\leq \sup_{\substack{x\in I_0\\0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}}\left\|\frac{1}{\delta}\left(\int_0^\delta s_t(|f'|)\,dt\right)\chi_{\widetilde{I}(x,r)}\right\|_{L^{p(\cdot)}(I_0)} \\
&\leq \sup_{\substack{x\in I_0\\0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}}\left\|\frac{1}{\delta}\int_0^\delta\left(s_t(|f'|)\,\chi_{\widetilde{I}(x,r)}\right)dt\right\|_{L^{p(\cdot)}(I_0)}.
\end{aligned} \tag{3.15}$$

Now, by Minkowski's integral inequality for variable exponent Lebesgue space, we have

$$\begin{aligned}
\left\|\frac{\sigma_\delta(f)}{\delta}\right\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} &\leq \sup_{\substack{x\in I_0\\0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}}\frac{1}{\delta}\int_0^\delta\|s_t(|f'|)\ \chi_{\widetilde{I}(x,r)}\|_{L^{p(\cdot)}(I_0)}dt \\
&\leq \sup_{\substack{x\in I_0\\0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}}\sup_{t>0}\left\|s_t(|f'|)\ \chi_{\widetilde{I}(x,r)}\right\|_{L^{p(\cdot)}(I_0)} \\
&= \sup_{t>0}\left\|s_t(|f'|)\right\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}
\end{aligned}$$

and, hence by applying Theorem 1.2, we complete the proof. □

We define

$$\|f\|_{\dot{\mathcal{M}}^{p(\cdot),\lambda(\cdot)}(I_0)} = \|f'\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}$$

for a function as long as the definition of f' makes sense as an element in $L^1(I_0)$ and $\dot{\mathcal{M}}^{p(\cdot),\lambda(\cdot)}(I_0)$ is the set of all f whose weak derivative f' satisfies $\|f\|_{\dot{\mathcal{M}}^{p(\cdot),\lambda(\cdot)}(I_0)} < \infty$.

Let $f \in \mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)$. The K-functional of $\dot{\mathcal{M}}^{p(\cdot),\lambda(\cdot)}(I_0)$ is defined as follows:

$$K(f,t)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} = \inf_{g \in W^{p(\cdot),\lambda(\cdot)}(I_0)} \left\{ \|f-g\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} + t\|g'\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \right\}$$

for $t > 0$. We recall

$$\Omega_{p(\cdot),\lambda(\cdot)}(f,t) = \sup_{|\delta|\le t} \|\sigma_\delta(f)\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}.$$

For $h > 0$, this K-functional $K(f,h)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}$ and $\Omega_{p(\cdot),\lambda(\cdot)}(f,h)$ are equivalent as the following lemma shows:

Lemma 3.5 *Let $p(\cdot)$ and $\lambda(\cdot)$ be measurable functions on I_0. Assume that $\lambda(\cdot)$ satisfies condition $0 \le \lambda_- \le \lambda_+ < 1$ and that $p(\cdot)$ satisfies conditions (1.2) and (1.4). Then*

$$c\,\Omega_{p(\cdot),\lambda(\cdot)}(f,h) \le K(f,h)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \le C\Omega_{p(\cdot),\lambda(\cdot)}(f,h),\ \ h > 0$$

for every $r \in \mathbb{N}^+$ and for all $f \in \mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)$ with constants $c, C > 0$ independent of f and h.

Proof Let $g \in W^{p(\cdot),\lambda(\cdot)}(I_0)$. If we write out the definition of $\sigma_\delta(g)(x)$ out in full, then we obtain

$$\sigma_\delta(g)(x) = \frac{2}{\delta}\int_{\frac{\delta}{2}}^{\delta} \Big|s_t(g)(x) - g(x)\Big| dt = \frac{2}{\delta}\int_{\frac{\delta}{2}}^{\delta} \Big|g_t\Big(x+\frac{t}{2}\Big) - g(x)\Big| dt.$$

Using Lemma 3.4, we have

$$\Omega_{p(\cdot),\lambda(\cdot)}(g,h) \le \sup_{\delta\le h} C(p(\cdot))\delta\ \|g'\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} = C(p(\cdot))h\ \|g'\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}.$$

Hence, taking into account the definition $K(f,t)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}$, by choosing g suitably, we obtain

$$\begin{aligned}\Omega_{p(\cdot),\lambda(\cdot)}(f,h) &\le \Omega_{p(\cdot),\lambda(\cdot)}(f-g,h) + \Omega_{p(\cdot),\lambda(\cdot)}(g,h)\\ &\le C(p(\cdot))\left(\|f-g\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} + h\|g'\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}\right)\\ &\le 2C(p(\cdot))K(f,h)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}\end{aligned}$$

for any $f \in \mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)$.

In order to prove the converse inequality, we introduce a Steklov-type transform for $f \in \mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)$ and $h > 0$:

$$(f_1)_h(x) = \frac{2}{h}\int_{\frac{h}{2}}^{h} \frac{2}{\delta}\int_{\frac{\delta}{2}}^{\delta} f_t\Big(x+\frac{t}{2}\Big)dtd\delta = \frac{4}{h\delta}\int_{\frac{h}{2}}^{h}\int_{\frac{\delta}{2}}^{\delta} f_t\Big(x+\frac{t}{2}\Big)dtd\delta,$$

where

$$f_v(x) = \frac{1}{v}\int_0^v f(x+w)dw.$$

Then

$$\begin{aligned}
&\|(f_1)_h - f\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \\
&\quad = \sup_{\substack{x\in I_0\\ 0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}} \left\| \Big((f_1)_h - f\Big)\chi_{\widetilde{I}(x,r)}\right\|_{L^{p(\cdot)}(I_0)} \\
&\quad = \sup_{\substack{x\in I_0\\ 0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}} \left\| \left\{\Big(\frac{2}{h}\int_{\frac{h}{2}}^{h} \frac{2}{\delta}\int_{\frac{\delta}{2}}^{\delta} \Big(f_t\Big(\cdot+\frac{t}{2}\Big)-f\Big)\,dt d\delta\right\}\chi_{\widetilde{I}(x,r)}\right\|_{L^{p(\cdot)}(I_0)}.
\end{aligned}$$

Now, by Minkowski's integral inequality for variable exponent Lebesgue space, we get

$$\begin{aligned}
&\|(f_1)_h - f\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \\
&\quad \le \sup_{\substack{x\in I_0\\ 0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}} \frac{2}{h}\int_{\frac{h}{2}}^{h} \left\| \Big(\frac{2}{\delta}\int_{\frac{\delta}{2}}^{\delta} \Big(f_t\Big(\cdot+\frac{t}{2}\Big)-f\Big)\,dt\Big)\chi_{\widetilde{I}(x,r)}\right\|_{L^{p(\cdot)}(I_0)} d\delta \\
&\quad \le 2 \sup_{\substack{x\in I_0\\ 0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}} \frac{2}{h}\int_{\frac{h}{2}}^{h} \left\| \Big\{\frac{1}{\delta}\int_0^{\delta} \Big(f_t\Big(\cdot+\frac{t}{2}\Big)-f(x)\Big)\,dt\Big\}\chi_{\widetilde{I}(x,r)}\right\|_{L^{p(\cdot)}(I_0)} d\delta \\
&\quad = 2 \sup_{\frac{h}{2}\le\delta\le h} \left\| \frac{1}{\delta}\int_0^{\delta} \Big\{f_t\Big(\cdot+\frac{t}{2}\Big)-f\Big\}dt\right\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}.
\end{aligned}$$

By the triangle inequality,

$$\begin{aligned}
\|(f_1)_h - f\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} &\le 2 \sup_{0\le\delta\le h} \left\| \frac{1}{\delta}\int_0^{\delta} \left|f_t\left(\cdot+\frac{t}{2}\right)-f\right| dt\right\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \\
&= 2 \sup_{0\le\delta\le h} \|\sigma_\delta(f)\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \\
&= 2\,\Omega_{p(\cdot),\lambda(\cdot)}(f,h). \qquad (3.16)
\end{aligned}$$

Meanwhile, by differentiating $(f_1)_h(x)$ in x, we have

$$(f_1)_h^{'}(x) = \frac{2}{h}\int_{\frac{h}{2}}^{h} \frac{2}{\delta}\int_{\frac{\delta}{2}}^{\delta} \frac{f(x+t)-f(x)}{t}dt d\delta.$$

Therefore

$$\|(f_1)_h'\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} = \sup_{\substack{x\in I_0\\ 0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}} \left\| \left((f_1)_h'\right)\chi_{\widetilde{I}(x,r)} \right\|_{L^{p(\cdot)}(I_0)}$$
$$= \sup_{\substack{x\in I_0\\ 0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}} \left\| \left(\frac{2}{h}\int_{\frac{h}{2}}^{h} \frac{2}{\delta}\int_{\frac{\delta}{2}}^{\delta} \frac{1}{t}\,(f(\cdot+t)-f)\,dt\,d\delta\right)\chi_{\widetilde{I}(x,r)} \right\|_{L^{p(\cdot)}(I_0)}.$$

Now, by Minkowski's integral inequality once again,

$$\|(f_1)_h'\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \le \sup_{\substack{x\in I_0\\ 0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}} \frac{2}{h}$$
$$\times \int_{\frac{h}{2}}^{h} \left\| \left(\frac{2}{\delta}\int_{\frac{\delta}{2}}^{\delta} \frac{f(\cdot+t)-f}{t}dt\right) \chi_{\widetilde{I}(x,r)} \right\|_{L^{p(\cdot)}(I_0)} d\delta$$

From the definition of $\Omega_{p(\cdot),\lambda(\cdot)}(f,h)$, we have

$$\|(f_1)_h'\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \le 8h^{-1} \sup_{\frac{h}{2}\le\delta\le h}$$
$$\times \sup_{\substack{x\in I_0\\ 0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}} \left\| \left(\frac{1}{\delta}\int_0^{\delta} (f(\cdot+t)-f)\,dt\right) \chi_{\widetilde{I}(x,r)} \right\|_{L^{p(\cdot)}(I_0)}$$
$$\le 8h^{-1} \sup_{0\le\delta\le h} \|\sigma_\delta(f)\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} = 8h^{-1}\Omega_{p(\cdot),\lambda(\cdot)}(f,h). \tag{3.17}$$

As a result, from (3.16) and (3.17), we deduce

$$K(f,t)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \le \|f-(f_1)_h\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}$$
$$+\, h\|(f_1)_h'\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \le C\,\Omega_{p(\cdot),\lambda(\cdot)}(f,h).$$

Thus, we obtain the reverse inequality. □

4 Jackson operators

To prove the direct theorem, we need some properties of Jackson operator. The Jackson kernel of order n is defined by:

$$J_n(t) = \frac{1}{2n(2n^2+1)} \sin^4\frac{nt}{2}\operatorname{cosec}^4\frac{t}{2}, \qquad n = 1, 2, 3, \dots .$$

The kernel J_n satisfies

$$\int_{-\pi}^{\pi} J_n(t)dt = \pi \tag{4.1}$$

and

$$\int_0^{\pi} tJ_n(t)dt \leq \frac{5\pi}{2n} \tag{4.2}$$

for each $n \in \mathbb{N}^+$; see [9, Page 144].

Let $n \in \mathbb{N}$. We consider the Jackson operator D_n defined by:

$$D_n(f)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)J_n(t)dt$$

due to the first property of Jackson kernels, we have

$$f(x) - D_n(f)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} [f(x) - f(x+t)]J_n(t)dt.$$

Lemma 4.1 *For all* $f \in W^{p(\cdot),\lambda(\cdot)}(I_0)$,

$$\|f - D_n(f)\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \leq \frac{C(p(\cdot))}{n} \|f^{'}\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}.$$

Proof Since $f \in W^{p(\cdot),\lambda(\cdot)}(I_0)$, for $x, t \in I_0$, we have

$$f(x+t) - f(x) = \int_x^{x+t} f^{'}(u)du.$$

Thus, we calculate

$$\begin{aligned}
&\|f - D_n(f)\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \\
&\quad = \sup_{\substack{x\in I_0 \\ 0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}} \left\| \Big(f - D_n(f)\Big)\chi_{\widetilde{I}(x,r)} \right\|_{L^{p(\cdot)}(I_0)} \\
&\quad = \sup_{\substack{x\in I_0 \\ 0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}} \left\| \Big(\frac{1}{\pi}\int_{-\pi}^{\pi} \Big[f - f(\cdot+t)\Big]\, J_n(t)dt\Big)\chi_{\widetilde{I}(x,r)} \right\|_{L^{p(\cdot)}(I_0)} \\
&\quad = \sup_{\substack{x\in I_0 \\ 0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}} \left\| \Big\{ \Big(\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{t}\int_{\cdot}^{\cdot+t} f^{'}(u)du\Big)\; tJ_n(t)dt\Big\}\chi_{\widetilde{I}(x,r)} \right\|_{L^{p(\cdot)}(I_0)} \\
&\quad = \sup_{\substack{x\in I_0 \\ 0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}} \left\| \Big(\frac{1}{\pi}\int_{-\pi}^{\pi} s_t(f^{'})(\cdot)tJ_n(t)\, dt\Big)\chi_{\widetilde{I}(x,r)} \right\|_{L^{p(\cdot)}(I_0)}.
\end{aligned}$$

Now, by Lemma 3.3 and (4.2), we obtain

$$\|f - D_n(f)\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \leq \sup_{\substack{x\in I_0 \\ 0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}} \frac{1}{\pi} \int_{-\pi}^{\pi} \left\| s_t(f') \, \chi_{\widetilde{I}(x,r)} \right\|_{L^{p(\cdot)}(I_0)} t J_n(t) \, dt$$

$$\leq \frac{5}{2n} \sup_{-\pi\leq t\leq\pi} \left(\sup_{\substack{x\in I_0 \\ 0<r<2\pi}} r^{-\frac{\lambda(x)}{p(x)}} \left\| s_t(f') \, \chi_{\widetilde{I}(x,r)} \right\|_{L^{p(\cdot)}(I_0)} \right).$$

Now using Theorem 1.2, we have

$$\|f - D_n(f)\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \leq \frac{5}{2n} \sup_{t\in I_0} \|s_t(f')\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \leq \frac{C(p(\cdot))}{n} \|f'\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)},$$

as was to be shown. □

5 Bernstein inequality for variable exponent Morrey spaces

Denote by $\mathcal{P}_n$ the set of trigonometric polynomials having degree not exceeding n. For $f \in \mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)$, $p(\cdot) \in L^\infty(I_0)$ and $\lambda(\cdot) \in L^\infty(I_0)$ satisfying conditions in Theorem 1.2, we define

$$E_n(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} = \inf\{\|f - T_n\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} : T_n \in \mathcal{P}_n\},$$

which is called the minimal error of approximation of f in the class $\mathcal{P}_n$.

Thanks to Lemma 4.1 the following estimate holds:

Lemma 5.1 *Let the exponents* $p(\cdot)$ *and* $\lambda(\cdot)$ *satisfy* $0 \leq \lambda_- \leq \lambda_+ < 1$, (1.2) *and* (1.4). *Then*

$$E_n(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \leq \frac{C}{n} \|f'\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}, \quad n \in \mathbb{N}^+$$

for all $f \in W^{p(\cdot),\lambda(\cdot)}(I_0)$ *with the constant* $C = C(p(\cdot))$ *independent of* f.

To prove the inverse theorem in Morrey spaces with variable exponents, we need a Bernstein type inequality in this space. To this end, we present the following lemma:

Lemma 5.2 *Let* $\lambda(\cdot)$ *be a measurable function on* I_0 *such that* $0 \leq \lambda_- \leq \lambda_+ < 1$, *and let* $p(\cdot)$ *satisfy conditions* (1.2) *and* (1.4). *Then for every trigonometric polynomial* T_n *in* $\mathcal{P}_n$ *and* $k \in \mathbb{N}^+$

$$\|T_n'\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \leq An\|T_n\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}, \quad n \in \mathbb{N}^+,$$

where

$$A = \sup_{t\in[0,2\pi]} \sup_{f\in\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0), \|f\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}=1} \|f(\cdot + t)\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}$$

is independent of n.

Proof If follows from [5, p. 99] that

$$T_n^{'}(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} T_n(x+t)F_{n-1}(t)\, n \sin nt dt \quad (x \in I_0),$$

where $F_n(t)$ is Fejer's kernel of order n. Then using Lemma 3.3 and

$$\int_{-\pi}^{\pi} F_{n-1}(t)\, dt = 2\pi,$$

we have

$$\begin{aligned}\|T_n^{'}\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} &\leq \frac{1}{\pi}\int_{-\pi}^{\pi} \|T_n(\cdot+t)\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} F_{n-1}(t)\, n|\sin nt|\, dt,\\ &\leq A\, n\|T_n\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \frac{1}{2\pi}\int_{-\pi}^{\pi} F_{n-1}(t)dt,\\ &= A\, n\, \|T_n\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}\end{aligned}$$

and the proof is complete. □

6 Direct and inverse theorems

Now we shall present the direct and inverse theorems in the Morrey space $\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)$ with variable exponents as follows:

Theorem 6.1 (Direct theorem) *Let* $n \in \mathbb{N}^+$. *Let* $p(\cdot)$ *and* $\lambda(\cdot)$ *be measurable functions on* I_0. *Assume that* $\lambda(\cdot)$ *satisfies condition* $0 \leq \lambda_- \leq \lambda_+ < 1$ *and that* $p(\cdot)$ *satisfies conditions* (1.2) *and* (1.4). *Then*

$$E_n(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \leq C\, \Omega_{p(\cdot),\lambda(\cdot)}\left(f, \frac{1}{n}\right)$$

for all $f \in \mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)$ *with the constant* $C > 0$ *independent of* f *and* n.

Proof Let $g \in W^{p(\cdot),\lambda(\cdot)}(I_0)$ be arbitrary. By Lemma 5.1 we have

$$\begin{aligned}E_n(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} &\leq E_n(f-g)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} + E_n(g)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}\\ &\leq \|f-g\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} + \frac{C}{n}\|g'\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}.\end{aligned}$$

Since this inequality holds for every $g \in W^{p(\cdot),\lambda(\cdot)}(I_0)$ thanks to the definition of the K-functional and by Lemma 3.5, we get

$$E_n(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \leq CK\left(f, \frac{1}{n}\right)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \leq C\Omega_{p(\cdot),\lambda(\cdot)}\left(f, \frac{1}{n}\right).$$

Thus, the proof of Theorem 6.1 is complete. □

Theorem 6.2 *(Inverse theorem) Let* $f \in \mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)$, *and* $n \in \mathbb{N}$. *Suppose that* $\lambda(\cdot)$ *is an exponent satisfying* $0 \le \lambda_- \le \lambda_+ < 1$, *and that the exponenent* $p(\cdot)$ *satisfies conditions* (1.2) *and* (1.4). *Then*

$$\Omega_{p(\cdot),\lambda(\cdot)}\left(f, \frac{1}{n}\right) \le \frac{4C}{n} \sum_{m=0}^{n} E_m(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}$$

with the constant $C > 0$ *independent of* f *and* n.

Proof Let $T_n = D_n(f) \in \mathcal{P}_n$ be the polynomial of the best approximation to f in $\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)$. For any integer $j = 1, 2, \ldots,$

$$\begin{aligned} K\left(f, \frac{1}{n}\right)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} &= \inf_{g \in W^{p(\cdot),\lambda(\cdot)}(I_0)} \left\{ \|f - g\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} + \frac{1}{n}\|g'\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \right\} \\ &\le \|f - T_{2^{j+1}}\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} + \frac{1}{n}\|T'_{2^{j+1}}\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \end{aligned}$$

from the definition of $K(f, n^{-1})$. Using Lemma 5.2, we get

$$\begin{aligned} \|T_{2^{j+1}}{}'\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} &\le \|T_1{}' - T_0{}'\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} + \sum_{i=0}^{j} \|T_{2^{j+1}}{}' - T_{2^i}{}'\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \\ &\le C\left\{ \|T_1 - T_0\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} + \sum_{i=0}^{j} 2^i \|T_{2^{i+1}} - T_{2^i}\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \right\} \\ &\le C E_1(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} + C E_0(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \\ &\quad + \sum_{i=0}^{j} 2^i \left\{ E_{2^{i+1}}(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} + E_{2^i}(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \right\} \\ &\le C\left\{ E_0(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} + \sum_{i=0}^{j} 2^i E_{2^i}(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \right\} \\ &= C\left\{ E_0(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} + 2E_1(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \right\} \\ &\quad + C\sum_{i=1}^{j} 2^i E_{2^i}(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}. \end{aligned}$$

Since

$$2^i E_{2^i}(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \le C \sum_{m=2^{i-1}+1}^{2^i} E_m(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} \tag{6.1}$$

for $i \geq 1$, we have

$$\begin{aligned}\|T_{2^{j+1}}{}'\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} &\leq C\Bigg\{E_0(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}\\ &\quad + E_1(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} + \sum_{m=2}^{2^j} E_m(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}\Bigg\}\\ &\leq C\left\{E_0(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} + \sum_{m=1}^{2^j} E_m(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}\right\}.\end{aligned}$$

Selecting $j \in \mathbb{Z}$ such that $2^j \leq n < 2^{j+1}$, from (6.1) we get

$$\begin{aligned}E_{2^{j+1}}(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} &= \frac{2^{(j+1)}E_{2^{j+1}}(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}}{2^{(j+1)}}\\ &\leq \frac{2^{(j+1)}}{n}E_{2^{j+1}}(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}\\ &\leq \frac{4}{n}\sum_{m=2^{j-1}+1}^{2^j} E_m(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}.\end{aligned}$$

Now from Lemma 3.5, we deduce that

$$\begin{aligned}\Omega_{p(\cdot),\lambda(\cdot)}\left(f,\frac{1}{n}\right) &\leq CK\left(f,\frac{1}{n}\right)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}\\ &\leq CE_{2^{j+1}}(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} + \frac{C}{n}\|T'_{2^{j+1}}\|_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}.\end{aligned}$$

Finally, we calculate

$$\begin{aligned}\Omega_{p(\cdot),\lambda(\cdot)}\left(f,\frac{1}{n}\right) &\leq \frac{C}{n}\sum_{m=2^{j-1}+1}^{2^j} E_m(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}\\ &\quad + \frac{C}{n}\left\{E_0(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} + \sum_{m=1}^{2^j} E_m(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}\right\}\\ &\leq \frac{C}{n}\left\{E_0(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)} + \sum_{m=1}^{n} E_m(f)_{\mathcal{M}^{p(\cdot),\lambda(\cdot)}(I_0)}\right\}.\end{aligned}$$

Thus, the proof of Theorem 6.2 is complete. □

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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