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The Fourier transforms for the spatially homogeneous Boltzmann equation and Landau equation

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Abstract In this paper, we study the Fourier transforms for two equations arising in the kinetic theory. The first equation is the spatially homogeneous Boltzmann equation. The Fourier transform of the spatially homogeneous Boltzmann equation has been first addressed by Bobylev (Sov Sci Rev C Math Phys 7:111–233, 1988) in the Maxwellian case. Alexandre et al. (Arch Ration Mech Anal 152(4):327–355, 2000) investigated the Fourier transform of the gain operator for the Boltzmann operator in the cut-off case. Recently, the Fourier transform of the Boltzmann equation is extended to hard or soft potential with cut-off by Kirsch and Rjasanow (J Stat Phys 129:483–492, 2007). We shall first establish the relation between the results in Alexandre et al. (2000) and Kirsch and Rjasanow (2007) for the Fourier transform of the Boltzmann operator in the cut-off case. Then we give the Fourier transform of the spatially homogeneous Boltzmann equation in the non cut-off case. It is shown that our results cover previous works (Bobylev 1988; Kirsch and Rjasanow 2007). The second equation is the spatially homogeneous Landau equation, which can be obtained as a limit of the Boltzmann equation when grazing collisions prevail. Following the method in Kirsch and Rjasanow (2007), we can also derive the Fourier transform for Landau equation.

Keywords Fourier transform · Boltzmann equation · Landau equation

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1 Introduction

This paper is concerned with the Fourier transforms of the spatially homogeneous Boltzmann equation and Landau equation. Before we state our main results in more details, let us introduce the problem in a precise way.

The spatially homogeneous Boltzmann equation describes the evolution (in time) of a rarefied gas, in which the velocity distribution of particles is assumed to be independent of the position x, it reads

$$\frac{\partial f}{\partial t} = Q_B(f, f), \quad v \in R^3, \quad t \geqslant 0,$$

where the unknown nonnegative function $f(t, v)$ stands for the density distributions of particles with velocity $v \in R^3$ at time t. $Q_B(f, f)$ is a quadratic non-local operator describing the collisions within the gas, which is defined as follows:

$$Q_B(f, f) = \iint_{R^3 \times S^2} B(|v - v_*|, \cos\theta)(f' f'_* - f f_*) d\sigma dv_*.$$

Here we have used the shorthands $f' = f(v')$, $f_* = f(v_*)$, and $f'_* = f(v'_*)$, where

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma$$

are the post-collisional velocities of particles which have velocities v and v_* before collision. In particular, they satisfy the conservations of mass, momentum and energy,

$$\begin{aligned} v' + v'_* &= v + v_*, \\ v'^2 + v'^2_* &= v^2 + v^2_*. \end{aligned}$$

$\theta \in [0, \pi]$ is the deviation angle between $v' - v'_*$ and $v - v_*$, and the collision kernel $B(|v - v_*|, \cos\theta)$ is given by physics and is related to the microscopic interactions between particles. In this paper, we shall make two different assumptions on the collision kernel B:

A1 $B(|v - v_*|, \cos\theta) = C_\lambda |v - v_*|^\lambda$ with $-3 < \lambda \leq 1$ and constant C_λ.
A2 $B(|v - v_*|, \cos\theta) = |v - v_*|^\lambda b(\cos\theta)$ with $-3 < \lambda \leq 1$, and where $b(\cos\theta)$ has a singularity of the form

$$b(\cos\theta) \sim \theta^{-(2+\nu)} \quad \text{as} \quad \theta \to 0, \quad 0 < \nu < 2.$$

In our first assumption, the collision kernel B is called variable hard spheres. Under this assumption, the collision operator $Q_B(f, f)$ can be split into the gain operator $Q_B^+(f, f)$ and the loss operator $Q_B^-(f, f)$ as follows:

$$Q_B(f, f) = Q_B^+(f, f) - Q_B^-(f, f),$$

where

$$Q_B^+(f, f) = C_\lambda \iint_{R^3 \times S^2} |v - v_*|^\lambda f' f'_* d\sigma dv_*,$$

and

$$Q_B^-(f, f) = C_\lambda \iint_{R^3 \times S^2} |v - v_*|^\lambda f f_* d\sigma dv_* = 4\pi C_\lambda f \cdot f * |v|^\lambda.$$

Here $*$ denotes the convolution. It is obvious that the Assumption **A1** covers the most important case (hard sphere collision kernel, $B(|v - v_*|, \cos\theta) = |v - v_*|$). We usually call hard potentials when $\lambda > 0$, Maxwell molecules when $\lambda = 0$, soft potentials when $\lambda < 0$,

In the second Assumption **A2**, the splitting of $Q_B(f, f)$ is impossible because of the singularity of $b(\cos\theta)$ in $\theta = 0$. The Assumption **A2** covers collision kernels deriving from interaction potentials behaving like inverse-power laws. More precisely, for an interaction potential $V(r)$=cst$\frac{1}{r^s}$, the kernel B satisfies the second Assumption **A2** with $\lambda = \frac{s-5}{s-1}$ and $\nu = \frac{2}{s-1}$.

The spatially homogeneous Landau equation [11] (also called Fokker-Planck-Landau) is another common model in kinetic theory. In the case of long- distance interactions, collisions occur mostly for very small θ. When all collisions become concentrated on $\theta = 0$, one obtains the spatially homogeneous Landau equation by the so-called grazing collision limit asymptotic (see for instance ([3,5,13,25]). It reads

$$\frac{\partial f}{\partial t} = Q_L(f, f),$$

where $f(t, v)$ is the density of particles which has the velocity v at time t. The Landau collision operator $Q_L(f, f)$ on the right-hand represents the effect of the (grazing) collisions between particles. It is given by the formula:

$$Q_L(f, f) = \frac{\partial}{\partial v_i}\left(\int_{R^3} a_{ij}(v - v_*)\left(\frac{\partial f}{\partial v_j}(v) f(v_*) - \frac{\partial f}{\partial v_j}(v_*) f(v)\right) dv_*\right),$$

where $a_{ij}(z) = |z|^{2+\gamma} P(z)$, $P(z)$ is the orthogonal projection on to $z^\perp$, i.e.,

$$P_{ij}(z) = \left(\delta_{ij} - \frac{z_i z_j}{|z|^2}\right).$$

To our purpose, we also give a weak form of the Landau collision operator $Q_L(f, f)$, which shall be used in the sequel. Let $\varphi = \varphi(v)$ be a regular test function, multiply the operator $Q_L(f, f)$ by $\varphi(v)$ and integrate over R^3, thanks to the change of variables with unit Jacobian,

$$(v, v_*) \to (v_*, v),$$

we have,

$$\int_{R^3} Q_L(f, f)(v)\varphi(v)dv = \frac{1}{2}\iint_{R^3\times R^3} f(v)f(v_*)a_{ij}(v-v_*)(\partial_{ij}\varphi(v) + \partial_{ij}\varphi(v_*))dvdv_* + \iint_{R^3\times R^3} f(v)f(v_*)b_i(v-v_*)(\partial_i\varphi(v) - \partial_i\varphi(v_*))dvdv_*,$$

where $b_i = \partial_j a_{ij}(z)$.

Up to our knowledge, there are a lot of various results about the Boltzmann equation and Landau equation in theories and numerics. However, in this paper, we only focus our attention on the results concerning the Fourier transform technique. The Fourier transform for the homogeneous Boltzmann equation has first been discovered by Bobylev [6] in the Maxwellian case. Alexandre et al. [1] studied the Fourier transform of the gain operator for arbitrary potential. In Kirsch and Rjasanow [16] found the Fourier transform of the Boltzmann equation for hard and soft potential with cut off by a technical method. On the other hand, as far as we know, there are no existing results of the Fourier transform for the homogeneous Landau equation.

The Fourier transform techniques for the Boltzmann equation have proven to be useful to obtain theoretical and numerical results. We refer the reader to [10] for a survey on the matter. Now we state the main works related to the Fourier transform technique for the Boltzmann equation.

For theoretical studies, Tanka [23] proved the existence and uniqueness of the measure valued solution with finite energy in the Maxwellian case, this result was simplified and generalized in [22,24]. Recently, measure valued solution with infinite energy was constructed in [8] for Maxwellian potential, and the smoothing effect was studied by Morimoto and Yang [17]. All these results listed above rely on the Fourier transform of the Boltzmann equation. The Fourier distance also has been successfully used to study the long time behavior for the solution in the Maxwellian case, see [9,12]. As for the numerics, a direct of use Fourier transformed Boltzmann equation was given in [7]. A Fourier Galerkin spectral method was introduced in [18] and further developments of this approach were given in [19–21].

Our goal in this work is to complete the theory of the Fourier transform for the spatially homogeneous Boltzmann equation and present the Fourier transform for the spatially homogeneous Landau equation.

The rest of this paper is organized as follows. In Sect. 2, we first collect previous works about Fourier transform on the Boltzmann equation, then we present our main results in this paper.

In Sect. 3, for the Fourier transform of the Boltzmann operator under the Assumption **A1** (cut-off case), since the Fourier transform of the gain operator has been studied in [1,16] with different methods, so that we first prove the two expressions in previous works are equal, then we give a new representation for the Fourier transform of the loss operator. The approach is elementary and based on change of variables. However, one should be more careful since the computations are complicated. We next turn to study the Fourier transform to the Boltzmann equation under the Assumption **A2** (non cut-off case). The method developed in [16] allows us to deal with such case, the main difficulty here is how to compute the kernel (3.5) in Sect. 3. The particular structure of the kernel (3.5) reminds us the Taylor expansion method, which has been used in [4,13,25] to prove the existence of weak solutions. Together with the Bobylev's equality (3.6), we arrive at our goal. It should be mentioned that our result covers previous results [6,16] by direct computation, thus our result can be considered as a general version for different kernels (including cut-off case and non cut-off case).

Finally, Sect. 4 is devoted to the Fourier transform of the spatially homogeneous Landau equation. We follow the spirit in [16], the main idea used here is based on the weak form of Landau equation with the shifted test function.

2 Preliminaries and main results

In this section, we shall recall previous works concerning transform of the Boltzmann collision operator in [1,6,16] and give the main results of this paper. We denote

$$\hat{f}(\xi) = \int_{R^3} f(v)e^{-i(v,\xi)}dv$$

the Fourier transform of f. The inverse Fourier transform of $\hat{f}(\xi)$ is

$$f(v) = F^{-1}(\hat{f}(\xi)) = \frac{1}{(2\pi)^3}\int_{R^3} \hat{f}(\xi)e^{i(v,\xi)}d\xi.$$

Let us first present a lemma which will be used systematically in our paper.

Lemma 2.1 *For any $\xi_* \in R^3$, it holds that:*

$$\int_{S^2} e^{-i(re\cdot\xi_*)}de = 4\pi\frac{\sin r|\xi_*|}{r|\xi_*|}, \tag{2.1}$$

$$\int_0^{+\infty} r^{\lambda+1}\sin r|\xi_*|dr = -\frac{\Gamma(\lambda+1)\sin(\frac{\pi\lambda}{2})(\lambda+1)}{|\xi_*|^{\lambda+2}}. \tag{2.2}$$

The next three Lemmas are concerned with the Fourier transform of the spatially homogeneous Boltzmann equation. These results can be found in [1,6] and [16].

Lemma 2.2 [6]

(i) *If the kernel B satisfies Assumption **A1** with $\lambda = 0$, then the Fourier transform of the spatially homogeneous Boltzmann equation is*

$$\frac{\partial\hat{f}}{\partial t} = \hat{Q}(f,f) = C_0\int_{S^2}(\hat{f}(\xi^-)\hat{f}(\xi^+) - \hat{f}(\xi)\hat{f}(0))d\sigma.$$

(ii) *If the kernel B satisfies Assumption **A2** with* $\lambda = 0$*, then the Fourier transform of the spatially homogeneous Boltzmann equation is*

$$\frac{\partial \hat{f}}{\partial t} = \hat{Q}(f, f) = \int_{S^2} (\hat{f}(\xi^-)\hat{f}(\xi^+) - \hat{f}(\xi)\hat{f}(0)) b\left(\frac{\xi}{|\xi|}\right) \cdot \sigma) d\sigma,$$

where

$$\xi^+ = \frac{\xi + |\xi|\sigma}{2}, \quad \xi^- = \frac{\xi - |\xi|\sigma}{2}.$$

Lemma 2.3 [1] *If the kernel B satisfies Assumption **A1**, then the Fourier transform of the gain operator* $Q_B^+(f, f)$ *is given as follows:*

$$\hat{Q}_B^+(f, f)(\xi) = \frac{1}{(2\pi)^3} \iint_{R^3 \times S^2} \hat{f}(\xi^- + \xi_*)\hat{f}(\xi^+ - \xi_*)\hat{B}\left(|\xi_*|, \frac{\xi}{|\xi|} \cdot \sigma\right) d\xi_* d\sigma.$$

where

$$\hat{B}\left(|\xi_*|, \frac{\xi}{|\xi|} \cdot \sigma\right) = \int_{R^3} B\left(|q|, \frac{\xi}{|\xi|} \cdot \sigma\right) e^{-iq\cdot\xi} dq$$

denotes the Fourier transform of B in the relative velocity variable, and ξ^+, ξ^- *are defined in Lemma* 2.2.

Lemma 2.4 [16] *If the kernel B satisfies Assumption **A1**, then*

(i) *the Fourier transform of the gain operator is*

$$\hat{Q}_B^+(f, f) = \frac{2^{\lambda-1}}{(2\pi)^3} \int_{R^3} \hat{f}\left(\frac{\xi+\eta}{2}\right) \hat{f}\left(\frac{\xi-\eta}{2}\right) T_+(\xi, \eta) d\eta,$$

where

$$T_+(\xi, \eta) = -16\pi^2 C_\lambda \Gamma(\lambda + 1) \sin\left(\frac{\pi\lambda}{2}\right) \frac{||\xi| - |\eta||^{-\lambda-1} - ||\xi| + |\eta||^{-\lambda-1}}{|\xi||\eta|}.$$

(ii) *the Fourier transform of the loss operator is*

$$\hat{Q}_B^-(f, f) = \frac{2^{\lambda-1}}{(2\pi)^3} \int_{R^3} \hat{f}\left(\frac{\xi+\eta}{2}\right) \hat{f}\left(\frac{\xi-\eta}{2}\right) T_-(\xi, \eta) d\eta,$$

where

$$T_-(\xi, \eta) = -16\pi^2 C_\lambda \Gamma(\lambda + 1)(\lambda + 1) \sin\left(\frac{\pi\lambda}{2}\right) \left(\frac{1}{|\xi - \eta|^{\lambda+3}} + \frac{1}{|\xi + \eta|^{\lambda+3}}\right).$$

Remark 2.5 Indeed, Kirsch and Rjasanow [16] gave the Fourier transform for the whole operator $Q_B(f, f)$, it is also easy to check the Fourier transforms of the gain operator and the loss operator by means of their proof.

We are ready to present our main results in this paper. The first result is about the Fourier transform of the Boltzmann operator satisfying Assumption **A1**. In view of Lemmas 2.3 and 2.4, there are two different representations for the Fourier transform of $Q_B^+(f, f)$. It is natural to ask weather these two representations are equal. We can give positive answer to this question. In addition, we can also establish another representation for the Fourier transform of the loss operator $Q_B^-(f, f)$.

Theorem 2.6 *If the kernel B satisfies Assumption **A1**, then*

(i) *two representations of $\hat{Q}_B^+(f, f)$ in Lemmas 2.3 and 2.4 are equal, i.e.,*

$$\hat{Q}_B^+(f, f)(\xi) = \frac{1}{(2\pi)^3} \iint_{R^3\times S^2} \hat{f}(\xi^- + \xi_*)\hat{f}(\xi^+ - \xi_*)\hat{B}\left(|\xi_*|, \frac{\xi}{|\xi|}\cdot\sigma\right) d\xi_* d\sigma$$
$$= \frac{2^{\lambda-1}}{(2\pi)^3} \int_{R^3} \hat{f}\left(\frac{\xi+\eta}{2}\right) \hat{f}\left(\frac{\xi-\eta}{2}\right) T_+(\xi, \eta) d\eta,$$

where $T_+(\xi, \eta)$ is defined in Lemma 2.4.

(ii) *the Fourier transform of $Q_B^-(f, f)$ is given as follows:*

$$\hat{Q}_B^-(f, f) = -\frac{2}{\pi} C_\lambda \Gamma(\lambda+1) \sin\left(\frac{\pi\lambda}{2}\right)(\lambda+1) \int_{R^3} \frac{\hat{f}(\xi-\xi_*)\hat{f}(\xi_*)}{|\xi_*|^{\lambda+3}} d\xi_*$$
$$= \frac{2^{\lambda-1}}{(2\pi)^3} \int_{R^3} \hat{f}\left(\frac{\xi+\eta}{2}\right) \hat{f}\left(\frac{\xi-\eta}{2}\right) T_-(\xi, \eta) d\eta,$$

where $T_-(\xi, \eta)$ is defined in Lemma 2.4.

Actually, in the case of non cut-off, the theory of the Fourier transform of the spatially homogeneous Boltzmann equation is unknown except for the Maxwellian case, see Lemma 2.2. Our next result is concerned with the Fourier transform of the spatially homogeneous Boltzmann equation in the general non cut-off cases.

Theorem 2.7 *If the kernel B satisfying Assumption **A2**, then the Fourier transform of the spatially homogeneous Boltzmann equation is given as follows:*

$$\frac{\partial \hat{f}}{\partial t} = \hat{Q}(f, f) = \frac{2^{\lambda-1}}{(2\pi)^3} \int_{R^3} \hat{f}\left(\frac{\xi+\eta}{2}\right) \hat{f}\left(\frac{\xi-\eta}{2}\right) T_B(\xi, \eta) d\eta.$$

where

$$T_B(\xi, \eta) = -4\pi\Gamma(\lambda+1)\sin\left(\frac{\pi\lambda}{2}\right)(\lambda+1)\int_{S^2} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\left(\frac{1}{|\eta-|\xi|\sigma|^{\lambda+3}}\right.$$
$$\left. + \frac{1}{|\eta+|\xi|\sigma|^{\lambda+3}} - \frac{1}{|\xi-\eta|^{\lambda+3}} - \frac{1}{|\xi+\eta|^{\lambda+3}}\right) d\sigma.$$

Corollary 2.8 *If the kernel B satisfying Assumption* ***A2*** *with* $\lambda = 0$, *our result can be reduced to the Bobylev's classical result in Lemma 2.2, i.e.,*

$$\frac{\partial \hat{f}}{\partial t} = \hat{Q}(f, f) = \int_{S^2} \left(\hat{f}\left(\frac{\xi + |\xi|\sigma}{2}\right) \hat{f}\left(\frac{\xi - |\xi|\sigma}{2}\right) - \hat{f}(\xi)\hat{f}(0) \right) b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) d\sigma.$$

Remark 2.9 Indeed, from the proof in Sect. 3, Theorem 2.7 holds in more general cut-off cases when $B(|v - v_*|, \cos\theta) = |v - v_*|^\lambda b(\cos\theta)$ with $\int_{S^2} b(\cos\theta) d\sigma < \infty$. It also covers the result in [16] if we set $b(\cos\theta) = 1$.

Remark 2.10 Theorem 2.7 shows that the Fourier transform does not reduce multiplicity of the integration in the Boltzmann collision operator for general collision kernel (from five to five) except for two special cases: (i) Maxwell molecules, i.e., $\lambda = 0$, see Lemma 2.2. (ii) the angular part of the collision kernel is a constant, i.e., $b(\cos\theta) = C$, see Lemma 2.4.

We last state the result on the Fourier transform for the spatially homogeneous Landau equation.

Theorem 2.11 *The Fourier transform of the spatially homogeneous Landau equation is given as follows:*

$$\frac{\partial \hat{f}}{\partial t} = \hat{Q}_L(f, f) = \frac{1}{(2\pi)^3} \int_{R^3} \hat{f}\left(\frac{\xi + \eta}{2}\right) \hat{f}\left(\frac{\xi - \eta}{2}\right) \cdot T_L(\xi, \eta) d\eta,$$

where

$$T_L(\xi, \eta) = -\frac{1}{2} \int_{R^3} a_{ij}(2y) e^{-i(y,\eta)} \xi_i \xi_j \left(e^{i(y,\xi)} + e^{-i(y,\xi)} \right) dy + i \int_{R^3} b_i(2y) e^{-i(y,\eta)} \xi_i \left[e^{i(y,\xi)} - e^{-i(y,\xi)} \right] dy.$$

Remark 2.12 It is believed that the kernel $T_L(\xi, \eta)$ can be computed explicitly, but the calculation is more complicate, so that we shall not to do so.

3 Fourier transform for the spatially homogeneous Boltzmann equation

In this section, we study the Fourier transform for the spatially homogeneous Boltzmann equation under Assumption **A1**. We prove Theorem 2.6 by elementary calculation and change of variables. However, the computations in the proof are complicate.

Proof of Theorem 2.6 We start by proving the first part of this Theorem, i.e., the Fourier transform of the gain operator.

When B takes the form $B = C_\lambda |v - v_*|^\lambda$, $\hat{B}(|\xi_*|, \frac{\xi}{|\xi|} \cdot \sigma)$ can be computed explicitly as follows, passing to the spherical coordinates and using (2.1) and (2.2), we find

$$\begin{aligned}\hat{B}(|\xi_*|) &= C_\lambda \int_0^{+\infty} r^{\lambda+2} dr \int_{S^2} e^{-i(re\cdot\xi_*)} de \\ &= \frac{4\pi C_\lambda}{|\xi_*|} \int_0^{+\infty} r^{\lambda+1} \sin r|\xi_*| dr \\ &= -\frac{4\pi C_\lambda \Gamma(\lambda+1) \sin\left(\frac{\pi\lambda}{2}\right)(\lambda+1)}{|\xi_*|^{\lambda+3}}.\end{aligned}$$

Inserting $\hat{B}(|\xi_*|)$ into $\hat{Q}^+(f, f)$, it yields

$$\begin{aligned}\hat{Q}^+(f, f)(\xi) = &-\frac{4\pi C_\lambda \Gamma(\lambda+1) \sin\left(\frac{\pi\lambda}{2}\right)(\lambda+1)}{(2\pi)^3} \\ &\times \iint_{R^3\times S^2} \hat{f}\left(\xi^- + \xi_*\right) \hat{f}(\xi^+ - \xi_*) \frac{1}{|\xi_*|^{\lambda+3}} d\xi_* d\sigma. \qquad (3.1)\end{aligned}$$

In order to complete the first part of the proof, it remains to compute the integral

$$\iint_{R^3\times S^2} \hat{f}\left(\xi^- + \xi_*\right) \hat{f}(\xi^+ - \xi_*) \frac{1}{|\xi_*|^{\lambda+3}} d\xi_* d\sigma.$$

For each fixed ξ and σ, we perform the change of the variables $\xi_* \to \eta$,

$$\xi_* = \frac{|\xi|\sigma + \eta}{2},$$

whose Jacobian of the transformation is $\frac{1}{8}$. In view of

$$\xi^+ = \frac{\xi + |\xi|\sigma}{2}, \quad \xi^- = \frac{\xi - |\xi|\sigma}{2},$$

it follows that,

$$\xi^- + \xi_* = \frac{\xi + \eta}{2}, \quad \xi^+ - \xi_* = \frac{\xi - \eta}{2}.$$

Applying this change of variables, we obtain

$$\begin{aligned}&\iint_{R^3\times S^2} \hat{f}(\xi^- + \xi_*) \hat{f}(\xi^+ - \xi_*) \frac{1}{|\xi_*|^{\lambda+3}} d\xi_* d\sigma \\ &= \iint_{R^3\times S^2} \hat{f}\left(\frac{\xi+\eta}{2}\right) \hat{f}\left(\frac{\xi-\eta}{2}\right) \frac{1}{|\frac{|\xi|\sigma+\eta}{2}|^{\lambda+3}} \frac{1}{8} d\eta d\sigma \\ &= 2^\lambda \iint_{R^3\times S^2} \hat{f}\left(\frac{\xi+\eta}{2}\right) \hat{f}\left(\frac{\xi-\eta}{2}\right) \frac{1}{|\eta + |\xi|\sigma|^{\lambda+3}} d\sigma d\eta. \qquad (3.2)\end{aligned}$$

For our purpose, we finally need to compute

$$\int_{S^2} \frac{1}{|\eta + |\xi|\sigma|^{\lambda+3}} d\sigma.$$

Let θ be the angle between η and σ, using spherical coordinates system and changing the variable $t = \cos\theta$, we have

$$\begin{aligned}
\int_{S^2} \frac{1}{|\eta + |\xi|\sigma|^{\lambda+3}} d\sigma &= \int_0^{2\pi} d\varphi \int_0^{\pi} \frac{\sin\theta}{\left(|\eta|^2 + |\xi|^2 + 2|\xi||\eta|\cos\theta\right)^{\frac{\lambda+3}{2}}} d\theta \\
&= 2\pi \int_{-1}^{1} \frac{dt}{\left(|\eta|^2 + |\xi|^2 + 2|\xi||\eta|t\right)^{\frac{\lambda+3}{2}}} \\
&= \frac{2\pi}{(2|\xi||\eta|)^{\frac{\lambda+3}{2}}} \int_{-1}^{1} \frac{dt}{\left(t + \frac{|\eta|^2+|\xi|^2}{2|\xi||\eta|}\right)^{\frac{\lambda+3}{2}}} \\
&= \frac{2\pi}{(2|\xi||\eta|)^{\frac{\lambda+3}{2}}} \cdot \left(-\frac{2}{\lambda+1}\right) \left(t + \frac{|\eta|^2 + |\xi|^2}{2|\xi||\eta|}\right)^{-\frac{1}{2}(\lambda+1)} \Big|_{-1}^{1} \\
&= \frac{4\pi}{(\lambda+1)2|\xi||\eta|} \left[||\xi| - |\eta||^{-\lambda-1} - ||\xi| + |\eta||^{-\lambda-1} \right].
\end{aligned}$$

Gathering above equality, (3.1) and (3.2), this completes the first part of Theorem 2.6.

We next turn to the Fourier transform of the loss operator $Q_B^-(f, f)$. Our method is different from the approach in [16], we will work on $Q_B^-(f, f)$ directly. With the help of some properties of the Fourier transform, it is possible to obtain the desired result. Recall that

$$Q_B^-(f, f) = 4\pi C_\lambda f \cdot f * (|v|^\lambda).$$

We perform the Fourier transform for $Q_B^-(f, f)$,

$$\hat{Q}_B^-(f, f) = \frac{4\pi C_\lambda}{(2\pi)^3} \hat{f} * (\widehat{f * |v|^\lambda}).$$

Thanks to the computation for $\hat{B}(|\xi_*|)$ in the first part,

$$\widehat{|v|^\lambda} = -\frac{4\pi\Gamma(\lambda+1)\sin\left(\frac{\pi\lambda}{2}\right)(\lambda+1)}{|\xi|^{\lambda+3}}.$$

It follows that

$$(\widehat{f * |v|^\lambda}) = \hat{f} \cdot \widehat{|v|^\lambda} = -\frac{4\pi \hat{f}(\xi)\Gamma(\lambda+1)\sin\left(\frac{\pi\lambda}{2}\right)(\lambda+1)}{|\xi|^{\lambda+3}}.$$

Therefore, we obtain

$$\hat{f} * (\widehat{f * |v|^{\lambda}}) = -\int_{R^3} \hat{f}(\xi - \xi_*)\hat{f}(\xi_*) \frac{4\pi \Gamma(\lambda+1)\sin\left(\frac{\pi\lambda}{2}\right)(\lambda+1)}{|\xi_*|^{\lambda+3}} d\xi_*.$$

Then, inserting above expression into $\hat{Q}_B^-(f, f)$, we get

$$\hat{Q}_B^-(f, f) = -\frac{2}{\pi} C_\lambda \Gamma(\lambda+1) \sin\left(\frac{\pi\lambda}{2}\right)(\lambda+1) \int_{R^3} \frac{\hat{f}(\xi - \xi_*)\hat{f}(\xi_*)}{|\xi_*|^{\lambda+3}} d\xi_*. \quad (3.3)$$

It remains to show the Fourier transform of the loss operator is equal to the expression in Lemma 2.3. To this end, for fixed ξ, we first do the change of variables $\xi_* = \frac{\xi+\eta}{2}$, then

$$\xi - \xi_* = \frac{\xi - \eta}{2}, \quad d\xi_* = \frac{1}{8} d\eta.$$

It yields that,

$$\int_{R^3} \frac{\hat{f}(\xi - \xi_*)\hat{f}(\xi_*)}{|\xi_*|^{\lambda+3}} d\xi_* = 2^\lambda \int_{R^3} \hat{f}\left(\frac{\xi+\eta}{2}\right) \hat{f}\left(\frac{\xi-\eta}{2}\right) \frac{1}{|\xi+\eta|^{\lambda+3}} d\eta.$$

We also do another change of variables $\xi_* = \frac{\xi-\eta}{2}$ for fixed ξ, then

$$\xi - \xi_* = \frac{\xi + \eta}{2}, \quad d\xi_* = \frac{1}{8} d\eta.$$

One has,

$$\int_{R^3} \frac{\hat{f}(\xi - \xi_*)\hat{f}(\xi_*)}{|\xi_*|^{\lambda+3}} d\xi_* = 2^\lambda \int_{R^3} \hat{f}\left(\frac{\xi+\eta}{2}\right) \hat{f}\left(\frac{\xi-\eta}{2}\right) \frac{1}{|\xi-\eta|^{\lambda+3}} d\eta.$$

Combining above equalities and inserting into (3.3),

$$\begin{aligned}
\hat{Q}_B^-(f, f) &= -\frac{2}{\pi} C_\lambda \Gamma(\lambda+1) \sin\left(\frac{\pi\lambda}{2}\right)(\lambda+1) \left(\frac{1}{2}\int_{R^3} \frac{\hat{f}(\xi - \xi_*)\hat{f}(\xi_*)}{|\xi_*|^{\lambda+3}} d\xi_* \right. \\
&\quad \left. + \frac{1}{2}\int_{R^3} \frac{\hat{f}(\xi - \xi_*)\hat{f}(\xi_*)}{|\xi_*|^{\lambda+3}} d\xi_* \right) \\
&= -\frac{2}{\pi} C_\lambda \Gamma(\lambda+1) \sin\left(\frac{\pi\lambda}{2}\right)(\lambda+1) \left(2^{\lambda-1} \right. \\
&\quad \times \int_{R^3} \hat{f}\left(\frac{\xi+\eta}{2}\right) \hat{f}\left(\frac{\xi-\eta}{2}\right) \frac{1}{|\xi+\eta|^{\lambda+3}} d\eta \\
&\quad \left. + 2^{\lambda-1} \int_{R^3} \hat{f}\left(\frac{\xi+\eta}{2}\right) \hat{f}\left(\frac{\xi-\eta}{2}\right) \frac{1}{|\xi-\eta|^{\lambda+3}} d\eta \right)
\end{aligned}$$

$$= \frac{2^{\lambda-1}}{(2\pi)^3} \int_{R^3} \hat{f}\left(\frac{\xi+\eta}{2}\right) \hat{f}\left(\frac{\xi-\eta}{2}\right) T_-(\xi,\eta) d\eta,$$

where

$$T_-(\xi,\eta) = -16\pi^2 C_\lambda \Gamma(\lambda+1)(\lambda+1)\sin\left(\frac{\pi\lambda}{2}\right)\left(\frac{1}{|\xi-\eta|^{\lambda+3}} + \frac{1}{|\xi+\eta|^{\lambda+3}}\right).$$

This finishes the proof. □

We now derive the Fourier transform for the Boltzmann collision operator under Assumption **A2**.

Proof of Theorem 2.7 It should be mentioned that we can follow the main path in [16] to derive the Fourier transform for the Boltzmann collision operator under Assumption **A2**. The first part of the proof is the same as the one in [16], so that we omit it. We are able to obtain

$$\hat{Q}(f,f)(\xi) = \frac{2^{\lambda-1}}{(2\pi)^3} \int_{R^3} \hat{f}\left(\frac{\xi+\eta}{2}\right) \hat{f}\left(\frac{\xi-\eta}{2}\right) T_B(\xi,\eta) d\eta, \tag{3.4}$$

with the kernel

$$T_B(\xi,\eta) = \int_{R^3} |y|^\lambda e^{i(y,\eta)} \int_{S^2} \Big[e^{-i(|y|\sigma.\xi)} + e^{i(|y|\sigma.\xi)} - e^{-i(y,\xi)} - e^{i(y,\xi)}\Big] b\left(\frac{y}{|y|}\cdot\sigma\right) d\sigma dy. \tag{3.5}$$

We emphasize that the main difference between the cut-off case [16] and non cut-off case is that the singular angular part $b(\frac{y}{|y|}\cdot\sigma)$ appears in the kernel $T_B(\xi,\eta)$, therefore we need first to show that the kernel $T_B(\xi,\eta)$ is well defined for the non cut-off case. In order to cancel the singularity of $b(\frac{y}{|y|}\cdot\sigma)$ in the kernel $T_B(\xi,\eta)$, we expand the term

$$e^{-i(|y|\sigma.\xi)} + e^{i(|y|\sigma.\xi)} - e^{-i(y,\xi)} - e^{i(y,\xi)}$$

to first order by Taylor formula. Such an idea is not new, it has been used to prove the existence of weak solutions for the spatially homogeneous Boltzmann equation in the non cut-off case, see [4,13,25]. Since

$$f(x) - f(x_0) = f'(x_0)(x-x_0) + \frac{f''(\zeta)}{2!}(x-x_0)^2,$$

where ζ lies between x and x_0, it holds

$$e^{-i(|y|\sigma.\xi)} - e^{-i(y,\xi)} = e^{-i(y,\xi)}(y-|y|\sigma,\xi) + C_1(y-|y|\sigma,\xi)^2$$

and

$$e^{i(|y|\sigma.\xi)} - e^{i(y,\xi)} = e^{i(y,\xi)}(y-|y|\sigma,\xi) + C_2(y-|y|\sigma,\xi)^2,$$

where C_1 and C_2 are constants.

Collecting above estimates, we get

$$e^{-i(|y|\sigma.\xi)} + e^{i(|y|\sigma.\xi)} - e^{-i(y,\xi)} - e^{i(y,\xi)} = O((y - |y|\sigma, \xi)).$$

It is expected that the term $(y-|y|\sigma, \xi)$ can help us to cancel the singularity of $b(\frac{y}{|y|}\cdot\sigma)$. To this end, we introduce a local sphere coordinate system $(\frac{y}{|y|}, i, j)$ attached t. Let θ and φ be the azimuthal and longitudinal angles of σ, as a consequence,

$$\sigma = \frac{y}{|y|}\cos\theta + i\sin\theta\cos\varphi + j\sin\theta\sin\varphi.$$

Let α be the azimuthal angle of ξ, then

$$\xi = |\xi|\left(\frac{y}{|y|}\cos\alpha + y^{\perp}\right),$$

where $y^{\perp}$ is orthogonal to y.

We compute the term $(y - |y|\sigma, \xi)$,

$$\begin{aligned}(y - |y|\sigma, \xi) &= \Bigg(y - |y|\left(\frac{y}{|y|}\cos\theta + i\sin\theta\cos\varphi\right.\\ &\quad \left. + j\sin\theta\sin\varphi\right), |\xi|\left(\frac{y}{|y|}\cos\alpha + y^{\perp}\right)\Bigg)\\ &= |\xi||y|(1-\cos\theta)\cos\alpha - |y||\xi|\left(i\sin\theta\cos\varphi + j\sin\theta\sin\varphi, y^{\perp}\right).\end{aligned}$$

We see that the second term above on the right hand side is a linear combination of $\cos\varphi$ and $\sin\varphi$, so that when we integrate over φ, the contributions vanishes, there only remains

$$\int_0^{2\pi}(y - |y|\sigma, \xi)d\varphi = |\xi||y|(1-\cos\theta)\cos\alpha.$$

Hence, it follows that

$$\begin{aligned}&\int_{S^2}\left[e^{-i(|y|\sigma.\xi)} + e^{i(|y|\sigma.\xi)} - e^{-i(y,\xi)} - e^{i(y,\xi)}\right] b\left(\frac{y}{|y|}\cdot\sigma\right)d\sigma\\ &\qquad = O(|\xi||y|\cos\alpha)\int_0^{\pi} b(\cos\theta)(1-\cos\theta)\sin\theta d\theta.\end{aligned}$$

Let us recall the singular order of $b(\cos\theta)$ in Assumption **A2**, hence the term

$$\int_{S^2}\left[e^{-i(|y|e.\xi)} + e^{i(|y|e.\xi)} - e^{-i(y,\xi)} - e^{i(y,\xi)}\right] b\left(\frac{y}{|y|}\cdot\sigma\right)d\sigma$$

is meaningful. This fact implies the kernel $T_B(\xi, \eta)$ is well defined.

In the sequel, we shall compute the kernel explicitly under Assumption **A2**. We shall do the calculation if B is integrable, and apply a limiting procedure to conclude the proof for non cut-off case. Indeed, the kernel

$$T_B(\xi,\eta) = \int_{R^3} |y|^\lambda e^{i(y,\eta)} \int_{S^2} \Big[e^{-i(|y|\sigma.\xi)} + e^{i(|y|\sigma.\xi)} \\ -e^{-i(y,\xi)} - e^{i(y,\xi)} \Big] b\left(\frac{y}{|y|}\cdot\sigma\right) d\sigma dy.$$

is well defined as showed above. We split the kernel into four terms as follows,

$$\begin{aligned} T_B(\xi,\eta) &= \int_{R^3} |y|^\lambda e^{i(y,\eta)} \int_{S^2} \Big[e^{-i(|y|\sigma.\xi)} + e^{i(|y|\sigma.\xi)} - e^{-i(y,\xi)} \\ &\quad - e^{i(y,\xi)} \Big] b\left(\frac{y}{|y|}\cdot\sigma\right) d\sigma dy \\ &= I + II + III + IV, \end{aligned}$$

where

$$\begin{aligned} I &= \int_{R^3} |y|^\lambda e^{i(y,\eta)} \int_{S^2} e^{-i(|y|\sigma.\xi)} b\left(\frac{y}{|y|}\cdot\sigma\right) d\sigma dy, \\ II &= \int_{R^3} |y|^\lambda e^{i(y,\eta)} \int_{S^2} e^{i(|y|\sigma.\xi)} b\left(\frac{y}{|y|}\cdot\sigma\right) d\sigma dy, \\ III &= \int_{R^3} |y|^\lambda e^{i(y,\eta)} \int_{S^2} e^{-i(y,\xi)} b\left(\frac{y}{|y|}\cdot\sigma\right) d\sigma dy, \\ IV &= \int_{R^3} |y|^\lambda e^{i(y,\eta)} \int_{S^2} e^{i(y,\xi)} b\left(\frac{y}{|y|}\cdot\sigma\right) d\sigma dy. \end{aligned}$$

We next shall compute every term separately. A key remark observed by Bobylev is that

$$\int_{S^2} e^{-i(|y|\sigma.\xi)} b\left(\frac{y}{|y|}\cdot\sigma\right) d\sigma = \int_{S^2} e^{-i(|\xi|\sigma.y)} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) d\sigma. \tag{3.6}$$

This equality will play an important role in our proof.

For the first term, using Bobylev's equality (3.6), letting $y = re_y$ and applying (2.1), (2.2), we get

$$\begin{aligned} I &= \int_{R^3} |y|^\lambda e^{i(y,\eta)} \int_{S^2} e^{-i(|\xi|\sigma.y)} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) d\sigma dy \\ &= \int_{S^2} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) d\sigma \int_0^\infty r^{\lambda+2} \int_{S^2} e^{i(re_y,\eta-|\xi|\sigma)} de_y dr \\ &= \int_{S^2} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) d\sigma \int_0^\infty r^{\lambda+2} 4\pi \frac{\sin r|\eta-|\xi|\sigma|}{r|\eta-|\xi|\sigma|} dr \\ &= -4\pi\Gamma(\lambda+1)\sin\left(\frac{\pi\lambda}{2}\right)(\lambda+1) \int_{S^2} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) \frac{1}{|\eta-|\xi|\sigma|^{\lambda+3}} d\sigma. \end{aligned}$$

The second term can be treated similarly,

$$II = -4\pi\Gamma(\lambda+1)\sin\left(\frac{\pi\lambda}{2}\right)(\lambda+1)\int_{S^2} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\frac{1}{|\eta+|\xi|\sigma|^{\lambda+3}}d\sigma.$$

For the third term, let $y = re_y$, and use (2.1), (2.2) and (3.6) again,

$$\begin{aligned} III &= \int_{S^2} b\left(\frac{y}{|y|}\cdot\sigma\right)d\sigma\int_{R^3}|y|^\lambda e^{i(y,\eta)-i(y,\xi)}dy \\ &= \int_{S^2} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)d\sigma\int_{R^3}|y|^\lambda e^{i(re_y,\eta-\xi)}dy \\ &= 4\pi\int_0^\infty r^{\lambda+2}\sin\frac{r|\xi-\eta|}{r|\xi-\eta|}dr\int_{S^2} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)d\sigma \\ &= \frac{4\pi}{|\xi-\eta|}\int_0^\infty r^{\lambda+1}\sin r|\xi-\eta|dr\int_{S^2} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)d\sigma \\ &= -\frac{4\pi\Gamma(\lambda+1)\sin\left(\frac{\pi\lambda}{2}\right)(\lambda+1)}{|\xi-\eta|^{\lambda+3}}\int_{S^2} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)d\sigma. \end{aligned}$$

The forth term can be calculated as the third term,

$$IV = -\frac{4\pi\Gamma(\lambda+1)\sin\left(\frac{\pi\lambda}{2}\right)(\lambda+1)}{|\xi+\eta|^{\lambda+3}}\int_{S^2} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)d\sigma.$$

Collecting the above estimates for I, II, III and IV, we obtain

$$\begin{aligned} T_B(\xi,\eta) &= \int_{R^3}|y|^\lambda e^{i(y,\eta)}\int_{S^2}\left[e^{-i(|y|\sigma.\xi)} + e^{i(|y|\sigma.\xi)} - e^{-i(y,\xi)}\right. \\ &\qquad \left. - e^{i(y,\xi)}\right] b\left(\frac{y}{|y|}\cdot\sigma\right)d\sigma dy \\ &= -4\pi\Gamma(\lambda+1)\sin\left(\frac{\pi\lambda}{2}\right)(\lambda+1)\int_{S^2} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\left(\frac{1}{|\eta-|\xi|e|^{\lambda+3}}\right. \\ &\qquad \left. + \frac{1}{|\eta+|\xi|e|^{\lambda+3}} - \frac{1}{|\xi-\eta|^{\lambda+3}} - \frac{1}{|\xi+\eta|^{\lambda+3}}\right)d\sigma, \end{aligned} \tag{3.7}$$

which concludes the proof of Theorem 2.7. □

Proof of Corollary 2.8 In the case of $\lambda = 0$, we do not let $\lambda = 0$ in (3.7). Instead we compute the kernel (3.5) with $\lambda = 0$, noticing that

$$\int_{R^3} e^{i(y,\xi)}dy = (2\pi)^3\delta(\xi),$$

and applying Bobylev's equality (3.6), we have

$$
\begin{aligned}
&\int_{R^3} e^{i(y,\eta)} \int_{S^2} \left[e^{-i(|y|\sigma.\xi)} + e^{i(|y|\sigma.\xi)} - e^{-i(y,\xi)} - e^{i(y,\xi)} \right] b\left(\frac{y}{|y|} \cdot \sigma \right) d\sigma dy \\
&= \int_{S^2} b(\frac{\xi}{|\xi|} \cdot \sigma) d\sigma \int_{R^3} e^{i(y,\eta-|\xi|\sigma)} dy + \int_{S^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma \right) d\sigma \int_{R^3} e^{i(y,\eta+|\xi|\sigma)} dy \\
&\quad - \int_{S^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma \right) d\sigma \int_{R^3} e^{i(y,\eta-\xi)} dy - \int_{S^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma \right) d\sigma \int_{R^3} e^{i(y,\eta+\xi)} dy \\
&= (2\pi)^3 \left[\int_{S^2} \delta(\eta - |\xi|\sigma) b\left(\frac{\xi}{|\xi|} \cdot \sigma \right) d\sigma + \int_{S^2} \delta(\eta + |\xi|\sigma) b\left(\frac{\xi}{|\xi|} \cdot \sigma \right) d\sigma \right. \\
&\quad \left. - \int_{S^2} \delta(\eta - \xi) b\left(\frac{\xi}{|\xi|} \cdot \sigma \right) d\sigma - \int_{S^2} \delta(\eta + \xi) b\left(\frac{\xi}{|\xi|} \cdot \sigma \right) d\sigma \right] \\
&= (2\pi)^3 \left[\delta(\eta - |\xi|\sigma) + \delta(\eta + |\xi|\sigma) - \delta(\eta - \xi) - \delta(\eta + \xi) \right].
\end{aligned}
$$

Inserting above expression into (3.4), one easily obtains the Bobylev's result in the Maxwellian case. □

4 Fourier transform for the spatially homogeneous Landau equation

This section is devoted to the Fourier transform of the spatially homogeneous Landau equation. It is remarkable the relation between the spatially homogeneous Boltzmann equation without cut-off and the spatially homogeneous Landau equation, see [3,5,13, 25]. Since the approach in [16] does work for the spatially homogeneous Boltzmann equation, it motivates us to apply this method to the Landau equation. In the sequel, we will follow the main steps in [16] to derive the Fourier transform for the spatially homogeneous Landau equation.

Proof of Theorem 2.11 Multiplying the Landau operator by a shifted test function $\varphi(z - v)$ for fixed z, and recalling the weak form of the Landau operator in Sect. 1, we see that

$$
\begin{aligned}
\int_{R^3} Q_L(f, f)(v)\varphi(z - v) dv = \frac{1}{2} \iint_{R^3 \times R^3} & f(v) f(v_*) a_{ij}(v - v_*)(\partial_{ij}\varphi(z - v) \\
& + \partial_{ij}\varphi(z - v_*)) dv dv_* \\
& + \iint_{R^3 \times R^3} f(v) f(v_*) b_i(v - v_*)(\partial_i \varphi(z - v) \\
& - \partial_i \varphi(z - v_*)) dv dv_*.
\end{aligned}
$$

For fixed z, changing the variables $z - v \to v$, $z - v_* \to v_*$, whose Jacobian is 1, we have

$$
\int_{R^3} Q_L(f, f)(v)\varphi(z - v) dv = \frac{1}{2} \iint_{R^3 \times R^3} f(z - v) f(z - v_*) a_{ij}(v - v_*)(\partial_{ij}\varphi(v)
$$

$$+\ \partial_{ij}\varphi(v_*))dvdv_*$$
$$+\ \iint_{R^3\times R^3} f(z-v)f(z-v_*)b_i(v-v_*)(\partial_i\varphi(v)$$
$$-\ \partial_i\varphi(v_*))dvdv_*.$$

Applying the inverse Fourier transform for $f(z-v)$ and $f(z-v_*)$, we have

$$f(z-v) = \frac{1}{(2\pi)^3}\int_{R^3}\hat{f}(\xi_1)e^{i(z-v,\xi_1)}d\xi_1 \text{ and}$$
$$f(z-v_*) = \frac{1}{(2\pi)^3}\int_{R^3}\hat{f}(\xi_2)e^{i(z-v_*,\xi_2)}d\xi_2.$$

Inserting above inverse Fourier transforms into $\int_{R^3} Q_L(f, f)(v)\varphi(z-v)dv$, we get

$$\int_{R^3} Q_L(f, f)\varphi(z-v)dv$$
$$= \frac{1}{(2\pi)^6}\iint_{R^3\times R^3}\hat{f}(\xi_1)\hat{f}(\xi_2)e^{i(z,\xi_1+\xi_2)}d\xi_1 d\xi_2\cdot(T_1(\xi_1,\xi_2)+T_2(\xi_1,\xi_2)),$$

where

$$T_1(\xi_1,\xi_2) = \frac{1}{2}\iint_{R^3\times R^3} a_{ij}(v-v_*)[\partial_{ij}\varphi(v)+\partial_{ij}\varphi(v_*)]e^{-i(v,\xi_1)}e^{-i(v_*,\xi_2)}dvdv_*$$

and

$$T_2(\xi_1,\xi_2) = \iint_{R^3\times R^3} b_i(v-v_*)[\partial_i\varphi(v)-\partial_i\varphi(v_*)]e^{-i(v,\xi_1)}e^{-i(v_*,\xi_2)}dvdv_*.$$

We next study $T_1(\xi_1,\xi_2)$ and $T_2(\xi_1,\xi_2)$. Let $x = \frac{v+v_*}{2}$, $y = \frac{v-v_*}{2}$, then $v = x+y$, $v_* = x-y$ and $dvdv_* = 8dxdy$, we deduce that

$$T_1(\xi_1,\xi_2) = 4\iint_{R^3\times R^3} a_{ij}(2y)[\partial_{ij}\varphi(x+y)$$
$$+\ \partial_{ij}\varphi(x-y)]e^{-i(x+y,\xi_1)}e^{-i(x-y,\xi_2)}dxdy$$
$$= 4\iint_{R^3\times R^3} a_{ij}(2y)[\partial_{ij}\varphi(x+y)$$
$$+\ \partial_{ij}\varphi(x-y)]e^{-i(x,\xi_1+\xi_2)}e^{-i(y,\xi_1-\xi_2)}dxdy$$
$$= 4\int_{R^3} a_{ij}(2y)e^{-i(y,\xi_1-\xi_2)}\int_{R^3}[\partial_{ij}\varphi(x+y)$$
$$+\ \partial_{ij}\varphi(x-y)]e^{-i(x,\xi_1+\xi_2)}dxdy$$

and

$$
\begin{aligned}
T_2(\xi_1,\xi_2) &= \iint_{R^3\times R^3} b_i(v-v_*)[\partial_i\varphi(v)-\partial_i\varphi(v_*)]e^{-i(v,\xi_1)}e^{-i(v_*,\xi_2)}dvdv_* \\
&= 8\iint_{R^3\times R^3} b_i(2y)[\partial_i\varphi(x+y) \\
&\quad - \partial_i\varphi(x-y)]e^{-i(x,\xi_1+\xi_2)}e^{-i(y,\xi_1-\xi_2)}dxdy \\
&= 8\int_{R^3} b_i(2y)e^{-i(y,\xi_1-\xi_2)}\int_{R^3}[\partial_i\varphi(x+y) \\
&\quad - \partial_i\varphi(x-y)]e^{-i(x,\xi_1+\xi_2)}dxdy.
\end{aligned}
$$

We first integrate the variable x for T_1 and T_2. For fixed y, let $x+y=u$, then $dx=du$,

$$
\begin{aligned}
\int_{R^3}\partial_{ij}\varphi(x+y)e^{-i(x,\xi_1+\xi_2)}dx &= \int_{R^3}\partial_{ij}\varphi(u)e^{-i(u-y,\xi_1+\xi_2)}du \\
&= -\,e^{i(y,\xi_1+\xi_2)}\hat{\varphi}(\xi_1+\xi_2)(\xi_1+\xi_2)_i(\xi_1+\xi_2)_j,
\end{aligned}
$$

where ξ_i denotes the i-th component of ξ. Similarly, for fixed y, let $x-y=u$, then $dx=du$,

$$
\begin{aligned}
\int_{R^3}\partial_{ij}\varphi(x-y)e^{-i(x,\xi_1+\xi_2)}dx &= \int_{R^3}\partial_{ij}\varphi(u)e^{-i(u+y,\xi_1+\xi_2)}du \\
&= -\,e^{-i(y,\xi_1+\xi_2)}\hat{\varphi}(\xi_1+\xi_2)(\xi_1+\xi_2)_i(\xi_1+\xi_2)_j.
\end{aligned}
$$

So that $T_1(\xi_1,\xi_2)$ becomes

$$
\begin{aligned}
T_1(\xi_1,\xi_2) &= 4\int_{R^3} a_{ij}(2y)e^{-i(y,\xi_1-\xi_2)}\int_{R^3}[\partial_{ij}\varphi(x+y) \\
&\quad + \partial_{ij}\varphi(x-y)]e^{-i(x,\xi_1+\xi_2)}dxdy \\
&= -4\int_{R^3} a_{ij}(2y)e^{-i(y,\xi_1-\xi_2)}\hat{\varphi}(\xi_1+\xi_2)(\xi_1+\xi_2)_i(\xi_1+\xi_2)_j(e^{i(y,\xi_1+\xi_2)} \\
&\quad + e^{-i(y,\xi_1+\xi_2)})dy.
\end{aligned}
$$

We use the same procedure to deal with $T_2(\xi_1,\xi_2)$,

$$
\begin{aligned}
T_2(\xi_1,\xi_2) = 8i\int_{R^3} & b_i(2y)e^{-i(y,\xi_1-\xi_2)}[e^{i(y,\xi_1+\xi_2)} \\
& -e^{-i(y,\xi_1+\xi_2)}]\hat{\varphi}(\xi_1+\xi_2)(\xi_1+\xi_2)_i dy.
\end{aligned}
$$

At last, we perform the change of variables $\xi_1+\xi_2=\xi$, $\xi_1-\xi_2=\eta$, then

$$
\xi_1=\frac{\xi+\eta}{2},\qquad \xi_1=\frac{\xi+\eta}{2},\qquad d\xi_1 d\xi_2=\frac{1}{8}d\xi d\eta.
$$

It follows that

$$\begin{aligned}
&\int_{R^3} Q(f,f)\varphi(z-v)dv \\
&= \frac{1}{(2\pi)^6}\int_{R^3\times R^3} \hat{f}\left(\frac{\xi+\eta}{2}\right)\hat{f}\left(\frac{\xi-\eta}{2}\right)e^{i(z,\xi)}\hat{\varphi}(\xi)d\xi d\eta\cdot T_L(\xi,\eta) \\
&= \frac{1}{(2\pi)^3}F^{-1}\left(\int_{R^3}\hat{f}\left(\frac{\xi+\eta}{2}\right)\hat{f}\left(\frac{\xi-\eta}{2}\right)\hat{\varphi}(\xi)d\eta\cdot T_L(\xi,\eta)\right),
\end{aligned}$$

where

$$\begin{aligned}
T_L(\xi,\eta) = &-\frac{1}{2}\int_{R^3} a_{ij}(2y)e^{-i(y,\eta)}\xi_i\xi_j(e^{i(y,\xi)}+e^{-i(y,\xi)})dy \\
&+ i\int_{R^3} b_i(2y)e^{-i(y,\eta)}\xi_i[e^{i(y,\xi)}-e^{-i(y,\xi)}]dy.
\end{aligned}$$

Let us consider the following weak form of the spatially homogeneous Landau equation:

$$(f_t,\varphi(z-v)) = \int_{R^3} Q_L(f,f)(v)\varphi(z-v)dv.$$

Applying the Fourier transform to both sides of the above equation, we see that

$$\frac{\partial\hat{f}}{\partial t} = \frac{1}{(2\pi)^3}\int_{R^3}\hat{f}\left(\frac{\xi+\eta}{2}\right)\hat{f}\left(\frac{\xi-\eta}{2}\right)\cdot T_L(\xi,\eta)d\eta.$$

This ends the proof. □

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Compliance with ethical standards

Conflict of interest No potential conflict of interest was reported by the authors.

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